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To cite this version:

Yu Li. Approximation of polynomial covariance functions of arbitrary degree. 2018. <hal-01896265>
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Abstract
Covariance function plays an important role in spatial modeling. For a function to be a valid covariance function, it must satisfy positive definiteness. However, checking for positive definiteness of a given function is, in general, a challenging task. In this paper, we present an approximation approach to characterize polynomial covariance functions of arbitrary degree.

Keywords. Covariance function, positive definiteness, Krein-Langer theorem, Fredholm determinant

1 Introduction
Covariance function provides a powerful tool in time series analysis and in spatial data analysis. In this paper we consider for a class of stationary process $X(u)$ with polynomial covariance function of

$$C(t) = \text{cov}(X(u + t), X(u)), \quad -1 \leq u < t + u \leq 1$$

where

$$C(t) = \sum_{i=0}^{n+1} b_i |t|^i, \quad -1 \leq t \leq 1$$

In general, an arbitrary polynomial will not be a valid covariance function and it is not evident which conditions a polynomial should be fulfilled for the existence of such a stationary process. Clearly, to be a valid covariance function, polynomial must be positive definite.

Polynomial covariance model finds various applications in Geostatistics [1, 11] and Hydrology [9] and attracts great interests of researchers. A polynomial covariance model of degree 4 is presented by a simulation study for the Stienen model spheres in three-dimensional Euclidean space [10]. Turning bands operator provides a convenient method to check for positive definiteness in the general case [5]. Polynomial covariance function of degree $k \leq 3$ was studied [2] and the quintic case was considered [6, 4]. A more
general form [7] was presented, which covers the quartic case except when the linear term vanishes.

The property of positive definiteness is, in general, difficult to check. Fortunately, Krein-Langer analytical technique [3] provides a powerful tool for characterization of covariance functions. In section 2 we review Krein-Langer theorem. In section 3 we study polynomial covariance function with vanishing odd terms. In section we present approximation theory for polynomial covariance functions of arbitrary degree.

2 Krein-Langer theorem

We begin this section by recalling the Krein-Langer theorem which is the key tool in this paper. Let $C$ denote an even continuous function such that

1. $C$ is second differentiable, i.e $C^2[-1,1]$;
2. the second $H = -C''$ exists for $t \neq 0$ and $H$ is absolutely integrable over $[-1,1]$;
3. $C'(0+) < 0$

Second derivative can be interpreted as accelerator. With the $H$ we define an operator on $L^2[0,1]$

$$Hf(t) = \int_0^1 H(t-\xi)f(\xi)d\xi, \quad 0 < t < 1$$

**Theorem 2.1.** (Krein-Langer theorem) Let $C$ be an even function with function $H$ and $C'(0+) = -\frac{1}{2}$. Furthermore, $-1$ is not an eigenvalue of $\mathcal{H}$. Then $C$ is a covariance function if and only if

1. the operator $\mathcal{H} + I$ has no negative eigenvalues;
2. $C(0) \geq \langle (I + \mathcal{H})^{-1}C', C' \rangle$

For calculating the inner product we can use integral equation approach. We determine first the resolvent kernel of $H$

$$\Gamma(t,s) + \int_0^1 H(t-u)\Gamma(u,s)du = H(t-s), \quad 0 < t, s < 1 \quad (1)$$

Here $\Gamma$ is a real function, otherwise $\Gamma - \overline{\Gamma}$ is an eigenfunction of $\mathcal{H}$ with eigenvalue $-1$. Let $\Gamma$ be the operator on $L^2[0,1]$

$$\Gamma \varphi(t) = \int_0^1 \Gamma(t,s)\varphi(s)ds, \quad 0 < t < 1$$

Then the equation (1) can be rewritten in the form

$$\Gamma + \Gamma \mathcal{H} = \mathcal{H}$$
Finally

\[(I + \mathcal{H})(I - \Gamma) = I\]

The first condition of Krein-Langer theorem leads to calculation of eigenvalues. Let \(-\mu\) be an eigenvalue of \(\mathcal{H}\)

\[\mu V(t) + \int_0^1 H(t-u)V(u)du = 0 \quad (0 < t < 1) \tag{2}\]

