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Abstract. Motivated by questions of deformations/moduli in foliation theory, we investigate the structure of some groups of diffeomorphisms preserving a foliation. We give an example of a C^∞ foliation whose diffeomorphism group is not a Lie group in any reasonable sense. On the positive side, we prove that the automorphism group of a transversely holomorphic foliation or a riemannian foliation is a strong ILH Lie goup in the sense of Omori.

1. Introduction

Let M be a closed smooth manifold. We denote by $\mathcal{D} = \mathcal{D}(M)$ the group of all C^∞-diffeomorphisms of M endowed with the C^∞ topology. It is a metrizable topological group. In [10], Leslie proved that $\mathcal{D}$ is a Fréchet Lie group; namely, a group with a structure of infinite dimensional manifold modeled on a Fréchet space such that the group operations are smooth. Fréchet manifolds are difficult to deal with as many of the fundamental results of calculus, like implicit function theorem or Frobenius theorem, are not valid in that category. However, Palais proved that $\mathcal{D}$ has a richer structure: it is a ILH Lie group, that is a topological group that is the inverse limit of Hilbert manifolds $\mathcal{D}^k$ each of which is a topological group [18, 14]. Later on, Omori introduced the notion of *strong* ILH Lie group, whose precise definition is recalled in 2.1. The interest of this stronger notion is that it allows to formulate an implicit function theorem and a Frobenius theorem in that setting (cf. [16]). In addition to demonstrate those results, Omori used them to show that $\mathcal{D}$ is a strong ILH Lie group.

It is natural to ask for similar results for groups of diffeomorphisms preserving a geometric structure. In [4], Ebin and Marsden proved the existence of a structure of a ILH Lie group on $\mathcal{D}_\eta$ and $\mathcal{D}_\omega$, the subgroups of smooth diffeomorphisms preserving a volume element η or a symplectic form ω on M respectively; and Omory showed that some subgroups of $\mathcal{D}$, including $\mathcal{D}_\eta$ and $\mathcal{D}_\omega$, are strong ILH Lie groups (cf. [16, 17]).

In this paper, we are interested in the structure of groups of diffeomorphisms preserving a foliation; and preserving a foliation with some additional geometric transverse structure. It turns out that the situation is much more complicated for those subgroups. We give in this paper a negative result but also two positive ones.

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Our motivation comes from deformation problems. We want to understand the deformations/moduli space of foliations with a geometric transverse structure. The existence of a "nice" moduli space is closely related to the existence of a "nice" Lie group structure on the associated diffeomorphism groups (see for example [12]). A strong ILH Lie group structure is a nice structure, especially because of the implict function and Frobenius Theorem proved by Omori in this context.

Let us first assume that $\mathcal{F}$ is a smooth foliation on M. Call $\mathcal{D}_{\mathcal{F}}$ the group of foliation preserving diffeomorphisms. In [11], Leslie asserted that $\mathcal{D}_{\mathcal{F}}$ is a Fréchet Lie group. Unfortunately there was a gap in his demonstration, which remains only valid for foliations with a transverse invariant connection. Indeed, for certain foliations the group $\mathcal{D}_{\mathcal{F}}$ is not a Lie group in any reasonable sense. This fact seems to be folklore although, as far as we know, there is no published example. In part 2, we construct and discuss in detail such a foliation.

Let us now consider a type of geometric structure Ξ having the property that if ϵ is a structure of type Ξ on a compact manifold M then the automorphism group $\mathrm{Aut}(M, \epsilon)$ is a Lie group in a natural way. Assume that the foliation $\mathcal{F}$ is endowed with a geometric transverse structure of type Ξ. Because of the previous negative result, there is no reason for the automorphism group $\mathrm{Aut}(\mathcal{F})$ (that is the group of the diffeomorphisms of M which preserve the foliation as well as its geometric transverse structure) to be a strong ILH Lie group. We show this is nevertheless true in the following two cases: $\mathcal{F}$ is a transversely holomorphic foliation (part 3) or a Riemannian foliation (part 4).

2. GROUPS OF DIFFEOMORPHISMS PRESERVING A FOLIATION

Throughout this article M will be a fixed closed smooth manifold. Let $\mathfrak{X} = \mathfrak{X}(M)$ be the Lie algebra of smooth (of class C^∞) vector fields on M. Endowed with the C^∞-topology, $\mathfrak{X}$ is a separable Fréchet space. We denote by $\mathcal{D} = \mathcal{D}(M)$ the group of smooth (of class C^∞) diffeomorphisms of M endowed with the C^∞ topology. It is a topological group, which is metrizable and separable; so it has second countable topology.

We recall that a Fréchet manifold is a Hausdorff topological space with an atlas of coordinate charts taking values in Fréchet spaces in such a way that the coordinate changes are smooth maps (i.e. they admit continuous Fréchet derivatives of all orders). A Fréchet Lie group is a Fréchet manifold endowed with structure of group such that multiplication and inverse maps are smooth. In [10], it was proved by Leslie that $\mathcal{D}$ is a Fréchet Lie group with Lie algebra $\mathfrak{X}$. In fact, $\mathcal{D}$ is endowed with a richer structure: it is the inverse limit of Hilbert manifolds $\mathcal{D}^k$ each of them being a topological group [4, 14]. Moreover $\mathcal{D}$ has in fact the structure of a *strong* ILH-Lie group [16], a notion introduced by Omori that can be defined as follows.

Recall that a Sobolev chain is a system $\{\mathbb{E}, E^k, k \in \mathbb{N}\}$ where E^k are Hilbert spaces, with E^{k+1} linearly and densely embedded in E^k, and $\mathbb{E} = \bigcap E^k$ has the inverse limit topology.

Definition 2.1. A topological group G is said to be a strong ILH-Lie group modeled on a Sobolev chain $\{\mathbb{E}, E^k, k \in \mathbb{N}\}$ if there is a family $\{G^k, k \in \mathbb{N}\}$ fulfilling

1) G^k is a smooth Hilbert manifold, modeled on E^k, and a topological group; moreover G^{k+1} is embedded as a dense subgroup of G^k with inclusion of class C^∞.
2) $G = \bigcap G^k$ has the inverse limit topology and group structure.
3) Right translations $R_g\colon G^k \to G^k$ are C^∞.
4) Multiplication and inversion extend to C^m mappings $G^{k+m} \times G^k \to G^k$ and $G^{k+m} \to G^k$ respectively.
5) Let $\mathfrak{g}^k$ be the tangent space of G^k at the neutral element e and let TG^k be the tangent bundle of G^k. The mapping $dR\colon \mathfrak{g}^{k+m} \times G^k \to TG^k$, defined by $dR(u, g) = dR_g u$, is of class C^m.
6) There is a local chart $\xi\colon \tilde{U} \subset \mathfrak{g}^1 \to G^1$, with $\xi(0) = e$ whose restriction to $\tilde{U} \cap \mathfrak{g}^k$ gives a local chart of G^k for all k.

Remarks 2.2. a) The strong ILH-Lie group $\mathcal{D}$ is modeled on the Sobolev chain $\{\mathfrak{X}, \mathfrak{X}^k, k \in \mathbb{N}\}$, where $\mathfrak{X}^k$ are the Sobolev completions of $\mathfrak{X}$. In that case $G^k = \mathcal{D}^k$ are the Sobolev completions of $\mathcal{D}$ and there are natural identifications $\mathfrak{g} \equiv \mathfrak{X}$ and $\mathfrak{g}^k \equiv \mathfrak{X}^k$ (cf. [16], Theorem 2.1.5).

b) Weaker definitions of ILH-Lie groups were considered in articles prior to [16], in particular in [4, 14]. In the present paper we will only consider ILH-Lie groups in the strong sense given by the above definition.

c) This strong definition ensures, by means of the last condition, that G is endowed with a structure of Fréchet Lie group. The space $\mathfrak{g} = \cap \mathfrak{g}^k$ with the inverse limit topology is a Fréchet space. Moreover, it is naturally endowed with a Lie algebra structure (cf. [16], I.3). We call $\mathfrak{g}$ the Lie algebra of G.

Suppose that H is a subgroup of a strong ILH-Lie group G and let $\mathfrak{B}$ be a basis of neighborhoods of e in G. For each $U \in \mathfrak{B}$, let $U_0(H)$ be the set of points $x \in U \cap H$ such that x and e can be joined by a piecewise smooth curve $c(t)$ in $U \cap H$. Here, piecewise smooth means that the mapping $c\colon [0, 1] \to G^k$ is piecewise of class C^1 for each k. Then the family $\mathfrak{B}_0(H) = \{U_0(H) \mid U \in \mathfrak{B}\}$ satisfies the axioms of neighborhoods of e of topological groups. As in [16], this topology will be called the LPSAC-topology of H, where LPSAC stands for "linear piecewise smooth arc-connected". In general the LPSAC-topology does not coincide with the induced topology.

Definition 2.3. Let G be a strong ILH-Lie group and let $\mathfrak{g}^k$ be the tangent space of G^k at e. A subgroup H of G is called a *strong ILH-Lie subgroup* of G if there is a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$ such that the following two conditions are fulfilled:

i) It induces splittings $\mathfrak{g}^k = \mathfrak{h}^k \oplus \mathfrak{f}^k$ for each k, where $\mathfrak{h}^k$ and $\mathfrak{f}^k$ are, respectively, the closures of $\mathfrak{h}$ and $\mathfrak{f}$ in $\mathfrak{g}^k$.
ii) There is a local chart $\xi\colon \tilde{U} \subset \mathfrak{g}^1 \to G^1$ fulfilling condition 6) in Definition 2.1 such that $\xi(\tilde{U} \cap \mathfrak{h}) = U_0(H)$, where $U = \xi(\tilde{U} \cap \mathfrak{g})$

Remark 2.4. Clearly, a strong ILH-Lie subgroup H of a strong ILH-Lie group G is itself a strong ILH-Lie group under the LPSAC-topology.

In [16], Omori was able to state some of the classical theorems of analysis, like the implicit function theorem or Frobenius theorem, in the setting of strong ILH-Lie groups. The following is a special case of Frobenius theorem that will be used later (cf. [16], Theorem 7.1.1 and Corollary 7.1.2).

Theorem 2.5 (Omori)**.** *Let E and F be Riemannian vector bundles over M and let $A\colon \Gamma(TM) \to \Gamma(E)$ and $B\colon \Gamma(E) \to \Gamma(F)$ be differential operators of order $r \geq 0$ with smooth coefficients. Assume that the following conditions are fulfilled:*

i) $B\,A = 0$.
ii) $A\,A^* + B^*\,B$ *is an elliptic differential operator, where* A^*, B^* *are formal adjoints of* A, B *respectively.*
iii) $\mathfrak{h} = \ker A$ *is a Lie subalgebra of* $\mathfrak{X} = \Gamma(TM)$.

