Approximation of polynomial convariance functions of arbitrary degree

Yu Li

To cite this version:

Yu Li. Approximation of polynomial convariance functions of arbitrary degree. 2018. hal-01896265

HAL Id: hal-01896265
https://hal.archives-ouvertes.fr/hal-01896265
Preprint submitted on 15 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Approximation of polynomial covariance functions of arbitrary degree

Yu Li*

*Faculty of Science, Technology and Communication, University of Luxembourg

Abstract

Covariance function plays an important role in spatial modeling. For a function to be a valid covariance function, it must satisfy positive definiteness. However, checking for positive definiteness of a given function is, in general, a challenging task. In this paper, we present an approximation approach to characterize polynomial covariance functions of arbitrary degree.

Keywords. Covariance function, positive definiteness, Krein-Langer theorem, Fredholm determinant

1 Introduction

Covariance function provides a powerful tool in time series analysis and in spatial data analysis. In this paper we consider for a class of stationary process \( X(u) \) with polynomial covariance function of

\[
C(t) = \text{cov}(X(u + t), X(u)), \quad -1 \leq u < t + u \leq 1
\]

where

\[
C(t) = \sum_{i=0}^{n+1} b_i |t|^i, \quad -1 \leq t \leq 1
\]

In general, an arbitrary polynomial will not be a valid covariance function and it is not evident which conditions a polynomial should be fulfilled for the existence of such a stationary process. Clearly, to be a valid covariance function, polynomial must be positive definite.

Polynomial covariance model finds various applications in Geostatistics [1, 11] and Hydrology [9] and attracts great interests of researchers. A polynomial covariance model of degree 4 is presented by a simulation study for the Stienen model spheres in three-dimensional Euclidean space [10]. Turning bands operator provides a convenient method to check for positive definiteness in the general case [5]. Polynomial covariance function of degree \( k \leq 3 \) was studied [2] and the quintic case was considered [6, 4]. A more
general form [7] was presented, which covers the quartic case except when the linear term vanishes.

The property of positive definiteness is, in general, difficult to check. Fortunately, Krein-Langer analytical technique [3] provides a powerful tool for characterization of covariance functions. In section 2 we review Krein-Langer theorem. In section 3 we study polynomial covariance function with vanishing odd terms. In section we present approximation theory for polynomial covariance functions of arbitrary degree.

2 Krein-Langer theorem

We begin this section by recalling the Krein-Langer theorem which is the key tool in this paper. Let \( C \) denote an even continuous function such that

1. \( C \) is second differentiable, i.e. \( C^2[-1,1] \);
2. the second \( H = -C'' \) exists for \( t \neq 0 \) and \( H \) is absolutely integrable over \([-1,1]\)
3. \( C'(0+) < 0 \)

Second derivative can be interpreted as accelerator. With the \( H \) we define an operator on \( L^2[0,1] \)

\[
Hf(t) = \int_0^1 H(t - \xi)f(\xi)d\xi, \quad 0 < t < 1
\]

**Theorem 2.1.** (Krein-Langer theorem) Let \( C \) be an even function with function \( H \) and \( C'(0+) = -\frac{1}{2} \). Furthermore, \(-1\) is not an eigenvalue of \( H \). Then \( C \) is a covariance function if and only if

1. the operator \( H + I \) has no negative eigenvalues;
2. \( C(0) \geq \langle (I + H)^{-1}C', C' \rangle \)

For calculating the inner product we can use integral equation approach. We determine first the resolvent kernel of \( H \)

\[
\Gamma(t,s) + \int_0^1 H(t - u)\Gamma(u,s)du = H(t - s), \quad 0 < t, s < 1
\]

Here \( \Gamma \) is a real function, otherwise \( \Gamma - \overline{\Gamma} \) is an eigenfunction of \( H \) with eigenvalue \(-1\).

