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Nonlinear Diffusion Models for Gravitational Wave Turbulence

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A fourth-order and a second-order nonlinear diffusion models in spectral space are proposed to describe gravitational wave turbulence in the approximation of strongly local interactions. We show analytically that the model equations satisfy the conservation of energy and wave action, and reproduce the power law solutions previously derived from the kinetic equations with a direct cascade of energy and an explosive inverse cascade of wave action. In the latter case, we show numerically by computing the second-order diffusion model that the non-stationary regime exhibits an anomalous scaling which is understood as a self-similar solution of the second kind with a front propagation following the law $k_f \sim (t_* - t)^{3.296}$, with $t < t_*$. These results are relevant to better understand the dynamics of the primordial universe where potent sources of gravitational waves may produce space-time turbulence.

Keywords: Wave turbulence – Diffusion model – Inverse cascade – Cosmology

I. INTRODUCTION

The nonlinear nature of the Einstein equations of general relativity suggests that space-time can be turbulent. Such a turbulence has been studied in the context of spinning black holes [1–4] by using the gravity-fluid correspondence. It is shown numerically that such a system can display a nonlinear parametric instability with transfers reminiscent of an inverse cascade; the precise mechanism is, however, not totally understood. The possibility of having a turbulent cascade within the metric perturbations was already discussed in the past [5] but it is only recently that a rigorous theory has been proposed for the regime of gravitational wave (GW) turbulence [6]. The presence of small nonlinearities has been exploited to derive a four-wave kinetic equation which describes the spectral transfers of energy and wave action. In other words, the theory explains the nonlinear evolution of weak ripples on the Poincaré-Minkowski flat space-time metric. The theory is limited to a 2.5+1 diagonal metric tensor [7], **which means that we only have a two-dimensional spatial dependence with no restriction on the metric components; the time variable is an additional variable (hence, the notation +1). This theory includes only one type (+) of GW (diagonal contribution); the \times GW corresponds to a non-diagonal metric**

perturbation which is excluded by construction. Besides the kinetic equations, the main results obtained are the derivation of its power-law (constant flux) solutions and the demonstration that we have a direct energy cascade and an explosive inverse cascade of wave action (a property of finite-capacity turbulence systems) with a priori the possibility to excite Fourier modes from the injection wavenumber k_I to $k = 0$ in a finite time. However, and as discussed in [6], such a transfer driven by GW turbulence stops at an intermediate scale where the turbulence regime becomes strong. Note that the change of regime does not preclude the possibility to extend such an inverse cascade to $k = 0$ in a finite time [8]. Generally speaking, we may say that space-time turbulence is likely to be a relevant regime for describing the very early universe soon after the Planck's time when the universe emerges from a quantum foam [9].

The main goal of the present paper is to study the properties of GW turbulence further and to compare it with other turbulent systems. The kinetic equation derived in [6] is, however, too complicated for detailed analytical or numerical studies. Therefore, our strategy is to derive nonlinear diffusion models (also called differential models) which correspond to strongly local interaction approximations of the kinetic equation. This type of reduction, first introduced by Leith in 1967 [10] to study three-dimensional Navier-Stokes turbulence, is quite common and plays an important role for achieving a qualitative and even quantitative understanding of turbulence in various physical situations, for both weak and strong turbulence [11–13]. For example, the anomalous scaling of the non-stationary spectrum, a property of finite-capacity turbulence systems first detected in the numerical simulations of some kinetic equations [14, 15], has been easily studied in other systems using differential models [16–21]. In particular, it was established that the

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anomalous exponent is independent of the initial conditions but varies for different models of the same physical system.

In the present paper, we first recall the main properties of the kinetic equation of GW turbulence (section II) derived in [6], and then we propose two diffusion models for GW turbulence: a fourth-order and a second-order model which are introduced in sections III and IV, respectively. Their derivation is based, in particular, on the phenomenology of wave turbulence that was introduced in [6]. Numerical simulations of the second-order diffusion model are then performed to study the form of the front propagation during the inverse cascade of wave action. The results are presented in section V. Conclusions are discussed in section VI.

