Commuter of operators in a Hilbert space

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Abstract

Commutativity between two stationary functions is a notion that generalizes the concept of stationary correlation. Two stationary functions commute if and only if their associated spectral measures and unitary operators commute. When the operators do not commute, we define the concept of commuter for two operators, and then derive it for two projectors, for a projector and a unitary operator, and for two unitary operators. We establish relations between these commuters and some other tools related to proximity between processes. Finally, we propose a method which retrieves the commuter for two stationary series of finite spectrum, and we study some convergence properties.

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AMS subject classification: 60G57, 60G10, 60B15, 60H05

1. Introduction

Methods for comparing sets of curves are largely rooted in the comparison of populations through factor analysis; see, e.g., [2, 12]. Spectral theory is well suited to the task when the curves stem from the observation of processes. This approach has been used successfully to study many issues dealing with, e.g., autoregressive processes [3], dimension reduction [4, 5], and large deviation theory [14]. To the best of our knowledge, however, the use of spectral elements for comparing two sets of functions has been scant. This is in spite of their obvious relevance for issues such as extracting common and specific features for two sets of curves, or retrieving all the curves with a common shape.

To avoid time dependence issues, curve comparison issues are typically addressed in the frequency domain, and harmonic analysis techniques can be used in this context. They do not rely on the Gaussian assumption commonly used in functional data analysis on functional processes; see [10] for a discussion. Dependent functional data analysis, including functional time series and spatial statistics, is treated in detail by Horváth and Kokoszka [13]. Part of these developments were stimulated by the early work of Bosq [3] for dependent data; see also Ramsay and Silverman [18].

According to Hsing and Eubank [15], functional data can be approached either from a random element perspective or from a stochastic process perspective. This paper does a bit of both. Specifically, we develop mathematical tools for stationary processes. For a stationary function, we know that there exists one and only one unitary operator, namely the shift operator. The latter can be expressed as a linear combination of projectors or more generally as an integral with respect to a projector-valued spectral measure, viz.

\[ UX = \sum_{j \in J} e^{i\lambda_j} P_j X \quad \text{or} \quad UX = \int e^{i\lambda} d\mathcal{E}(\lambda)X, \]

where \( \mathcal{E} \) is a projector-valued spectral measure. A spectral measure is associated with a unitary operator \( U \) in a unique way; we denote it \( \mathcal{E}_U \). Our objective is to study the relations between the spectral measures \( \mathcal{E}_U, \mathcal{E}_V, \mathcal{E}_{UV} \) and \( \mathcal{E}_{VU} \), where \( U \) and \( V \) are unitary operators. The comparison of two random functions can then be transposed in the frequency domain through the comparison of their associated unitary operators.
Commutativity between two unitary operators makes complete sense in this context. When two random functions are stationarily correlated, they show some similarity. The notion of stationarity correlation can be generalized by the commutativity of these functions; see [8]. In the frequency domain, this notion corresponds to the commutativity of the associated spectral measures. When two unitary operators $U$ and $V$ commute, there exists a simple relation between $E_U$, $E_V$ and $E_{UV} = E_{VU}$. By analogy with measure theory, it is natural to say that $E_{UV}$ is the convolution product of the spectral measures $E_U$ and $E_V$. We emphasize that the product of convolution of spectral measures, as defined in [6], relies on a commutativity assumption. When the operators do not commute, the relation is more complex.

These considerations lead us to introduce here the notion of commutator, which retrieves the part that commutes between two operators. The study of this commutator then allows us to define how close two random functions can be. Projections play an important role in our study; for, $E_{UV}A$, $E_{VU}A$ and the commutator are projectors. We also rely on a partial order relation defined on the set of projectors [9]. Obviously, the commutator of two projectors is linked with the canonical analysis of the spaces they generate. These concepts are developed here in a general Hilbertian framework, and when the $C$-Hilbert space is of the $L^2(Ω, μ_A, P)$ type, our results apply to stationary processes. When the operators do not commute, we can define the convolution product on a subspace generated by the commutator. The study of unitary operators, and of their associated random functions, contributes to the development of operator-based statistical theory, with potential application to functional data analysis.

This paper is structured as follows. Prerequisites and notation are first reviewed in Section 2. The maximal commutator of two operators is then defined in Section 3, where it is discussed in three special cases: two unitary operators, a projector and a unitary operator, and two projectors. In Section 4, we examine the relations between the operators do not commute, we can define the convolution product on a subspace generated by the commutator. The study of unitary operators, and of their associated random functions, contributes to the development of operator-based statistical theory, with potential application to functional data analysis.

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A spectral measure $E$ on $\xi$ for $H$ is an application from $\xi$ into $\mathcal{P}(H)$ such that $AE = I_H$, $E(A \cup B) = EA + EB$ for any pair $(A, B)$ of disjoint elements of $\xi$, and $E_A X \to 0$ as $n \to \infty$ for any $X \in H$ and any decreasing sequence $(A_n : n \in \mathbb{N})$ of elements of $\xi$ converging to $\emptyset$.

It can be shown that for any $X \in H$, the application $Z_X^E : A \in \xi \mapsto EAX H$ is a random measure. If $(F, \mathcal{F})$ is a second measurable space and $f$ is a measurable application from $E$ to $F$, then the application $f(E) : A \in \mathcal{F} \mapsto E(f^{-1}A) \in \mathcal{P}(H)$ is a spectral measure on $\mathcal{F}$ for $H$ called the spectral measure image of $E$ by $f$.

With any unitary operator $U$ of $H$, we can associate a unique spectral measure on $B$ for $H$, denoted $E_U$, and called the spectral measure associated with the unitary operator $U$, such that $UX = \int e^{i\lambda}dZ_X^E(\lambda) \in H$ for all $X \in H$.

Conversely, if $E$ is a spectral measure on $B$ for $H$, then the application $X \in H \mapsto \int e^{i\lambda}dZ_X^E(\lambda) \in H$ is a unitary operator of associated spectral measure $E$. If $U$ is a unitary operator, then, for any $\lambda \in \Pi$, $\text{Im} E_U(\langle \lambda \rangle) = \{X \in H : UX = e^{i\lambda}X\}$.

Let $(P_j : j \in J)$ be a finite family of projectors such that $\sum_{j \in J} P_j = I_H$ and such that $P_j P_l = 0$ for all pairs $(j, l)$ of distinct elements of $J$. If $\{\lambda_j : j \in J\}$ is a (finite) family of distinct elements of $\Pi$, then $U = \sum_{j \in J} e^{i\lambda_j}P_j$ is a unitary operator and $E_U = \sum_{j \in J} \delta_{\lambda_j} P_j$.

For any $n \in \mathbb{Z}$, the application $\hat{h}_n : \lambda \in \Pi \mapsto n\lambda - 2\pi[(n\lambda + \pi)/(2\pi)] \in \Pi$, where $[x]$ denotes the integer part of $x$, is measurable. It can be shown that if $U$ is a unitary operator, the spectral measure associated with the unitary operator $U^n$ is the image by $\hat{h}_n$ of the spectral measure associated with $U$. With the notational convention we have adopted, $\hat{h}_n(E_U) = E_{U^n}$.

A bounded endomorphism $T$ of $H$ commutes with a unitary operator $U$ if and only if $(E_U A)T = T(E_U A)$ for all $A \in B$. Thus, two unitary operators $U$ and $V$ commute if and only if, for any pair $(A, B)$ of elements of $B$, the projectors $E_U A$ and $E_V B$ commute. Then we say that the spectral measures $E_U$ and $E_V$ commute and it can be shown that there exists a unique spectral measure, denoted $E_{UV}$, on $B \otimes B$ for $H$, such that $E_{UV} \otimes E_{UV}$, on $(B \otimes B) \otimes (B \otimes B)$ for $H$, is a unitary operator of $B \otimes B$ elements of $B$. Note that $(E_U A)(E_V B)$ is a projector because $E_U A$ and $E_V B$ commute. The image of $E_{UV} \otimes E_{UV}$ by the measurable application $S : (\lambda, \lambda') \in \Pi \times \Pi \mapsto \lambda + \lambda' - 2\pi[(\lambda + \lambda' + \pi)/(2\pi)] \in \Pi$, denoted $E_{UV} \circ E_{UV}$, is called the product of convolution of the spectral measures $E_U$ and $E_V$; it is associated with the unitary operator $UV$, i.e., $E_{UV} = E_U \circ E_V = S(E_U \otimes E_V)$.

When $C$ is a projector of $H$, we denote by $L_C$ the application $X \in \text{Im} C \mapsto X \in H$. It can be seen that $L_C^*(X) = CX$ for all $X \in H$, that $L_C C = C$, and that $C L_C = L_C C = L_C^2$. Any family $(P_\lambda : \lambda \in \Lambda)$ of projectors has a greatest lower bound, denoted $\inf \{P_\lambda : \lambda \in \Lambda\}$, and a least upper bound, denoted $\sup \{P_\lambda : \lambda \in \Lambda\}$. In what follows, the following properties will often be used:

(i) $P \ll Q$ if and only if $Q^+ \ll P^+$. (ii) If $(P, Q)$ is a pair of projectors which commute, then $\text{inf}(P, Q) = PQ$. (iii) $\text{Im} \text{inf} \{P_\lambda : \lambda \in \Lambda\} = \cap_{\lambda \in \Lambda} \text{Im} P_\lambda$. (iv) $\text{inf} \{P_\lambda : \lambda \in \Lambda\}^+ = \sup \{P_\lambda^+ : \lambda \in \Lambda\}$. (v) $\text{inf} \{P_\lambda : \lambda \in \Lambda\}^+ = \sup \{P_\lambda^+ : \lambda \in \Lambda\}$.

If a projector $C$ commutes with projectors $P$ and $Q$, then $C$ commutes with $\text{inf}(P, Q)$, and $C \text{inf}(P, Q) = \text{inf}(CP, CQ)$.

Let $(P_n : n \in \mathbb{N})$ be a sequence of projectors. It is then possible to define its upper and lower limits as

$$\lim \sup\{P_n : n \in \mathbb{N}\} = \inf\{\sup\{P_m : m \geq n\} : n \in \mathbb{N}\}, \quad \lim \inf\{P_n : n \in \mathbb{N}\} = \sup\{\inf\{P_m : m \geq n\} : n \in \mathbb{N}\},$$

respectively. We always have $\lim \sup\{P_n : n \in \mathbb{N}\} \ll \lim \sup\{P_n : n \in \mathbb{N}\}$. When there is equality, we say that $(P_n : n \in \mathbb{N})$ r-converges: $\lim_n P_n = P$ if and only if $\lim \inf\{P_n : n \in \mathbb{N}\} = \lim \sup\{P_n : n \in \mathbb{N}\} = P$. 

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If a sequence of projections \( r \)-converges, then it converges point-wise, but the converse is not true. If \( (P_n, n \in \mathbb{N}) \) is an increasing (resp. decreasing) sequence of projectors, it is \( r \)-convergent and \( \lim_n P_n = \sup\{P_n : n \in \mathbb{N}\} \) (resp. \( \lim_n P_n = \inf\{P_n : n \in \mathbb{N}\} \)).

For any pair \( (P, Q) \) of projectors, we define \( d(P, Q) = \sup(P, Q) - \inf(P, Q) \). This notion looks like a distance but it is not, as \( d(P, Q) \) is a projector. It is interesting nevertheless because \( \lim_n P_n = P \) if and only if \( \lim_n d(P_n, P) = 0 \). Note that \( d(P, Q) = 0 \) if and only if \( P = Q \). It can also be checked that \( \text{Im}(d(P, Q)) = \overline{\text{Ker}(P - Q)} \).

Given spectral measures \( E \) and \( a \) on \( \mathcal{B} \) for \( H \), their associated gap is given by \( E_{E, a} = \sup\{d(EA, a\lambda) : \lambda \in \mathcal{B}\} \). An equalizer of two unitary operators \( U \) and \( V \) is a projector \( K \) which commutes with \( U \) and \( V \) and such that \( UK = VK \).

The upper bound of the family of the equalizers of the unitary operators \( U \) is an equalizer of two unitary operators \( U \) and \( V \). It is interesting nevertheless because \( \lim_n P_n = P \) if and only if \( \lim_n d(P_n, P) = 0 \). Note that \( d(P, Q) = 0 \) if and only if \( P = Q \). It can also be checked that \( \text{Im}(d(P, Q)) = \overline{\text{Ker}(P - Q)} \).

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The notion of commuter studied here is different from the notion of commutator described in other papers, e.g., Laustsen [16], who defines the commutator of two operators \( A \) and \( B \) as \( [A, B] = AB - BA \). We define a commuter as a projector which will be equal to the identity when the operators commute.

3. Commuters

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3.1. Maximal commuter

We begin by stating some preliminary results pertaining to commutativity.

**Lemma 1.** If the projector \( P \) commutes with \( A \in \mathcal{L}(H) \), then \( PA^* = A^* P \) and \( P^+ A = AP^+ \).

**Proof.** From \( AP = PA \), we get \( PA^* = A^* P \) using the adjoint. Commutativity between \( P^+ \) and \( A \) is easily checked. □

The following result is somewhat more involved.

**Lemma 2.** If \( A \in \mathcal{L}(H) \) commutes with each projector in \( \{ P_\lambda : \lambda \in \Lambda \} \), then \( MAM = AM \) with \( M = \inf\{ P_\lambda : \lambda \in \Lambda \} \).