Let \( \Gamma \) be the operator on \( L^2[0,1] \)

\[
\Gamma \varphi(t) = \int_0^1 \Gamma(t,s)\varphi(s)ds, \quad 0 < t < 1
\]

Then the equation (1) can be rewritten in the form

\[
\Gamma + \Gamma H = H
\]
Finally

\[(I + \mathcal{H})(I - \Gamma) = I\]

The first condition of Krein-Langer theorem leads to calculation of eigenvalues. Let \(-\mu\) be an eigenvalue of \(\mathcal{H}\)

\[\mu V(t) + \int_0^1 H(t - u)V(u)du = 0 \quad (0 < t < 1)\]  \hspace{1cm} (2)

Differentiating \(l\) times yields

\[\mu V^{(l-1)}(t) + \int_0^1 H^{(l-1)}(t - u)V(u)du - \sum_{m=0}^{l-2} b_{l-m}(1 - (-1)^{l-m})V^{(m)}(t) = 0\]  \hspace{1cm} (3)

for \(l = 1, 2, ..., n\). To calculate the inner product, we define a function \(G\)

\[G(t) := (\Gamma C')(t) = \int_0^1 \Gamma(t, s)C'(s)ds\]

It holds obviously

\[((I + \mathcal{H})^{-1}C', C')_{L^2[0,1]} = (C' - G, C')_{L^2[0,1]}\]

Differentiating the equation \(l\) times

\[G^{(l-1)}(t) + \int_0^1 H^{(l-1)}(t - s)(G(s) - C'(s))ds - \sum_{m=0}^{l-2} b_{l-m}(1 - (-1)^{l-m})(G^{(m)}(t) - C'^{(m)}(t)) = 0\]  \hspace{1cm} (4)

3 Polynomial covariance functions with vanishing odd terms

We consider polynomials with vanishing odd terms

\[C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6 + ... + b_{n+1}|t|^{n+1}, \quad -1 \leq t \leq 1\]

From (3) und (4) we see

\[V^{(n)}(t) = 0\]

\[G^{(n)}(t) = 0\]  \hspace{1cm} (5)

The solutions are

\[V(t) = \sum_{j=1}^{n} \frac{c_j}{(j-1)!}t^{j-1}\]  \hspace{1cm} (6)

\[G(t) = \sum_{j=1}^{n} \frac{d_j}{(j-1)!}t^{j-1}\]  \hspace{1cm} (7)

where \(c_j\) and \(d_j\) are constants.
Calculation of eigenvalues of $H$

Setting (6) into (3) with $t = 1$ yields

$$\sum_{j=l}^{n} \mu \frac{c_j}{(j-l)!} - \sum_{j=1}^{n} \sum_{k=1+l}^{n+1} \frac{b_k c_j}{(k-l+j-1)!} = 0$$

and

$$- \sum_{j=1}^{l-1} c_j \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \sum_{j=l}^{n} c_j \left( - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!} \right) = 0$$

This equations system can rewritten as matrix equation with respect to $c = [c_1, ..., c_n]$

$$Bc = 0$$

where

$$b_{l,j} = \begin{cases} - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\ - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

For non-trivial $V$ it must hold

$$\det(B) = 0$$

Calculation of inner product

We set (7) in (4) with $t = 1$, then

$$- \sum_{j=1}^{l-1} d_j \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \sum_{j=l}^{n} d_j \left( \frac{1}{(j-l)!} - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} \right)$$

$$= - \sum_{j=1}^{n} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!}$$

We rewrite in matrix form

$$ED = G$$

where

$$c_{l,j} = \begin{cases} - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\ - \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{1}{(j-l)!} & \text{for } l \leq j \leq n \end{cases}$$

and $g_l = - \sum_{j=1}^{n} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!}$

and $D$ is the vector of coefficients of function $V$ in (7).
Theorem 3.1. Let $C$ be a polynomial with vanishing odd terms of the form

$$C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6 + \ldots + b_{n+1}|t|^{n+1}, \quad 0 < t < 1$$