Differentiating \(l\) times yields

\[\mu V^{(l-1)}(t) + \int_0^1 H^{(l-1)}(t-u)V(u)du - \sum_{m=0}^{l-2} b_{l-m}(1 - (-1)^{l-m})V^{(m)}(t) = 0 \tag{3}\]

for \(l = 1, 2, \ldots n\). To calculate the inner product, we define a function \(G\)

\[G(t) := (\Gamma C')(t) = \int_0^1 \Gamma(t,s)C'(s)ds\]

It holds obviously

\[((I + \mathcal{H})^{-1}C', C')_{L^2[0,1]} = (C' - G, C')_{L^2[0,1]}\]

Differentialting the equation \(l\) times

\[G^{(l-1)}(t) + \int_0^1 H^{(l-1)}(t-s)(G(s) - C'(s))ds - \sum_{m=0}^{l-2} b_{l-m}(1 - (-1)^{l-m})(G^{(m)}(t) - C'^{(m)}(t)) = 0 \tag{4}\]

### 3 Polynomial covariance functions with vanishing odd terms

We consider polynomials with vanishing odd terms

\[C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6 \ldots + b_{n+1}|t|^{n+1}, \quad -1 \leq t \leq 1\]

From (3) and (4) we see

\[V^{(n)}(t) = 0\]
\[G^{(n)}(t) = 0 \tag{5}\]

The solutions are

\[V(t) = \sum_{j=1}^{n} \frac{c_j}{(j-1)!} t^{j-1} \tag{6}\]
\[G(t) = \sum_{j=1}^{n} \frac{d_j}{(j-1)!} t^{j-1} \tag{7}\]

where \(c_j\) and \(d_j\) are constants.
Calculation of eigenvalues of $\mathcal{H}$

Setting (6) into (3) with $t = 1$ yields

$$\sum_{j=l}^{n} \mu \frac{c_j}{(j-l)!} - \sum_{j=l}^{n+1} \sum_{k=1+l}^{n} \frac{b_k c_j}{(k-l+j-1)!} = 0$$

and

$$\sum_{j=1}^{l-1} c_j \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \sum_{j=l}^{n+1} c_j (- \sum_{k=1+l}^{n} \frac{b_k}{(k-l+j-1)!} + \mu \frac{1}{(j-l)!}) = 0$$

This equations system can rewritten as matrix equation with respect to $c = [c_1, ..., c_n]$

$$Bc = 0$$

where

$$b_{l,j} = \begin{cases} - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l - 1 \\ - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

For non-trivial $V$ it must hold

$$\det(B) = 0$$

Calculation of inner product

We set (7) in (4) with $t = 1$, then

$$- \sum_{j=1}^{l-1} d_j \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \sum_{j=l}^{n} d_j \left( \frac{1}{(j-l)!} - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} \right)$$

$$= - \sum_{j=1}^{n+1} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!}$$

We rewrite in matrix form

$$ED = G$$

where

$$c_{l,j} = \begin{cases} - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l - 1 \\ - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{1}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

$$g_l = - \sum_{j=1}^{n+1} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!}$$

and $D$ is the vector of coefficients of function $V$ in (7).
Theorem 3.1. Let $C$ be a polynomial with vanishing odd terms of the form

$$C(t) = r - \frac{1}{2} |t| + b_2 |t|^2 + b_4 |t|^4 + b_6 |t|^6 + \ldots + b_{n+1} |t|^{n+1}, \quad 0 < t < 1$$

Then $C$ is a covariance function if and only if

1. the equation admits only solution $\mu < 1$
   $$\det B = 0$$

   where
   $$b_{l,j} = \begin{cases} -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l-j-1)!} & \text{for } 1 \leq j \leq l-1 \\ -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l-j-1)!} + \frac{\mu}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