**Proof.** Fix an arbitrary \( X \in H \). For any \( \lambda \in \Lambda \), we have \( P_\lambda A M X = A P_\lambda M X = A M X \). This means that \( A M X \) belongs to \( \bigcap_{\lambda \in \Lambda} \text{Im} P_\lambda = \text{Im} \inf\{ P_\lambda : \lambda \in \Lambda \} = \text{Im} M \). So \( M A M = A M X \), which completes the proof. □

The commutativity of a family of projectors implies the commutativity of the upper and the lower bounds.

**Lemma 3.** If \( A \in \mathcal{L}(H) \) commutes with each projector in \( \{ P_\lambda : \lambda \in \Lambda \} \), then \( A \) commutes with \( M = \inf\{ P_\lambda : \lambda \in \Lambda \} \) and \( S = \sup\{ P_\lambda : \lambda \in \Lambda \} \).

**Proof.** From Lemma 2, we have \( M A M = A M \). Furthermore, from Lemma 1, \( A^* \) commutes with each projector in \( \{ P_\lambda : \lambda \in \Lambda \} \). Lemma 2 also allows us to write \( M A^* M = A^* M \), and hence \( M M A = M M A \). We can then conclude that \( M A = A M \). Finally, from Lemma 1, \( A \) commutes with each projector in \( \{ P^\lambda_\lambda : \lambda \in \Lambda \} \), i.e., with \( \inf\{ P^\lambda_\lambda : \lambda \in \Lambda \} \), and hence with \( [\inf\{ P^\lambda_\lambda : \lambda \in \Lambda \}]^\perp = \sup\{ P_\lambda : \lambda \in \Lambda \} = S \). □

We are now ready to introduce the notion of commuter.

**Definition 1.** A projector \( K \) is a commuter of \( A, B \in \mathcal{L}(H) \) if \( K \) commutes with \( A \) and \( B \) and \( A K B = B K A \).

The upper bound of a family of commuters is a commuter.

**Proposition 1.** If each projector in \( \{ K_\lambda : \lambda \in \Lambda \} \) is a commuter of \( A, B \in \mathcal{L}(H) \), then \( K = \sup\{ K_\lambda : \lambda \in \Lambda \} \) is a commuter of \( A \) and \( B \).

**Proof.** From Lemma 3, \( A \) (resp. \( B \)) commutes with \( K \). Denote by \( D \) the projector on the closed vector subspace \( \text{Ker}(A B - BA) \). For any \( \lambda \in \Lambda \), we have

\[
(AB - BA)K_\lambda = ABK_\lambda - BAK_\lambda = AK_\lambda B - BK_\lambda A = BK_\lambda A - BK_\lambda A = 0.
\]
This means that Im $K_A \subset \text{Im } D = \text{Ker } (AB - BA)$, and hence $K_A \ll D$. The projector $D$ being an upper bound of the family $\{K_A : A \in \Lambda\}$, we have $K \ll D$. For any $X \in H$, we have $XX \in \text{Im } K \subset \text{Im } D = \text{Ker } (AB - BA)$, so $ABXX = BAX$. In other words, because $K$ commutes with $A$ and $B$, $ABXX = BAX$. Since $X \in H$ was arbitrary, we have $AB = BKA$. Hence the proof is complete. □

As the set of the commutators of a pair $(A, B)$ of elements of $\mathcal{L}(H)$ is never empty, because 0 is a commutator of $A$ and $B$, Proposition 1 allows us to define the maximal commutator.

**Definition 2.** If $A$ and $B$ are elements of $\mathcal{L}(H)$, the upper bound of the family of all commutators of $A$ and $B$ is called the maximal commutator of $A$ and $B$, and it is denoted $C_{A,B}$.

In the following subsections, we consider in turn the maximal commutator of two unitary operators, of a projector and a unitary operator, and of two projectors.

### 3.2. Maximal commutator of two unitary operators

A commutator of two unitary operators can be defined by means of their associated spectral measures. For this, we establish the following preliminary result.

**Lemma 4.** A projector $K$ is a commutator of the unitary operators $U$ and $V$ if and only if $L_K^*UL_K$ and $L_K^*VL_K$ are unitary operators which commute.

**Proof.** If $K$ is a commutator of $U$ and $V$, $K$ commutes with $U$, so $L_K^*UL_K$ is a unitary operator of $\text{Im } K$. For the same reasons, $L_K^*VL_K$ is a unitary operator of $\text{Im } K$. Moreover,


so $L_K^*UL_K$ and $L_K^*VL_K$ are unitary operators of $\text{Im } K$ which commute.

Conversely, if $L_K^*UL_K$ and $L_K^*VL_K$ are unitary operators of $\text{Im } K$ which commute, we have that $K$ commutes with $U$ and $V$. In addition,

$$UKV = KUKV = L_KL_K^*UL_KL_K^*VL_KL_K = L_KL_K^*VL_KL_K^*UL_K = KVUK = VKU.$$

Indeed, the projector $K$ is a commutator of the unitary operators $U$ and $V$. □

We can now establish a relationship between the commutator of two unitary operators and the commutator of a family of pairs of projectors.

**Proposition 2.** A projector $K$ is a commutator of the unitary operators $U$ and $V$ if and only if, for all pairs $(A, B)$ of elements of $\mathcal{B}$, $K$ is a commutator of the projectors $E_UA$ and $E_VB$.

**Proof.** Let $K$ be a commutator of $U$ and $V$. As $K$ commutes with $U$ (resp. $V$), $L_K^*UL_K$ (resp. $L_K^*VL_K$) is a unitary operator of $\text{Im } K$, for all $A \in \mathcal{B}$, $L_K^*E_UAL_K$ (resp. $L_K^*E_VAL_K$) is a projector of $\text{Im } K$ and the application $A \in \mathcal{B} \mapsto L_K^*E_UAL_K \in \mathcal{P}(\text{Im } K)$ (resp. $A \in \mathcal{B} \mapsto L_K^*E_VAL_K \in \mathcal{P}(\text{Im } K)$) is the spectral measure associated with $L_K^*UL_K$ (resp. $L_K^*VL_K$). From the previous lemma, $K$ being a commutator of $U$ and $V$, the unitary operators $L_K^*UL_K$ and $L_K^*VL_K$ commute, so, considering a pair $(A, B)$ of elements of $\mathcal{B}$, $K$ commutes with $E_UA$ and $E_VB$ and the projectors of $\text{Im } K$, $L_K^*E_UAL_K$ and $L_K^*E_VBL_K$, commute. In order to prove that $K$ is a commutator of the projectors $E_UA$ and $E_VB$, it is then sufficient to note that


Conversely, assume that $K$ is a commutator of the projectors $E_UA$ and $E_VB$, for any pair $(A, B)$ of elements of $\mathcal{B}$. As $K$ commutes with $E_UA$ for all $A \in \mathcal{B}$, $K$ commutes with $U$. So, $L_K^*E_UAL_K$ is a projector of $\text{Im } K$ and the application $A \in \mathcal{B} \mapsto L_K^*E_UAL_K \in \mathcal{P}(\text{Im } K)$ is the spectral measure associated with the unitary operator $L_K^*UL_K$. The same property stands for $V$, viz. $KV = VK$. and the application $B \in \mathcal{B} \mapsto L_K^*(E_VB)L_K \in \mathcal{P}(\text{Im } K)$ is the spectral measure...
associated with the unitary operator $L^*_K VL_K$. Let us consider a pair $(A, B)$ of elements of $B$. As $K$ is a commuter of \( E_U A \) and \( E_V B \), we can write

\[
L^*_K(E_U A)L_K L^*_K(E_V B)L_K = L^*_K(E_U A)K(E_V B)L_K = L^*_K(E_V B)K(E_U A)L_K = L^*_K(E_V B)L_K L^*_K(E_U A)L_K.
\]

Thus the projectors \( L^*_K(E_U A)L_K \) and \( L^*_K(E_V B)L_K \) commute whatever \((A, B) \in B \times B \). We conclude that the unitary operators \( L^*_K U L_K \) and \( L^*_K V L_K \) commute, so (see Lemma 4) \( K \) is a commuter of the unitary operators \( U \) and \( V \).

Proposition 2 has the following consequence.

**Corollary 1.** If \( K \) is a commuter of the unitary operators \( U \) and \( V \), then, for any \((n, m) \in \mathbb{Z}^2\), \( K \) is a commuter of the unitary operators \( U^n \) and \( V^m \).

**Proof.** From the background material in Section 2, \( E_U^n = E_U \) and \( E_V^m = E_V \). From Proposition 2, \( K \) will be a commuter of \( U^n \) and \( V^m \) if \( K \) is a commuter of the projectors \( E_U \) and \( E_V \) for all \((A, B) \in B \times B \). So, let \((A, B) \) be a pair of elements of \( B \). As \( K \) is a commuter of \( U \) and \( V \), Proposition 2 implies that \( K \) is a commuter of the projectors \( E_U \) and \( E_V \), and so of the projectors \( E_U A \) and \( E_V B \).

If \( U \) and \( V \) are two unitary operators, denote by \( C \) the projector on the closed vector subspace

\[
\cap_{(n, m) \in \mathbb{Z}^2} \ker(U^n V^m - V^m U^n).
\]

**Proposition 3.** \( C \) is a commuter of the unitary operators \( U \) and \( V \).

**Proof.** First note that \( X \in \text{Im} C \) if and only if \( U^n V^m X = V^m U^n X \) for all \((n, m) \in \mathbb{Z}^2\). For any \((p, X) \in \mathbb{Z} \times \text{Im} C \), write

\[
U^n V^m(U^p X) = U^n(V^m U^p X) = U^n(U^p V^m X) = U^{n+p} V^m X = V^m U^n(U^p X)
\]

for any \( n, m \in \mathbb{Z} \). We can then assert that, for any \( X \in H \), \( U C X \) and \( U^{-1} C X \) belong to \( \text{Im} C \), so \( U C X = U C X \) and \( C U^{-1} C X = U^{-1} C X \). Then \( U C = U C \) and \( U C = U C \). This involves the commutativity of \( C \) and \( U \), because \( U = (U^{-1})^* = (C U^{-1})^* = C U = U C \). Similarly, we can prove that \( V C = C V \). Finally, these properties of commutativity and the fact that \( C X \) belongs to \( \text{Im} C \) allow us to write \( U C V X = U V C X = V U C X = V C U X \) whenever \( X \in H \). So \( U C V = V C U \), which allows us to conclude.

With the same notation, we have the following.

**Proposition 4.** If \( K \) is a commuter of the unitary operators \( U \) and \( V \), then \( K \Leftrightarrow C \).

**Proof.** Take \( X \in \text{Im} K \), i.e., such that \( X = K X \). As \( K \) is a commuter of the operators \( U^n \) and \( V^m \) (see Corollary 1), we have

\[
U^n V^m X = U^n V^m K X = U^n K V^m X = V^m K U^n X = V^m U^n K X = V^m U^n X
\]

for any \( n, m \in \mathbb{Z} \). Thus \( X \in \cap_{(n, m) \in \mathbb{Z}^2} \ker(U^n V^m - V^m U^n) \), i.e., \( X \in \text{Im} C \). Hence \( \text{Im} K \subset \text{Im} C \), i.e., \( K \Leftrightarrow C \).

With these results, we can describe explicitly the maximal commuter of two unitary operators.

**Proposition 5.** If \( U \) and \( V \) are two unitary operators of \( H \), then \( \text{Im} C_{U,V} = \cap_{(n, m) \in \mathbb{Z}^2} \ker(U^n V^m - V^m U^n) \).

The maximal commuter can also be defined by the way of a family of equalizers.

**Proposition 6.** For any pair \((U, V)\) of unitary operators, we have \( C_{U,V} = \text{inf} \{ R_{U^n V^m : n \in \mathbb{Z}} \} \).

**Proof.** By mathematical induction, we can show that for any \( m \in \mathbb{N} \), \( U^{-m} V U^m = (U^{-m} V U^m)^m \). Then, extend this property for any \( m \in \mathbb{Z} \); if the property is verified for \( m \geq 0 \), then when \( m < 0 \), we can write \( (U^{-m} V U^m)^m = U^{-m} V U^m \), and taking the adjoint we get \( (U^{-m} V U^m)^m = U^{-m} V U^m \), i.e., \( (U^{-m} V U^m)^m = U^{-m} V U^m \). Finally,

\[
\text{Im} C_{U,V} = \cap_{(n, m) \in \mathbb{Z}^2} \ker(U^n V^m - V^m U^n) = \cap_{m \in \mathbb{Z}} \cap_{n \in \mathbb{Z}} \ker(V^m - U^{-m} V U^m) = \cap_{m \in \mathbb{Z}} \text{Im} R_{U^{-m} V U^m} = \text{Im} \text{inf} \{ R_{U^n V^m : n \in \mathbb{Z}} \},
\]

so the claim holds.
3.3. Maximal commuter of a projector and of a unitary operator

When \( A \in \mathcal{L}(H) \), \( \{A^n/n! : n \in \mathbb{N}\} \) is a summable family of the \( \mathbb{C} \)-Banach space \( \mathcal{L}(H) \). The sum of this family is denoted \( e^A \); see [19]. Of course, the formula \( e^A = \sum_{n \in \mathbb{N}} A^n/n! \) is reminiscent of the formula \( e^z = \sum_{n \in \mathbb{Z}} z^n/n! \), when \( z \in \mathbb{R} \) or \( \mathbb{C} \). In the particular case where \( P \) is a projector of \( H \), we have

\[
e^P = \sum_{n \in \mathbb{N}} \frac{P^n}{n!} = I + \sum_{n \in \mathbb{N}} \frac{P^n}{n!} = I + (e^1 - 1)P = e^P + P_{\bot}.
\]

It is clear that \( e^P \) is a unitary operator with spectral measure \( E_{e^P} = \delta_0P_{\bot} + \delta_1P \). This yields the following result.