Then $C$ is a covariance function if and only if

1. the equation admits only solution $\mu < 1$

$$\det B = 0$$

where

$$b_{l,j} = \begin{cases} 
- \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l - 1 \\
- \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(j-l)!} & \text{for } l \leq j \leq n
\end{cases}$$

2. $C(0) \geq \langle C' - G, C' \rangle$ where $G$ is a polynomial of the form

$$G(t) = \sum_{j=1}^{n} \frac{d_j}{(j-1)!} t^{j-1}$$

The coefficients vector $D = [d_1, d_2, ..., d_n]$ of $G$ can be determined by

$$ED = G$$

with

$$e_{l,j} = \begin{cases} 
- \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l - 1 \\
- \sum_{k=1+l}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{1}{(j-l)!} & \text{for } l \leq j \leq n
\end{cases}$$

$$g_l = - \sum_{j=1}^{n+1} \sum_{k=1+l}^{n+1} \frac{b_k b_j}{(k-1-l)!}$$

Example 3.1. For $n = 5$

$$C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6$$

We calculate the determinant of matrix $\det(b_{i,j})_{n \times n}$ and the inner product $\langle (I - \Gamma)C', C' \rangle$

$$\det(b_{i,j})_{n \times n} = -\frac{1}{22122558259200000} (1209600\mu^2 + 100800b_4\mu + 5040\mu b_6 - b_6^2)$$

$$\quad + 25401600a_4^2\mu + 15240960a_4\mu^2 + 907200b_4\mu b_6 - 15389152000\mu^3$$

$$\quad + 76204800\mu^2 b_6 + 18289152000b_2\mu^2 + 20160\mu b_6^2 - 25401600b_2 b_6 - a_6^3) \quad (8)$$

5
\[ (I - \Gamma)C', C' = \frac{1}{10059033600(-b_6^2 + 5040b_6 + 1209600 + 100800b_4)} \]
\[ (1013950556880000b_6^2 - 279417600b_2b_6^2 - 20118067200b_2b_6^2 - 60354201600b_6^2b_2 + 5069752934000b_4b_2 - 304185176064000b_2) \]
\[ -b_6^4 + 60480b_6^3 + 4536000b_4b_6^2 + 533433600b_6^2 + 4224794112000b_6 + 279417600b_4^2b_6 + 83825280000b_4b_6 + 1810626048000b_4^2 \]
\[ + 10059033600b_4^3 + 3041851760640000 \leq r \] (9)

The polynomial \( C \) is a valid covariance function if and only the equation (8) only admits solutions \( \mu < 1 \) and the inequality (9) holds

**Example 3.2.** For \( n = 7 \)

\[ C(t) = r - \frac{1}{2}|t| + b_2|t|^2 + b_4|t|^4 + b_6|t|^6 + b_8|t|^8 \]

Analog we calculate \( \det(b_{i,j})_{n \times n} \) and \( (C' - G, C')_{L^2[0,1]} \)