2. $C(0) \geq \langle C' - G, C' \rangle$ where $G$ is a polynomial of the form

   $$G(t) = \sum_{j=1}^{n} \frac{d_j}{(j-1)!} t^{j-1}$$

   The coefficients vector $D = [d_1, d_2, \ldots, d_n]$ of $G$ can be determined by

   $$ED = G$$

   with

   $$c_{l,j} = \begin{cases} -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l-j-1)!} & \text{for } 1 \leq j \leq l-1 \\ -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l-j-1)!} + \frac{1}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

   $$g_l = -\sum_{j=1}^{n+1} \sum_{k=1+l}^{n+1} \frac{b_kb_j}{(k-L-j)!}$$

Example 3.1. For $n = 5$

$$C(t) = r - \frac{1}{2} |t| + b_2 |t|^2 + b_4 |t|^4 + b_6 |t|^6$$

We calculate the determinant of matrix $\det(b_{i,j})_{n \times n}$ and the inner product $\langle (I - \Gamma)C', C' \rangle$

$$\det(b_{i,j})_{n \times n} = \frac{1}{22122558259200000} (1209600\mu^2 + 100800b_4\mu + 5040b_6 - b_6^2)$$

$$(2540160a_4^2\mu + 152409600b_4\mu^2 + 907200b_4^2\mu b_6 - 18289152000\mu^3$$

$$+ 76204800\mu^2b_6 + 18289152000b_2\mu^2 + 20160b_6^2 - 2540160b_2^2\mu b_6 - a_6^3)$$

(8)
\[
\langle (I - \Gamma)C', C' \rangle = \frac{1}{10059033600(-b_6^2 + 5040b_6 + 1209600 + 100800b_4)}
\]

\[
(1013950556880000b_2^2 - 279417600b_2b_4^2 - 2011867200b_2b_4b_6 - 603542016000b_4b_2 + 50697529344000b_4b_2 - 3041851760640000b_2
\]

\[-b_6^4 + 60480b_6^3 + 533433600b_4b_6^2 + 37426105280000b_4b_2 + 1810626048000b_4^2
\]

\[+279417600b_2^2b_6 + 83825280000b_4b_6 + 4536000b_2^2b_6^2 + 533433600b_4b_6^2 + 4224794112000b_6
\]

\[+10059033600b_4^3 + 3041851760640000) \leq r
\]

(9)

The polynomial \(C\) is a valid covariance function if and only the equation (8) only admits solutions \(\mu < 1\) and the inequality (9) holds

**Example 3.2.** For \(n = 7\)

\[
C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6 + b_8|t|^8
\]

Analog we calculate \(\det(b_{i,j})_{n \times n}\) and \((C' - G, C')_{L^2[0,1]}\).

\[
\det(b_{i,j}) = \frac{1}{128039477702525013027172279910400000000000}
\]

(10)

\[
(-10059033600 \mu b_6^2 + 50697529344000 \mu^2 b_6
\]

\[-279417600 \mu b_6b_8 + 1216740704256000 \mu^3 + 1207084032000 b_4 \mu^2 + 10059033600 b_8 \mu b_4
\]

\[-4536000 \mu b_8^2 - b_8^3 + 1013950586880000 b_4 \mu^2)(105231523244566118400000 \mu^4
\]

\[-105231523244566118400000 \mu^2 b_2 - 10439635242516480000 \mu^3 b_2 - 438464680185692160000 \mu^3 b_6
\]

\[-8769293603713843200000 \mu^3 b_4 + 28998986784768000 \mu^2 a_8 b_4 - 115995947139072000 \mu^2 b_6
\]

\[-5219817621528240000 \mu^2 b_6b_4 + 5219817621528240000 \mu^2 b_6b_2 - 40437315072000 \mu^2 b_2^2
\]

\[+146154893395230720000 \mu^2 b_6b_2 - 14615489339523072000 \mu^2 b_4^2 - 2855960819712000 \mu^2 b_6b_8
\]

\[+130767436800 \mu b_6^2b_8 + 5753767219200 \mu b_6^3 + 16632000 b_8^3 \mu + 16765056000 b_8^2 \mu b_6
\]