**Lemma 5.** If \( P \) is a projector and \( U \) a unitary operator, then \( C_{P,U} = C_{e^P,U} \).

**Proof.** It is easy to check that a projector \( K \) is a committer of \( P \) and \( U \) if and only if \( K \) is committer of the unitary operators \( e^P = I + (e^1 - 1)P \) and \( U \). Therefore, the families of the commuters of \( P \) and \( U \) and of the commuters of \( e^P \) and \( U \) are equal, which is why \( C_{P,U} = C_{e^P,U} \). □

This lemma allows us to use the results of Section 3.2 for the study of the maximal committer of a projector and a unitary operator.

**Proposition 7.** For any projector \( P \) of \( H \) and any unitary operator \( U \) of \( H \), we have \( \text{Im} C_{P,U} = \cap_{n \in \mathbb{Z}} \text{Ker}(PU^n - U^nP) \).

**Proof.** We first notice that for any \( m \in \mathbb{Z} \), we have \( (e^P)^m = e^{im}P + P_{\bot} = I + (e^m - 1)P \). This can be proved by induction for any \( m \in \mathbb{N} \), and then can be extended to any \( m \in \mathbb{Z} \), by using the adjoint form. Then, for any \( n,m \in \mathbb{Z} \), we have \( (e^P)^mU^nU^*(e^P)^m = (e^mP - U^n(U^*)^m = (e^mP - U^n - U^nP) \). It follows from Proposition 5 that

\[
\text{Im}C_{P,U} = \text{Im}C_{e^P*,U} = \cap_{(n,m) \in \mathbb{Z}^2} \text{Ker}(e^{iP}U^n - U^n(e^P)^m = \cap_{n \in \mathbb{Z}} \text{Ker}(PU^n - U^nP),
\]

which completes the proof. □

Noting that \( U^nPU^n \) is a projector, we have the following ergodic property.

**Proposition 8.** If \( P \) is a projector and \( U \) a unitary operator, then \( C_{P,U} = \inf\{d(P,U^nPU^n)^+ : n \in \mathbb{Z}\} \).

**Proof.** As \( \text{Ker}(PU^n - U^nP) = \text{Ker}(U^nPU^n - P) = \text{Im}(d(P,U^nPU^n)^+) \), we have \( \text{Im} C_{P,U} = \cap_{n \in \mathbb{Z}} \text{Im} \{d(P,U^nPU^n)^+ \} = \text{Im} \inf\{d(P,U^nPU^n) : n \in \mathbb{Z}\} \), and the property stands. □

**Remark 1.** The maximal commuter \( C_{P,U} \) of the projector \( P \) and of the unitary operator \( U \) measures the level of commutativity between \( P \) and \( U \). We can easily verify that they commute if and only if \( C_{P,U} = I \). As for the formula \( C_{P,U} = \inf\{d(P,U^nPU^n)^+ : n \in \mathbb{Z}\} \), it leads to the following interpretation: if \( PU^n \) and \( U^nP \) are close together, that is if the projectors \( P \) and \( U^nPU^n \) are close together, then \( d(P,U^nPU^n) \) is of high rank, and so \( d(P,U^nPU^n)^+ \) is of high rank, together with the lower bound, \( C_{P,U} \), of the family \( d(P,U^nPU^n)^+ : n \in \mathbb{Z} \).

3.4. Maximal commuter of two projectors

If \( P \) and \( D \) are two projectors, it is easy to check that any committer of \( P \) and \( D \) is a committer of \( P \) and \( e^D \). Conversely, any committer of \( P \) and \( e^D \) is a committer of \( P \) and \( D \). Thus, the family of the committers of \( P \) and \( D \) coincides with the family of the committers of \( P \) and \( e^D \). We deduce the following result.

**Lemma 6.** If \( P \) and \( D \) are two projectors, then \( C_{P,D} = C_{P,e^D} \).

We can complete this observation with the following.

**Proposition 9.** If \( P \) and \( D \) are two projectors, then \( \text{Im} C_{P,D} = \text{Ker}(PD - DP) \).
Proof. For any \( n \in \mathbb{Z} \), we have \( P(e^{iD})^n = (e^{iD})^nP = (e^{in} - 1)(PD - DP) \). Lemma 6 and Proposition 7 yield \( \text{Im } C_{P,D} = \text{Im } C_{P,e^D} = \cap_{n \in \mathbb{Z}} \text{Ker}(e^{in} - 1)(PD - DP) = \text{Ker } (PD - DP) \).

Let \((P, D)\) be a pair of projectors. For any \( X \in H \), we have
\[
\|PDX - D PX\| \leq \|PDX - D PC_{P,D}X\| + \|DPC_{P,D}X - D PX\|.
\]
As \( DPC_{P,D}X = PC_{P,D}DX = DCP_{P,D}X = DPC_{P,D}X \), we find
\[
\|PDX - D PX\| \leq \|X - C_{P,D}X\| + \|C_{P,D}X - X\| = 2\|C_{P,D}X\|.
\]
which allows us to conclude. \( \square \)

**Proposition 10.** Given any pair \((P, D)\) of projectors, we have \( \|PDX - D PX\| \leq 2 \|C_{P,D}X\| \) for all \( X \in H \).

**Remark 2.** (a) The term “maximal commuter” is justified by the fact that \( C_{P,D} \) is the projector on the closed vector subspace of the elements \( X \) such that \( PDX = DPX \).

(b) The maximal commuter measures the degree of commutativity of two projectors. We can easily check that \( C_{P,D} = I \) if and only if \( P \) and \( D \) commute.

(c) From Proposition 10, when \( X \) is close to \( \text{Im } C_{P,D} \), i.e., when \( \|C_{P,D}X\| \) is small, then \( PDX \) and \( D PX \) are close together.

(d) For any pair \((P, D)\) of projectors, \( C_{P,D} = C_{P^+, D} \) because \( \text{Ker } (PD - DP) = \text{Ker } (P^+D - DP^+) \).

### 3.5. Links between the three families of commuters

In this section, we will establish and study various relations between the maximal commuters of (a) two unitary operators; (b) a projector and a unitary operator; (c) two projectors. In particular, the maximal commuter of two unitary operators \( U \) and \( V \) can be expressed through a family of maximal commuters of projectors \( \{C_{E_U;E_V} : (A, B) \in \mathcal{B} \times \mathcal{B}\} \).

**Proposition 11.** If \( U \) and \( V \) are two unitary operators, we have \( C_{U,V} = \inf \{C_{E_U;E_V} : (A, B) \in \mathcal{B} \times \mathcal{B}\} \).

**Proof.** Let us consider an element \( X \) of \( \cap_{(A, B) \in \mathcal{B} \times \mathcal{B}} \text{Ker}[(E_U;A)(E_V;B) - (E_U;B)(E_V;A)] \). For any \((B, n) \in \mathcal{B} \times \mathbb{Z}\), we have
\[
Z_{E_U}^{E_V}X = (E_V;B) \circ Z_{E_U}X.
\]
As \( \text{vec}(1_A : A \in \mathcal{B}) = L^2(\mathbb{P}, \mathcal{B}, \mu_{\mathbb{P}, \mathbb{B}}, \mu_{\mathbb{B}, \mathbb{P}} + \mu_{\mathbb{B}, \mathbb{B}}) \), we can write
\[
e^{i\alpha} = \lim_{m \to \infty} \sum_{j=1}^{k_m} \alpha_j \lambda_j \text{ in } L^2(\mu_{\mathbb{P}, \mathbb{B}}, \mu_{\mathbb{B}, \mathbb{P}} + \mu_{\mathbb{B}, \mathbb{B}}),
\]
where \( A_{\lambda j} \in \mathcal{B} \), \( \alpha_j \in \mathbb{C} \), and \( k_m \in \mathbb{N}^+ \). Relation (2) is also true in \( L^2(\mu_{\mathbb{P}, \mathbb{B}}, \mu_{\mathbb{B}, \mathbb{P}}) \) and in \( L^2(\mu_{\mathbb{B}, \mathbb{B}}) \) because
\[
\|\cdot\mid L^2(\mu_{\mathbb{P}, \mathbb{B}}, \mu_{\mathbb{B}, \mathbb{P}}) + \|\cdot\|_{L^2(\mu_{\mathbb{B}, \mathbb{B}})}.
\]
Integrating (2) with respect to the random measure \( Z_{E_V}X \) yields
\[
U^nX = \lim_{m \to \infty} \sum_{j=1}^{k_m} \alpha_j Z_{E_U}X A_{\lambda j} \quad \text{and} \quad (E_V;B)U^nX = \lim_{m \to \infty} \sum_{j=1}^{k_m} \alpha_j (E_V;B)Z_{E_U}X A_{\lambda j}.
\]
Taking into account (1), we then get
\[
(E_V;B)U^nX = \lim_{m \to \infty} \sum_{j=1}^{k_m} \alpha_j Z_{E_U}^{E_V}X A_{\lambda j}.
\]
Integrating (2) with respect to the random measure $Z_{E_n}^{(E_n)X}$, we find
\[ U^n(E_n)B X = \lim_{m \to \infty} \sum_{j=1}^{k_n} \alpha_{jm} Z_{E_n}^{(E_n)B X} A_{jm}. \]

Thus, from what precedes, $(E_n)B U^n X = U^n(E_n)B X$ for any $B \in \mathcal{B}$, from which we deduce that $Z_{E_n}^{U^n X} = U^n \circ Z_{E_n}^X$. Keeping in mind the background material, we have, for all $n, m \in \mathbb{Z}$,
\[ \int e^{i m} dZ_{E_n}^{U^n X} = \int e^{i m} dU^n \circ Z_{E_n}^X = U^n \left( \int e^{i m} dZ_{E_n}^X \right). \]

As $Z_{E_n}^{U^n X}$ and $Z_{E_n}^X$ are the random measures respectively associated with the stationary series $(V^m U^n X, m \in \mathbb{Z})$ and $(V^m X, m \in \mathbb{Z})$, we see that $V^m U^n X = U^n V^m X$. This means that $X$ belongs to $\text{Ker} (V^m U^n - U^n V^m)$ whatever $n, m \in \mathbb{Z}$, so that $X \in \cap_{(u, m) \in \mathbb{Z}} \text{Ker} (V^m U^n - U^n V^m) = \text{Im } C_{U,V}$. We have just proved that
\[ \cap_{(A, B) \in \mathcal{B} \times \mathcal{B}} \text{Ker } ((E_U A)(E_V B) - (E_V B)(E_U A)) \subset \text{Im } C_{U,V}, \]

and hence
\[ \text{Im } \inf \{ C_{E_{A,E_{E_n}}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \} \subset \text{Im } C_{U,V}, \]

so that $\inf \{ C_{E_{A,E_{E_n}}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \} < C_{U,V}$.

As $C_{U,V}$ is a commutator of $U$ and $V$, Proposition 2 implies that it is a commutator of the projectors $E_U A$ and $E_V B$ for all $(A, B) \in \mathcal{B} \times \mathcal{B}$. So we have $C_{U,V} < C_{E_{A,E_{E_n}}, B}$. This means that $C_{U,V}$ is a lower bound of the family of projectors $\{ C_{E_{A,E_{E_n}}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \}$. Therefore, $C_{U,V} < \inf \{ C_{E_{A,E_{E_n}}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \}$, which allows us to conclude. \(\square\)

Let us examine what this last result becomes in the particular case where $U = e^{i \theta}$, $P$ being a projector.

**Proposition 12.** If $P$ is a projector and $V$ a unitary operator of $H$, then we have $C_{P,V} = \inf \{ C_{P,E_{E_n}, B} : B \in \mathcal{B} \}$.

**Proof.** We know that $E_U = \delta_0 P + \delta_1 P$. Depending on whether an element $A$ of $\mathcal{B}$ includes 0 and 1, 0 and not 1, 1 and not 0, or neither 0 neither 1, $E_U A$ takes the value $I$, $P_+$, $P$ or 0, respectively. Thus, for any $A, B \in \mathcal{B}$, $C_{E_{U}, A, E_{E_n}, B}$ equals either $C_{E_{U}, E_{E_n}, B} = I = C_{P, E_{E_n}, B}$, either $C_{P, E_{E_n}, B} = C_{E_{U}, E_{E_n}, B}$, or $C_{E_{U}, E_{E_n}, B} = 1 = C_{E_{U}, E_{E_n}, B}$, or $C_{E_{U}, E_{E_n}, B} = 1 = C_{E_{U}, E_{E_n}, B}$. This means that $\{ C_{E_{U}, E_{E_n}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \} \subset \{ C_{P, E_{E_n}, B} : B \in \mathcal{B} \}$. However, we also have $\{ C_{P, E_{E_n}, B} : B \in \mathcal{B} \} \subset \{ C_{E_{U}, E_{E_n}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \}$, because $C_{P, E_{E_n}, B} = C_{E_{U}, E_{E_n}, B}$. Thus we can write $\{ C_{E_{U}, E_{E_n}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \} = \{ C_{P, E_{E_n}, B} : B \in \mathcal{B} \}$. In view of Lemma 6 and Proposition 11, we conclude that $C_{P,V} = C_{E_U,V} = \inf \{ C_{E_{U}, E_{E_n}, B} : (A, B) \in \mathcal{B} \times \mathcal{B} \} = \inf \{ C_{P, E_{E_n}, B} : B \in \mathcal{B} \}$. \(\square\)

The following is a consequence of Proposition 12.