II. KINETIC EQUATION

Let us consider the metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\eta_{\mu\nu}$ the Poincaré-Minkowski flat space-time metric and $|h_{\mu\nu}| \sim h \ll 1$ a weak perturbation. Then, the three-dimensional wave action spectrum $n(\mathbf{k})$ scales like h^2/k and the kinetic equation of GW turbulence writes [6]

$$\begin{aligned} \frac{\partial n(\mathbf{k})}{\partial t} &= 4\pi \int |T_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}\mathbf{k}_3}|^2 \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}) \\ &(n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3} + n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_2} - n_{\mathbf{k}}n_{\mathbf{k}_2}n_{\mathbf{k}_3} - n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_3}) \\ &\delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \end{aligned} \quad (1)$$

where $\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$ (c is the speed of light) and $T_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}\mathbf{k}_3}$ is the coefficient describing the 4-wave interactions; it is characterized by the homogeneous property

$$T_{\xi\mathbf{k}_1\xi\mathbf{k}_2}^{\xi\mathbf{k}\xi\mathbf{k}_3} = T_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}\mathbf{k}_3}. \quad (2)$$

Equation (1) conserves the wave action $N = \int n(\mathbf{k})d\mathbf{k}$ and energy $E = \int \omega_{\mathbf{k}}n(\mathbf{k})d\mathbf{k}$. It has the zero flux (thermodynamic) solution $n_{\mathbf{k}} = T/(k + \mu)$ (with T and μ two constants), and the Kolmogorov-Zakharov isotropic constant flux solutions $N(k) \sim kn_{\mathbf{k}} \sim k^{-2/3}$ and $E(k) \sim k\omega_{\mathbf{k}}n_{\mathbf{k}} \sim k^0$ which are the wave action and the energy isotropic spectra, respectively [6]. The correct dimensions can be recovered by noticing that the energy density follows in fact the relation [22]

$$E \sim \frac{c^4}{32\pi G} h^2 k^2. \quad (3)$$

Complexity of equation (1) does not allow to derive analytically a simple model – a nonlinear diffusion equation – corresponding to the approximation of strongly local interactions. Below, we will assume that this type of interactions is dominant in GW turbulence and we shall derive two models by using phenomenological arguments.

III. FOURTH-ORDER DIFFUSION MODEL

Nonlinear diffusion models have proved to be a very useful tool in the analysis of both strong [10, 16] and wave turbulence [12, 13, 19, 23]. Here, we shall derive such a model for GW turbulence. Since the leading nonlinear interaction of GW is four-wave interaction [6], the model is a fourth-order diffusion equation of the type

$$\frac{\partial N(k)}{\partial t} = \frac{\partial^2}{\partial k^2} \left[D_{\mathbf{k}} N^4(k) \frac{\partial^2(k^2/N(k))}{\partial k^2} \right], \quad (4)$$

where $N(k)$ is the one-dimensional (isotropic) wave action spectrum, k the wavenumber and $D_{\mathbf{k}}$ a diffusion coefficient that eventually depends on the wave number k . This equation is constructed in such a way that it preserves the nonlinearity degree with respect to the spectrum (cubic in our case) and, its cascade and thermodynamic solutions. It is only in rare situations that one can derive a diffusion equation directly from the kinetic equation of wave turbulence (by taking the strongly local interactions limit). These exceptions concern nonlinear optics [11] and magnetohydrodynamics [13]. In our case, we will also use the phenomenology of wave turbulence [6]. A dimensional analysis of expression (4) gives the following information about $D_{\mathbf{k}}$

$$\frac{N(k)}{\tau} \sim \frac{D_{\mathbf{k}} N^3(k)}{k^2}, \quad (5)$$

and thus

$$D_{\mathbf{k}} \sim \frac{k^2}{\tau N^2(k)}. \quad (6)$$

We also have the scaling relation [6, 22]

$$E(k) \sim h^2 k \sim \omega N(k) \sim k N(k), \quad (7)$$

and therefore $N(k) \sim h^2$. This gives

$$D_{\mathbf{k}} \sim \frac{k^2}{\tau h^4} \sim \frac{k^2}{(\tau_{GW}/\epsilon^4) h^4} \sim \frac{k^2}{\tau_{GW}} \sim k^3, \quad (8)$$

where the GW time is given by the relation $\tau_{GW} \sim 1/\omega$ and where $\epsilon \sim h_{\mu\nu}/\eta_{\mu\nu} \sim h \ll 1$ is a small parameter. The introduction of expression (8) into (4) leads to the following fourth-order diffusion equation for isotropic three-dimensional GW turbulence

$$\frac{\partial N(k)}{\partial t} = A \frac{\partial^2}{\partial k^2} \left[k^3 N^4(k) \frac{\partial^2(k^2/N(k))}{\partial k^2} \right], \quad (9)$$

where A is a positive constant (a dimensionless number presumably of order one). Note that equation (9) is valid in the inertial range of weak GW turbulence. In the context of the primordial universe, the Planck length scale ($\sim 1/k_P$) can be seen as a small-scale limit for the diffusion model because the Einstein equations are not valid below, ie. when $k > k_P$. On the other side, the large-scale limit will be given by the scale k_s at which GW

turbulence becomes strong: it corresponds to space-time fluctuations $h \sim 1$.

