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Approximation of polynomial convariance functions of arbitrary degree

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Abstract

Covariance function plays an important role in spatial modeling. For a function to be a valid covariance function, it must satisfy positive definiteness. However, checking for positive definiteness of a given function is, in general, a challenging task. In this paper, we present an approximation approach to characterize polynomial covariance functions of arbitrary degree.

Keywords. Covariance function, positive definiteness, Krein-Langer theorem, Fredholm determinant

1 Introduction

Covariance function provides a powerful tool in time series analysis and in spatial data analysis. In this paper we consider for a class of stationary process $X(u)$ with polynomial covariance function of

$$C(t) = \mathrm{cov}(X(u+t), X(u), \quad -1 \leq u < t+u \leq 1$$

where

$$C(t) = \sum_{i=0}^{n+1} b_i |t|^i, \quad -1 \leq t \leq 1$$

In general, an arbitrary polynomial will not be a valid covariance function and it is not evident which conditions a polynomial should be fulfilled for the existence of such a stationary process. Clearly, to be a valid covariance function, polynomial must be positive definite.

Polynomial covariance model finds various applications in Geostatistics [1, 11] and Hydrology [9] and attracts great interests of researchers. A polynomial covariance model of degree 4 is presented by a simulation study for the Stienen model spheres in three-dimensional Euclidean space [10]. Turning bands operator provides a convenient method to check for positive definiteness in the general case [5]. Polynomial covariance function of degree $k \leq 3$ was studied [2] and the quintic case was considered [6, 4]. A more

general form [7] was presented, which covers the quartic case except when the linear term vanishes.

The property of positive definiteness is, in general, difficult to check. Fortunately, Krein-Langer analytical technique [3] provides a powerful tool for characterization of covariance functions. In section 2 we review Krein-Langer theorem. In section 3 we study polynomial covariance function with vanishing odd terms. In section we present approximation theory for polynomial covariance functions of arbitrary degree.

2 Krein-Langer theorem

We begin this section by recalling the Krein-Langer theorem which is the key tool in this paper. Let C denote an even continuous function such that

1. C is second differentialble, i.e $C^2[-1,1]$;
2. the second $H = -C''$ exists for $t \neq 0$ and H is absolutely integrable over $[-1,1]$
3. $C'(0+) < 0$

Second derivative can be interpreted as accelerator. With the $\mathcal{H}$ we define an operator on $L^2[0,1]$

$$\mathcal{H}f(t) = \int_0^1 H(t-\xi)f(\xi)d\xi, \quad 0 < t < 1$$

Theorem 2.1. (Krein-Langer theorem) *Let C be an even function with function H and $C'(0+) = -\frac{1}{2}$. Furthermore, -1 is not an eigenvalue of $\mathcal{H}$. Then C is a covariance function if and only if*

1. *the operator $\mathcal{H} + I$ has no negative eigenvalues;*
2. *$C(0) \geq \langle (I+\mathcal{H})^{-1}C', C' \rangle$*

For calculating the inner product we can use integral equation approach. We determine first the resolvent kernel of H

$$\Gamma(t,s) + \int_0^1 H(t-u)\Gamma(u,s)du = H(t-s), \quad 0 < t,s < 1 \tag{1}$$

Here Γ is a real function, otherwise $\Gamma - \overline{\Gamma}$ is an eigenfunction of $\mathcal{H}$ with eigenvalue -1. Let $\mathit{\Gamma}$ be the operator on $L^2[0,1]$

$$\mathit{\Gamma}\varphi(t) = \int_0^1 \mathit{\Gamma}(t,s)\varphi(s)ds, \quad 0 < t < 1$$