Then there is a strong ILH-Lie subgroup H of $\mathcal{D}$ whose Lie algebra is $\mathfrak{h}$. Moreover, H is an integral submanifold of the distribution $\mathcal{H}$ in $T\mathcal{D}$ given by right translation of $\mathfrak{h}$, i.e. $\mathcal{H} = \{dR_f\mathfrak{h} \mid f \in \mathcal{D}\}$.

Here, the notation $\Gamma(E)$ stands for the space of smooth sections of class C^∞ of a given vector bundle E.

As already remarked, the group H in the above theorem is a strong ILH-Lie group with the LPSAC-topology. However that topology on H can be stronger that the topology induced by G, even if H is closed in G. Therefore the above theorem is not sufficient by itself to assure that H, endowed with the induced topology, is a strong ILH-Lie group, or even a Fréchet Lie group. The following Proposition gives sufficient conditions for the above two topologies to coincide. (cf. [16], VII.1).

Proposition 2.6 (Omori)**.** *Let H be a strong ILH-Lie subgroup of G satisfying the conditions of the above theorem and assume that the following conditions are also fulfilled*

a) H *is closed in* G,
b) G *is second countable and the LPSAC-topology of* H *is also second countable.*

Then the topology on H induced by G coincides with the LPSAC-topology.

From now on we assume that M is endowed with a smooth foliation $\mathcal{F}$ of dimension p and codimension m. We denote by $T\mathcal{F}$ its tangent bundle and by $\nu\mathcal{F} = TM/T\mathcal{F}$ the normal bundle of $\mathcal{F}$. The foliation $\mathcal{F}$ decomposes the manifold into the disjoint union of p-dimensional submanifolds that are called the leaves of the foliation.

An atlas adapted to $\mathcal{F}$ is a smooth atlas $\{(U_i, \varphi_i)\}$ of M, with $\varphi_i\colon U_i \to \mathbb{R}^p \times \mathbb{R}^m$ homeomorphisms, such that the leaves L of $\mathcal{F}$ are defined on U_i by the level sets $\phi_i = constant$ of the submersions $\phi_i = pr_2 \circ \varphi_i$. Here $pr_2\colon \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ denotes the projection onto the second factor. In that case there is a cocycle $\{\gamma_{ij}\}$ of local smooth transformations of $\mathbb{R}^m$ such that

$$\phi_j = \gamma_{ji} \circ \phi_i. \tag{1}$$

The subsets $\phi_i = constant$ are called the slices of $\mathcal{F}$ in U_i.

We consider now the following subgroups of $\mathcal{D}$ of foliation preserving diffeomorphisms

$$\begin{aligned} \mathcal{D}_{\mathcal{F}} &= \{f \in \mathcal{D} \mid f_*\mathcal{F} = \mathcal{F}\}, \\ \mathcal{D}_L &= \{f \in \mathcal{D}_{\mathcal{F}} \mid f(L) = L, \ \text{ for each leaf } L\}. \end{aligned}$$

We denote by $\mathfrak{X}_{\mathcal{F}}$ the Lie algebra of foliated vector fields (i.e. of vector fields whose flows preserve the foliation) and by $\mathfrak{X}_L$ the Lie algebra of vector fields that are tangent to $\mathcal{F}$ (i.e. vector fields that are smooth sections of $T\mathcal{F}$). Notice that the flows associated to vector fields of $\mathfrak{X}_{\mathcal{F}}$ and $\mathfrak{X}_L$ define one-parameter subgroups of $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}_L$ respectively. The following result was proved by Omori in [16]. We include a sketch of proof for later use.

Proposition 2.7 (Omori)**.** *The group $\mathcal{D}_L$ is a strong ILH-Lie subgroup of $\mathcal{D}$ with Lie algebra $\mathfrak{X}_L = \Gamma(T\mathcal{F})$.*

Proof. Let $\pi\colon \Gamma(TM) \to \Gamma(\nu\mathcal{F})$ be the linear mapping induced by the natural projection $TM \to \nu\mathcal{F}$. We can regard π as differential operator of order 0 and its adjoint operator π^*, constructed using Riemannian metrics on TM and $\nu\mathcal{F}$, is also a differential operator of order 0 such that $\pi\pi^*$ is elliptic (i.e. an isomorphism). Theorem 2.5 implies now that there is a strong ILH-Lie subgroup $\mathcal{D}'_L$ of $\mathcal{D}$ with Lie algebra $\mathfrak{X}_L$. Let $\mathcal{D}^k_L$ and $\mathcal{D}'^k_L$ be the Sobolev completions of $\mathcal{D}_L$ and $\mathcal{D}'_L$ respectively. Clearly $\mathcal{D}'_L \subset \mathcal{D}_L$ and $\mathcal{D}'^k_L \subset \mathcal{D}^k_L$. Since $\mathcal{D}'_L$ is obtained by the Frobenius theorem, if a piecewise C^1-curve $c(t)$ satisfies $c(0) = e$ and $c(t) \in \mathcal{D}^k_L$, then $c(t) \in \mathcal{D}'^k_L$. Using the exponential mapping with respect to a connection under which $\mathcal{F}$ is parallel, one can see that $\mathcal{D}^k_L$ has a structure of Hilbert manifold, hence it is LPSAC. This implies that $\mathcal{D}'^k_L = \mathcal{D}^k_{L,0}$ (the connected component of $\mathcal{D}^k_L$ containing e) and therefore that $\mathcal{D}'_L$ is the connected component of $\mathcal{D}_L$ containing the identity. □

Remark 2.8. It follows from the above Proposition that $\mathcal{D}_L$ is a strong ILH-Lie group with the LPSAC-topology and, in particular, it is endowed with a structure of Fréchet Lie group. In general, however, $\mathcal{D}_L$ is not closed in $\mathcal{D}$. This is the case for instance if $\mathcal{F}$ is a linear foliation on the 2-torus T^2 with irrational slope.

In [11], Leslie asserted that the group $\mathcal{D}_{\mathcal{F}}$ is a Fréchet Lie group. Unfortunately there was an error in his demonstration, which remains valid only for Riemann's foliations or, more generally, for foliations with a transverse invariant connection. In fact, the group $\mathcal{D}_{\mathcal{F}}$ may not be a Lie group in any reasonable sense. That fact seems to be folklore although, as far as we know, there is no published example. We exhibit here a concrete one. More precisely we prove

Theorem 2.9. *There is a foliation $\mathcal{F}$ on the 2-torus T^2 such that $\mathcal{D}_{\mathcal{F}}$ cannot be endowed with a structure of a topological group fulfilling*

i) The inclusion $\mathcal{D}_{\mathcal{F}} \hookrightarrow \mathcal{D}(T^2)$ is continuous.
ii) $\mathcal{D}_{\mathcal{F}}$ has second countable topology.
iii) $\mathcal{D}_{\mathcal{F}}$ is locally path-connected.

We dedicate the rest of the section to prove that statement.

Assume that an orientation preserving diffeomorphism $h \in \mathcal{D}(S^1)$ has been given. Let $\tau\colon \mathbb{R} \to \mathbb{R}$ be the translation $\tau(t) = t+1$ and let $\tilde{h}\colon \mathbb{R}\times S^1 \to \mathbb{R}\times S^1$ denote the map defined by

$$\tilde{h}(t,x) = (\tau(t), h(x)) = (t+1, h(x)).$$

Since h is isotopic to the identity, the quotient manifold $M = \mathbb{R}\times S^1/\langle\tilde{h}\rangle$ is the 2-torus T^2. The foliation $\tilde{\mathcal{F}}$ of $\mathbb{R}\times S^1$ whose leaves are $\mathbb{R}\times\{x\}$ is preserved by $\tilde{h}$. Hence, $\tilde{\mathcal{F}}$ induces a well defined foliation $\mathcal{F}$ on M, which is transverse to the fibres of the natural projection

$$\pi\colon \mathbb{R}\times S^1/\langle\tilde{h}\rangle \longrightarrow S^1 = \mathbb{R}/\langle\tau\rangle.$$

We say that $\mathcal{F}$ is the foliation on T^2 obtained by suspension of h.

We fix a circle $C = \pi^{-1}(t_0)$ inside the 2-torus T^2. In a neighborhood U of C, the foliation $\mathcal{F}$ is a product $(t_0-\epsilon, t_0+\epsilon)\times C$ and each element $f \in \mathcal{D}_{\mathcal{F}}$ close enough to the identity map Id sends $C = \{t_0\}\times C$ into U. Moreover, in the adapted coordinates (t,x), it is written

$$f(t,x) = (f_1(t,x), f_2(x)).$$

Therefore, there is a neighborhood $\mathcal{W}$ of Id in $\mathcal{D}_{\mathcal{F}}$ such that the map

$$\begin{array}{rccc} P\colon & \mathcal{W} & \to & \mathcal{D}(S^1) \\ & f & \mapsto & f_2 \end{array} \tag{2}$$

is well-defined. Notice that, if we consider on $\mathcal{D}(S^1)$ the topology induced by $\mathcal{D}$, then P is a continuous map; it is also a morphism of local groups.

Lemma 2.10. *The map P sends $\mathcal{W}$ into the centralizer $Z^\infty(h)$ of h in $\mathcal{D}(S^1)$. Moreover, P has a continuous right-inverse map σ defined on a neighborhood $\mathcal{V}$ of the identity in $Z^\infty(h)$.*

Remark 2.11. It follows from the above statement that the image of the map P contains a neighborhood of the identity in $Z^\infty(h)$.