Equation (9) conserves the wave action $\int N(k)dk$ and energy $\int \omega N(k)dk$. Indeed, if we define

$$\frac{\partial N(k)}{\partial t} = \frac{\partial^2 K(k)}{\partial k^2} = -\frac{\partial Q(k)}{\partial k}, \quad (10)$$

with $Q(k)$ being the wave action flux, then (for simplicity $c = 1$)

$$\begin{aligned} \frac{\partial E(k)}{\partial t} &= k \frac{\partial N(k)}{\partial t} = -\frac{\partial}{\partial k} \left[K(k) - k \frac{\partial K(k)}{\partial k} \right] \\ &= -\frac{\partial P(k)}{\partial k}, \end{aligned} \quad (11)$$

with P being the energy flux. We can check that the thermodynamic (zero flux) solution $N(k) \sim k^2 T / (k + \mu)$ [6] is satisfied by equation (9).

We can also find the constant (non-zero) flux solutions and find the cascade directions. Let us introduce $N(k) = C_N k^\alpha$ into equation (9); after simple calculations we obtain

$$Q(\alpha) = -(1 - \alpha)(2 - \alpha)(3 + 3\alpha)k^{2+3\alpha} AC_N^3. \quad (12)$$

Therefore, a constant wave action flux solution corresponds to $\alpha = -2/3$. For this value we find $Q_0 \equiv Q(-2/3) = -(40/9)AC_N^3 < 0$, which means that this solution corresponds to an inverse cascade (the wave action spectrum is positive definite thus $C_N > 0$). Let us now substitute $N(k) = C_E k^\beta$ into equation (11); after some calculations we obtain

$$P(\beta) = (1 - \beta)(2 - \beta)(-2 - 3\beta)k^{3+3\beta} AC_E^3. \quad (13)$$

The constant energy flux solution corresponds to $\beta = -1$; in this case we have $P_0 \equiv P(-1) = 6AC_E^3 > 0$, which means that we have a direct cascade of energy (because $C_E > 0$). This analysis gives a prediction for the Kolmogorov constants C_Q and C_P , which depend, however, on A . We find

$$N(k) = (-Q_0)^{1/3} C_Q k^{-2/3}, \quad (14)$$

with the Kolmogorov constant $C_Q = (9/(40A))^{1/3}$ and

$$E(k) = P_0^{1/3} C_P k^0, \quad (15)$$

with the Kolmogorov constant $C_P = (1/(6A))^{1/3}$. Interestingly, the ratio of the Kolmogorov constants becomes independent of A :

$$\frac{C_Q}{C_P} = \left(\frac{27}{20} \right)^{1/3} \simeq 1.105. \quad (16)$$

Note that a similar situation was also found with a differential model for two-dimensional hydrodynamic turbulence [18].

IV. SECOND-ORDER DIFFUSION MODEL

The nonlinear diffusion model of GW turbulence gets simplified if we do not include the thermodynamic solutions: it becomes a second-order diffusion equation which is easier to simulate numerically. We start with a diffusion equation of the type

$$\frac{\partial N(k)}{\partial t} = \frac{\partial}{\partial k} \left[\tilde{D}_k N^2(k) \frac{\partial(kN(k))}{\partial k} \right], \quad (17)$$

where \tilde{D}_k is a diffusion coefficient which may depend on k . As for the fourth-order diffusion equation, this equation is constructed in such a way that it preserves the nonlinearity degree with respect to the spectrum and its cascade solutions. A dimensional analysis of expression (17) gives

$$\frac{N(k)}{\tau} \sim \frac{\tilde{D}_k N^3(k)}{k}, \quad (18)$$

and thus (with $N(k) \sim h^2$)

$$\tilde{D}_k \sim \frac{k}{\tau N^2(k)} \sim \frac{k}{(\tau_{GW}/\epsilon^4) N^2(k)} \sim k^2. \quad (19)$$

Then, we obtain

$$\frac{\partial N(k)}{\partial t} = B \frac{\partial}{\partial k} \left[k^2 N^2(k) \frac{\partial(kN(k))}{\partial k} \right], \quad (20)$$

where B is a positive constant (a dimensionless number presumably of order one). (See Section III for a discussion about the domain of validity of weak GW turbulence.)