Then the equation (1) can be rewritten in the form

$$\mathit{\Gamma} + \mathit{\Gamma}\mathcal{H} = \mathcal{H}$$

Finally

$$(I+\mathcal{H})(I-\Gamma)=I$$

The first condition of Krein-Langer theorem leads to calculation of eigenvalues. Let $-\mu$ be an eigenvalue of $\mathcal{H}$

$$\mu V(t)+\int_0^1 H(t-u)V(u)du=0 \qquad (0<t<1) \tag{2}$$

Differentiating l times yields

$$\mu V^{(l-1)}(t)+\int_0^1 H^{(l-1)}(t-u)V(u)du-\sum_{m=0}^{l-2} b_{l-m}(1-(-1)^{l-m})V^{(m)}(t)=0 \tag{3}$$

for $l=1,2,...n$. To calculate the inner product, we define a function G

$$G(t):=(\Gamma C')(t)=\int_0^1 \Gamma(t,s)C'(s)ds$$

It holds obviously

$$((I+\mathcal{H})^{-1}C',C')_{L^2[0,1]}=(C'-G,C')_{L^2[0,1]}$$

Differntialting the equation l times

$$\begin{aligned} G^{(l-1)}(t) \quad &+ \quad \int_0^1 H^{(l-1)}(t-s)(G(s)-C'(s))ds \\ &- \quad \sum_{m=0}^{l-2} b_{l-m}(1-(-1)^{l-m})(G^{(m)}(t)-C'^{(m)}(t))=0 \end{aligned} \tag{4}$$

3 Polynomial covariance functions with vanishing odd terms

We consider polynomials with vanishing odd terms

$$C(t)=r-\frac{1}{2}|t|+b_2|t|^2+b_4|t|^4+b_6|t|^6...+b_{n+1}|t|^{n+1}, \quad -1\le t\le 1$$

From (3) und (4) we see

$$\begin{aligned} V^{(n)}(t)&=0 \\ G^{(n)}(t)&=0 \end{aligned} \tag{5}$$

The solutions are

$$V(t)=\sum_{j=1}^{n}\frac{c_j}{(j-1)!}t^{j-1} \tag{6}$$

$$G(t)=\sum_{j=1}^{n}\frac{d_j}{(j-1)!}t^{j-1} \tag{7}$$

where c_j and d_j are constants.

Calculation of eigenvalues of $\mathcal{H}$

Setting (6) into (3) with $t = 1$ yields

$$\sum_{j=l}^{n} \mu \frac{c_j}{(j-l)!} - \sum_{j=1}^{n} \sum_{k=1+l}^{n+1} \frac{b_k c_j}{(k-l+j-1)!} = 0$$

and

$$-\sum_{j=1}^{l-1} c_j \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \sum_{j=l}^{n} c_j (-\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!}) = 0$$

This equations system can rewritten as matrix equation with respect to $c = [c_1, ..., c_n]$

$$Bc = 0$$

where

$$b_{l,j} = \begin{cases} -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\ -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

For non-trivial V it must hold

$$\det(B) = 0$$

Calculation of inner product

We set (7) in (4) with $t = 1$, then

$$-\sum_{j=1}^{l-1} d_j \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \sum_{j=l}^{n} d_j (\frac{1}{(j-l)!} - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!})$$

$$= -\sum_{j=1}^{n+1} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!}$$

We rewrite in matrix form

$$ED = G$$

where

$$\begin{aligned} e_{l,j} &= \begin{cases} -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\ -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{1}{(j-l)!} & \text{for } l \leq j \leq n \end{cases} \\ g_l &= -\sum_{j=1}^{n+1} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!} \end{aligned}$$

and D is the vector of coefficients of function V in (7).