Proof. The first assertion follows from an easy computation. Given an element $g_1 \in Z^\infty(h)$ close enough to the identity, the map $\tilde{g}\colon \mathbb{R}\times S^1 \to \mathbb{R}\times S^1$ defined by $\tilde{g}(t,x) = (t, g_1(x))$ commutes with $\tilde{h}$ and preserves the foliation $\mathcal{F}$. Hence, it induces an element $g \in \mathcal{D}_{\mathcal{F}}$ which is close to the identity. The map $g_1 \mapsto \sigma(g_1) = g$ is well-defined on a neighborhood $\mathcal{V}$ of the identity in $Z^\infty(h)$. Clearly, the map σ is continuous and fulfills $P\circ\sigma = \mathrm{Id}$. □

Lemma 2.12. *There is a neighborhood $\mathcal{W}'$ of the identity in $\mathcal{D}(T^2)$ and a smooth map $\Psi\colon \mathcal{W}' \to \mathcal{D}(S^1)$ such that P is the composition*

$$P\colon \mathcal{W}'\cap\mathcal{D}_{\mathcal{F}} \hookrightarrow \mathcal{W}' \xrightarrow{\Psi} \mathcal{D}(S^1)$$

Proof. Let $\mathcal{W}'$ be a neighborhood of the identity in $\mathcal{D}(T^2)$ such that P is defined on $\mathcal{W}'\cap\mathcal{D}_{\mathcal{F}}$. If $\mathcal{W}'$ is small enough, we can also assume that, for each $f \in \mathcal{W}'$, $f(\{t_0\}\times C)$ is contained in U and it is transverse to the foliation $\mathcal{F}$. We define $\Psi(f)$ as the map

$$\Psi(f) = \mathrm{pr}_2\circ f|_C\colon C \longrightarrow C$$

where $\mathrm{pr}_2\colon U = (t_0-\epsilon, t_0+\epsilon)\times C \to C$ is the projection onto the second factor. We can think of $\Psi(f)$ as an element of $\mathcal{D}(S^1)$. Clearly the map Ψ is smooth and fulfills the required conditions. □

Denote by $\mathcal{D}^r(S^1)$ the group of diffeomorphisms of S^1 of class r, where $0 \le r \le \infty$, endowed with the C^r-topology. For a given $h \in \mathcal{D}(S^1) = \mathcal{D}^\infty(S^1)$, we denote by $Z^r(h)$ the centralizer of h in $\mathcal{D}^r(S^1)$, and we denote by $Z_0^r(h)$ the closure in $\mathcal{D}^r(S^1)$ of the group $\langle h\rangle$ generated by h. Notice that $Z_0^r(h) \subset Z^r(h)$.

Let

$$\rho\colon \mathcal{D}^0(S^1) \to S^1$$

be the map that associates to each element $h \in \mathcal{D}^0(S^1)$ its rotation number $\rho(h)$. We assume that $h \in \mathcal{D}^\infty(S^1)$ has been given and that its rotation number $\alpha = \rho(h)$ is irrational. Because of Denjoy's theorem, there is a homeomorphism φ of S^1 conjugating h to the rotation R_α, i.e. $\varphi \circ h \circ \varphi^{-1} = R_\alpha$. Since α is irrational, the centralizer of R_α in $\mathcal{D}^0(S^1)$ is the group of rotations. Hence the centralizer $Z^0(h)$ is the set $\{\varphi^{-1} \circ R_\beta \circ \varphi\}$ with $\beta \in S^1$. The natural inclusion

$$\iota\colon Z^\infty(h) \hookrightarrow Z^0(h) \cong S^1 \tag{3}$$

is a continuous group morphism mapping $Z^\infty(h)$ onto a subgroup of S^1 (cf. [22], p. 185). Note that the map ι is just the restriction of ρ to $Z^\infty(h)$. Notice also that the identification of $Z^\infty(h)$ with a subgroup of S^1 given by ι is algebraic but not topological.

Let d_∞ be a distance defining the topology of $\mathcal{D}^\infty(S^1)$ and set

$$K_\infty(f) = \inf_{n\ge 1} d_\infty(f^n, \mathrm{Id}).$$

Lemma 2.13. *$K_\infty^{-1}(0) = A \cup B$ where A is the subset of $\mathcal{D}^\infty(S^1)$ of elements f such that $Z_0^\infty(f)$ has the cardinality of the continuum and B is the subset of $\mathcal{D}^\infty(S^1)$ of elements of finite order.*

Proof. Let f be an element of $K_\infty^{-1}(0)\backslash B$. Each neighborhood of the identity contains non trivial elements of $\langle f\rangle$ and therefore of its closure $Z_0^\infty(f)$ with respect to the C^∞-topology. Hence the group $Z_0^\infty(f)$ is closed and non-discrete. As $Z_0^\infty(f)$ is perfect, it contains a Cantor set and its cardinality is bigger or equal to the cardinality of the continuum $2^{\aleph_0}$. In fact $Z_0^\infty(f)$ is separable, as it is a space of smooth functions on a compact manifold, and therefore its cardinality is that of the continuum. This shows that $K_\infty^{-1}(0) \subset B \cup A$.

Clearly B is a subset of $K_\infty^{-1}(0)$. Let us show that each element $f \in A$ belongs to $K_\infty^{-1}(0)$. The group $Z_0^\infty(f)$ cannot be discrete. Otherwise it would be countable, since it is separable, getting a contradiction. We deduce that there is a sequence $(f^{n_k})_{k\ge 1}$ of pairwise different elements that converges. Notice that $K_\infty^{-1}(0)$ is closed under inversion. Hence we can assume that $n_k > 0$. Moreover we can also assume that the sequence n_k is strictly increasing. We choose a sequence $m_k = n_{q(k)} - n_k$ such that $q(k) > k$ for each $k \in \mathbb{N}$ and such that $(m_k)_{k\ge 1}$ is strictly increasing. Then $(f^{m_k})_{k\ge 1}$ is a sequence of non-trivial elements converging to the identity and therefore f belongs to $K_\infty^{-1}(0)$. $\square$

Given an irrational number $\alpha \in S^1 - \mathbb{Q}/\mathbb{Z}$, set

$$Z_0^\infty(\alpha) = \bigcap_{\rho(f)=\alpha} Z_0^\infty(f)$$

where the groups $Z_0^\infty(f)$ are thought as subsets of S^1. The following result was proved by Yoccoz (cf. [22], Théorème 3.5, p. 190)

Theorem 2.14 (Yoccoz)**.** *There is a residual set $\mathcal{C}$ of numbers $\alpha \in S^1 - \mathbb{Q}/\mathbb{Z}$ such that $Z_0^\infty(\alpha)$ has the cardinality of the continuum.*

Corollary 2.15. *There is an element $h_0 \in \mathcal{D}^\infty(S^1)$ whose rotation number α is Liouville and such that Z_0^∞ has the cardinality of the continuum. Moreover, each continuous path $\gamma\colon [0,1] \to Z^\infty(h_0)$ is constant.*

Remark 2.16. We recall that for each Liouville number α there is a non-C^∞-linearizable diffeomorphism of S^1 whose rotation number is α. This result was proved by Herman in [8, Chapter XI]. Yoccoz proved that, on the contrary, all diffeomorphisms with a diophantine rotation number are C^∞-linearizable [21].

Proof. Since the set of Liouville numbers is residual and the intersection of two residual sets is residual, we can find an element $h_0 \in \mathcal{D}^\infty(S^1)$ whose rotation number $\alpha = \rho(h_0)$ is Liouville and belongs to the set $\mathcal{C}$ in the above theorem, and such that h_0 is not linearizable. Since $\alpha \in \mathcal{C}$ we deduce that there are non trivial elements of $Z^\infty(h_0)$ arbitrarily close to the identity in the C^∞-topology. Let $\gamma\colon [0,1] \to Z^\infty(h_0)$ be a continuous path. Since the C^∞-topology is finer that the C^0-topology, it is sufficient to prove that if γ is continuous when we consider on $Z^\infty(h_0)$ the topology of the uniform convergence, then the path γ is constant. But this is equivalent to say that the composition

$$\rho \circ \gamma\colon [0,1] \to Z^\infty(h_0) \hookrightarrow Z^0(h_0) \cong S^1$$

is constant. If this were not the case the image $(\rho \circ \gamma)[0,1]$ would contain diophantine rotation numbers. In particular there would be an element f in $Z^\infty(h_0)$ which is C^∞-conjugated to a rotation, but then the same conjugation would also linearize h_0 leading to a contradiction. □

Proof of Theorem 2.9. Let h_0 be the element of $\mathcal{D}^\infty(S^1)$ given by the above corollary and let $\mathcal{F}$ be the foliation on T^2 obtained by suspension of h_0. Assume that $\mathcal{D}_\mathcal{F}$ is endowed with a structure of locally path-connected topological group such that the natural inclusion $\mathcal{D}_\mathcal{F} \hookrightarrow \mathcal{D}(T^2)$ is continuous. Let U be a neighborhood of the identity in $\mathcal{D}(T^2)$ such that the map $f \to P(f)$, given in (2), is defined for $f \in U \cap \mathcal{D}_\mathcal{F}$. Because of Lemma 2.12, $P(U \cap \mathcal{D}_\mathcal{F})$ contains a neighborhood of the identity in $Z^\infty(h_0)$. This implies that $U \cap \mathcal{D}_\mathcal{F}$ has an uncountable number of path-connected components. Therefore $\mathcal{D}_\mathcal{F}$ cannot be second countable. □

Remark 2.17. Theorem 2.9 provides an example of a closed subgroup of $\mathcal{D}(T^2)$ for which the LPSAC-topology and the induced topology do not coincide. More precisely, the induced topology of the subgroup is not LPSAC and its LPSAC-topology is not countable. This shows in particular that condition b) in Proposition 2.6 is not always fulfilled.

3. The automorphism group of a transversely holomorphic foliation

Throughout this section we assume that the foliation $\mathcal{F}$ on the compact manifold M is transversely holomorphic, of (real) dimension p and complex codimension q. This means that the atlas of adapted local charts $\{(U_i, \varphi_i)\}$ can be chosen as taking values in $\mathbb{R}^p \times \mathbb{C}^q$ and that the maps $\{\gamma_{ij}\}$ fulfilling the cocycle condition (1) are local holomorphic transformations of $\mathbb{C}^q$.

The transverse complex structure of $\mathcal{F}$ induces a complex structure on the normal bundle $\nu\mathcal{F} = TM/T\mathcal{F}$ of the foliation, which is invariant by the holonomy.

We denote by $\mathrm{Aut}(\mathcal{F})$ the group of automorphisms of the foliation. That is, $\mathrm{Aut}(\mathcal{F})$ is the group of elements $f \in \mathcal{D}_{\mathcal{F}}$ which are transversely holomorphic in the sense that, in adapted local coordinates they are of the form

$$f(x, z) = (f_1(x, z, \bar{z}), f_2(z))$$

with $f_2 = f_2(z)$ holomorphic transformations.

Let $\mathfrak{aut}(\mathcal{F})$ denote the Lie algebra of (real) vector fields ξ on M whose flows are one-parameter subgroups of $\mathrm{Aut}(\mathcal{F})$. In particular, $\mathfrak{aut}(\mathcal{F})$ is the Lie subalgebra of $\mathfrak{X}_{\mathcal{F}}$.

In adapted local coordinates (x, z), a (real) vector field $\xi \in \mathfrak{X}$ is written in the form

$$\xi = \sum_i a^i(x, z, \bar{z}) \frac{\partial}{\partial x^i} + \sum_j b^j(x, z, \bar{z}) \frac{\partial}{\partial z^j} + \sum_j \overline{b}^j(x, z, \bar{z}) \frac{\partial}{\partial \bar{z}^j}, \tag{4}$$

where the functions a^i are real whereas b^j are complex-valued. Then ξ is a foliated vector field, i.e. it belongs to $\mathfrak{X}_{\mathcal{F}}$, if b^i are basic functions, that is if $b^i = b^i(z, \bar{z})$. And ξ is a holomorphic foliated vector field, i.e. it belongs to $\mathfrak{aut}(\mathcal{F})$, if b^i are holomorphic basic functions, that is if $b^i = b^i(z)$.