**Corollary 2.** If $U$ and $V$ are any two unitary operators, then $C_{U,V} = \inf \{ C_{E_{U}, B,U} : B \in \mathcal{B} \}$.

**Proof.** From Propositions 11 and 12, we get
\[ \text{Im } C_{U,V} = \cap_{B \in \mathcal{B}} \cap_{A \in \mathcal{B}} \text{Im } C_{E_{A,E_{E_n}, B}} = \cap_{B \in \mathcal{B}} \text{Im } \inf \{ C_{E_{A,E_{E_n}, B}} : A \in \mathcal{B} \} = \cap_{B \in \mathcal{B}} \text{Im } C_{E_{B,U}} = \text{Im } \inf \{ C_{E_{B,U}, B} : B \in \mathcal{B} \}, \]

which completes the proof. \(\square\)

In general, if $U$ and $V$ are two unitary operators which do not commute, the series $(U^n V^m X, (n,m) \in \mathbb{Z}^2)$ is not stationary. Nevertheless, if $X$ belongs to $\text{Im } C_{U,V}$, i.e., if $U^n V^m X = U^m V^n X$ for any $n, m \in \mathbb{Z}$, then for any pair $((n,m), (n', m'))$ of elements of $\mathbb{Z}^2$, we have $U^n V^m X, U^{n'} V^{m'} X$, $< U^{n-n'} V^{m-m'} X, X >$ < $U^{n-n'} V^{m-m'} X, X >$. So we get the following result.

**Proposition 13.** Let $U$ and $V$ be two unitary operators. Then, for any $X \in \text{Im } C_{U,V}$, the series $(U^n V^m X, (n,m) \in \mathbb{Z}^2)$ and $(V^m U^n X, (n,m) \in \mathbb{Z}^2)$ are equal and stationary.
Proposition 14. Let $U$ and $V$ be two unitary operators. Then, for any $X \in H$, we have

(i) $\|(|E_U(A)(E_V B)X - (E_V B)(E_U A)X)| \leq 2 \|C_{U,V}^1\|$, for all $A, B \in \mathcal{B}$.

(ii) $\|U^n V^m X - V^m U^n X\| \leq 2 \|C_{U,V}^1\|$ for all $n, m \in \mathbb{Z}$.

(iii) $|<U^n V^m X, U^{n'-m'} V^{m'-n'} X, X>| \leq 4 \|C_{U,V}^1\| \|X\|$, for all $n, m, n', m' \in \mathbb{Z}$.

Proof. From Proposition 11, we have $C_{E_U A, E_V B}^1 \ll C_{U,V}^1$. Proposition 10 then allows us to get Part (i), viz.

$\|(|E_U(A)(E_V B)X - (E_V B)(E_U A)X)| \leq 2 \|C_{E_U A, E_V B}^1\| \leq 2 \|C_{U,V}^1\|$.

Turning to Part (ii), for any $X \in H, C_{U,V} X$ belongs to $\cap_{(m,n) \in \mathbb{Z}^2} \text{Ker}(U^n V^m - V^n U^m)$ and then $U^n V^m C_{U,V} X = V^m U^n C_{U,V} X$.

We can then write

$\|U^n V^m X - V^m U^n X\| \leq \|U^n V^m X - U^n V^m C_{U,V} X + \|U^n V^m C_{U,V} X - V^m U^n X\|$

$= \|X - C_{U,V} X\| + \|V^m U^n C_{U,V} X - V^n U^m X\| = \|C_{U,V} X\| + \|C_{U,V} X - X\| = 2 \|C_{U,V}^1\|$.

As for Part (iii), it is a consequence of Part (ii). Indeed, for any pair $((n, m), (n', m'))$ of elements of $\mathbb{Z}^2$, we have

$|<U^n V^m X, U^{n'} V^{m'} X| < U^n V^m V^{-n'} X, X>| = |<V^{-m'} U^{-n'} U^n V^m X, X > + U^n V^m X - V^n U^m X, X>|$

$\leq \|V^{-m'} U^{-n'} X - U^n V^m X - U^n V^m X\| \|X\| + \|U^n V^m X - V^n U^m X\| \|X\| \leq 4 \|C_{U,V}^1\| \|X\|$.

Thus the proof is complete. □

Remark 3. (a) Parts (i) and (ii) of Proposition 14 respectively concern the frequentional and temporal expressions of the same phenomenon: the proximity between $X$ and $\text{Im} C_{U,V}$.

(b) If $X$ is close to $\text{Im} C_{U,V}$, i.e., if $\|C_{U,V}^1\|$ is small, then the series $(U^n V^m X, (n, m) \in \mathbb{Z}^2)$ is almost stationary: for any $n, m, n', m' \in \mathbb{Z}^2, |<U^n V^m X, U^{n'} V^{m'} X| < U^n V^m X, X>|$ is small.

(c) For any $(X, X') \in H \times \text{Im} C$, we have the following commutativity property:

$\|U^n V^m X' - U^n V^m X\| = \|X' - X\| = \|V^m U^n X' - V^m U^n X\|

$ for any $n, m \in \mathbb{Z}$. Recall that the series $(U^n V^m X', (n, m) \in \mathbb{Z}^2) = (V^m U^n X', (n, m) \in \mathbb{Z}^2)$ is stationary.

(d) It can be checked that $C_{U,V}$ is an equalizer of $U V$ and $V U$. If $R$ denotes the maximal equalizer of the unitary operators $U V$ and $V U$, then $d(E_{U,V} A, E_{U,V} A) \ll E_{E_{U,V} E_{U,V}} = R^+ \ll C_{U,V}^1$. Note that this equalizer can be different from the maximal equalizer.

Of course, we can obtain similar results to the last proposition for the maximal commutator of a projector and a unitary operator.

Corollary 3. If $P$ is a projector and $V$ a unitary operator, then, for any $X \in H$, we have

(i) $\|P(E_V B)X - (E_V B)PX\| \leq 2 \|C_{P,V}^1\|$ for all $B \in \mathcal{B}$.

(ii) $\|P V^m X - V^m PX\| \leq \|C_{P,V}^1\| |\sin(1)|$ for all $m \in \mathbb{Z}$.
In a similar way, we have

\[ \lim_{n \to \infty} \delta P + \delta P = I + (e^{i} - 1)P. \]

We recall that \( E_{\text{cor}} = \delta P + \delta P. \) Part (i) of this proposition implies that, for any \( B \in \mathcal{B}, \)

\[
\|P(E_{\text{cor}}B)X - (E_{\text{cor}}B)PX\| = \|E_{\text{cor}}(1)(E_{\text{cor}}B)X - (E_{\text{cor}}B)E_{\text{cor}}(1)X\| \leq 2\|C_{P_{\text{cor}}}^{2}(X)\| = 2\|C_{P_{\text{cor}}}^{2}X\|,
\]

which proves Part (i). As for Part (ii), it can be deduced from Part (ii) of Proposition 14, which implies

\[
\|(e^{i} - 1)(PV^{m} - V^{m}P)X\| = \|UT^{2}V^{m}X - V^{m}UT^{2}X\| \leq 2\|C_{P_{\text{cor}}}^{2}(X)\| = 2\|C_{P_{\text{cor}}}^{2}X\|.
\]

Thus, for any \( m \in \mathbb{Z}, \) \( \|PV^{m}X - V^{m}PX\| \leq \|C_{P_{\text{cor}}}^{2}X\|/\sin(1) \) and hence the argument is complete. \( \square \)

**Remark 4.** These inequalities are respectively the temporal and the frequency expressions of the same phenomenon. They can be interpreted as follows: when \( X \) is close to \( \text{Im } C_{PV}, \) that is when \( \|C_{P_{\text{cor}}}^{2}X\| \) is small, then \( (PV^{m}X, m \in \mathbb{Z}) \) is an almost stationary series, close to the series \( (V^{m}PX, m \in \mathbb{Z}). \) As for \( P \circ Z_{P}^{x}, \) it is almost a random measure, close to \( Z_{P_{\text{cor}}}^{X}, \) which is the random measure associated with the stationary series \( (V^{m}PX, m \in \mathbb{Z}). \)

### 3.6. Commute and history of a series

When \( (X_{n}, n \in \mathbb{Z}) \) is a stationary series (of elements of \( H \)), we call \( P_{n} \) the projector on \( \text{vect}(X_{n} : m \leq n), \) as per [11, 17]. Obviously, if \( n' \leq n, \) from \( \text{vect}(X_{n} : m \leq n') \subseteq \text{vect}(X_{n} : m \leq n), \) we have \( P_{n'} \ll P_{n}. \) It can be shown [11] that the applications \( P_{-\infty} : X \in H \mapsto \lim_{n \to -\infty} P_{n}X \in H \) and \( P_{+\infty} : X \in H \mapsto \lim_{n \to +\infty} P_{n}X \in H \) are projectors.

From the property \( P_{n'} \ll P_{n} \) when \( n' \leq n, \) we deduce that the sequences of projectors \( (P_{k+n}, n \in \mathbb{N}) \) and \( (P_{k-n}, n \in \mathbb{N}) \) are increasing and decreasing, respectively. In view of the background material in Section 2, we can write \( \lim_{n \to +\infty} P_{k+n} = \sup\{P_{k+n} : n \in \mathbb{N}\} \) and \( \lim_{n \to -\infty} P_{k-n} = \inf\{P_{k-n} : n \in \mathbb{N}\}. \) Noting that \( \sup\{P_{k+n} : n \in \mathbb{N}\} = \sup\{P_{n} : n \in \mathbb{N}\} \) and \( \inf\{P_{k-n} : n \in \mathbb{N}\} = \inf\{P_{n} : n \in \mathbb{N}\}, \) one can then make the following assertion:

- For any \( k \in \mathbb{Z}, \) the sequence \( (P_{k+n}, n \in \mathbb{N}) \) r-converges increasingly to \( \sup\{P_{n} : n \in \mathbb{N}\}, \) and the sequence \( (P_{k-n}, n \in \mathbb{N}) \) r-converges decreasingly to \( \inf\{P_{n} : n \in \mathbb{N}\}. \)

Thus, when \( k = 0, \) \( \lim_{n \to +\infty} P_{n} = \sup\{P_{n} : n \in \mathbb{N}\} \) and \( \lim_{n \to -\infty} P_{n} = \inf\{P_{n} : n \in \mathbb{N}\}. \) As r-convergence implies point-wise convergence, on one hand we have \( \lim_{n \to +\infty} P_{n}X = (\sup\{P_{n} : n \in \mathbb{N}\})X \) and hence \( P_{+\infty} = \sup\{P_{n} : n \in \mathbb{N}\}. \) Furthermore, \( \lim_{n \to -\infty} P_{n}X = (\inf\{P_{n} : n \in \mathbb{N}\})X \) and hence \( P_{-\infty} = \inf\{P_{n} : n \in \mathbb{N}\}. \)

Let \( U \) be a unitary operator of \( H \) such that \( UX_{n} = X_{n+1} \) for all \( n \in \mathbb{Z}. \) There exists at least one \( U. \) If \( H = \text{vect}(X_{n} : n \in \mathbb{Z}), \) then \( U \) is unique, and is often associated with the shift operator. So we have the following property.

**Lemma 7.** For any \( k \in \mathbb{Z}, \) \( C_{P_{k}, U} = \inf\{d(P_{k}, P_{k+n})^{1} : n \in \mathbb{N}\}. \)

**Proof.** As stated in [11], \( U^{n}(\text{Im } P_{k}) = \text{Im } P_{k+n}. \) Hence, \( U^{n}P_{k} = P_{k+n}U^{n}P_{k} \) for any \( (n, k \in \mathbb{Z}^{2}. \) We also can write \( U^{+n}P_{k+n} = P_{k}U^{-n}P_{k+n} \), because \((-n, k+n) \in \mathbb{Z}^{2}. \) This implies, taking the adjoint, that \( P_{k+n}U^{n} = P_{k+n}U^{-n}P_{k}. \) So we have \( P_{k+n}U^{n} = U^{n}P_{k} \) or otherwise \( P_{k+n}U^{-n} = U^{-n}P_{k}. \) Proposition 8 then implies

\[
C_{P_{k}, U} = \inf\{d(P_{k}, U^{n}P_{k}U^{n})^{1} : n \in \mathbb{N}\} = \inf\{d(P_{k}, U^{n}P_{k}U^{-n})^{1} : n \in \mathbb{N}\} = \inf\{d(P_{k}, P_{k+n})^{1} : n \in \mathbb{N}\},
\]

whence the proof is complete. \( \square \)

We are now in a position to state the main result of this section.