\[-1150753438400 \mu b_6b_4b_8 - 130767436800 b_8^2 \mu b_4 + 5753767219200 b_8^2 \mu b_2 + b_8^4) = 0
\]
\[(I - \Gamma)C'(C') = (448793843097600100590336006_4b_8 + 1013950586880000b_4 - b_8^3 - 4536000b_8^2 - 279417600b_8^6 + 120708432000b_8 + 12167407042560000 - 10059033600b_8^2 + 50697593140000b_8)^{-1}(-17253981501521200000b_8^4b_8^2 - 36643119701232844800000b_2b_6b_8 - 125400898531079577600000b_2b_4b_8 - 13651643417908264713584640000000b_2 + 13660618588259232133120000000 + 677164852078782971904000000b_8 + 1346381529228000b_4b_6^2b_8 - 90288646943857729587200000b_2b_4b_6 + 172613016576000b_4b_6b_8^2 + 22752739028947107859744000000b_4b_2 + 8125978224945395662848000000b_4^2 + 3762026959932387328000000b_4b_6 + 4788034307627758387200000b_4b_8 - 2708659408315131887616000000b_6b_2 + 23940171538138791936000000b_6^2 + 394618212167122944000000b_6b_2 - 6315979321722470400000b_6^2 - 300962156479459098624000000b_8b_2 - 8975876861952000000b_8b_2^2 - 1254008985331079577600000b_6b_2^2 - 991765348039065600000b_2b_8^2 + 1254008985331079577600000b_2b_6^2 - 4071457744581427200000b_4^2b_6^2 + 2035728872290713600000b_4b_2^2 - 30448936124006400000b_4b_8^2 + 45144323471918864793600000b_4^3 + 2714305163054284800000b_6^3 + 4236864952320000b_6^3 - 4487938430976000b_6^4 + 6089787224801280000b_6^2b_8 + 86996960354304000b_6b_8^2 - 4487938430976000b_4b_8^2 + 9153720576000b_4b_8^2 - 863065082880000b_6^3b_8 - 9153720576000b_6^2b_8^2 - 72648576000b_6b_8^3 - b_8^5 - 86306508288000b_3^2b_8^2 - 49896000b_8^4 + 4550547805969421571194880000000b_2^2 + 45144323471918864793600000b_6b_8^2 + 136516434179082647135846400000000) \leq r \quad (11)

The C is a valid covariance function if equation (10) admits only \( \mu < 1 \) and inequality (11) holds.

4 Approximation theory for polynomial covariance functions

In this section we present a numerical approach to calculate inner product. For a function \( C \) be a function defined on interval \([-1, 1] \), Toeplitz matrix is defined as

\[
M_n(C) = \left( C(\frac{i-j}{n}) \right)_{i,j=0}^n
\]

And if \( C \) is a covariance function, then Toeplitz matrix \( M_n(C) \) is positive definite for all natural number \( n \). The Toeplitz matrix is symmetric and all the eigenvalues are positive. Moreover we define a function \( C_r \)

\[
C_r(t) = C(t) - C(0) + r, \quad (|t| \leq 1)
\]

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Let \( r_\infty \) denote the inferium such that \( C_r \) is a positive definite function
\[
r_\infty = \inf \{ r \in \mathbb{R} : C_r \text{ is positive definite} \}
\] (12)

A sequence \( \{ r_n \} \) is defined
\[
r_n = \sup \{ r \in \mathbb{R} : M_n(C_r) \text{ has a negative eigenvalue} \}
\] (13)

The following theorem provides a rigorous approximation of \( r_\infty \).

**Theorem 4.1** (Mitra-Gneiting-Sasvari). If \( C \) is a continuous function on \([-1, 1]\) and there exists a positive number \( r_\infty \) such that \( C_r \) is a covariance function if and only if \( r \geq r_\infty \), then
\[
\lim_{n \to \infty} r_n = r_\infty
\]

We define a function \( L_n \)
\[
L_n(r) = \det(M_n(C_r))
\]

**Lemma 4.2.** The function \( L_n \) is a linear function of \( r 
\]
\[
L_n(r) = ar + b
\] (14)

where \( a \) and \( b \) are two constants.