\[ \det(b_{i,j}) = \frac{1}{128039477702525013027172799104000000000000(-10059033600\mu b_6^2 + 5069752934000\mu^2b_6)} \]
\[ -279417600\mu b_6b_8 + 1216740742560000\mu b_3 + 1207084032000b_3\mu b_4 + 10059033600b_8\mu b_4 \]
\[ -4536000\mu b_3^2 - b_8^3 + 1013950556880000b_4\mu^2/(10523152324456611840000\mu^4) \]
\[ -10523152324456611840000\mu^3b_2 - 1043963524516480000\mu^3b_2 - 438466680185692160000\mu^3b_6 \]
\[ -876929360371384320000\mu^3b_4 + 28998986784768000\mu^2a_8b_4 - 115995947139072000\mu^2b_6^2 \]
\[ -5219817621528240000\mu^2b_6b_4 + 5219817621528240000\mu^2b_6b_2 - 404373150720000\mu^2b_8^2\mu^2 \]
\[ +14615489339523072000\mu^2b_2b_6^2 - 14615489339523072000\mu^2b_4^2 - 2855960819712000\mu^2b_6b_8 \]
\[ +130767436800\mu b_6b_8 + 5753767219200\mu b_6^3 + 16632000b_3\mu + 1676505600b_8^2\mu b_6 \]
\[ -11507534438400\mu b_6b_4b_8 - 130767436800b_8^2\mu b_4 + 5753767219200b_8^2\mu b_2 + b_8^4 = 0 \] (10)
\((I - \Gamma) C', \Gamma') = (448793843097600010059033600b_4 - b_8^3 - 4536000b_8^2
-279417600b_8 + 1207084032000b_8 + 1216740704256000 - 10059033600b_8^2
+ 50697529344000b_8)^{-1}(-1724759815015212920000b_4b_8 - 3664311970123284800000b_8b_8
- 12540089533107957760000b_2b_4b_8 - 136516434179082647135846000000b_2b_8
+ 189601685858205923213312000000b_8 + 67716485207872971904000000b_8
+ 13463815292958720000b_4b_8
+ 172613016576000b_4b_8^2 + 227523902984710785974400000b_4b_2b_8
+ 812597822494539566284800000b_4^2 + 37620269559932387328000000b_4b_6b_8
+ 4789943307627758387200000b_4b_8 - 2708659408315131887616000000b_6b_2b_8
+ 23940171538138791936000000b_6b_2^2 + 394618212167122944000000b_6b_2
- 631597932172247040000b_8^2 - 300962156479459098624000000b_8b_2b_8
- 897587688619520000b_6a_2b_8^2 - 12540089533107957760000b_6b_2^4
- 9917653483906560000b_2b_8^2 + 12540089533107957760000b_2b_8b_6
- 407145774581427200000b_4b_2b_8 + 203572887229071360000b_4b_8^2
- 3044893612400640000b_4b_8^2 + 45144323471918864793600000b_4^3
+ 2714035163054284800000b_6^3 + 4236864952320000b_6^3
- 4487938430976000b_6^4 + 6089787224801280000b_6^2b_8 + 86996960354304000b_6b_8^2
- 4487938430976000b_6^2b_8^2 + 915372057600b_4b_8^3 - 863065082880000b_6b_8^3
- 915372057600b_6^2b_8^2 - 7264857600b_8b_8^3 - b_8^5
- 86306508288000b_6^3b_8^2 - 49896000b_8^4 + 4550547805969421571194880000000b_2^2
+ 45144323471918864793600000b_8b_8^2 + 13651643417908264713584600000000) \leq r \tag{11}

The \( C \) is a valid covariance function if equation (10) admits only \( \mu < 1 \) and inequality (11) holds.

4 Approximation theory for polynomial covariance functions

In this section we present a numerical approach to calculate inner product. For a function \( C \) be a function defined on interval \([-1, 1]\), Toeplitz matrix is defined as

\[
M_n(C) = \left( C \left( \frac{i-j}{n} \right) \right)_{i,j=0}^n
\]

And if \( C \) is a covariance function, then Toeplitz matrix \( M_n(C) \) is positive definite for all natural number \( n \). The Toeplitz matrix is symmetric and all the eigenvalues are positive.

Moreover we define a function \( C_r \)

\[
C_r(t) = C(t) - C(0) + r, \quad (|t| \leq 1)
\]
Let $r_\infty$ denote the inferium such that $C_r$ is a positive definite function

$$r_\infty = \inf \{ r \in \mathbb{R} : C_r \text{ is positive definite} \}$$  \hspace{1cm} (12)

A sequence $\{r_n\}$ is defined

$$r_n = \sup \{ r \in \mathbb{R} : M_n(C_r) \text{ has a negative eigenvalue} \}$$  \hspace{1cm} (13)

The following theorem provides a rigorous approximation of $r_\infty$.