Theorem 3.1. *Let C be a polynomial with vanishing odd terms of the form*

$$C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6 ... + b_{n+1}|t|^{n+1}, \quad 0 < t < 1$$

Then C is a covariance function if and only if

1. *the equation admits only solution $\mu < 1$*

$$\det B = 0$$

where

$$b_{l,j} = \begin{cases} -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\ -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

2. *$C(0) \geq \langle C' - G, C' \rangle$ where G is a polynomial of the form*

$$G(t) = \sum_{j=1}^{n} \frac{d_j}{(j-1)!} t^{j-1}$$

The coefficients vector $D = [d_1, d_2, ..., d_n]$ of G can be determined by

$$ED = G$$

with

$$\begin{aligned} e_{l,j} &= \begin{cases} -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\ -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{1}{(j-l)!} & \text{for } l \leq j \leq n \end{cases} \\ g_l &= -\sum_{j=1}^{n+1} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!} \end{aligned}$$

Example 3.1. For $n = 5$

$$C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6$$

We calculate the determinant of matrix $\det(b_{i,j})_{n\times n}$ and the inner product $\langle (I - \Gamma)C', C' \rangle$

$$\begin{aligned} &\det(b_{i,j})_{n\times n} = -\frac{1}{22122558259200000} \left(1209600\mu^2 + 100800 b_4 \mu + 5040 \mu b_6 - {b_6}^2\right) \\ &(25401600 {a_4}^2 \mu + 1524096000 b_4 \mu^2 + 907200 b_4 \mu b_6 - 18289152000 \mu^3 \\ &+76204800 \mu^2 b_6 + 18289152000 b_2 \mu^2 + 20160 \mu {b_6}^2 - 25401600 b_2 \mu b_6 - {a_6}^3) \end{aligned} \tag{8}$$

$$
\begin{aligned}
&\langle (I-\Gamma)C', C'\rangle = \frac{1}{10059033600(-{b_6}^2+5040b_6+1209600+100800b_4)}\\
&(1013950586880000{b_2}^2-279417600b_2{b_6}^2-20118067200b_2b_4b_6\\
&-603542016000b_6b_2+50697529344000b_4b_2-3041851760640000b_2\\
&-{b_6}^4+60480{b_6}^3+4536000b_4{b_6}^2+533433600\,{b_6}^2+4224794112000b_6\\
&+279417600{b_4}^2b_6+83825280000\,b_4b_6+1810626048000{b_4}^2\\
&+10059033600{b_4}^3+3041851760640000)\leq r
\end{aligned}
\tag{9}
$$

The polynomial C is a valid covariance function if and only the equation (8) only admits solutions $\mu < 1$ and the inequality (9) holds

Example 3.2. For $n = 7$

$$
C(t) = r - \frac{1}{2}\,|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6 + b_8|t|^8
$$