**Proof.** We denote \( z_i \) the \( i \)-th row and \( s_j \) the \( j \)-th column.
\[
L_n(r) = \det(M_n(C_r))
\]

\[
= \det \begin{pmatrix}
C(0) - C(0) + r & \cdots & C(-\frac{i}{n}) - C(0) + r & \cdots & C(1) - C(0) + r \\
\vdots & \ddots & \vdots & & \vdots \\
C(\frac{i}{n}) - C(0) + r & \cdots & C(\frac{i-1}{n}) - C(0) + r & & C(\frac{n-1}{n}) - C(0) + r \\
C(1) - C(0) + r & & \cdots & C(0) - C(0) + r \\
\end{pmatrix}_{(n+1) \times (n+1)}
\]

\[
= \det \begin{pmatrix}
r & \cdots & C(-\frac{i}{n}) - C(0) + r & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
C(\frac{i}{n}) - C(0) & \cdots & C(\frac{i-1}{n}) - C(-\frac{i}{n}) & \cdots \\
\end{pmatrix}_{n \times n}
\]

\[
\Rightarrow \det \begin{pmatrix}
r & \cdots & C(-\frac{i}{n}) - C(0) + r & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
C(\frac{i}{n}) - C(0) & \cdots & C(\frac{i-1}{n}) - C(-\frac{i}{n}) & \cdots \\
\end{pmatrix}_{n \times n}
\]

\[
\Rightarrow \det \begin{pmatrix}
r & \cdots & C(-\frac{i}{n}) - C(0) + r & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
C(\frac{i}{n}) - C(0) & \cdots & C(\frac{i-1}{n}) - C(-\frac{i}{n}) & \cdots \\
\end{pmatrix}_{n \times n}
\]

\[
= \begin{pmatrix}
r & \cdots & C(-\frac{i}{n}) - C(0) + r & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
C(\frac{i}{n}) - C(0) & \cdots & C(\frac{i-1}{n}) - C(-\frac{i}{n}) & \cdots \\
\end{pmatrix}_{n \times n}
\]
\[ s_j^{i+1} - s_j^i \leq \det \left( \begin{pmatrix} r \\ \vdots \\ C\left(\frac{i}{n}\right) - C(0) \end{pmatrix}_{n \times 1} \begin{pmatrix} \cdots & C\left(\frac{i}{n}\right) - C(0) & \cdots \end{pmatrix}_{1 \times n} \right) \quad (n+1) \times (n+1) \] 

(15)

In this matrix, \( r \) appears only once, then \( L_n(r) \) is a linear function of \( r \). Furthermore, \( L_n \) is a monotone increasing function from Krein-Langer theorem, then \( a > 0. \) \( \square \)

**Proposition 4.3.** \( r_n \) is the zero point of \( L_n \), i.e.

\[ L_n(r_n) = 0 \]

**Proof.** Let \( r_n^* \) be the zero point of \( L_n \) and \( \bar{r} \) be the middle point \( \bar{r} := \frac{r_n^* + r_n}{2} \)

If \( r_n < r_n^* \) and recall that \( L_n \) is a linear function

\[ L_n(\bar{r}) < L_n(r_n^*) = \det(M_n(C_{r_n^*})) = 0 \]

Then the matrix \( M_nC_{\bar{r}} \) has at least one negative eigenvalue, which contradicts the definition of \( r_n \).

If \( r_n^* < r_n \), then for middle point \( \bar{r} \), there exists a unit vector \( x \)

\[ x^T M_n(C_{\bar{r}})x < 0 \]

Fix this unit vector \( x \), due to continuity, there exists a \( r_2 > \bar{r} \) such that

\[ x^T M_n(C_{r_2})x = 0 \]

then

\[ \det(M_n(C_{r_2})) = 0 \]

This implies that \( r_2 \) is a second zero point of \( L_n \), which contradicts the uniqueness of zero point of linear function. \( \square \)

**Corollary.** The \( r_n \) can be expressed

\[ r_n = \frac{-\det(\beta_{i,j})_{i,j=0}^n}{\det(\alpha_{i,j})_{i,j=1}^n} \quad (16) \]

where

\[ \beta_{i,j} = C\left(\frac{i-j}{n}\right) - C(0) \]
\[ \alpha_{i,j} = C\left(\frac{i-j}{n}\right) - C\left(\frac{i}{n}\right) - C\left(\frac{j}{n}\right) + C(0) \]