We can check that the constant flux solutions discussed above are recovered by this equation, and that the wave action and the energy are conserved (the relation $dP = kdQ$ can be used). By using similar notation as above, we can demonstrate that $Q_0 \equiv Q(-2/3) = -(1/3)BC_N^3$ and $P_0 \equiv P(-1) = (1/3)BC_E^3$, which means that the directions of the cascades are recovered. For this model we obtain

$$N(k) = (-Q_0)^{1/3} C_Q k^{-2/3}, \quad (21)$$

and

$$E(k) = P_0^{1/3} C_P k^0, \quad (22)$$

with equal Kolmogorov constants, $C_Q = C_P = (3/B)^{1/3}$.

Time-dependent solutions of equation (20) will be studied in section V. In particular we shall find a non-stationary solution with the wave action spectrum propagating towards small wavenumbers. This system is of finite capacity, ie. the integral

$$\int_0^{k_I} N(k) dk, \quad (23)$$

is finite when the solution (21) is used. This leads to an anomalous scaling with a non-trivial power-law. This non-stationary spectrum may be modelled as a self-similar solution of the second kind (see eg. [21, 24]) taking the form

$$N(k) = \frac{1}{\tau^a} N_0 \left(\frac{k}{\tau^b} \right), \quad (24)$$

where $\tau = t_* - t$, and t_* is a finite time at which the wave action spectrum reached the smallest available wavenumber. By introducing the above expression into (20) we find the condition

$$2a - b = 1. \quad (25)$$

A second condition can be found by assuming that $N_0(\xi) \sim \xi^m$ far behind the front. Then, the stationarity condition gives the following relation

$$a + mb = 0. \quad (26)$$

Finally, the combination of both relations gives

$$m = -\frac{a}{b} = -\frac{1}{2} - \frac{1}{2b}. \quad (27)$$

The latter expression means that we have a direct relation between the power law index m of the spectrum and the law of the front propagation which follows $k_f \sim \tau^b$. For example, if we assume that the stationary solution – the Kolmogorov-Zakharov (KZ) spectrum – is established immediately during the front propagation [24], then $m = -2/3$ and $b = 3$ (and $a = 2$). In this case, the prediction for the front propagation is

$$k_f \sim (t_* - t)^3. \quad (28)$$

Any deviation from this prediction has to be considered as an anomalous scaling which is sometimes difficult to observe numerically because the deviation is often tiny [25, 26].

V. NUMERICAL SIMULATION

In this section we shall investigate numerically the time evolution of the wave action spectrum described by the second-order diffusion equation (20) with $B = 1$. Linear hyper-viscous and hyper-hypoviscous terms are added in order to introduce sinks at small scale and large scale, respectively, to avoid the development of numerical instabilities. Then, the following equation is simulated

$$\begin{aligned} \frac{\partial N(k)}{\partial t} = & \frac{\partial}{\partial k} \left[k^2 N^2(k) \frac{\partial(kN(k))}{\partial k} \right] \\ & - \nu k^4 N(k) - \eta \frac{N(k)}{k^4}, \end{aligned} \quad (29)$$

with $\nu = 10^{-85}$ and $\eta = 10^{20}$. A logarithmic subdivision of the k -axis is used with $k_i = 2^{i/10}$ and i an