**Theorem 4.1** (Mitra-Gneiting-Sasvari). *If $C$ is a continuous function on $[-1, 1]$ and there exists a positive number $r_\infty$ such that $C_r$ is a covariance function if and only if $r \geq r_\infty$, then*

$$\lim_{n \to \infty} r_n = r_\infty$$

We define a function $L_n$

$$L_n(r) = \det(M_n(C_r))$$

**Lemma 4.2.** *The function $L_n$ is a linear function of $r$

$$L_n(r) = ar + b$$  \hspace{1cm} (14)

*where $a$ and $b$ are two constants.*

**Proof.** We denote $z_i$ the $i$-th row and $s_j$ the $j$-th column.

$$L_n(r) = \det(M_n(C_r))$$

\[
\begin{vmatrix}
C(0) - C(0) + r & \cdots & C(-\frac{j}{n}) - C(0) + r & \cdots & C(-1) - C(0) + r \\
\vdots & \ddots & \vdots & & \vdots \\
C(\frac{i}{n}) - C(0) + r & \cdots & C(\frac{i-1}{n}) - C(0) + r & \cdots & \vdots \\
C(1) - C(0) + r & \cdots & C(\frac{n-1}{n}) - C(0) + r & \cdots & \vdots \\
\end{vmatrix}_{(n+1) \times (n+1)}
\]

\[
\begin{vmatrix}
C(\frac{1}{n}) - C(0)
\vdots
\end{vmatrix}_{n \times 1}
\begin{vmatrix}
\vdots
C(\frac{-1}{n}) - C(0) + r & \cdots & C(-\frac{j}{n}) - C(0) + r & \cdots
\end{vmatrix}_{1 \times n}
\begin{vmatrix}
C(\frac{i}{n}) - C(0) + r & \cdots & C(\frac{i-1}{n}) - C(0) + r & \cdots
\end{vmatrix}_{n \times n}
\begin{vmatrix}
\vdots
C(\frac{n-1}{n}) - C(0) + r
\end{vmatrix}_{n \times 1}
\begin{vmatrix}
\end{vmatrix}_{(n+1) \times (n+1)}
\]

$$z_{z+1-z}^\frac{1}{2}$$

8
\[
\begin{equation}
\frac{s_{j+1} - s_1}{s_2} = \det \left( \begin{array}{cccc}
r & \cdots & C\left(-\frac{j}{n}\right) & C(0) \\
\vdots & \ddots & \vdots & \vdots \\
C\left(\frac{j}{n}\right) - C(0) & & C\left(\frac{j}{n}\right) - C\left(-\frac{j}{n}\right) & \cdots \\
\end{array} \right)_{n \times 1} \cdot \left( \begin{array}{cccc}
\cdots & C\left(-\frac{i}{n}\right) - C\left(-\frac{j}{n}\right) + C(0) \\
& & & \\
& & & \\
& & & \\
\end{array} \right)_{n \times n} \right)
\end{equation}
\]

(15)

In this matrix, \( r \) appears only once, then \( L_n (r) \) is a linear function of \( r \). Furthermore, \( L_n \) is a monotone increasing function from Krein-Langer theorem, then \( a > 0 \). \( \Box \)

**Proposition 4.3.** \( r_n \) is the zero point of \( L_n \), i.e.
\[
L_n (r_n) = 0
\]

**Proof.** Let \( r_n^* \) be the zero point of \( L_n \) and \( \bar{r} \) be the middle point \( \bar{r} := \frac{r_n^* + r_n}{2} \)

If \( r_n < r_n^* \) and recall that \( L_n \) is a linear function
\[
L_n (\bar{r}) < L_n (r_n^*) = \det (M_n (C_{r_n^*})) = 0
\]

Then the matrix \( M_n C_{\bar{r}} \) has at least one negative eigenvalue, which contradicts the definition of \( r_n \).