Analog we calculate $\det(b_{i,j})_{n\times n}$ and $(C'-G, C')_{L^2[0,1]}$

$$
\begin{aligned}
&\det(b_{i,j}) = \frac{1}{1280394777025250130271722799104000000000}(-10059033600\,\mu\,{b_6}^2+50697529344000\,\mu^2b_6\\
&-279417600\,\mu\,b_6b_8+12167407042560000\,\mu^3+1207084032000\,b_8\mu^2+10059033600\,b_8\mu\,b_4\\
&-4536000\,\mu\,{b_8}^2-{b_8}^3+1013950586880000\,b_4\mu^2)(1052315232445661184000000\,\mu^4\\
&-1052315232445661184000000\,\mu^3b_2-10439635242516480000\,\mu^3b_8-438464680185692160000\,\mu^3b_6\\
&-8769293603713843200000\,\mu^3b_4+28998986784768000\,\mu^2a_8b_4-115995947139072000\,\mu^2{b_6}^2\\
&-5219817621258240000\,\mu^2b_6b_4+5219817621258240000\,\mu^2b_8b_2-40437315072000\,{b_8}^2\mu^2\\
&+146154893395230720000\,\mu^2b_6b_2-146154893395230720000\,\mu^2{b_4}^2-2855960819712000\,\mu^2b_6b_8\\
&+130767436800\,\mu\,{b_6}^2b_8+5753767219200\,\mu\,{b_6}^3+16632000\,{b_8}^3\mu+1676505600\,{b_8}^2\mu\,b_6\\
&-11507534438400\,\mu\,b_6b_4b_8-130767436800\,{b_8}^2\mu\,b_4+5753767219200\,{b_8}^2\mu\,b_2+{b_8}^4)=0
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\langle (I-\Gamma)C', C'\rangle = (4487938430976000100590336 00 b_4 b_8 + 1013950586880000 b_4 - {b_8}^3 - 4536000 {b_8}^2 \\
&-279417600 b_6 b_8 + 1207084032000\, b_8 + 12167407042560000 - 10059033600 {b_6}^2 \\
&+506975293440 00 b_6)^{-1}(-1722539815015219200 00 b_4 b_6 b_8 - 36643119701232844800000\, b_2 b_6 b_8 \\
&-1254008985331079577600000 b_2 b_4 b_8 - 136516434179082647135846400000 00\, b_2 \\
&+1896061585820592321331200 00\, b_6 + 677164852078782971904000000\, b_8 \\
&+13463815292928000\, b_4 {b_6}^2 b_8 - 902886469438377295872000 00\, b_2 b_4 b_6 \\
&+172613016576000\, b_4 b_6 {b_8}^2 + 2275273902984710785597440 00 00\, b_4 b_2 \\
&+8125978224945395662848000000\, {b_4}^2 + 37620269559932387328000 00 00\, b_4 b_6 \\
&+4788034307627758387200000\, b_4 b_8 - 2708659408315131887616000000\, b_6 b_2 \\
&+2394017153813879193600000\, {b_6}^2 + 3946182121671229440000 00\, b_6 b_8 \\
&-631597932172247040000\, {b_8}^2 - 3009621564794590986240000 00\, b_8 b_2 \\
&-8975876861952000\, b_6 a_2 {b_8}^2 - 1254008985331079577600000\, b_2 {b_6}^2 \\
&-99176534803906560000\, b_2 {b_8}^2 + 1254008985331079577600000\, {b_4}^2 b_6 \\
&-4071457744581427200000\, {b_4}^2 b_8 + 2035728872290713600000 0\, b_4 {b_6}^2 \\
&-3044893612400640000\, b_4 {b_8}^2 + 4514432347191886479360000 0\, {b_4}^3 \\
&+2714305163054284800 00\, {b_6}^3 + 423686495232000\, {b_8}^3 \\
&-4487938430976000\, {b_6}^4 + 6089787224801280000\, {b_6}^2 b_8 + 86996960354304000\, b_6 {b_8}^2 \\
&-4487938430976000\, {b_4}^2 {b_8}^2 + 915372057600\, b_4 {b_8}^3 - 86306508288000\, {b_6}^3 b_8 \\
&-915372057600\, {b_6}^2 {b_8}^2 - 7264857600\, b_6 {b_8}^3 - {b_8}^5 \\
&-86306508288000\, {b_8}^3 b_2 - 49896000\, {b_8}^4 + 4550547805969421571194880000000\, {b_2}^2 \\
&+4514432347191886479360000 0\, b_8 {b_2}^2 + 136516434179082647135846400000 00) \leq r
\end{aligned}
\tag{11}
$$

The C is a valid covariance function if equation (10) admits only $\mu < 1$ and inequality (11) holds.

4 Approximation theory for polynomial covariance functions

In this section we present a numerical approach to calculate inner product. For a function C be a function defined on interval $[-1, 1]$, Toeplitz matrix is defined as

$$
M_n(C) = \left(C(\frac{i-j}{n}) \right)_{i,j=0}^{n}
$$

And if C is a covariance function, then Toeplitz matrix $M_n(C)$ is positive definite for all natural number n. The Toeplitz matrix is symmetric and all the eigenvalues are positive.