Notice that $\mathcal{D}_L$ is a normal subgroup of $\mathrm{Aut}(\mathcal{F})$ and that $\mathfrak{X}_L$ is an ideal of $\mathfrak{aut}(\mathcal{F})$. We denote by $\mathcal{G} = \mathfrak{aut}(\mathcal{F})/\mathfrak{X}_L$ the quotient Lie algebra. Notice also that $\mathcal{D}_L$ is not necessarily closed in $\mathrm{Aut}(\mathcal{F})$ as far as the leaves of $\mathcal{F}$ are not closed.

We will suppose that a Riemannian metric g on M, that will play an auxiliary role, has also been fixed.

We recall now some facts concerning transversely holomorphic foliations that we will use in the sequel (for the omitted details we refer to [3]). The transverse complex structure of $\mathcal{F}$ induces an almost complex structure on the normal bundle $\nu\mathcal{F}$ and therefore a splitting of the complexified normal bundle in the standard way

$$\nu^{\mathbb{C}}\mathcal{F} = \nu^{1,0} \otimes \nu^{0,1}.$$

There is a short exact sequence of complex vector bundles

$$0 \longrightarrow L \longrightarrow T^{\mathbb{C}}M \xrightarrow{\pi^{1,0}} \nu^{1,0} \longrightarrow 0$$

where $\pi^{1,0}$ denotes the composition of natural projections

$$T^{\mathbb{C}}M \to \nu^{\mathbb{C}}\mathcal{F} \to \nu^{1,0}$$

and L is the kernel of $\pi^{1,0}$. In local adapted coordinates (x, z), the bundle $\nu^{1,0}$ is spanned by the (classes of) vector fields $[\partial/\partial z^j]$. In a similar way, the bundle L is spanned by the vector fields $\partial/\partial x^i, \partial/\partial \bar{z}^j$, and the dual bundle L^* is spanned by the (classes of) 1-forms $[dx^i, d\bar{z}^j]$. For a given $k \in \mathbb{N}$, denote

$$\mathcal{A}^k = \Gamma\Big(\bigwedge^k L^* \otimes \nu^{1,0}\Big).$$

The exterior derivative d induces differential operators of order 1 (cf. [3])

$$d^k_{\mathcal{F}} \colon \mathcal{A}^k \longrightarrow \mathcal{A}^{k+1}.$$

In adapted local coordinates, the operators $d^k_{\mathcal{F}}$ act as follows

$$\begin{aligned} d^k_{\mathcal{F}}\Big[\sum \alpha^\ell_{IJ} dx^I \wedge d\bar{z}^J \otimes \frac{\partial}{\partial z^\ell}\Big] &= \sum \Big[d(\alpha^\ell_{IJ} dx^I \wedge d\bar{z}^J)\Big] \otimes \Big[\frac{\partial}{\partial z^\ell}\Big] \\ &= \sum \Big[\big(\frac{\partial \alpha^\ell_{IJ}}{\partial x^i} dx^i + \frac{\partial \alpha^\ell_{IJ}}{\partial \bar{z}^j} d\bar{z}^j\big) \wedge dx^I \wedge d\bar{z}^J \otimes \frac{\partial}{\partial z^\ell}\Big]. \end{aligned}$$

Notice that $\mathcal{A}^0$ is just the space of smooth sections of the vector bundle $\nu^{1,0}$ and that $\mathcal{G}$ is the kernel of the map $d^0_{\mathcal{F}} \colon \mathcal{A}^0 \longrightarrow \mathcal{A}^1$. One has

Proposition 3.1 ([3])**.** *The sequence*

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{d^0_{\mathcal{F}}} \mathcal{A}^1 \xrightarrow{d^1_{\mathcal{F}}} \mathcal{A}^2 \xrightarrow{d^2_{\mathcal{F}}} \dots \longrightarrow \mathcal{A}^{p+q} \longrightarrow 0 \tag{5}$$

is an elliptic complex.

The elements of the quotient Lie algebra $\mathcal{G} = \mathfrak{aut}(\mathcal{F})/\mathfrak{X}_L$ are naturally identified to the holomorphic basic sections of the normal bundle $\nu^{1,0}$, and are called holomorphic basic vector fields. In particular $\mathcal{G}$ is a complex Lie algebra in a natural way. Using the Riemannian metric on M, we can identify $\mathcal{G}$ to the vector space $\mathfrak{X}_N$ of those vector fields in $\mathfrak{X}$ that are orthogonal to $\mathcal{F}$. Notice that, although $\mathfrak{X}_N$ is not a Lie subalgebra of $\mathfrak{X}$, there is a vector space decomposition $\mathfrak{X} = \mathfrak{X}_L \oplus \mathfrak{X}_N$.

As a corollary of Proposition 3.1 one obtains the following result, which was also proved by a different method by X. Gómez-Mont [5].

Proposition 3.2 (Gómez-Mont, and Duchamp and Kalka)**.** *Let $\mathcal{F}$ be a transversely holomorphic foliation on a compact manifold M. The Lie algebra $\mathcal{G}$ of holomorphic basic vector fields has finite dimension.*

We denote by G the simply-connected complex Lie group whose Lie algebra is $\mathcal{G}$.

Let $\mathcal{D}_{L,0}$ denote the connected component of $\mathcal{D}_L$ containing the identity. Our main result concerning the structure of $\mathrm{Aut}(\mathcal{F})$ is the following

Theorem 3.3. *Let $\mathcal{F}$ be a transversely holomorphic foliation on a compact manifold M. Endowed with the topology induced by $\mathcal{D}$, the automorphism group $\mathrm{Aut}(\mathcal{F})$ of automorphisms of $\mathcal{F}$ is a closed, strong ILH-Lie subgroup of $\mathcal{D}$ with Lie algebra $\mathfrak{aut}(\mathcal{F})$. Moreover, the left cosets of the subgroup $\mathcal{D}_{L,0}$ define a Lie foliation $\mathcal{F}_{\mathcal{D}}$ on $\mathrm{Aut}(\mathcal{F})$, which is transversely modeled on the simply-connected complex Lie group G associated to the Lie algebra $\mathcal{G}$.*

We recall that a Lie foliation, modeled on a Lie group G, is defined by local submersions Φ_i with values on G fulfilling $\Phi_j = L_{\gamma_{ji}} \circ \Phi_i$, where γ_{ij} is a locally constant function with values in G and $L_{\gamma_{ij}}$ denotes left translation by γ_{ij}.

The proof of the above theorem will be given in several steps:

1) First, we use Theorem 2.5 to show that there is a strong ILH-Lie subgroup $\mathrm{Aut}'(\mathcal{F})$ of $\mathcal{D}$, contained in $\mathrm{Aut}(\mathcal{F})$ and whose Lie algebra is $\mathfrak{aut}(\mathcal{F})$. Note that this is not sufficient to assure that $\mathrm{Aut}'(\mathcal{F})$ coincides with the connected component of the identity of $\mathrm{Aut}(\mathcal{F})$.
2) We consider the group $\mathrm{Aut}(\mathcal{F})$ endowed with the topology induced by $\mathcal{D}$ and the group $\mathcal{D}_L$ with the Fréchet topology given by Proposition 2.7, and we fix small enough neighborhoods of the identity $\mathcal{V}$ and $\mathcal{V}_L$ of these two groups. We construct a continuous map $\Phi\colon \mathcal{V} \to \mathcal{D}$ with image $\tilde{\Sigma} = \Phi(\mathcal{V})$ contained in $\mathrm{Aut}(\mathcal{F})$ and with the property that $\mathcal{V}$ is homeomorphic to the product $\mathcal{V}_L \times \tilde{\Sigma}$.
3) We show that $\tilde{\Sigma}$ is naturally identified to a neigbourhood Σ of e in the Lie group G. This implies that the induced topology of $\mathrm{Aut}(\mathcal{F})$ is LPSAC and reasoning as in the proof of Proposition 2.7 we conclude that, with the induced topology, $\mathrm{Aut}(\mathcal{F})$ is a closed strong ILH-Lie subgroup of $\mathcal{D}$, proving the first part of the Theorem.
4) Finally, we prove that the map Φ is smooth with respect to the Fréchet structure of $\mathrm{Aut}(\mathcal{F})$ and that the identification of $\mathcal{V}$ with $\mathcal{V}_L \times \Sigma$ is a local chart of $\mathrm{Aut}(\mathcal{F})$ defining the Lie foliation in a neighborhood of the identity. A similar construction of local charts around any element of $\mathrm{Aut}(\mathcal{F})$ completes the proof.

Thus we begin with showing the following statement.

Proposition 3.4. *There is a strong ILH-Lie subgroup* $\mathrm{Aut}'(\mathcal{F})$ *of* $\mathcal{D}$ *whose Lie algebra is* $\mathfrak{aut}(\mathcal{F})$*. It is contained in* $\mathrm{Aut}(\mathcal{F})$*.*

Proof. Let $\pi\colon \mathfrak{X} = \Gamma(TM) \to \mathcal{A}^0 = \Gamma(\nu^{0,1})$ be the natural projection. That is, if $\xi \in \mathfrak{X}$ is written in local adapted coordinates as in (4), then

$$\pi(\xi) = \sum_j b^j(x, z, \bar{z}) \frac{\partial}{\partial z^j}.$$

Then π is a differential operator of order 0 and its adjoint operator π^*, constructed using suitable Riemannian metrics on TM and $\nu^{0,1}$, is also a differential operator of order 0 such that $\pi\pi^* = \mathrm{Id}$.