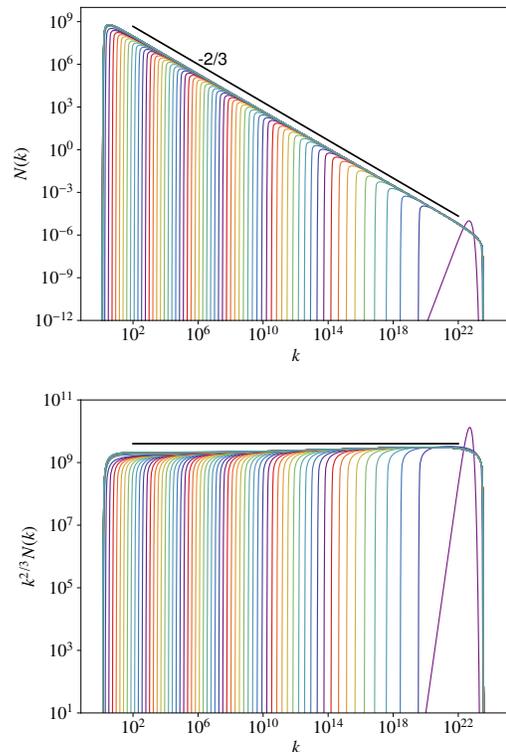


FIG. 1. Time evolution of the wave action spectrum with (bottom) and without (top) a compensation by $k^{2/3}$. The initial spectrum is localized at small scales ($k \sim 10^{22}$). **The comparison with the solution $k^{-2/3}$ (black segments) reveals a slight difference.**

integer varying between 0 and 799. A Crank-Nicholson numerical scheme is implemented for the nonlinear term and an adaptive time-step is used. No forcing term is added. **At the boundary cells a zero flux of wave action is imposed, then the (virtual) cells out of the numerical domain have a constant zero wave action.** The code is publicly available from <https://git.ias.u-psud.fr/ebuchlin/nldiffus-gw>.

Figure 1 shows the time evolution (every 6000 time-steps) of the wave action spectrum with an initial injection at $k_I \sim 10^{22}$. As expected, an inverse cascade appears with a spectrum reaching the smallest scale available ($k \sim 1$). **An extended power law spectra (over more than 20 decades!) is formed with, however, a power law exponent slightly different from $-2/3$ as we can see at the bottom of Fig. 1 where compensated spectra are displayed.**

The anomalous exponent can be determined very precisely by solving numerically an eigenvalue problem characterizing the existence of self-similar solutions of the second kind to the evolution equation. Such solutions were described in [21] for a broad class of second-order diffusion equations, exhibiting (forward) flux solutions with finite ultra-violet capacity. In the present case, a similar analysis carries through provided one maps the

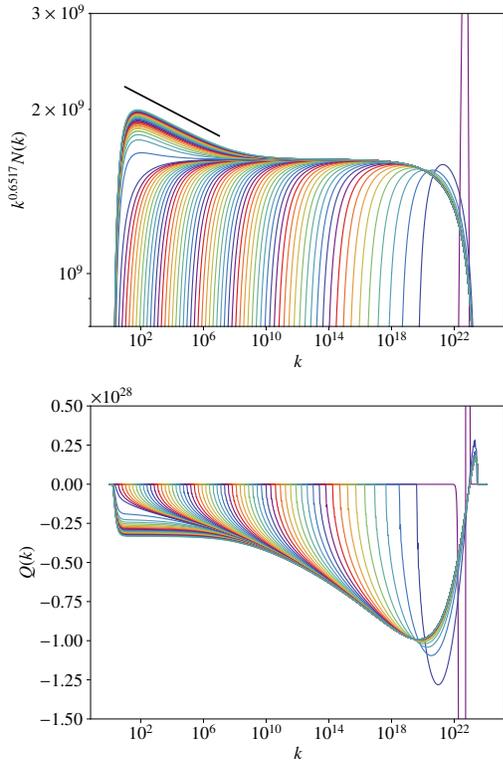


FIG. 2. Top: time evolution of the compensated (by $k^{0.6517}$) wave action spectrum; the Kolmogorov-Zakharov solution is established only after the hyper-hypoviscous scale is reached (the black segment corresponds to a spectrum in $k^{-2/3}$). Bottom: time evolution of the wave action flux $Q(k)$.

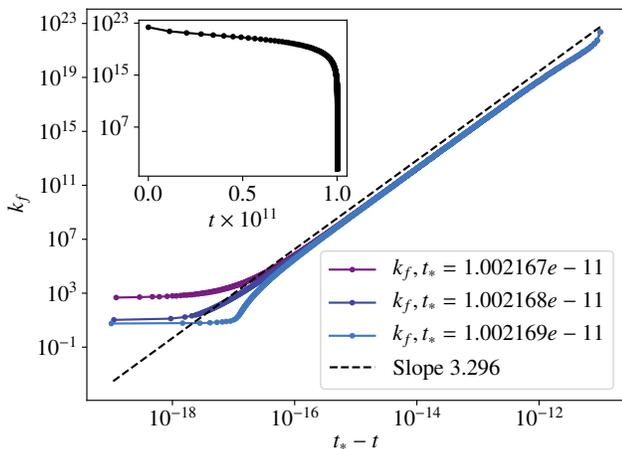


FIG. 3. Temporal evolution of the spectral front k_f for $t \leq t_*$ in linear-logarithmic coordinates (inset) and as a function of $t_* - t$ in double logarithmic coordinates for three values of t_* . The black dashed line corresponds to $(t_* - t)^{3.296}$.