Moreover we define a function C_r

$$
C_r(t) = C(t) - C(0) + r, \quad (|t| \leq 1)
$$

Let r_∞ denote the inferium such that C_r is a positive definite function

$$r_\infty = \inf\{r \in \mathbb{R} : C_r \text{ is positive definite}\} \tag{12}$$

A sequence $\{r_n\}$ is defined

$$r_n = \sup\{r \in \mathbb{R} : M_n(C_r) \text{ has a negative eigenvalue}\} \tag{13}$$

The following theorem provides a rigorous approximation of r_∞.

Theorem 4.1 (Mitra-Gneiting-Sasvari)**.** *If C is a continuous function on $[-1,1]$ and there exists a positive number r_∞ such that C_r is a covariance function if and only if $r \geq r_\infty$, then*

$$\lim_{n\to\infty} r_n = r_\infty$$

We define a function L_n

$$L_n(r) = \det(M_n(C_r))$$

Lemma 4.2. *The function L_n is a linear function of r*

$$L_n(r) = ar + b \tag{14}$$

where a and b are two constants.

Proof. We denote z_i the i-th row and s_j the j-th column.

$$L_n(r) = \det(M_n(C_r))$$

$$= \det \begin{pmatrix} C(\frac{0-0}{n}) - C(0) + r & \cdots & C(-\frac{j}{n}) - C(0) + r & \cdots & C(-1) - C(0) + r \\ \vdots & \ddots & \vdots & & \vdots \\ C(\frac{i}{n}) - C(0) + r & \cdots & C(\frac{i-j}{n}) - C(0) + r & & \\ \vdots & & & \ddots & \\ C(1) - C(0) + r & & & & C(\frac{n-n}{n}) - C(0) + r \end{pmatrix}_{(n+1)\times(n+1)}$$

$$\overset{z_{i+1}-z_1}{=} \det \begin{pmatrix} r & \begin{pmatrix} \cdots & C(-\frac{j}{n}) - C(0) + r & \cdots \end{pmatrix}_{1\times n} \\ \begin{pmatrix} \vdots \\ C(\frac{i}{n}) - C(0) \\ \vdots \end{pmatrix}_{n\times 1} & \begin{pmatrix} \ddots & & \\ & C(\frac{i-j}{n}) - C(-\frac{j}{n}) & \\ & & \ddots \end{pmatrix}_{n\times n} \end{pmatrix}_{(n+1)\times(n+1)}$$

$$
\overset{s_{j+1}-s_1}{=} \det \begin{pmatrix} r & \begin{pmatrix} \cdots & C(-\frac{j}{n}) - C(0) & \cdots \end{pmatrix}_{1\times n} \\ \begin{pmatrix} \vdots \\ C(\frac{i}{n}) - C(0) \\ \vdots \end{pmatrix}_{n\times 1} & \begin{pmatrix} \ddots & & \\ & C(\frac{i-j}{n}) - C(\frac{i}{n}) - C(-\frac{j}{n}) + C(0) & \\ & & \ddots \end{pmatrix}_{n\times n} \end{pmatrix}_{(n+1)\times(n+1)} \tag{15}
$$

In this matrix, r appears only once, then $L_n(r)$ is a linear function of r. Furthermore, L_n is a monotone increasing function from Krein-Langer theorem, then $a > 0$. □

Proposition 4.3. *r_n is the zero point of L_n, i.e.*

$$
L_n(r_n) = 0
$$

Proof. Let r_n^* be the zero point of L_n and $\bar{r}$ be the middle point $\bar{r} := \frac{r_n^* + r_n}{2}$

If $r_n < r_n^*$ and recall that L_n is a linear function

$$
L_n(\bar{r}) < L_n(r_n^*) = \det(M_n(C_{r_n^*})) = 0
$$

Then the matrix $M_n C_{\bar{r}}$ has at least one negative eigenvalue, which contradicts the definition of r_n.