We define the operator $A\colon \Gamma(TM) \to \mathcal{A}^1$ as the composition $A = d^0_{\mathcal{F}} \circ \pi$ and we set $B = d^1_{\mathcal{F}}$. Then one has $BA = 0$ and $\ker A = \mathfrak{aut}(\mathcal{F})$. Moreover the operator

$$A\,A^* + B^*\,B = d^0_{\mathcal{F}} \circ \delta^0 + \delta^1 \circ d^1_{\mathcal{F}},$$

where δ^k are formal adjoints of $d^k_{\mathcal{F}}$, is elliptic because of Proposition 3.1. Then Theorem 2.5 proves the existence of a strong ILH-Lie subgroup $\mathrm{Aut}'(\mathcal{F})$ of $\mathcal{D}$ with Lie algebra is $\mathfrak{aut}(\mathcal{F})$. The fact that $\mathrm{Aut}'(\mathcal{F})$ is an integral submanifold of the distribution $T\mathcal{D}$ obtained by right translation of $\mathfrak{aut}(\mathcal{F})$ implies that each smooth curve $c = c(t)$ in $\mathrm{Aut}'(\mathcal{F})$ starting at the identity fulfills an equation of type

$$\dot{c}(t) = \xi_t \circ c(t)$$

where ξ_t is a curve in $\mathfrak{aut}(\mathcal{F})$. This implies that each diffeomorphism $c(t)$ preserves the transversely holomorphic foliation $\mathcal{F}$ showing that $\mathrm{Aut}'(\mathcal{F})$ is contained in $\mathrm{Aut}(\mathcal{F})$. □

Remark 3.5. The above argument also shows that the Sobolev completion $\mathrm{Aut}'^k(\mathcal{F})$ of $\mathrm{Aut}'(\mathcal{F})$ is also a subgroup of $\mathrm{Aut}^k(\mathcal{F})$.

Proposition 3.6. *Let us consider* $\mathrm{Aut}(\mathcal{F})$ *endowed with the topology induced by* $\mathcal{D}$*, and* $\mathcal{D}_L$ *endowed with the Fréchet topology given by Proposition 2.7. There are open neighborhoods of the identity* $\mathcal{V} \subset \mathrm{Aut}(\mathcal{F})$ *and* $\mathcal{V}_L \subset \mathcal{D}_L$*, and a continuous map* $\Phi \colon \mathcal{V} \to \mathcal{D}$*, with image* $\tilde{\Sigma} = \Phi(\mathcal{V})$ *contained in* $\mathrm{Aut}(\mathcal{F})$*, such that the multiplication map*

$$\begin{array}{cccc} \Psi\colon & \mathcal{V}_L \times \tilde{\Sigma} & \longrightarrow & \mathcal{V} \\ & (f, s) & \mapsto & f \circ s \end{array} \tag{6}$$

is a homeomorphism and the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_L \times \tilde{\Sigma} & \xrightarrow{\Psi} & \mathcal{V} \\ {\scriptstyle pr_2}\downarrow & \swarrow{\scriptstyle \Phi} & \\ \tilde{\Sigma} & & \end{array} \tag{7}$$

Proof. The idea for proving this proposition is to associate, to any given diffeomorphism $f \in \mathrm{Aut}(\mathcal{F})$ close enough to the identity, the element $\bar{f} \in \mathrm{Aut}(\mathcal{F})$ fulfilling the following property: for each w, $\bar{f}(w)$ is the point of the local slice of the foliation through $f(w)$ which is at the minimal distance from w. The mapping $\bar{f}$ is uniquely determined by that condition and the diffeomorphisms f and $\bar{f}$ differ by an element of $\mathcal{D}_L$ close to the identity. Then $\tilde{\Sigma}$ will be the set of the diffeomorphisms $\bar{f}$ obtained in this way. In order to carry out this construction in a rigorous way we need to introduce appropriate families of adapted local charts.

We say that a local chart (U, φ) of M adapted to $\mathcal{F}$ is cubic if its image $\varphi(U) \subset \mathbb{R}^p \times \mathbb{C}^q$ is the product $C \times \Delta$ of a cube $C \subset \mathbb{R}^p$ and a polydisc $\Delta \subset \mathbb{C}^q$, and we say that it is centered at $w_0 \in U$ if $\varphi(w_0) = (0, 0)$. The submersion $\phi = pr_2 \circ \varphi$ maps U onto Δ and the slices $S_z = \phi^{-1}(z)$ in U are copies of C.

Let $N\mathcal{F} \cong \nu\mathcal{F}$ be the subbundle of TM orthogonal to $T\mathcal{F}$ and let $p \colon N\mathcal{F} \to M$ be the natural projection. We denote by $\exp \colon N\mathcal{F} \to M$ the geodesic exponential map associated to the Riemannian metric g and we set $W_\epsilon = \{(w, v) \in N\mathcal{F} \mid |v| < \epsilon\}$. Since M is compact there is $\epsilon_1 > 0$ such that $T_w = \exp(p^{-1}(w) \cap W_{\epsilon_1})$ is a submanifold transverse to $\mathcal{F}$ for each $w \in M$.

Given a cubic adapted local chart (U, φ), we set $W_{\epsilon,U} = W_\epsilon \cap p^{-1}(U)$ and we define $\psi_U \colon W_{\epsilon,U} \to \Delta \times M$ as the map

$$\psi_U(w, v) = (\phi(w), \exp(w, v)). \tag{8}$$

Then there are constants $0 < \epsilon_0 < \epsilon_1$ and $0 < \delta_0$ such that, if the diameter of U fulfills $\mathrm{diam}\, U < \delta_0$, then ψ_U is a diffeomorphism from $W_{\epsilon_0,U}$ onto its image. We fix once and for all the values ϵ_0, δ_0 and we set $W_U = W_{\epsilon_0, U}$. We denote by W_U^z the restriction of W_U to the slice S_z, i.e. $W_U^z = \{(w, v) \in W_U \mid \phi(w) = z\}$.

Let (U, φ) be a cubic adapted chart centered at w_0 and U' an open subset of U. We say that the pair (U, U') is *regular* if the following conditions are fulfilled:

(R1) $\operatorname{diam} U < \delta_0$,

(R2) $w_0 \in U' \subset \overline{U}' \subset U$ and $(U', \varphi|_{U'})$ is a cubic chart,

(R3) $\overline{U}'$ is contained in the image $\exp(W_U^z)$ for each $z \in \Delta$.

If the pair (U, U') is regular then the intersection of a slice S_z of U with U' is either empty or coincides with a slice of U'. Condition (R3) implies that, for each point $w \in U'$ and each slice S_z of U meeting U', there is a unique point $w' \in S_z$ which is at the minimal distance between w and S_z, namely

$$w' = p \circ \psi_U^{-1}(z, w). \tag{9}$$

Notice also that the submersion ϕ maps the intersection $T_w^{U'} = T_w \cap U'$ diffeomorphically onto the open polydisc $\phi(U') \subset \mathbb{C}^q$.

Since M is compact, we can find positive numbers $0 < \delta' < \delta < \delta_0$, a finite family of adapted and cubic local charts $\{(U_i, \varphi_i)\}_{i=1,\dots,m}$, centered at points w_i, and a family of 4-tuples

$$\mathcal{U} = \{(U_i, U_i', V_i, V_i')\}_{i=1,\dots,m} \tag{10}$$

of open sets such that

(C1) (U_i, U_i') is a regular pair fulfilling $\operatorname{diam} U_i < \delta$ and $\delta' < d(w_i, \partial U_i')$,

(C2) $\overline{V}_i \subset U_i'$, $(V_i, \varphi_i|_{V_i})$ is a cubic adapted local chart centered at w_i, $\operatorname{diam} V_i < \delta'/4$ and (V_i, V_i') is a regular pair,

(C3) the family $\{V_i'\}_{i=1,\dots,m}$ is an open covering of M.

We then set $\psi_i = \psi_{U_i}$, $W_i = W_{U_i}$ and $W_i^z = W_{U_i}^z$. We remark that such a covering has the following properties:

(P1) If $V_i \cap V_j \neq \emptyset$ then $\overline{V}_i \cup \overline{V}_j \subset U_i' \cap U_j'$,

(P2) given a point $w \in V_i'$ and a slice S_z in U_i which meets V_i, the point $w' \in S_z$ that is at the minimal distance from w belongs to V_i.

We fix from now on such a family $\mathcal{U}$. We denote by $\mathcal{V}$ the open neighborhood of the identity in $\operatorname{Aut}(\mathcal{F})$ defined by $f \in \mathcal{V}$ if and only if $f(\overline{V}_i') \subset V_i$ for each $i = 1, \dots m$. We also define the subset $\mathcal{V}_L$ of $\mathcal{D}_L$ of those elements $f \in \mathcal{D}_L$ fulfilling $f(\overline{V}_i') \subset V_i$ for each $i = 1, \dots m$ and keeping fixed each slice of V_i' (recall that a leaf of $\mathcal{F}$ can cut V_i in many slices). The set $\mathcal{V}_L$ is open in the Fréchet topology of $\mathcal{D}_L$. We are now ready to define the map $\Phi\colon \mathcal{V} \to \operatorname{Aut}(\mathcal{F}) \subset \mathcal{D}$ and the set $\tilde{\Sigma} = \Phi(\mathcal{V})$.

Let f be a fixed element in $\mathcal{V}$. For a given point $w \in M$ we choose $i = 1, \dots, k$ with $w \in V_i'$. Then $\hat{w} = f(w) \in V_i$ and we denote by S_z and $S_{\hat{z}}$ the slices of U_i through the points w and $\hat{w}$ respectively, i.e. $z = \phi_i(w)$ and $\hat{z} = \phi_i(f(w))$. Then $S_{\hat{z}}$ meets $V_i \subset U_i'$ and, since the pair (U_i, U_i') is regular, there is a uniquely determined point $w' \in S_{\hat{z}}$ which is at the minimal distance from $S_{\hat{z}}$ to w, moreover $w' \in V_i$. We then define a local map $\bar{f}\colon V_i' \to V_i \subset M$ by $\bar{f}(w) = w'$. It follows from properties (P1) and (P2) that the above definition does not depend on the choice of $i = 1, \dots, m$.

Indeed, if $w \in V'_i \cap V'_j$ then the points w' constructed using U_i or U_j belong to $V_i \cap V_j$ and are necessarily the same. This fact implies that the map $\bar{f}$ is globally defined in the entire manifold M.

Since f preserves leaves and is transversely holomorphic, it induces a local holomorphic transformation $\hat{f}_i$ between open subsets of $\Delta_i = \phi_i(U_i) \subset \mathbb{C}^n$. Notice that, by construction, $\bar{f}$ sends leaves into leaves and that the local transverse part of $\bar{f}$ in U_i is just $\hat{f}_i$.

Let us prove that $\bar{f}$ is a smooth diffeomorphism. We first notice that, if $w' = \bar{f}(w)$ with $w \in V'_i$, then the geodesic joining w and w' and giving the minimal distance between w and the slice $S_{z'}$, where $z' = \phi_i(w')$, meets $S_{z'}$ orthogonally. By applying (9), we have

$$\bar{f}(w) = w' = p \circ \psi_i^{-1}(\phi_i(f(w)), w), \tag{11}$$

Using the local transverse part $\hat{f}_i$ of f the above equality can be written as

$$\bar{f}(w) = w' = p\big(\psi_i^{-1}(\hat{f}_i(\phi_i(w)), w)\big). \tag{12}$$

This equality proves that $\bar{f}$ is smooth and that $\bar{f}$ is entirely determined by the set of local transformations $\{\hat{f}_i\}$.