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Proof. \( r_n \) is the zero point of \( L_n \)
\[ ar_n + b = 0 \]
From (15)
\[ a = \det(\alpha_{i,j})_{i,j=1}^n \]
and set \( r = 0 \) into (14)
\[ b = \det(\beta_{i,j})_{i,j=0}^n \]

In the following table we present the coefficients of polynomial
\[ C(t) = r - \frac{1}{2} |t| + \frac{b_2}{2!} |t|^2 + \frac{b_4}{4!} |t|^4 + \frac{b_6}{6!} |t|^6 + \frac{b_8}{8!} |t|^8 \]
In the table are the values of \( r_n \) for \( n = 10, 20, 30, 40, 50 \) and the analytical limit \( r_\infty \).

| Table 1: Numerical values of \( b_k \) |
|---|---|---|---|
| case | (i) | (ii) | (iii) | (iv) |
| \( b_2 \) | 0 | -1 | 6 | 0.08084 |
| \( b_4 \) | 1 | 0 | 1 | 3.04749 |
| \( b_6 \) | 4 | 2 | 0 | 5.43217 |
| \( b_8 \) | 3 | 5 | -4 | -11.35261 |

| Table 2: \( r_n \) and \( r_\infty \) |
|---|---|---|---|
| case | (i) | (ii) | (iii) | (iv) |
| \( r_{10} \) | 0.2286409660 | 0.5783240166 | 1.612114043 | 0.1837180146 |
| \( r_{20} \) | 0.2286639511 | 0.5789134327 | 1.632410822 | 0.183883736 |
| \( r_{30} \) | 0.2286682283 | 0.5790225524 | 1.636167163 | 0.1839200018 |
| \( r_{40} \) | 0.2286697268 | 0.5790607420 | 1.637481711 | 0.1839310802 |
| \( r_{50} \) | 0.2286704207 | 0.5790784180 | 1.638090129 | 0.1839361972 |
| \( r_\infty \) | 0.2286716547 | 0.5791098412 | 1.639171713 | 0.1839453225 |

From the table we see that all zero points \( r_n \) tend to the lower bound \( r_\infty \).

**Theorem 4.4.** Let \( C \) be a polynomial with vanishing odd terms of the form
\[ C(t) = r - \frac{1}{2} |t| + b_2 |t|^2 + b_4 |t|^4 + b_6 |t|^6 + b_8 |t|^8, -1 \leq t \leq 1 \]
Then \( C \) is a covariance function if and only if
1. the equation admits only solution \( \mu < 1 \)
\[ \det B = 0 \]
where
\[ b_{i,j} = \begin{cases} 
-\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\
-\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{\mu-1} & \text{for } l \leq j \leq n 
\end{cases} \]
2. the inequality holds

\[ C(0) \geq r_n, \quad \forall n \in \mathbb{N} \]

where

\[ r_n = -\frac{\det(\beta_{i,j})_{i,j=0}^n}{\det(\alpha_{i,j})_{i,j=1}^n} \]

with

\[ \beta_{i,j} = C\left(\frac{i-j}{n}\right) - C(0) \]
\[ \alpha_{i,j} = C\left(\frac{i-j}{n}\right) - C\left(\frac{i}{n}\right) - C\left(-\frac{j}{n}\right) + C(0) \]

For polynomial with non-vanishing odd terms, there is no simple numerical formula to calculate the eigenvalue of \( \mathcal{H} \). But as \( \mathcal{H} \) is a compact operator, we can use Fredholm determinant [8].

**Theorem 4.5.** Let \( C \) be a polynomial of the form

\[ C(t) = r - \frac{1}{2}|t| + \sum_{i=2}^{n+1} b_i |t|^i, \quad -1 \leq t \leq 1 \]

Then \( C \) is a covariance function if and only if

1. the following equation admits only \( \mu < 1 \)

\[ \sum_{n=0}^{\infty} \frac{1}{\mu^n} \text{tr} \left( \Lambda^n \left( \mathcal{H} \right) \mu \right) = 0 \]