**Proposition 15.** For any \( k \in \mathbb{Z}, \) we have \( C_{P_{k}, U} = P_{+\infty} + P_{-\infty}. \)

**Proof.** Let \( X \) be an element of \( \text{Im } C_{P_{k}, U}, \) i.e., \( \cap_{n \in \mathbb{Z}} \text{Im } d(P_{k}, P_{k+n})^{1} \cap \ker(P_{k} - P_{k+n}). \) This means that \( P_{k}X = P_{k+n}X \) for all \( n \in \mathbb{Z}. \) So, as \( \lim_{n \to +\infty} P_{k+n}X = P_{+\infty}X, \)

\[
P_{+\infty}X = \lim_{n \to +\infty} P_{k+n}X = P_{k}X.
\]

In a similar way, we have \( P_{k}X = P_{k+n}X \) for all \( n \in \mathbb{Z}, \) and hence \( P_{k}X = P_{k+n}X \) for all \( n \in \mathbb{Z}. \) Therefore, as \( \lim_{n \to -\infty} P_{k+n} = P_{-\infty}, \)

\[
P_{-\infty}X = \lim_{n \to -\infty} P_{k+n}X = P_{k}X.
\]

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Combining (3) and (4), we have $P_{-\infty} = P_{+\infty}$, which brings us to conclude that $X$ belongs to $\text{Ker}(P_{-\infty} - P_{+\infty}) = \text{Im}(P_{-\infty})$. Thus we have proved that $\text{Im}(C_{P, U}) \subseteq \text{Im}(P_{+\infty} - P_{-\infty})$.

Conversely, consider an element $X$ of $\text{Im}(P_{+\infty} - P_{-\infty}) = \text{Ker}(P_{-\infty} - P_{+\infty})$. Then $P_{+\infty}X = P_{-\infty}X$. Fix $m \in \mathbb{Z}$. From $P_{-\infty} = \inf\{P_n : n \in \mathbb{Z}\}$ and $P_{+\infty} = \sup\{P_n : n \in \mathbb{Z}\}$, we deduce that $||P_{-\infty}X|| \leq ||P_mX|| \leq ||P_{+\infty}X||$. Given that $||P_{-\infty}X|| = ||P_{+\infty}X||$, and $||P_mX|| = ||P_{-\infty}X||$, we have $P_{-\infty}X = P_{+\infty}X$. Indeed,

$$||P_{m}X||^2 = ||P_{-\infty} + (P_m - P_{-\infty})||^2 = ||P_{-\infty}X||^2 + ||(P_m - P_{-\infty})X||^2.$$ 

As this is true for arbitrary $m \in \mathbb{Z}$, we have $P_{-\infty}X = P_{+\infty}X = P_{k+n}X$ for all $n \in \mathbb{Z}$. We can thus conclude that $X$ belongs to $\cap_{n \in \mathbb{Z}} \text{Ker}(P_{k} - P_{k+n}) = \cap_{n \in \mathbb{Z}} \text{Im}(d(P_{k}, P_{k+n}) = \text{Im}(C_{P, U})$. Hence, $\text{Im}(P_{-\infty} + P_{+\infty}) \subseteq \text{Im}(C_{P, U})$. This, combined with the previous inclusion, gives $\text{Im}(C_{P, U}) = \text{Im}(P_{+\infty} + P_{-\infty})$, and finally we have $C_{P, U} = P_{+\infty} + P_{-\infty}$. □

**Remark 5.** From [11, 17], the series $(X_n, n \in \mathbb{Z})$ is said to be regular or purely non-deterministic when $P_{-\infty} = 0$, and to be singular or purely deterministic when $P_{+\infty} = 0$. In order to simplify these definitions, let us consider the case where $H = \text{vect}(X_n, n \in \mathbb{Z})$. So $P_{-\infty} = I$. The series then will be said to be singular when $P_{+\infty} = I$. As for the formula $C_{P, U} = P_{+\infty} + P_{-\infty}$, it becomes $C_{P, U} = P_{+\infty}$. $P_{-\infty}$ is the maximal commuter of the projector $P_{k}$ and the unitary operator $U$. This gives us a new definition of the projector $P_{\pm\infty}$, and makes the maximal commuter relevant once again. Then we can assert that the series is singular if and only if $C_{P, U} = I$, i.e., if and only if the projector on the history to the instant $k$ and the shift operator commute.

4. Relations between the spectral measures $\mathcal{E}_U, \mathcal{E}_V, \mathcal{E}_{UV}$ and $\mathcal{E}_W$

When two unitary operators $U$ and $V$ commute, we can compute the spectral measure associated with $UV = VU$, thanks to the convolution product of the spectral measures respectively associated with $U$ and $V$. Now what happens when the operators do not commute? The maximal commuter will help us to address this question. Let us first look at some preliminary results.

**Lemma 8.** If $P$ and $Q$ are two projectors on $H$ which commute, then $L_P^* Q L_P$ is a projector of $\text{Im} P$.

**Proof.** This result holds because $(L_P^* Q L_P)^* = L_P^* Q L_P$ and $(L_P^* Q L_P)(L_Q^* Q L_P) = L_P^* Q L_P^* Q L_P = L_P^* Q L_P$.

**Lemma 9.** If $P$ is a projector of $H$ and $Q$ a projector of $\text{Im} P$, then $L_P^* Q L_P$ is a projector of $H$.

**Proof.** This is because $L_P^* Q L_P$ is self-adjoint and $L_P^* Q L_P^* = L_P^* Q L_P = L_P^* Q L_P^* = L_P$.

**Lemma 10.** If $C$ is a projector of $H$ which commutes with the projectors $P$ and $P'$, then $\text{inf}(L_C^* P L_C, L_C^* P' L_C) = L_C^* \text{inf}(P, P') L_C$.

**Proof.** As $C$ commutes with each of the elements of the family of projectors $\{P, P'\}$, it commutes with $\text{inf}(P, P')$. As $C$ commutes with the projectors $P$ and $P'$, $\text{inf}(P, P')$, from Lemma 8, $L_C^* P L_C, L_C^* P' L_C$, and $L_C^* \text{inf}(P, P') L_C$ are projectors of $\text{Im} C$. Given that

$$L_C^* \text{inf}(P, P') L_C L_C^* P L_C = L_C^* \text{inf}(P, P') C P L_C = L_C^* \text{inf}(P, P') P L_C = L_C^* \text{inf}(P, P') L_C,$$

we deduce that $L_C^* \text{inf}(P, P') L_C \ll L_C^* P L_C$. In the same way, we establish that $L_C^* \text{inf}(P, P') L_C \ll L_C^* P' L_C$, so that $L_C^* \text{inf}(P, P') L_C$ is a lower bound of the family of projectors $\{L_C^* P L_C, L_C^* P' L_C\}$.

Let us consider a lower bound $K$ of the family $\{L_C^* P L_C, L_C^* P' L_C\}$. From Lemma 9, $L_C K L_C^* P L_C$ is a projector of $H$. From $K \ll L_C^* P L_C$, we deduce that $KL_C^* P L_C = K$ and hence $L_C K L_C^* P L_C = L_C K L_C^* P C = L_C K L_C^* C P = L_C K L_C^* P$. This means that $K L_C^* \ll P$. In the same way, we can show that $L_C K L_C^* P \ll P'$ and hence $L_C K L_C^* P$ is a lower bound of the family $\{P, P'\}$, so $L_C K L_C^* P \ll \text{inf}(P, P')$. Then we can write $L_C^* L_C K L_C^* \text{inf}(P, P') L_C = L_C^* L_C K L_C^* L_C$, that is $KL_C^* \ll \text{inf}(P, P') L_C = K$. This means that $L_C^* \text{inf}(P, P') L_C$ is the largest lower bound of the family of projectors $\{L_C^* P L_C, L_C^* P' L_C\}$. Thus the proof is complete. □

In what follows, $U$ and $V$ are two unitary operators, and we denote by $C$ their maximal commuter. As $C$ commutes with $U$ and $V$, it follows from the background material in Section 2 that $U' = L_C^* U L_C$ and $V' = L_C^* V L_C$ are unitary operators of $\text{Im} C$. Moreover, $\mathcal{E}_{EU} A = L_C^* (\mathcal{E}_U A) L_C$ and $\mathcal{E}_{EV} A = L_C^* (\mathcal{E}_V A) L_C$ for all $A \in \mathcal{B}$. As $C$ is a commuter of the unitary operators $U$ and $V$, the unitary operators $U'$ and $V'$ commute (see Lemma 4), yielding the following result.
Proposition 16. There exists a spectral measure $E_{UV} \otimes E_{V'}$, and only one, on $B \otimes B$ for $\text{Im} C$, such that $E_{UV} \otimes E_{V'}(A \times B) = L_C^* \inf(E_{UV}(A), E_{V'}(B))L_C$ for all $A, B \in B$.

Proof. As $U'V' = V'U'$, the spectral measures $E_{UV}$ and $E_{V'}$ commute and we can consider $E_{UV} \otimes E_{V'}$, which is a spectral measure on $B \otimes B$ for $\text{Im} C$ such that, for all $A, B \in B$,

$$E_{UV} \otimes E_{V'}(A \times B) = (E_{UV}(A))(E_{V'}(B)) = L_C^* (E_{UV}(A))L_C L_C^* (E_{V'}(B))L_C = \inf[L_C^* (E_{UV}(A))L_C, L_C^* (E_{V'}(B))L_C].$$

As $C$ commutes with $U$ and $V$, it commutes with the projectors $E_{UV}$ and $E_{V'}$. So we can use Lemma 10, and write

$$\inf[L_C^* (E_{UV}(A))L_C, L_C^* (E_{V'}(B))L_C] = L_C^* \inf(E_{UV}(A), E_{V'}(B))L_C,$$

which leads us to conclude that the spectral measure $E_{UV} \otimes E_{V'}$ is the only one on $B \otimes B$ for $\text{Im} C$ such that, for all $A, B \in B, E_{UV} \otimes E_{V'}(A \times B) = L_C^* \inf(E_{UV}(A), E_{V'}(B))L_C$.

Remark 6. When $U$ and $V$ commute, we know how to define a spectral measure on $B \otimes B$ for $H$. When $U$ and $V$ do not commute, we know how to define a spectral measure on $B \otimes B$, but for $\text{Im} C$, which is a closed vector subspace of $H$. Of course, when $U$ and $V$ commute, we recover the definition of the spectral measure $E_{UV} \otimes E_{V'}$ such that $E_{UV} \otimes E_{V'}(A \times B) = \inf(E_{UV}(A), E_{V'}(B)) = (E_{UV}(A))(E_{V'}(B))$ for all $A, B \in B$.

Now we are going to study some relations between the spectral measures $E_{UV}$, $E_{VU}$ and $E_{UV} \otimes E_{V'} = E_{V'} \otimes E_{V'}$.

Proposition 17. For any $A \in B$, we have

(i) $L_C \circ (E_{UV} \ast E_{V'}(A)) \circ L_C^* = CE_{UV}A = CE_{VU}A = C \inf(E_{UV}(A), E_{VU}(A)) = \inf(C, E_{UV}(A), E_{VU}(A))$.

(ii) $L_C \circ (E_{UV} \ast E_{V'}(A)) \circ L_C^* = \inf(E_{UV}(A), E_{VU}(A)) \ll \sup(E_{UV}(A), E_{VU}(A)) \ll C^+ + L_C \circ (E_{UV} \ast E_{V'}(A)) \circ L_C^*.$

Proof. $E_{UV} \ast E_{V'} = E_{UV} \ast E_{V'}$ is the spectral measure associated with the unitary operator $V'U'$. But $U'V' = L_C^* ULC = L_C^* UVL_C V' = V'U' = L_C^* VUL_C$. As $C$ commutes with the unitary operators $UV$ and $VU$, we can conclude that $L_C^* UVC$ and $L_C^* VUL_C$ are unitary operators of $\text{Im} C$ which are equal, and of associated spectral measure the application $A \in B \mapsto L_C^* (E_{UV}(A))L_C \in \mathcal{P}(\text{Im} C)$ or equivalently $A \in B \mapsto L_C^* (E_{UV}(A))L_C \in \mathcal{P}(\text{Im} C)$.

As $U'V' = L_C^* UVC$, $E_{VU} \ast E_{UV} = L_C^* VUL_C$, we have, for any $A \in B$,

$$E_{UV} \ast E_{V'}(A) = L_C^* \circ (E_{UV}(A)) \circ L_C = L_C^* \circ (E_{VU}(A)) \circ L_C.$$

Therefore,

$$L_C \circ (E_{UV} \ast E_{V'}(A)) \circ L_C^* = C(E_{UV}(A))C = C(E_{VU}(A))C = CE_{UV}A = CE_{VU}A.$$

We recall that $C$ commutes with the unitary operators $UV$ and $VU$, so with the projectors $E_{UV}$ and $E_{VU}$. As $C$ commutes with $E_{UV}$ and $E_{VU}$, it commutes with $\inf(E_{UV}(A), E_{VU}(A))$ and $C \inf(E_{UV}(A), E_{VU}(A)) = \inf(CE_{UV}A, CE_{VU}A) = CE_{UV}A = CE_{VU}A$.