Restricted to V'_i the map $\bar{f}$ has an inverse that can be described as follows. Given $w' = \bar{f}(w) \in V_i$ we put $z' = \phi_i(w')$ and $z = \phi_i(w) = \hat{f}_i^{-1}(z')$. As pointed out before, $\exp(N_{w'}\mathcal{F} \cap W_i)$ meets the slice S_z and the intersection is precisely the point $w \in S_z$. In fact, there is a unique vector $v = v(w') \in N_{w'}\mathcal{F} \cap W_i$ such that $w = \exp(w', v)$. Therefore we can write

$$w = \bar{f}^{-1}(w') = \exp(w', v) = pr_2 \circ \psi_i(w', v(w'))$$

and, since ψ_i is a diffeomorphism, the vector $v = v(w')$ depends smoothly on w'. Indeed, if we set $T = \exp(N_{w'}\mathcal{F} \cap W_i) \cap V_i$ then $\phi_i|_T$ maps T isomorphically onto $\phi_i(V_i) \subset \Delta_i$ and we have

$$(w', v(w')) = \psi_i^{-1}\big(z, (\phi_i|_T)^{-1}(z)\big).$$

We deduce in particular that $\bar{f}$ is locally injective and that its (local) inverse is smooth. This also proves that the map $\bar{f}$ is open and, as the manifold M is compact, $\bar{f}$ is necessarily surjective.

We remark that formula (12) also shows that $\bar{f}$ is entirely determined by the set of local transformations $\{\hat{f}_i\}$. That formula also shows that $\bar{f}$ depends continuously on $f \in \mathcal{U}$. Since the transverse part of $\bar{f}$ is just $\hat{f}$, the map $\bar{f}$ is transversely holomorphic. It remains to prove that $\bar{f}$ is globally injective and therefore an element of $\mathrm{Aut}(\mathcal{F}) \subset \mathcal{D}$.

Assume that $w' = \bar{f}(w_1) = \bar{f}(w_2)$ and choose $i, j \in \{1, \ldots, m\}$ with $w_1 \in V'_i$ and $w_2 \in V'_j$. Then $V_i \cap V_j \neq \emptyset$ and therefore $V_i \cap V_j \subset U'_i$. We deduce that w_1, w_2 belong to the same slice S_z of U_i where $z = \hat{f}_i^{-1}(\phi_i(w'))$. Now w_1, w_2 are in the intersection of S_z with the submanifold $\exp(N_{w'}\mathcal{F} \cap W_i) \cap V_i$. But this intersection reduces to a unique point, which shows that $w_1 = w_2$.

Summarizing the above considerations, we see that the correspondence $f \mapsto \bar{f} = \Phi(f)$ given by (12) defines a continuous map $\Phi \colon \mathcal{V} \to \mathcal{D}$ with image $\tilde{\Sigma} = \Phi(\mathcal{V})$ contained in $\mathrm{Aut}(\mathcal{F})$. By construction, the diffeomorphism $f_L := f \circ \bar{f}^{-1}$ is an element of $\mathcal{V}_L$ and Φ maps $\mathcal{V}_L$ to the identity. It is clear from the construction that each element $f \in \mathcal{V}$ decomposes in a unique way as the composition

$$f = f_L \circ \bar{f}$$

with $\bar{f} = \Phi(f) \in \tilde{\Sigma}$ and $f_L \in \mathcal{V}_L$ and that $\bar{f}$ and f_L depend continuously on f, with respect to the Fréchet topology. This tells us that the composition map $\Psi\colon \mathcal{V}_L \times \tilde{\Sigma} \to \mathcal{V}$ defined in (6) has a continuous inverse. Therefore, by shrinking the set $\mathcal{V}_L$ if it is necessary, the map Φ is a homeomorphism and the diagram (7) is commutative. □

Remark 3.7. Notice that $\tilde{\Sigma}$ is not necessarily closed under composition or taking inverses, even for elements close to the identity.

Remark 3.8. The construction carried out in the proof of the above proposition also provides a commutative diagram of continuous maps

$$\begin{array}{ccc} \mathcal{V}_L^k \times \tilde{\Sigma}^k & \xrightarrow{\Psi} & \mathcal{V}^k \\ {\scriptstyle pr_2}\big\downarrow & \swarrow{\scriptstyle \Phi} & \\ \tilde{\Sigma}^k & & \end{array} \qquad (13)$$

where $\mathcal{V}^k$ and $\mathcal{V}_L^k$ are open neighborhoods of the identity in the Sobolev completions $\mathrm{Aut}^k(\mathcal{F})$ and $\mathcal{D}_L^k$ respectively, the maps Φ and Ψ are defined in the same way and Ψ is a homeomorphism.

Remark 3.9. Notice that the correspondence $f \mapsto \bar{f} = \Phi(f)$ carried out in the above theorem is well defined for any $f \in \mathcal{D}_{\mathcal{F}}$ close enough to the identity since the transverse holomorphy of the diffeomorphisms plays no role in the construction of Φ. However, for a general foliation, nothing can be said about the properties of the image $\tilde{\Sigma}$ of the map so constructed.

The following proposition states that, after shrinking it, if necessary, the set $\tilde{\Sigma}$ is naturally identified with an open neighborhood of the unity in the simply-connected Lie group G associated to the Lie algebra $\mathcal{G}$. More precisely, if we identify $\mathcal{G}$ with the space $\mathfrak{X}_{N,b}$ of transversely holomorphic basic vector fields on M that are orthogonal to $\mathcal{F}$, then we have:

Proposition 3.10. *There is a neighborhood Σ of the unity in G and a homeomorphism $\sigma\colon \Sigma \to \tilde{\Sigma}$ such that the composition*

$$\hat{\Phi} := \sigma^{-1} \circ \Phi\colon \mathcal{V} \to \Sigma$$

preserves multiplication and inverses, i.e. $\hat{\Phi}$ is a morphism of local topological groups. Moreover, the composition

$$U \subset \mathcal{G} \equiv \mathfrak{X}_{N,b} \to \Sigma,$$

defined as $[\xi] \to \hat{\Phi} \circ \varphi_1^{\xi}$ for vector fields ξ in $\mathfrak{X}_{N,b}$ close to zero, is the restriction to a neighborhood of zero of the natural exponential map $\mathcal{G} \to G$.

Here, φ_t^{ξ} stands for the one-parameter group associated to ξ.

Proof. We assume, as in the proof of the above proposition, that a finite family of adapted and cubic local charts $\{(U_i, \varphi_i)\}_{i=1,\dots,m}$, and a family of 4-tuples $\{(U_i, U_i', V_i, V_i')\}_{i=1,\dots,m}$ fulfilling conditions C1, C2 and C3, have been fixed. As before we set $\phi = pr_2 \circ \varphi$ and we denote by Δ_i the polydisc

$\phi_i(U_i)$ of $\mathbb{C}^q$. We also set $D_i = \phi_i(V_i)$ and $D'_i = \phi_i(V'_i)$. Let B_i be the Banach space of continuous maps $f\colon \bar{D}_i \to \mathbb{C}^q$ that are holomorphic on D_i with the norm

$$|f|_i = \max_{z\in\bar{D}_i} |f(z)|.$$

Then the space $\mathcal{B} = \oplus_i B_i$ with the norm

$$\|F\| = \|(f_1, \dots, f_m)\| = \sum_i |f_i|$$

is also a Banach space. Notice that $I = (\mathrm{Id}_1, \dots, \mathrm{Id}_m)$, where Id_i denotes the identity map of D_i, is an element of $\mathcal{B}$.

We claim that there is an open neighborhood $\mathcal{I}_\epsilon = \{F \in \mathcal{B} \mid \|F - I\| < \epsilon\}$ of I in $\mathcal{B}$ such that each $F = (f_1, \dots, f_m) \in \mathcal{I}_\epsilon$ fulfills: (i) $f_i(\bar{D}'_i) \subset D_i$ and (ii) $f_i|_{D'_i}$ is a biholomorphism onto its image for $i = 1, \dots, m$. Indeed, we know from Cauchy's inequalities that, for a given $\epsilon' > 0$ there is $\epsilon > 0$ such that $\|F - I\| < \epsilon$ implies that $\|f_i - \mathrm{Id}_i\|_{C^1} < \epsilon'$, where $\|\cdot\|_{C^1}$ stands for the C^1-norm on the polydisc D'_i. Using the mean value theorem we see that, for $z, z' \in D'_i$, one has

$$|f_i(z) - f_i(z')| \geq c\,|z - z'| \tag{14}$$

for a suitable constant $c > 0$. Inequality (14) implies that the restriction of f_i to the polydisc D'_i is injective and it is known that a one-to-one holomorphic map is necessarily a biholomorphism onto its image. Therefore, there is $\epsilon < 0$ such that condition (ii) holds for $F \in \mathcal{I}_\epsilon$ and condition (i) is also fulfilled if ϵ is small enough. We assume that such an ϵ, and therefore the open subset $\mathcal{I}_\epsilon$ of $\mathcal{B}$, have been fixed.

We consider now the holonomy pseudogroup $\mathcal{H}$ of the transversely holomorphic foliation $\mathcal{F}$. As a total transversal of $\mathcal{F}$ we choose the union $T = \cup T_i$ of disjoint submanifolds T_i of X of dimension $2q$ and transverse to $\mathcal{F}$ with the property that $T_i \subset V'_i$ and $\phi_i(T_i) = D'_i$. Notice that T is naturally endowed with a complex structure. We identify T, through the maps ϕ_i, with the disjoint union of open polydiscs $T \equiv \amalg D'_i$. The holonomy of the foliation $\mathcal{F}$ induces local transformations between open subsets of the total transversal T. They are holomorphic and $\mathcal{H}$ is the pseudogroup generated by these holonomy transformations. Since the manifold X is compact, $\mathcal{H}$ is a compactly generated pseudogroup. This means that $\mathcal{H}$ is generated by a finite number of transformations of the pseudogroup, $h_\mu\colon W_\mu \to T$ for $\mu = 1, \dots, \ell$, with the following properties: W_μ is a relatively compact subset of some D'_i and h_μ has an extension to an open neigborhood of $\overline{W}_\mu$ in D'_i such that $h_\mu(\overline{W}_\mu) \subset D'_j$ for some $1 \leq j \leq m$ (cf. [6]).