If $r_n^* < r_n$, then for middle point $\bar{r}$, there exists a unit vector x

$$
x^T M_n(C_{\bar{r}}) x < 0
$$

Fix this unit vector x, due to continuity, there exists a $r_2 > \bar{r}$ such that

$$
x^T M_n(C_{r_2}) x = 0
$$

then

$$
\det(M_n(C_{r_2})) = 0
$$

This implies that r_2 is a second zero point of L_n, which contradicts the uniqueness of zero point of linear function. □

Corollary. *The r_n can be expressed*

$$
r_n = -\frac{\det(\beta_{i,j})_{i,j=0}^n}{\det(\alpha_{i,j})_{i,j=1}^n} \tag{16}
$$

where

$$
\begin{aligned}
\beta_{i,j} &= C(\frac{i-j}{n}) - C(0) \\
\alpha_{i,j} &= C(\frac{i-j}{n}) - C(\frac{i}{n}) - C(-\frac{j}{n}) + C(0)
\end{aligned}
$$

Proof. r_n is the zero point of L_n

$$ar_n + b = 0$$

From (15)

$$a = \det(\alpha_{i,j})_{i,j=1}^n$$

and set $r = 0$ into (14)

$$b = \det(\beta_{i,j})_{i,j=0}^n$$

□

In the following table we present the coefficients of polynomial

$$C(t) = r - \frac{1}{2}|t| + \frac{b_2}{2!}|t|^2 + \frac{b_4}{4!}|t|^4 + \frac{b_6}{6!}|t|^6 + \frac{b_8}{8!}|t|^8$$

In the table are the values of r_n for $n = 10, 20, 30, 40, 50$ and the analytical limit r_∞.

Table 1 : Numerical values of b_k

case	(i)	(ii)	(iii)	(iv)
b_2	0	-1	6	0.08084
b_4	1	0	1	3.04749
b_6	4	2	0	5.43217
b_8	3	5	-4	-11.35261

Table 2 : r_n and r_∞

case	(i)	(ii)	(iii)	(iv)
r_{10}	0.2286409660	0.5783240166	1.612114043	0.1837180146
r_{20}	0.2286639511	0.5789134327	1.632410822	0.1838883736
r_{30}	0.2286682283	0.5790225524	1.636167163	0.1839200018
r_{40}	0.2286697268	0.5790607420	1.637481711	0.1839310802
r_{50}	0.2286704207	0.5790784180	1.638090129	0.1839361972
r_∞	0.2286716547	0.5791098412	1.639171713	0.1839453225

From the table two we see that all zero points r_n tend to the lower bound r_∞.

Theorem 4.4. *Let C be a polynomial with vanishing odd terms of the form*

$$C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4, ... + b_{n+1}|t|^{n+1}, \quad -1 \le t \le 1$$

Then C is a covariance function if and only if

1. *the equation admits only solution $\mu < 1$*

$$\det B = 0$$

where

$$b_{l,j} = \begin{cases} -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \le j \le l-1 \\ -\sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!} & \text{for } l \le j \le n \end{cases}$$

2. *the inequality holds*

$$C(0) \geq r_n, \quad \forall n \in \mathbb{N}$$

where

$$r_n = -\frac{\det(\beta_{i,j})_{i,j=0}^n}{\det(\alpha_{i,j})_{i,j=1}^n}$$

with

$$\begin{aligned} \beta_{i,j} &= C(\frac{i-j}{n}) - C(0) \\ \alpha_{i,j} &= C(\frac{i-j}{n}) - C(\frac{i}{n}) - C(-\frac{j}{n}) + C(0) \end{aligned}$$

For polynomial with non-vanishing odd terms, there is no simple numerical formula to calculate the eigenvalue of $\mathcal{H}$. But as $\mathcal{H}$ is a compact operator, we can use Fredholm determinant [8].