Let $A$ be an element of $B$. Then

$$L_C^* \circ (E_{UV} \ast E_{V'}(A)) \circ L_C^* = CE_{UV}A = CE_{VU}A = \inf(CE_{UV}A, CE_{VU}A) = \inf[C, E_{UV}(A), E_{VU}(A)].$$

However, the right-hand term is a lower bound of the family $\{C, E_{UV}(A), E_{VU}(A)\}$, so

$$\inf[C, E_{UV}(A), E_{VU}(A)] \ll \inf(C, E_{UV}(A), E_{VU}(A)).$$

Moreover, $\inf[C, E_{UV}(A), E_{VU}(A)]$ is a lower bound of the families $[C, E_{UV}(A)]$ and $[C, E_{VU}(A)]$, so $\inf[C, E_{UV}(A), E_{VU}(A)] \ll \inf(C, E_{UV}(A))$ and $\inf(C, E_{UV}(A), E_{VU}(A)) \ll \inf(C, E_{UV}(A))$ and then

$$\inf[C, E_{UV}(A), E_{VU}(A)] \ll \inf[C, E_{UV}(A), E_{VU}(A)],$$

which completes the proof of claim (i). Furthermore, we have

$$L_C \circ (E_{UV} \ast E_{V'}(A)) \circ L_C^* = CE_{UV}A = CE_{VU}A = \inf[C, E_{UV}(A), E_{VU}(A)].$$
Let $A$ be an element of $B$, $L_C \circ (E_U \circ E_V(A)) \circ L_C^\perp = CE_{UV}A = CE_{UV}A = \inf(C, E_{UV}A, E_{UV}A)$ is then a lower bound of the family $\{C, E_{UV}A, E_{UV}A\}$ and then of the family $\{E_{UV}A, E_{UV}A\}$. Thus
\[
L_C \circ (E_U \circ E_V(A)) \circ L_C^\perp \ll \inf(E_{UV}A, E_{UV}A) \ll \sup(E_{UV}A, E_{UV}A).
\]
(5)

For any $A \in B$, we can also write
\[
L_C \circ (E_U \circ E_V(\bar{A})) \circ L_C^\perp \ll \inf(E_{UV}(\bar{A}), E_{UV}(\bar{A}))
\]
and then, taking the orthogonal, $\sup(E_{UV}A, E_{UV}A) \ll I - L_C \circ (E_U \circ E_V(\bar{A})) \circ L_C^\perp$, i.e.,
\[
\sup(E_{UV}A, E_{UV}A) \ll C^+ + L_C L_C^\perp - L_C \circ (E_U \circ E_V(\bar{A})) \circ L_C^\perp.
\]
Therefore,
\[
\sup(E_{UV}A, E_{UV}A) \ll C^+ + L_C \circ [I_{\text{im}C} - (E_U \circ E_V(\bar{A}))] \circ L_C^\perp = C^+ + L_C \circ (E_U \circ E_V(A)) \circ L_C^\perp.
\]
Thus, taking into account (5), we can conclude that
\[
L_C \circ (E_U \circ E_V(A)) \circ L_C^\perp \ll \inf(E_{UV}A, E_{UV}A) \ll \sup(E_{UV}A, E_{UV}A) \ll C^+ + L_C \circ (E_U \circ E_V(A)) \circ L_C^\perp
\]
for all $A \in B$. \qed

From Part (i) of Proposition 17, we deduce the following.

**Corollary 4.** When $A \in B$, we have $E_U \circ E_V(A) = 0$ if and only if $\inf(C, E_{UV}A, E_{UV}A) = 0$.

**Proof.** If $0 = \inf(C, E_{UV}A, E_{UV}A) = L_C \circ (E_U \circ E_V(A)) \circ L_C^\perp$, then $0 = L_C \circ 0 \circ L_C = L_C \circ (E_U \circ E_V(A)) \circ L_C = E_U \circ E_V(A)$ because $L_C^\perp \circ L_C = I_{\text{im}C}$. As for the converse, it is obvious. \qed

As $\inf(C, E_{UV}A, E_{UV}A) \ll \inf(E_{UV}A, E_{UV}A)$, we also have the following result.

**Corollary 5.** If $A \in B$ is such that $\inf(E_{UV}A, E_{UV}A) = 0$, then $E_U \circ E_V(A) = 0$.

This is a way to obtain a sufficient condition in order to have $E_U \circ E_V(A) = 0$. Let us now examine a sufficient condition in order to have $E_U \circ E_V(A) = I$.

**Corollary 6.** If $A \in B$ is such that $\sup(E_{UV}A, E_{UV}A) = I$, then $E_U \circ E_V(A) = I$.

**Proof.** From $I = \sup(E_{UV}A, E_{UV}A)$, taking the orthogonal, we have $0 = \inf(E_{UV}(\bar{A}), E_{UV}(\bar{A}))$ and then, from Corollary 5, $E_U \circ E_V(A) = 0$, i.e., $E_U \circ E_V(A) = I$. \qed

Part (ii) of Proposition 17 gives an upper bound and a lower bound of the projectors $\inf(E_{UV}A, E_{UV}A)$ and $\sup(E_{UV}A, E_{UV}A)$. These projectors give an idea of the difference between the spectral measures $E_{UV}$ and $E_{UV}$. Note that the inequalities of Part (ii) of Proposition 17 are optimal in the sense that the inequalities become equalities in some particular cases: $L_C \circ (E_U \circ E_V(\emptyset)) \circ L_C^\perp = \inf(E_{UV}(\emptyset), E_{UV}(\emptyset))$ and $\sup(E_{UV}(\Pi), E_{UV}(\Pi)) = C^+ + L_C \circ (E_U \circ E_V(\Pi)) \circ L_C^\perp$. The following result enables us to better understand the conditions for such equalities.

**Proposition 18.** For arbitrary $A \in B$, the following assertions are equivalent.

(i) $L_C \circ (E_U \circ E_V(A)) \circ L_C^\perp = \inf(E_{UV}A, E_{UV}A)$.

(ii) $\inf(E_{UV}A, E_{UV}A) \ll C$.

(iii) $C^+ \ll \sup(E_{UV}(\bar{A}), E_{UV}(\bar{A}))$.

(iv) $\sup(E_{UV}(\bar{A}), E_{UV}(\bar{A})) = C^+ + L_C \circ (E_U \circ E_V(\bar{A})) \circ L_C^\perp$. 

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Proof. We will prove successively that (i) implies (ii), (ii) implies (iii), (iii) implies (iv), and (iv) implies (i).

Assume (i) holds. If $L_C \circ [\mathcal{E}_U + \mathcal{E}_V(A)] \circ L_C^* = \inf(\mathcal{E}_U(A), \mathcal{E}_V(A))$, from Proposition 17, we have $C \inf(\mathcal{E}_U(A), \mathcal{E}_V(A)) = \inf(\mathcal{E}_U(A), \mathcal{E}_V(A))$, i.e., $\inf(\mathcal{E}_U(A), \mathcal{E}_V(A)) \ll C$, and hence (ii) holds.

We prove the second implication by considering orthogonal elements. If $C^\perp \ll \sup(\mathcal{E}_U(A), \mathcal{E}_V(A))$, we have

\[ C^\perp = C^\perp \sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)) = \sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)) - C \sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)). \]

So

\[ \sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)) = C^\perp + C \sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)). \]  \hspace{1cm} (6)

But, taking into account Proposition 17,

\[ C \sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)) = C[I - \inf(\mathcal{E}_U(A), \mathcal{E}_V(A))] = C - C \inf(\mathcal{E}_U(A), \mathcal{E}_V(A)) \]

\[ = L_C L_C^* - L_C [\mathcal{E}_U \ast \mathcal{E}_V(A)] L_C^* = L_C \{I_{\text{im}C} - \mathcal{E}_U \ast \mathcal{E}_V(A)\} L_C^* = L_C \circ [\mathcal{E}_U \ast \mathcal{E}_V(CA)] L_C^*. \]

In the light of (6), this gives

\[ \sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)) = C^\perp + L_C \circ [\mathcal{E}_U \ast \mathcal{E}_V(CA)] \circ L_C^*, \]

which proves the third implication.

Finally, if $\sup(\mathcal{E}_U(CA), \mathcal{E}_V(CA)) = C^\perp + L_C \circ [\mathcal{E}_U \ast \mathcal{E}_V(CA)] \circ L_C^*$, taking the orthogonal, we get

\[ \inf(\mathcal{E}_U(A), \mathcal{E}_V(A)) = C - L_C \circ [\mathcal{E}_U \ast \mathcal{E}_V(CA)] \circ L_C^* = L_C L_C^* - L_C \circ [\mathcal{E}_U \ast \mathcal{E}_V(CA)] \circ L_C^* \]

\[ = L_C \circ [I_{\text{im}C} - \mathcal{E}_U \ast \mathcal{E}_V(CA)] \circ L_C^* = L_C \circ [\mathcal{E}_U \ast \mathcal{E}_V(CA)] \circ L_C^*, \]

which completes the proof. \hfill \square

Let us now examine a relation which evokes a convolution formula.

Proposition 19. For any pair $(U, V)$ of unitary operators and for any $A \in \Pi$, we have $\sup(\inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) : \lambda' \in \Pi) \ll \mathcal{E}_V([A])$.

Proof. Let $X$ be an element of $\text{Im} \ inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) = \text{Im} \mathcal{E}_U([A] \ominus \lambda') \cap \text{Im} \mathcal{E}_V([A'])$. From $X \in \text{Im} \mathcal{E}_U([A] \ominus \lambda')$, we deduce that

\[ UX = e^{i\lambda X} X. \]  \hspace{1cm} (7)

From $X \in \text{Im} \mathcal{E}_V([A'])$, we also deduce that

\[ VX = e^{i\lambda X}. \]  \hspace{1cm} (8)

Combining (7) and (8), we can write $UVX = e^{i\lambda X}$, so $X \in \text{Im} \mathcal{E}_U([A])$. We have just proved that $\text{Im} \ inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) \subset \text{Im} \mathcal{E}_U([A])$, i.e., $\inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) \ll \mathcal{E}_U([A])$. The projector $\mathcal{E}_U([A])$ being an upper bound of the family $[\inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) : \lambda' \in \Pi]$, we can write $\sup(\inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) : \lambda' \in \Pi) \ll \mathcal{E}_U([A])$, and the proposition is proved. \hfill \square

Remark 7. (a) An upper bound of projectors can be a sum, e.g., if the projectors $P_1$ and $P_2$ commute, then $\sup(P_1, P_2) = P_1 + P_2$. Then the first term of the inequality of the proposition becomes similar to the expression of a convolution product.

(b) In some particular cases, the formula may become $\sup(\inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) : \lambda' \in \Pi) = 0 \ll \mathcal{E}_U([A])$, and in other cases, it becomes $\sup(\inf(\mathcal{E}_U([A] \ominus \lambda'), \mathcal{E}_V([A'])) : \lambda' \in \Pi) = \mathcal{E}_U([A])$.

We conclude this section with a final relation between the spectral measures $\mathcal{E}_UV$ and $\mathcal{E}_VU$. The unitary operators $UV$ and $VU$ are unitary equivalent: $U^{-1}(UV)U = VU$. We are going to see that this can be transposed to the spectral measures $\mathcal{E}_UV$ and $\mathcal{E}_VU$.

Proposition 20. (i) For any $A \in \mathcal{B}$, $\alpha A = U^{-1}(\mathcal{E}_UV(A)) U$ is a projector. (ii) the application $\alpha : A \in \mathcal{B} \mapsto U^{-1}(\mathcal{E}_UV(A)) U \in \mathcal{P}(H)$ is a spectral measure. (iii) For any $A \in \mathcal{B}$, $\mathcal{E}_VU A = U^{-1}(\mathcal{E}_UV(A)) U$.
Proof. It is easy to check the first two parts. Now denote by \( W \) the unitary operator of associated spectral measure \( \alpha \). For a given \( X \) of \( H \), we have \( WX = \int \! e^{i\lambda} d\zeta_{\alpha}(\lambda) \). But \( \zeta_{\alpha} = U^{-1} \circ \zeta^{\alpha}_X \), because for any \( A \in \mathcal{B} \),

\[
Z^X_A = (\alpha A)X = U^{-1}(E_{UV}A)UX = U^{-1} \circ Z^{\alpha}_X A.
\]

As a result

\[
WX = \int \! e^{i\lambda} dU^{-1} \circ Z^{\alpha}_X = \int \! e^{i\lambda} dZ_{\alpha} = U^{-1} UVX,
\]

because \( E_{UV} \) is the spectral measure associated with \( UV \). The relation \( WX = U^{-1} UVX \), for any \( X \in H \), allows us to write \( W = U^{-1} \circ UV \circ U \). This means that \( U^{-1} \circ UV \circ U \), i.e., \( V U \), has \( \alpha \) as associated spectral measure. And then, for any \( A \in \mathcal{B} \), \( E_{UV}A = \alpha A = U^{-1}(E_{UV}A)U \). \( \square \)

5. The finite spectrum case and convergence properties

In this section, we study the structure of the commutators \( C_{PU} \) and \( C_{UV} \) when the spectral measures which are associated with the unitary operators \( U \) and \( V \) are concentrated on a finite number of elements of \( \Pi \). This will allow us to specify the nature of the projectors \( C_{PU} \) and \( C_{UV} \). Then, when this assumption no longer holds, we will establish properties of \( r \)-convergence of the form: \( C_{PU} = \lim_{n} C_{PU_n} \) and \( C_{UV} = \lim_{n} C_{UV_n} \), where \( U_n \) and \( V_n \) are unitary operators, respectively defined from \( U \) and \( V \), thanks to a partition of \( \Pi \) for which the associated spectral measures are concentrated on a finite number of elements of \( \Pi \).