We define $\mathcal{S}$ as the subset of $\mathcal{I}_\epsilon$ of those $F = (f_1, \dots, f_m)$ fulfilling

$$f_j^{-1} \circ h_\mu \circ f_i = h_\mu \tag{15}$$

in the common domain of definition. Clearly, $\mathcal{S}$ is closed in $\mathcal{I}_\epsilon$. We notice that each element $F = (f_1, \dots, f_m) \in \mathcal{S}$ induces a well-defined diffeomorphism $\sigma(F)$ belonging to $\tilde{\Sigma}$. This correspondence $\sigma\colon \mathcal{S} \to \tilde{\Sigma}$ can be constructed as follows. Formula (12), where we replace $\hat{f}_i$ by the corresponding component f_i of F, determines a (local) foliation preserving diffeomorphism $\bar{f}_i\colon V'_i \to V_i$ whose transverse part is just f_i. The identities (15) guarantee that formula (12) applied to the family $\{f_i\}$ provides a well defined diffeomorphism $f = \sigma(F)$ of M belonging to $\tilde{\Sigma}$. Clearly σ maps $\mathcal{S}$ homeomorphically onto its image $\tilde{\Sigma}' = \sigma(\mathcal{S})$. The fact that $f = \sigma(F)$ is a diffeomorphism globally defined over M implies in particular that each one of the maps f_i extends holomorphically to the polydisc $\Delta_i = \phi_i(U_i)$ and maps $\phi_i(U_i')$ biholomorphically onto an open subset of Δ_i. From that fact one deduces easily that $\mathcal{S}$ is a local topological group (for the general properties of local topological groups we refer to [19]). Moreover, if we set $\mathcal{V}' = \Phi^{-1}(\tilde{\Sigma}')$ then the composition

$$\hat{\Phi} := \sigma^{-1} \circ \Phi \colon \mathcal{V}' \to \mathcal{S}$$

can be seen as the map which associates its transverse part $\hat{f}$ to each diffeomorphism f in $\mathrm{Aut}(\mathcal{F})$ which is close enough to the identity. It follows in particular that the map $\hat{\Phi}$ preserves multiplication and inverses. It is therefore a morphism of local groups.

The end of the proof will be based on the following fact.

Lemma 3.11. *The local group $\mathcal{S}$ is locally compact.*

Proof. Let us fix $0 < \epsilon' < \epsilon$. It is sufficient to prove that the neighborhood $\mathcal{S}_{\epsilon'}$ of I defined by $\|F - I\| \leq \epsilon'$ is compact. Let $\{F_k = (f_{k,1}, \dots, f_{k,m})\}$ be a sequence of elements of $\mathcal{S}_{\epsilon'}$. As noticed before, we can think of each $f_{k,i}$ as a holomorphic transformation defined on Δ_i that sends $\phi_i(U_i')$ into Δ_i. By taking ϵ' small enough, we can also assume that $f_{k,i}(\phi_i(U_i'))$ contains D_i. It follows from Montel's theorem that there is a subsequence $\{F_{k_j}\}$ of $\{F_k\}$ such that each $f_{k_j,i}$ converges uniformly to a holomorphic map $\tilde{f}_i$. Since $f_{k,i}$ are invertible, we can suppose that the sequences $\{f_{k_j}^{-1}\}$, that are defined on D_i, also converge to a limit which necessarily is $\tilde{f}_i^{-1}$. This proves that $\{F_{k_j}\}$ has a limit that belongs to $\mathcal{S}_{\epsilon'}$. □

End of proof of Proposition 3.10. Now we are in position to apply Theorem A in [1] to the local group $\mathcal{S}$. Although the result by Bochner and Montgomery is stated in the context of topological groups, the arguments used there are purely local and therefore they still remain valid for locally compact local groups of transformations. It implies in particular that $\mathcal{S}$ is a local Lie group, i.e. $\mathcal{S}$ is isomorphic to a neighborhood of the identity of a certain Lie group.

Finally, we claim that this Lie group is just the 1-connected Lie group G associated to the Lie algebra $\mathcal{G}$. Indeed, each element ξ of $\mathfrak{X}_{N,b} \equiv \mathcal{G}$ induces a one parameter group φ_t^{ξ} that is sent by $\hat{\Phi}$ (and for small t) into a local one parameter subgroup of $\mathcal{S}$ which does not reduce to the identity unless $\xi = 0$. Conversely, each local one parameter subgroup of $\mathcal{S}$ is obtained in this way. Hence $\mathcal{S}$ is naturally identified to a neighborhood Σ of the unity in G. The last assertion of the proposition is clear from the above discussion. □

Remark 3.12. The map σ constructed in the above proposition also identifies Σ with the set $\tilde{\Sigma}^k = \Phi(\mathcal{V}^k)$ considered in Remark 3.8

Remark 3.13. The following criterium is useful to assure that certain maps between Fréchet manifolds are smooth (cf. [7]). Let V and V' be fibre bundles over a compact manifold X, eventually with boundary, and let $\Gamma(V)$

and $\Gamma(V')$ denote the Fréchet manifolds of the smooth sections of V and V' respectively. Assume that there is an open subset $U \subset V$ meeting all the fibres of V and a fibrewise smooth map $F \colon U \to V'$ projecting onto the identity of X. Let $\tilde{U}$ be the subset of $\Gamma(V)$ of those sections s having image in U. Then the map $\chi \colon \tilde{U} \subset \Gamma(V) \to \Gamma(V')$ given by $s \mapsto \chi(s) = F \circ s$ is a smooth map of Fréchet manifolds. Moreover, assume that $\mathcal{E}$ and $\mathcal{E}'$ are Fréchet submanifolds of $\Gamma(V)$ and $\Gamma(V')$ respectively with $\mathcal{E} \subset \tilde{U}$ and such that $\chi(\mathcal{E}) \subset \mathcal{E}'$, then the restricted map $\chi \colon \mathcal{E} \to \mathcal{E}'$ is also smooth.

Combining the above two propositions we obtain the commutative diagram

(16)
$$\begin{array}{ccc} \mathcal{V}_L \times \Sigma & \xrightarrow{\hat{\Psi}} & \mathcal{V} \\ {\scriptstyle pr_2}\downarrow & \swarrow {\scriptstyle \hat{\Phi}} & \\ \Sigma & & \end{array}$$

As a corollary we deduce the following

Proposition 3.14. *With the induced topology,* $\mathrm{Aut}(\mathcal{F})$ *is a closed strong ILH-Lie subgroup of* $\mathcal{D}$ *with Lie algebra* $\mathfrak{aut}(\mathcal{F})$*.*

Proof. Notice first that $\mathrm{Aut}(\mathcal{F})$ is a closed subgroup of $\mathcal{D}$. Using the above commutative diagram as well as Remarks 3.8 and 3.12, and arguing as in the proof of Proposition 2.7, we see that $\mathrm{Aut}'^k(\mathcal{F})$ coincides with $\mathrm{Aut}_0^k(\mathcal{F})$ (the connected component of $\mathrm{Aut}^k(\mathcal{F})$ containing the identity). This implies that $\mathrm{Aut}'(\mathcal{F})$ is the connected component of the identity of $\mathrm{Aut}(\mathcal{F})$ concluding the proof. □

End of proof of Theorem 3.3. We first prove that the map $\hat{\Phi}$ in diagram (16) is smooth. Let $\{(U_i, \varphi_i)\}_{i=1,\dots,m}$ be a finite family of adapted and cubic local charts of M and let $\{(U_i, U_i')\}$ be a family of regular pairs with the properties that $\{U_i'\}$ is an open cover of M and $\phi_i(U_i) = D_i$ and $\phi_i(U_i') = D_i'$ are open polydiscs. We choose a family $\{T_i\}$ of disjoint transversals $T_i \subset U_i$ of $\mathcal{F}$ such that $\phi_i(T_i) = D_i$. We set $T_i' = T_i \cap \bar{U}_i'$ and we identify $T = \amalg T_i'$, through the maps ϕ_i, with the disjoint union of closed polydiscs $T = \amalg \bar{D}_i'$. The manifolds $V = \amalg (T_i' \times U_i)$ and $W = \amalg (T_i' \times D_i)$ are fibre bundles over T taking as projection the natural projections onto the first factor. The map $F \colon V \to W$ given by $F(z_i, w_i) = (z_i, \phi_i(w_i))$ is fibrewise and smooth, and it follows from the criterium stated in Remark 3.13 that the map $\chi \colon \Gamma(V) \to \Gamma(W)$, given by $\chi(s) = F \circ s$, is smooth. Notice that diffeomorphisms of M close to the identity can be thought of sections of the fibre bundle $pr_1 \colon M \times M \to M$ with image close to the diagonal. Moreover, for a small enough neighborhood $\mathcal{W}$ of the identity in $\mathcal{D}$, the map

$$\begin{array}{cccc} \tau \colon \mathcal{W} & \longrightarrow & \Gamma(V) \\ f & \mapsto & \{z_i \mapsto (z_i, f(z_i))\} \end{array}$$

is well defined and smooth. If $f \in \mathcal{W}$ belongs to $\mathrm{Aut}(\mathcal{F})$ the family $\{z_i, \phi_i(f(z_i))\} = \chi(\tau(f))$ is an element of the local group $\mathcal{S} \equiv \Sigma$ constructed in the proof of Proposition 3.10. Therefore the map $\hat{\Phi} \colon \mathcal{V} \to \Sigma$ can be written as the restriction to $\mathcal{V} \subset \mathrm{Aut}(\mathcal{F})$ of the composition $\chi \circ \tau$, which proves that it is smooth.

Let us consider now the map $\sigma\colon \Sigma \equiv \mathcal{S} \to \tilde{\Sigma} \subset \mathrm{Aut}(\mathcal{F})$. A reasoning similar to the previous one, using now the local expression of the correspondence $s = \{f_i\} \in \mathcal{S} \mapsto \bar{f} \in \tilde{\Sigma}$ given by the equality (12), shows that σ is smooth and therefore that $\hat{\Psi}\colon \mathcal{V}_L \times \Sigma \to \mathcal{V}$, which is given by the multiplication $\hat{\Psi}(f, s) = f \circ \sigma(s)$, is also smooth. Its inverse $\hat{\Psi}^{-1}$ is just the correspondence $f \mapsto (f_L, f_N)$, where $f_N = \hat{\Phi}(f)$ and $f_L = f \circ f_N^{-1}$, which proves that $\hat{\Psi}$ is a diffeomorphism.

Therefore, the map $\hat{\Psi}\colon \mathcal{V}_L \times \Sigma \to \mathcal{V}$ can be regarded as a local chart of $\mathrm{Aut}(\mathcal{F})$ and the map $\hat{\Phi}\colon \mathcal{V} \to \Sigma$ as a submersion defining the foliation $\mathcal{F}_{\mathcal{D}}$ in a neighborhood of the identity.