2. the inequality holds

\[ C(0) \geq r_n, \quad \forall n \in \mathbb{N} \]

where

\[ r_n = -\frac{\det(\beta_{i,j})_{i,j=0}^n}{\det(\alpha_{i,j})_{i,j=1}^n} \]

with

\[ \beta_{i,j} = C\left(\frac{i-j}{n}\right) - C(0) \]
\[ \alpha_{i,j} = C\left(\frac{i-j}{n}\right) - C\left(\frac{i}{n}\right) - C\left(-\frac{j}{n}\right) + C(0) \]
Proof. \( \mathcal{H} \) is a finite rank operator by definition, then it is trace class operator. The Fredholm determinant is well-defined. \(-\mu\) is an eigenvalue of \( \mathcal{H} \), then the function \( \varphi(\lambda) = \det(I + \lambda \mathcal{H}) \) has a zero of order \( n \) at \( \lambda = \frac{1}{\mu} \), where \( n \) is the algebraic multiplicity of \(-\mu\). Then the Fredholm determinant can be expressed

\[
\det(I + \frac{\mathcal{H}}{\mu}) = \sum_{n \geq 0} \frac{1}{\mu^n} \text{tr} \left( \Lambda^n(\frac{\mathcal{H}}{\mu}) \right)
\]

where tr denotes operator trace and \( \Lambda \) denotes exterior product.

\[ \square \]

A  Maple Program for calculation of eigenvalues of \( \mathcal{H} \)

restart;

with(LinearAlgebra);

with(CodeGeneration);

n:=; 
expr:={};

CL:=t->eval(-1/2+sum(a[k]*t^(k-1)/(k-1)!,k=2..n+1),expr);

b:=(l,j)->piecewise(j<l,eval(-sum(a[k]/(k-l+j-1)!,k=1+l..n+1),expr),
j>=l,eval(mu/(j-l)!-sum(a[k]/(k-l+j-1)!,k=1+l..n+1),expr));

B:=Matrix(n,n,(x,y)->b(x,y));

factor(eval(Determinant(B)))=0;

B  Maple Program for calculation of inner product

restart;

with(LinearAlgebra);

with(CodeGeneration);

n:=;
\begin{align*}
\text{expr:} & = \{a[1]=-1/2,a[3]=0\}; \\
\text{CL:} & = t\rightarrow \text{eval}(-1/2 + \text{sum}(a[k]*t^{k-1}/(k-1)!,k=2..n+1),\text{expr}); \\
a & = (l,j)\rightarrow \text{piecewise}(j<l,\text{eval}(-\text{sum}(a[k]/(k-l+j-1)!,k=1+1..n+1),\text{expr}), \\
& \quad j\geq l, \text{eval}(1/(j-1)!-\text{sum}(a[k]/(k-l+j-1)!,k=1+1..n+1),\text{expr})); \\
f & = l\rightarrow \text{eval}(-\text{sum}(a[k]*a[j]/(k-l+j-1)!,k=1+1..n+1),j=1..n+1),\text{expr}); \\
A & = \text{Matrix}(n,n,(x,y)\rightarrow a(x,y)); \\
F & = \text{Matrix}(n,1,x\rightarrow f(x)); \\
p & = t\rightarrow \text{Determinant}(\text{Transpose}(\text{Matrix}(n,1,j\rightarrow t^{j-1}/(j-1)!)).A^{-1}.F); \\
factor & (\text{eval}(\text{int}((\text{CL}(t)-p(t))*\text{CL}(t),t=0..1))); \\
\end{align*}

\textbf{C Maple Programm for calculation of } r_m

restart;

with(LinearAlgebra);

with(CodeGeneration);

n:=;

m:=;

expr:=\{};

c:=t\rightarrow \text{piecewise}(t\geq 0,\text{eval}(r-1/2*t+\text{sum}(a[k]*t^k/k!,k=2..n+1),\text{expr}),t<0,\text{eval}(r+1/2*t+
\quad +\text{sum}(a[k]*(-t)^k/k!,k=2..n+1),\text{expr})));

A:=\text{Matrix}(m,m,(x,y)\rightarrow c((x-y)/m)+c(0)-c(x/m)-c(-y/m)); \\
B:=\text{Matrix}(m+1,m+1,(x,y)\rightarrow c((x-y)/m)-r);
evalf[6](Determinant(B)/Determinant(A));

References