To this end, we first give some reminders and specific notation for this section. For any \( n \in \mathbb{N} \), set \( A_{n,k} = [-\pi + k2\pi/n, -\pi + (k+1)2\pi/n) \) for all \( k \in \{0, \ldots, 2^n - 1\} \). It is clear that \( \{A_{n,k} : k = 0, \ldots, 2^n - 1\} \) is a family of elements of \( \mathcal{B} \) which constitutes a partition of \( \Pi \). The application \( L_n = \sum_{k=0}^{2^n-1} (-\pi + k2\pi/n) 1_{A_{n,k}} \) from \( \Pi \) into itself is measurable and such that \( L_n \circ L_{n+1} = L_n \). If \( U \) is a unitary operator and \( U_n \) the unitary operator of associated spectral measure \( L_n E_{U} = L_n E_{U} \), then

\[
L_n E_{U} = \sum_{k=0}^{2^n-1} \delta_{-\pi + k2\pi/n} E_{U} A_{n,k}, \quad U_n = \sum_{k=0}^{2^n-1} e^{i(-\pi + k2\pi/n)} E_{U} A_{n,k}, \quad UX = \lim_{n} \sum_{k=0}^{2^n-1} e^{i(-\pi + k2\pi/n)} (E_{U} A_{n,k})X = \lim_{n} U_n X.
\]

We first examine the expression of \( C_{UV} \) when the spectra of \( U \) and \( V \) are finite.

Proposition 21. Let \( \{P_j : j \in J\} \) and \( \{D_\ell : \ell \in L\} \) be two finite families of projectors such that \( I = \sum_{j} P_j = \sum_{\ell} D_\ell \) and such that \( P_j P_j = D_\ell D_\ell = 0 \), for any pairs \((j,j')\) and \((\ell, \ell')\) of distinct elements of \( J \) and \( L \), respectively. If \( \{\lambda_j : j \in J\} \) and \( \{\mu_\ell : \ell \in L\} \) are families of distinct elements of \( \Pi \), then (i) \( U = \sum_{j} \phi^{\lambda_j} P_j \) and \( V = \sum_{\ell} \phi^{\mu_\ell} D_\ell \) are unitary operators of \( H \); (ii) \( C_{UV} = \sum_{j} \sum_{\ell} \sum_{k} \inf(P_j, D_\ell) \).

Proof. Let \( C = \sum_{j} \sum_{\ell} \sum_{k} \inf(P_j, D_\ell) \). For any pair \((j, \ell)\) of distinct elements of \( J \times L \), we have \( \inf(P_j, D_\ell) = \inf(P_j, D_\ell) = 0 \), so \( C \) is a projector. It is easy to verify that \( UC = CU \), that \( VC = CV \) and that \( VCU = UCV \), so that \( C \) is a commutator of the unitary operators \( U \) and \( V \). Therefore,

\[
C \ll C_{UV}.
\]

Let \( X \) be an element of \( \text{Im} C_{UV} \subset \bigcap_{(A,B) \in \mathcal{B} \times \mathcal{B}} \text{Ker}[(E_{U}A)(E_{V}B) - (E_{V}B)(E_{U}A)] \). For any \((A, B) \in \mathcal{B} \times \mathcal{B} \), we have \( (E_{U}A)(E_{V}B)X = (E_{V}B)(E_{U}A)X \). Let \((j, \ell)\) be an element of \( J \times L \). As \((\lambda_j), (\mu_\ell)\) is an element of \( \mathcal{B} \times \mathcal{B} \), we can write

\[
P_j D_\ell X = (E_{U}A(\lambda_j))(E_{V}B(\mu_\ell))X = (E_{V}B(\mu_\ell))(E_{U}A(\lambda_j))X = D_\ell P_j X.
\]

So as \( P_j D_\ell X = D_\ell P_j X \in \text{Im} P_j \cap \text{Im} D_\ell = \text{Im} \inf(P_j, D_\ell) \), we have \( P_j D_\ell X = \inf(P_j, D_\ell) P_j D_\ell X = \inf(P_j, D_\ell) X \). As this holds for all \((j, \ell) \in J \times L \), we deduce that

\[
CX = \sum_{j \in J} \sum_{\ell \in L} \inf(P_j, D_\ell) X = \sum_{j \in J} \sum_{\ell \in L} P_j D_\ell X = \sum_{j \in J} P_j X = X.
\]

We have just proved that \( \text{Im} C_{UV} \subset \text{Im} C \), so that \( C_{UV} \ll C \), which, combined with (9), gives \( C_{UV} = C = \sum_{j \in J} \sum_{\ell \in L} \inf(P_j, D_\ell) \).

\[\square\]
Remark 8. If \( P \) is a projector, as \( \epsilon^p = \epsilon^p + P \), so as \( \mathcal{E}_n^p = \delta_0 P^\perp + \delta_1 P \), we have
\[
C_{PV} = C_{\varphi^p V} = \sum_{\ell \in \mathbb{L}} \inf(\|P^\perp, D_\ell\|) + \sum_{\ell \in \mathbb{L}} \inf(\|P, D_\ell\|) = \sum_{\ell \in \mathbb{L}} \inf(\|P, D_\ell\| + \inf(\|P^\perp, D_\ell\|)).
\]

We now state a final preliminary result before the major result of this section.

Lemma 11. Let \( \mathcal{L} \) be a measurable application from \( \Pi \) into itself, \( U \) and \( V \) be two unitary operators, and \( U' \) and \( V' \) the unitary operators of the corresponding spectral measures \( \mathcal{L}(\mathcal{E}_U) \) and \( \mathcal{L}(\mathcal{E}_V) \). Then we have \( C_{UV} \ll C_{U'V'} \).

Proof. As \( \{C_{E_n A E_p B} : (A, B) \in \mathcal{B} \times \mathcal{B}\} \subset \{C_{E_n A E_p B} : (A, B) \in \mathcal{B} \times \mathcal{B}\} = \{C_{E_n A E_p B} : (A, B) \in \mathcal{B} \times \mathcal{B}\} \subset \inf\{C_{E_n A E_p B} : (A, B) \in \mathcal{B} \times \mathcal{B}\} = C_{U'V'} \). \( \square \)

Here are some consequences of these results.

Proposition 22. If \( (U, V) \) is a pair of unitary operators, then \( (\sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} \inf(\mathcal{E}_U(A_{n,k}, \mathcal{E}_V A_{n,l}), n \in \mathbb{N}) \)  is a decreasing sequence of projectors which converges to \( C_{UV} \).

Proof. For any \( n \in \mathbb{N} \), denote by \( U_n \) and \( V_n \) the unitary operators of respective associated spectral measures \( \mathcal{L}_n \mathcal{E}_U \) and \( \mathcal{L}_n \mathcal{E}_V \). Then we have \( E_{U_n} = \mathcal{L}_n \mathcal{E}_U \) and \( E_{V_n} = \mathcal{L}_n \mathcal{E}_V \). So \( E_{U_n} = \mathcal{L}_n \mathcal{E}_U \) and \( E_{V_n} = \mathcal{L}_n \mathcal{E}_V \). From Lemma 11, we deduce that \( C_{U_n V_n} \ll C_{U_n V_n} \). So \( (C_{U_n V_n}, n \in \mathbb{N}) \) is a decreasing sequence of projectors. From Section 2, it is \( r \)-convergent, and \( \lim_n C_{U_n V_n} = \inf\{C_{U_n V_n}, n \in \mathbb{N}\} \). Let \( C = \lim_n C_{U_n V_n} \). Still from Lemma 11, we have \( C_{UV} \ll C_{U_n V_n} \) for all \( n \in \mathbb{N} \). Therefore,
\[
C_{UV} \ll C. \tag{10}
\]

The projector \( C \) commutes with \( U \) because
\[
\|UCX - CUX\| \leq \|UCX - U_n CX\| + \|U_n CX - U_n C_{U_n V_n} X\| + \|C_{U_n V_n} UX - C_{U_n V_n} U X\| + \|C_{U_n V_n} UX - CUX\|
\]

but
\[
\lim_{n \to \infty} \|UCX - U_n CX\| = \lim_{n \to \infty} \|C_{U_n V_n} UX - C_{U_n V_n} U X\| = \lim_{n \to \infty} \|C_{U_n V_n} UX - CUX\| = 0.
\]

Hence \( \|UCX - CUX\| = 0 \) for every \( X \in H \). That is, \( C \) and \( U \) commute. In the same way, we can prove that \( VC = CV \).

For any \( X \in H \), we also can write
\[
\|UCVX - VCUX\| \leq \|UCVX - U_n CVX\| + \|U_n CVX - U_n C_{U_n V_n} VX\| + \|U_n C_{U_n V_n} VX - U_n C_{U_n V_n} V_n X\|
\]

but
\[
\lim_{n \to \infty} \|UCVX - U_n CVX\| = \lim_{n \to \infty} \|C_{U_n V_n} VX - CVX\| = \lim_{n \to \infty} \|V_n VX - VX\| = 0.
\]

So \( \|UCVX - VCUX\| = 0 \), i.e., \( UCVX = VCUX \) for any \( X \in H \). Hence \( UCV = VCU \), and we can say that \( C \) is a commuter of \( U \) and \( V \). Then \( C \ll C_{UV} \), which, combined with (10), implies that \( C_{UV} = C = \inf\{C_{U_n V_n} \colon n \in \mathbb{N}\} \). As
\[
U_n = \sum_{k=0}^{2^n-1} e^{i(2\pi(2n)^2)} \mathcal{E}_U A_{n,k}, \quad V_n = \sum_{l=0}^{2^n-1} e^{i(2\pi(2n)^2)} \mathcal{E}_V A_{n,l},
\]
it follows from Proposition 17 that
\[
C_{U_n V_n} = \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} \inf(\mathcal{E}_U A_{n,k}, \mathcal{E}_V A_{n,l}).
\]

So we can conclude that \( (\sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} \inf(\mathcal{E}_U A_{n,k}, \mathcal{E}_V A_{n,l}), n \in \mathbb{N}) \) is a sequence of projectors which decreasingly converges to \( C_{UV} \). \( \square \)
Proposition 23. \[ \lim_{n \to \infty} [\inf(P, E_V A_{n,t}) + \inf(P^+, E_V A_{n,t})] = \inf_{n \in \mathbb{N}} [\lim_{t \to \infty} (P, E_V A_{n,t}) + \lim_{t \to \infty} (P^+, E_V A_{n,t})]. \]

6. Application to the study of a class of families of stationary correlated series

Let \((X_n, n \in \mathbb{Z})\) be a stationary series. We propose to characterize all the stationary series \((X'_n, n \in \mathbb{Z})\), stationarily correlated with \((X_n, n \in \mathbb{Z})\), and such that \((X'_n, n \in \mathbb{Z}) \subset \text{Im} P\), where \(P\) is a projector. Of course, if \(H = L^2(\Omega, \mathcal{A}, Q)\), \(\text{Im} P\) can be of \(L^2(\Omega, \mathcal{A}', Q)\) type, where \(\mathcal{A}'\) is a sub-\(\sigma\)-field of \(\mathcal{A}\). A series solution is a series of \(\mathcal{A}'\)-measurable random variables. In order to solve this problem, we need some preliminary results.

**Lemma 12.** If \(V\) is a unitary operator of \(H\), \(K\) being a projector of \(H\), then \(U = K^+ + L_K V^n L_K^*\) is a unitary operator of \(H\) such that \(U^n = K^+ + L_K V^n L_K^*\) for all \(n \in \mathbb{Z}\).

**Proof.** It is easily seen that \(U = K^+ + L_K V^n L_K^*\) is a unitary operator. It can then be shown by induction that, for all \(n \in \mathbb{N}\), \(U^n = K^+ + L_K V^n L_K^*\). This property extends to \(\mathbb{Z}\). Indeed, when \(n < 0\), we have \(U^{|n|} = K^+ + L_K V^{|n|} L_K^*\) and considering the adjoint, \(U^{-|n|} = K^+ + L_K V^{-|n|} L_K^*\), i.e., \(U^n = K^+ + L_K V^n L_K^*\). \(\square\)

**Lemma 13.** If \((X'_n, n \in \mathbb{N})\) and \((X''_n, n \in \mathbb{N})\) are two stationary series that are stationarily correlated, then there exists a unitary operator \(U\) of \(H\) such that, for all \(n \in \mathbb{Z}\), \(U^n X'_n = X'_n\) and \(U^n X''_n = X''_n\).