inverse cascade behavior of the evolution equation (20) into a direct cascade in physical space. Explicitly, with the change of variables $k \rightarrow 1/\ell$, $N \rightarrow \tilde{N} = N/\ell^2$, the evolution equation (20) becomes

$$\frac{\partial \tilde{N}(\ell)}{\partial t} = \frac{\partial}{\partial \ell} \left[\ell^4 \tilde{N}^2(\ell) \frac{\partial(\ell \tilde{N}(\ell))}{\partial \ell} \right]. \quad (30)$$

This renormalized system has the equilibrium solution $\tilde{N} \sim \ell^{-1}$ and the flux solution $\tilde{N} \sim \ell^{-4/3}$, which has finite ultra-violet capacity: the general framework of [21] then predicts the existence of a self-similar solution $\tilde{N}(\ell, t) = (t_* - t)^{-x/(2x-1)} \phi[\ell/\ell_*]$, where ϕ is a shape function describing the spectrum profile. The front evolves as $\ell^*(t) \sim (t_* - t)^{1/(2x-3)}$ for a specific and uniquely determined exponent $x = x^*$, that characterizes the spectrum at small scales, $\tilde{N}(\ell, t) \sim \ell^{-x^*}$ for $\ell \ll 1$.

The anomalous exponent x^* is determined from a dichotomic search, a prescribed in [21]: We find $x_* \simeq 1.3483$, and this in turn implies the scaling exponent of the original system to be $m = x^* - 2 \simeq -0.6517$. Quite remarkably, this is indeed the anomalous behavior observed in Fig. 2 (top) which shows the compensated spectra. Clearly the KZ solution is established only at $t > t_*$, ie. after the largest (hyper-hypoviscous) scales are reached. This agrees with the time evolution of the wave action flux (bottom): a plateau corresponding to the KZ solution appears at the latest times only. Note that the simulation is stopped before the formation of the KZ spectrum over the entire range of scales.

We may also investigate the front propagation towards small wavenumbers and check if the power-law

$$k_f \sim (t_* - t)^{3.296}, \quad (31)$$

corresponding to the anomalous scaling $m \simeq -0.6517$ (and also $a \simeq -2.148$ and $b \simeq 3.296$) is observed. For that, one needs to follow the front propagation $k_f(t)$ which will be defined by using the compensated spectra in Fig. 1 (bottom): the value 1.4×10^9 for the compensated spectra is chosen to define the front $k_f(t)$. The result (Fig. 3; inset) displays a sharp decrease of the wavenumber of the front at a time close to 10^{-11} , which will be used to define t_* . The value $t_* = 1.002168 \times 10^{-11}$ is taken because it corresponds to the most extended power law (dark blue curve). We see (Fig. 3) that the expected power-law (31) is well observed over six decades. This result illustrates the explosive character of the inverse cascade of wave action in GW turbulence. Note that the 24 decades in wavenumbers used for the simulation are necessary to detect without ambiguity the anomalous scaling. However, as explained above, it is expected that GW turbulence becomes strong at large scale, which prevents the formation of such an extended power-law range.

VI. CONCLUSION

In this paper we have proposed two nonlinear diffusion models for GW turbulence which reproduce the prop-

erties previously derived from the kinetic equation [6] (power-law solutions, cascade directions). We have also derived some specific properties like the Kolmogorov constants. Additionally, we have performed a numerical simulation of the second-order diffusion model to illustrate the existence of an explosive inverse cascade of wave action, and have demonstrated the existence of an anomalous scaling which is typical to finite-capacity turbulence systems. This finding leads to a non-trivial power-law behind the propagating front in the inverse cascade spec-

trum. This analysis of the anomalous spectrum in the case of an inverse cascade is the first made with a diffusion model. Note, however, that a similar analysis based on kinetic equations has already been done by [15, 27, 28] to study the formation of a Bose-Einstein condensate. **The numerical simulation of the kinetic equation derived in [6] is much heavier to perform but not impossible. We can already predict the probable existence of an anomalous scaling for the wave action spectrum with, however, a possible difference in the power law index, as previously observed for other systems [14, 21].**

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