Theorem 4.5. *Let C be a polynomial of the form*

$$C(t) = r - \frac{1}{2}|t| + \sum_{i=2}^{n+1} b_i |t|^i, \quad -1 \leq t \leq 1$$

Then C is a covariance function if and only if

1. *the following equation admits only $\mu < 1$*

$$\sum_{n \geq 0} \frac{1}{\mu^n} tr\left(\Lambda^n(\frac{\mathcal{H}}{\mu})\right) = 0$$

2. *the inequality holds*

$$C(0) \geq r_n, \quad \forall n \in \mathbb{N}$$

where

$$r_n = -\frac{\det(\beta_{i,j})_{i,j=0}^n}{\det(\alpha_{i,j})_{i,j=1}^n}$$

with

$$\begin{aligned} \beta_{i,j} &= C(\frac{i-j}{n}) - C(0) \\ \alpha_{i,j} &= C(\frac{i-j}{n}) - C(\frac{i}{n}) - C(-\frac{j}{n}) + C(0) \end{aligned}$$

Proof. $\mathcal{H}$ is a finite rank operator by definition, then it is trace class operator. The Fredholm determinant is well-defined. $-\mu$ is an eigenvalue of $\mathcal{H}$, then the function $\varphi(\lambda) = \det(I + \lambda\mathcal{H})$ has a zero of order n at $\lambda = \frac{1}{\mu}$, where n is the algebraic multiplicity of $-\mu$. Then the Fredholm determinant can be expressed

$$\det(I + \frac{\mathcal{H}}{\mu}) = \sum_{n\geq 0} \frac{1}{\mu^n} \mathrm{tr}\left(\Lambda^n(\frac{\mathcal{H}}{\mu})\right)$$

where tr denotes operator trace and Λ denotes exterior product. □

A Maple Program for calculation of eigenvalues of $\mathcal{H}$

```
restart;

with(LinearAlgebra);

with(CodeGeneration);

n:=;

expr:={};

CL:=t->eval(-1/2+sum(a[k]*t^(k-1)/(k-1)!,k=2..n+1),expr);

b:=(l,j)->piecewise(j<l,eval(-sum(a[k]/(k-l+j-1)!,k=1+l..n+1),expr),

j>=l,eval(mu/(j-l)!-sum(a[k]/(k-l+j-1)!,k=1+l..n+1),expr));

B:=Matrix(n,n,(x,y)->b(x,y));

factor(eval(Determinant(B)))=0;
```

B Maple Program for calculation of inner product

```
restart;

with(LinearAlgebra);

with(CodeGeneration);

n:=;
```

```
expr:={a[1]=-1/2,a[3]=0};

CL:=t->eval(-1/2+sum(a[k]*t^(k-1)/(k-1)!,k=2..n+1),expr);

a:=(l,j)->piecewise(j<l,eval(-sum(a[k]/(k-l+j-1)!,k=1+l..n+1),expr),

j>=l,eval(1/(j-l)!-sum(a[k]/(k-l+j-1)!,k=1+l..n+1),expr));

f:=l->eval(-sum(sum(a[k]*a[j]/(k-1-l+j)!,k=1+l..n+1),j=1..n+1),expr);

A:=Matrix(n,n,(x,y)->a(x,y));

F:=Matrix(n,1,x->f(x));

p:=t->Determinant(Transpose(Matrix(n,1,j->t^(j-1)/(j-1)!)).A^(-1).F);

factor(eval(int((CL(t)-p(t))*CL(t),t=0..1)));
```

C Maple Programm for calculation of r_m

```
restart;

with(LinearAlgebra);

with(CodeGeneration);

n:=;

m:=;

expr:={};

c:=t->piecewise(t>=0,eval(r-1/2*t+sum(a[k]*t^k/k!,k=2..n+1),expr),t<0,eval(r+1/2*t

+sum(a[k]*(-t)^k/k!,k=2..n+1),expr));

A:=Matrix(m,m,(x,y)->c((x-y)/m)+c(0)-c(x/m)-c(-y/m)):

B:=Matrix(m+1,m+1,(x,y)->c((x-y)/m)-r):
```

```
-evalf[6](Determinant(B)/Determinant(A));
```

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