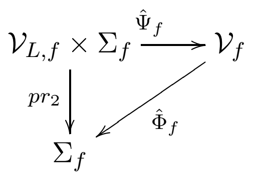
Let $f \in \mathrm{Aut}_0(\mathcal{F})$ be given. As each connected topological group, $\mathrm{Aut}_0(\mathcal{F})$ is generated by a given neighborhood of the identity, in particular by $\mathcal{V} \cong \mathcal{V}_L \times \tilde{\Sigma}$. It follows, using that $\mathcal{D}_L$ is a normal subgroup of $\mathrm{Aut}_0(\mathcal{F})$, that we can write

$$f = f_L \circ f_N = f_L \circ \exp(v_1) \circ \cdots \circ \exp(v_k)$$

where $f_L \in \mathcal{D}_L$ and $v_i \in \mathfrak{X}_{N,b}$ are vector fields with the property that, if we denote $\hat{v}_i = \vartheta(v_i) \in \mathcal{G}$, where $\vartheta\colon \mathfrak{X}_{N,b} \to \mathcal{G}$ is the natural identification, then $\exp(\hat{v}_i) \in \Sigma$. We denote by $\hat{f}_N$ the element of G given by

$$\hat{f}_N = \exp(\hat{v}_1) \circ \cdots \circ \exp(\hat{v}_k)$$

Now, proceeding as before we can prove that there exist neighborhoods $\mathcal{V}_f \subset \mathrm{Aut}(\mathcal{F})$, $\mathcal{V}_{L,f} \subset \mathcal{D}_L$ and $\Sigma_f \subset G$, of f, f_L and $\hat{f}_N$ respectively, and a commutative diagram



where $\hat{\Psi}_f$ is a diffeomorphism providing a local chart of $\mathrm{Aut}(\mathcal{F})$ and $\hat{\Phi}_f$ is a smooth submersion defining $\mathcal{F}_{\mathcal{D}}$ in $\mathcal{V}_f$. Finally, it also follows from the above discussion that, if $\mathcal{V}_{f_i}$ and $\mathcal{V}_{f_j}$ have non-empty intersection, then there are locally constant G-valued functions γ_{ij} fulfilling

$$\hat{\Phi}_{f_i} = L_{\gamma_{ij}} \circ \hat{\Phi}_{f_j}.$$

This ends the proof of the theorem □

4. The automorphism group of a Riemannian foliation

We suppose in this section that the foliation $\mathcal{F}$ on M is Riemannian. This means that the local submersions ϕ_i defining the foliation take values in a Riemannian manifold T and that the transformations γ_{ij} fulfilling the cocycle condition (1) are local isometries of T. In an equivalent way, the foliation $\mathcal{F}$ is Riemannian if there is a Riemannian metric g on M which is bundle-like with respect to $\mathcal{F}$, that is a metric that in local adapted coordinates (x, y) is written

$$g = \sum g_{ij}(x,y)\omega^i\omega^j + \sum g_{ab}(y)dy^a dy^b,$$

where $\{\omega^i, dy^a\}$ is a local basis of 1-forms such that ω^i vanish on the bundle of vectors orthogonal to $\mathcal{F}$.

We suppose that such a bundle-like metric g has been fixed. Let p and q stand for the dimension and the codimension of $\mathcal{F}$ respectively.

In this section $\mathrm{Aut}(\mathcal{F})$ will denote the automorphism group of the Riemannian foliation $\mathcal{F}$, that is the group of elements of $\mathcal{D}_{\mathcal{F}}$ which preserve the transverse Riemannian metric, and by $\mathfrak{aut}(\mathcal{F})$ the Lie algebra of vector fields whose flows are one-parameter subgroups of $\mathrm{Aut}(\mathcal{F})$. Also in this case, $\mathcal{D}_L$ is a normal subgroup of $\mathrm{Aut}(\mathcal{F})$ and $\mathfrak{X}_L$ is an ideal of $\mathfrak{aut}(\mathcal{F})$. We denote by $\mathcal{G} = \mathfrak{aut}(\mathcal{F})/\mathfrak{X}_L$ the quotient Lie algebra. The elements of $\mathcal{G}$ generate local isometries on T, therefore they are called basic Killing vector fields.

Proposition 4.1. *The Lie algebra $\mathcal{G}$ of basic Killing vector fields has finite dimension.*

The above proposition can be proved in the same way as the classical theorem of Myers and Steenrod (cf. [9, 13]). Namely, let $\pi\colon P \to M$ be the $O(q)$-principal fibre bundle of transverse orthonormal frames of the Riemannian foliation $\mathcal{F}$. The total space P is endowed with a foliation $\mathcal{F}_P$ of dimension p that is projected by π to $\mathcal{F}$. The foliation $\mathcal{F}_P$ is transversely parallelizable and the elements of $\mathrm{Aut}(\mathcal{F})$ are in one-to-one correspondence with the automorphisms of the principal fibre bundle which preserve the transverse parallelism and the Riemannian connection. The finiteness of $\mathcal{G}$ then follows from the general fact according to which the automorphism group of a parallelism is finite dimensional.

In particular we can consider the simply connected Lie group associated to the Lie algebra $\mathcal{G}$ that we denote G.

As we already mentioned Leslie proved in [11] that, in the case of a Riemannian foliation, the group $\mathcal{D}_{\mathcal{F}}$ is a Fréchet Lie group. Moreover, using the connection provided by the Riemannian metric, Omori's proof that $\mathcal{D}$ is a strong ILH-Lie group can be adapted to show the following

Proposition 4.2. *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. Then $\mathcal{D}_{\mathcal{F}}$ is an ILH-Lie group with Lie algebra $\mathfrak{X}_{\mathcal{F}}$.*

Our main result in this section is the following theorem.

Theorem 4.3. *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. The automorphism group* $\mathrm{Aut}(\mathcal{F})$ *of $\mathcal{F}$ is closed in $\mathcal{D}_{\mathcal{F}}$ and, with the induced topology, it is a strong ILH-Lie group with Lie algebra* $\mathfrak{aut}(\mathcal{F})$. *Moreover, the left cosets of the subgroup $\mathcal{D}_L$ define a Lie foliation $\mathcal{F}_{\mathcal{D}}$ on* $\mathrm{Aut}(\mathcal{F})$, *which is transversely modeled on the simply-connected complex Lie group G associated to the Lie algebra $\mathcal{G}$.*

Remark 4.4. The above theorem is weaker than Theorem 3.3 as we are not able to exhibit a supplementary to the Lie subalgebra $\mathfrak{aut}(\mathcal{F})$ inside the Lie algebra $\mathfrak{X}$ of $\mathcal{D}$. Consequently we do not show that $\mathrm{Aut}(\mathcal{F})$ is a strong ILH-Lie subgroup of $\mathcal{D}$. Notice however that that Theorem 4.3 implies in particular that the topology of $\mathrm{Aut}(\mathcal{F})$ is second countable and LPSAC.

The proof of the above result is parallel to that of Theorem 3.3, hence we just indicate the differences.

Sketch of proof. We recall first that the correspondence $f \mapsto \bar{f} = \Phi(f)$ given by Proposition 3.6 is well defined for each $f \in \mathcal{D}_{\mathcal{F}}$ close enough to the identity (cf. Remark 3.9). The proof of that proposition shows that there is a commutative diagram of continuous maps

$$\begin{array}{ccc} \mathcal{V}_L \times \tilde{\Sigma} & \xrightarrow{\Psi} & \mathcal{V} \\ {\scriptstyle pr_2}\downarrow & \swarrow{\scriptstyle \Phi} & \\ \tilde{\Sigma} & & \end{array} \tag{17}$$

where $\mathcal{V}$ and $\mathcal{V}_L$ are neighborhoods of the identity in $\mathrm{Aut}(\mathcal{F})$ and $\mathcal{D}_{\mathcal{F}}$ respectively, $\tilde{\Sigma}$ is the image of Φ and the map Ψ, that is given by multiplication, is a homeomorphism. As stated in Remark 3.8 the same proof also provides commutative diagrams of continuous maps between the Sobolev completions of the spaces in (17).

Proceeding as in Proposition 3.10, we consider a complete transversal T to $\mathcal{F}$ which now is endowed with a Riemannian metric. The families of local isometries of T that are close to the identity and that commute with the holonomy pseudogroup of $\mathcal{F}$ are the elements of a local group $\mathcal{S}$ that can be identified to $\tilde{\Sigma}$. We claim that $\mathcal{S}$ is locally compact. In the present setting this fact follows from the following observation that replaces Lemma 3.11. A local isometry is determined, on a connected domain, by its 1-jet at a given point, therefore $\mathcal{S}$ can be embedded in a space of 1-jets over T, which is finite dimensional. Moreover, $\mathcal{S}$ is locally closed in this space of jets and therefore it is locally compact.

The arguments used by Salem in [20] can be applied here to show that $\mathcal{S}$ is in fact a local Lie group, more precisely each element of $\mathcal{S}$ is in the flow of a Killing vector field over T that commutes with the holonomy pseudogroup of $\mathcal{F}$. These Killing vector fields form a Lie algebra naturally isomorphic to $\mathcal{G}$ and therefore $\tilde{\Sigma}$ is identified, through a suitable map σ, to a neighborhood Σ of the neutral element of the simply connected Lie group G. Notice that isometries of a smooth Riemannian metric are necessarily of class C^∞. Therefore, and as it happens in Remark 3.12, the map σ also identifies Σ with the set $\tilde{\Sigma}^k = \Phi(\mathcal{V}^k)$ for k big enough.

The maps $\hat{\Psi}\colon \mathcal{V}_L \times \Sigma \to \mathcal{V}$ and $\hat{\Psi}\colon \mathcal{V}_L^k \times \Sigma \to \mathcal{V}^k$ can be regarded as local charts of $\mathrm{Aut}(\mathcal{F})$ and $\mathrm{Aut}^k(\mathcal{F})$ respectively. Moreover the map $\hat{\Phi}\colon \mathcal{V} \to \Sigma$ is a submersion defining the foliation $\mathcal{F}_{\mathcal{D}}$ in a neighborhood of the identity. In a similar way as in the proof of Theorem 3.3, one can construct atlases of adapted local charts for $\mathrm{Aut}(\mathcal{F})$ and $\mathrm{Aut}^k(\mathcal{F})$. In particular the groups $\mathrm{Aut}^k(\mathcal{F})$ turn out to be Hilbert manifolds showing that $\mathrm{Aut}(\mathcal{F})$ is an ILH-Lie group. This ends the proof. □

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Laurent Meersseman, LAREMA, Université d'Angers, F-49045 Angers, France, meersseman@univ-angers.fr

Marcel Nicolau, Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Cerdanyola, Spain, nicolau@mat.uab.cat

Javier Ribón, Instituto de Matemática, Universidade Federal Fluminense, Niterói, Rio de Janeiro, Brasil, jribon@id.uff.br