**Proof.** For any pair \(((i, n), (i', n'))\) of elements of \([1, 2] \times \mathbb{Z}\), we have \(<X'_i, X''_{i'}> = <X'_{i+1}, X''_{i'+1}>, \) so from \([1]\), there exists an isometry \(V\) from \(\text{vec}(X'_n) : (i, n) \in [1, 2] \times \mathbb{Z}\) onto \(\text{vec}(X''_{n+1}) : (i, n) \in [1, 2] \times \mathbb{Z}\) such that \(V X'_n = X''_{n+1}\), for any \((i, n) \in [1, 2] \times \mathbb{Z}\). Noting that \(\text{vec}(X'_n) : (i, n) \in [1, 2] \times \mathbb{Z}\) is a sequence of projectors which completes the proof. For any \((i, n) \in [1, 2] \times \mathbb{Z}\), we can then write \(U^n X'_0 = X'_0\) and \(U^n X''_0 = X''_0\), which completes the proof. \(\square\)

We now examine the notion of unitary operator compatible with a stationary series.

**Definition 3.** We say that a unitary operator \(U\) of \(H\) is compatible with a stationary series \((X_n, n \in \mathbb{Z})\) if \(U^n X_0 = X_n\) for all \(n \in \mathbb{Z}\).

Of course, the stationary series \((X_n, n \in \mathbb{Z})\) is stationarily correlated with itself, and Lemma 13 allows us to express the following.

**Proposition 23.** If \((X_n, n \in \mathbb{Z})\) is a stationary series, then there exists at least one unitary operator of \(H\) which is compatible with it.

We now have the tools we need in order to solve the problem stated at the beginning of this section.

**Definition 4.** Let \((X_n, n \in \mathbb{Z})\) be a stationary series and \(P\) be a projector of \(H\). A series solution is any stationary series \((X'_n, n \in \mathbb{Z})\), stationarily correlated with \((X_n, n \in \mathbb{Z})\), and such that \((X'_n, n \in \mathbb{Z}) \subset \text{Im} P\).
Then we have the following property.

**Proposition 24.** Let \((X_n,n \in \mathbb{Z})\) be a stationary series and \(P\) be a projector of \(H\). If \(U\) is a unitary operator compatible with \((X_n,n \in \mathbb{Z})\), for any \(X \in \mathcal{H}\), then the series \(\{U^n \inf(P,C_{PU})X,n \in \mathbb{Z}\}\) is a solution. Conversely, any solution is of this type.

**Proof.** Let \(U\) be a unitary operator, compatible with the stationary series \((X_n,n \in \mathbb{Z})\). The series \(\{U^n \inf(P,C_{PU})X,n \in \mathbb{Z}\}\) is stationary, stationarily correlated with \(\{U^nX_0,n \in \mathbb{Z}\}\), i.e., with \((X_n,n \in \mathbb{Z})\) because \((U^nX_0,n \in \mathbb{Z}) = (X_n,n \in \mathbb{Z})\). Moreover, for any \(n \in \mathbb{Z}\), we have

\[
U^n \inf(P,C_{PU})X = U^nC_{PU}P \inf(P,C_{PU})X.
\]  

(11)

As \(C_{PU}\) is a commutator of \(P\) and \(U\), it is a commutator of \(P\) and \(U^n\) \((C_{PU} = C_PU\) is a commutator of the unitary operators \(e^{ip}\) and \(U^n\), so of the unitary operators \(e^{ip}\) and \(U^n\), and then of \(P\) and \(U^n\)), so (11) can be completed by

\[
U^n \inf(P,C_{PU})X = PC_{PU}U^n \inf(P,C_{PU})X.
\]

We have then \(\{U^n \inf(P,C_{PU})X,n \in \mathbb{Z}\} \subset \text{Im } P\), which allows us to conclude that \(\{U^n \inf(P,C_{PU})X,n \in \mathbb{Z}\}\) is a series solution.

Conversely, let \((X'_n,n \in \mathbb{Z})\) be a series solution. As the stationary series \((X_n,n \in \mathbb{Z})\) and \((X'_n,n \in \mathbb{Z})\) are stationarily correlated, from Lemma 13, there exists a unitary operator \(U\) such that \(U^nX_0,n \in \mathbb{Z}\) = \((X_n,n \in \mathbb{Z})\) and such that \(U^nX'_0,n \in \mathbb{Z}\) = \((X'_n,n \in \mathbb{Z})\). The equality \(U^nX_0,n \in \mathbb{Z}\) = \((X_n,n \in \mathbb{Z})\) means that \(U\) is compatible with \((X_n,n \in \mathbb{Z})\); see Definition 3. As \((X'_n,n \in \mathbb{Z}) \subset \text{Im } P\), for any \(n \in \mathbb{Z}\), we can write \(PU^nX'_0 = PX'_0 = X'_n = U^nX_0 = U^nPX_0\) so \(X'_0\) belongs to \(\text{Ker}(PU^n - U^nP)\). This being exact for any \(n \in \mathbb{Z}\), \(X'_0\) belongs to \(\bigcap \text{Ker}(PU^n - U^nP) = \text{Im } C_{PU}\), and so \(X'_0 = P_{C_{PU}}X'_0\). Moreover, \(\inf(P,C_{PU})X'_0 = PC_{PU}U^n \inf(P,C_{PU})X'_0\). We deduce that \(X'_n = U^nX_n' = U^n \inf(P,C_{PU})X'_0\). It is then clear that the series \((X'_n,n \in \mathbb{Z})\) can be written as \(\{U^n \inf(P,C_{PU})X'_0,n \in \mathbb{Z}\}\), which concludes the proof. □

7. **Graphical illustrations**

As an illustration, we consider two series \((X_n,n \in \mathbb{Z})\) and \((Y_n,n \in \mathbb{Z})\), where \(X_n = U^nX_0\) and \(Y_n = V^nX_0\), \(U\) and \(V\) being two unitary operators expressed as follows

\[
U = e^{-i\lambda}P + e^{i\lambda}P_1 + e^{i\lambda'}P_2, \quad V = e^{-i\lambda}P + e^{i\lambda'}D_1 + e^{i\lambda''}D_2,
\]

where \(P, P_1, P_2, D_1\) and \(D_2\) are projectors of \(H = \mathbb{C}^4\) such that \(P_1D_1 \neq 0, P + P_1 + P_2 = P + D_1 + D_2 = I, PP_1 = PP_2 = P_1D_1 = PD_2 = D_1D_2 = 0,\) and \(\lambda, \lambda'\) and \(\lambda''\) are values of \(\Pi\). Then, for any \(n,m \in \mathbb{Z}\), \(U^n = e^{-i\lambda}P + e^{i\lambda}P_1 + e^{i\lambda'}P_2, V^n = e^{-i\lambda}P + e^{i\lambda'}D_1 + e^{i\lambda''}D_2\) and \(U^nV^n - V^nU^n = (e^{-i\lambda} - e^{i\lambda})(e^{i\lambda'} - e^{i\lambda''})(P_1D_1 - D_1P_1)\). From the latter equality, we deduce that \(C_{UV} = C_P(D_1)\).

For illustration purposes, we took \(H = \mathbb{C}^3, \lambda = \sqrt{2}/2, \lambda' = \sqrt{3} \) and \(\lambda'' = -\sqrt{5}/5\). The fact that \(\lambda, \lambda'\) and \(\lambda''\) are not elements of \(2\pi\mathbb{Q}\) guarantees the non-periodicity of the series \((X_n,n \in \mathbb{Z})\) and \((Y_n,n \in \mathbb{Z})\). We choose \(P_1 = a \perp a\) and \(D_1 = b \perp b\), where \(a\) and \(b\) are two linearly independent and non orthogonal normed elements of \(H\). In this case, \(\text{Im } C_{UV} = \{\text{vect}(a,b)\}^\perp\).

Let \(X_0 = \varepsilon_0X_0^0 + \varepsilon_1X_0^1\) be an element of \(H\) such that \(X_0^0 \in \text{Im } C_{UV}\) and \(X_0^1 \in \text{Im } C_{UV}^\perp\). If \(\varepsilon_1 = 0\), then \(X_0 \in \text{Im } C_{UV}\), and \((U^nV^n)X_0, n \in \mathbb{Z}\) = \((V^nU^n)X_0, n \in \mathbb{Z}\). If \(\varepsilon_1\) is small (resp. large), we say that \(X_0\) is close to (resp. far from) \(\text{Im } C_{UV}\).

Figure 1 shows that when \(X_0\) is of high coefficient in \(\text{Im } C_{UV}\), the two sets of curves \(\{(U^nV^n)X_0 : n \in \mathbb{Z}\}\) and \(\{(V^nU^n)X_0 : n \in \mathbb{Z}\}\) have very close shapes. When \(X_0\) is close to \(\text{Im } C_{UV}^\perp\), then the two sets of curves have different shapes. With the same notation, for any \(\lambda, \lambda'\) and \(\lambda''\), we have

\[
\|(U^nV^nX - V^nU^nX)\| = |e^{i\lambda} - e^{i\lambda'}| \times |e^{i\lambda''} - e^{i\lambda'''}| < b, a > < X, b > a < a, b > < X, a > b|.
\]

The proximity between the series \(\{(U^nV^n)X_0, n \in \mathbb{Z}\}\) and \(\{(V^nU^n)X_0, n \in \mathbb{Z}\}\) can also be illustrated by the norms of the differences \(\|U^nV^nX_0 - V^nU^nX_0\|\), when \(n\) and \(m\) vary. In Figure 2, we see that, even though this norm varies, it becomes smaller when \(X_0\) gets closer to \(\text{Im } C_{UV}\).
Finally, we illustrate Proposition 24 by a simulation of \((X_n, n \in \mathbb{Z})\) and a solution \((Y_n, n \in \mathbb{Z})\) of the form 

\[ (U_n \inf(P, C P_U) X, n \in \mathbb{Z}) \]

which is a unique way to write a stationary and stationarily correlated series with \((X_n, n \in \mathbb{Z})\).

Let \((v_1, \ldots, v_5)\) be an orthonormal basis of \(C^5\). Then let us consider 

\[ Q_1 = v_1 \otimes v_1, \quad Q_2 = v_2 \otimes v_2 + v_3 \otimes v_3 \quad \text{and} \quad Q_3 = v_4 \otimes v_4 + v_5 \otimes v_5. \]

We denote by \(U\) the unitary operator 

\[ U = e^{i\lambda} Q_1 + e^{i\lambda'} Q_2 + e^{i\lambda''} Q_3, \]

which is the shift operator of the series \((X_n, n \in \mathbb{Z})\) with \(X_0 = U^n X_0, X_0\) being any element of \(C^5\).

Let us choose a projector \(P = v_2 \otimes v_2 + v_4 \otimes v_4\). As \(\inf(P, Q_1) = 0, \inf(P, Q_2) = v_2 \otimes v_2\) and \(\inf(P, Q_3) = v_4 \otimes v_4\), for this projector, the series solution of the form 

\[ (U^n \inf(P, C P_U) X, n \in \mathbb{Z}) \]

is \((Y_n, n \in \mathbb{Z}) = (e^{i\lambda} v_2 \otimes v_2 X_0 + e^{i\lambda'} v_4 \otimes v_4 X_0, n \in \mathbb{Z})\).

In Figure 3, the series \((Y_n, n \in \mathbb{Z})\) appears to have a similar shape as that of the series \((X_n, n \in \mathbb{Z})\), with less amplitude variations.

Figure 1: First dimension curves \(|(U^n V)^n X_0, n \in \mathbb{Z}|\) (Group 1) and \(|(V^n U)^n X_0, n \in \mathbb{Z}|\) (Group 2) when \(X_0\) is close to \(\text{Im} C_{UV}\) (left: \(\varepsilon_0 = 1, \varepsilon_1 = 0.05\)) and when \(X_0\) is far from \(\text{Im} C_{UV}\) (right: \(\varepsilon_0 = 0.05, \varepsilon_1 = 1\)).

Figure 2: Norms \(||(U^n V)^n X_0 - V^n U^n X_0||\) when \(X_0\) is close to \(\text{Im} C_{UV}\) (left: \(\varepsilon_0 = 1, \varepsilon_1 = 0.05\)) and when \(X_0\) is far from \(\text{Im} C_{UV}\) (right: \(\varepsilon_0 = 0.05, \varepsilon_1 = 1\)), for \(n\) and \(m\) varying from 1 to 20.
8. Conclusion

In closing, we wish to emphasize the similarity between the three types of commuter. The maximal commuters have analogous properties. In particular, \( \text{Im } C_{PD} = \text{Ker } (PD - DP) \), \( \text{Im } C_{PU} = \cap_{n \in \mathbb{Z}} \text{Ker } (PU^n - U^n P) \), and \( \text{Im } C_{V,U} = \cap_{n,m \in \mathbb{Z}} \text{Ker } (V^m U^n - U^n V^m) \). Note also that \( C_{PU} \) can be written by means of the maximal commuters of projectors, viz. \( C_{PU} = \inf \{ C_{E_U A} : A \in \mathcal{B} \} \), that \( C_{V,U} \) can be written by means of a family of maximal commuters \( C_{P,U} \) and of a family of commuters of two projectors, i.e., \( C_{V,U} = \inf \{ C_{E_{V,A,U}} : A \in \mathcal{B} \} = \inf \{ C_{E_{V,A,U,B}} : (A, B) \in \mathcal{B} \times \mathcal{B} \} \). In the future, one could consider, for two functional processes, the estimation of common and specific features for their associated shift operators, more precisely, the maximal commuter of the two shift operators.

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