If \( r_n^* < r_n \), then for middle point \( \bar{r} \), there exists a unit vector \( x \)
\[
x^T M_n (C_{\bar{r}}) x < 0
\]

Fix this unit vector \( x \), due to continuity, there exists a \( r_2 > \bar{r} \) such that
\[
x^T M_n (C_{r_2}) x = 0
\]
then
\[
\det (M_n (C_{r_2})) = 0
\]
This implies that \( r_2 \) is a second zero point of \( L_n \), which contradicts the uniqueness of zero point of linear function. \( \Box \)

**Corollary.** The \( r_n \) can be expressed
\[
r_n = -\frac{\det (\beta_{i,j})_{i,j=0}^n}{\det (\alpha_{i,j})_{i,j=1}^n}
\]
(16)

where
\[
\beta_{i,j} = C\left(\frac{i-j}{n}\right) - C(0) \\
\alpha_{i,j} = C\left(\frac{i-j}{n}\right) - C\left(\frac{i}{n}\right) - C\left(\frac{j}{n}\right) + C(0)
\]

9
Proof. $r_n$ is the zero point of $L_n$

$$ar_n + b = 0$$

From (15)

$$a = \det(\alpha_{i,j})_{i,j=1}^n$$

and set $r = 0$ into (14)

$$b = \det(\beta_{i,j})_{i,j=0}^n$$

In the following table we present the coefficients of polynomial

$$C(t) = r - \frac{1}{2} |t| + \frac{b_2}{2!} |t|^2 + \frac{b_4}{4!} |t|^4 + \frac{b_6}{6!} |t|^6 + \frac{b_8}{8!} |t|^8$$

In the table are the values of $r_n$ for $n = 10, 20, 30, 40, 50$ and the analytical limit $r_\infty$.

<table>
<thead>
<tr>
<th>Table 1 : Numerical values of $b_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>case</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>$b_2$</td>
</tr>
<tr>
<td>$b_4$</td>
</tr>
<tr>
<td>$b_6$</td>
</tr>
<tr>
<td>$b_8$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2 : $r_n$ and $r_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>case</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>$r_{10}$</td>
</tr>
<tr>
<td>$r_{20}$</td>
</tr>
<tr>
<td>$r_{30}$</td>
</tr>
<tr>
<td>$r_{40}$</td>
</tr>
<tr>
<td>$r_{50}$</td>
</tr>
<tr>
<td>$r_\infty$</td>
</tr>
</tbody>
</table>

From the table two we see that all zero points $r_n$ tend to the lower bound $r_\infty$.

**Theorem 4.4.** Let $C$ be a polynomial with vanishing odd terms of the form

$$C(t) = r - \frac{1}{2} |t| + b_2 |t|^2 + b_4 |t|^4 + ... + b_{n+1} |t|^{n+1}, \quad -1 \leq t \leq 1$$

Then $C$ is a covariance function if and only if

1. the equation admits only solution $\mu < 1$

$$\det B = 0$$

where

$$b_{i,j} = \begin{cases} 
-\sum_{k=1}^{n+1} \frac{b_k}{(k-l+j-1)!} & \text{for } 1 \leq j \leq l-1 \\
-\sum_{k=1}^{n+1} \frac{b_k}{(k-l+j-1)!} + \frac{\mu}{(n-j)!} & \text{for } l \leq j \leq n
\end{cases}$$
2. the inequality holds

\[ C(0) \geq r_n, \quad \forall n \in \mathbb{N} \]

where

\[ r_n = -\frac{\det(\beta_{i,j})_{i,j=0}}{\det(\alpha_{i,j})_{i,j=1}} \]

with

\[ \beta_{i,j} = C\left(\frac{i-j}{n}\right) - C(0) \]
\[ \alpha_{i,j} = C\left(\frac{i-j}{n}\right) - C\left(\frac{i}{n}\right) - C\left(\frac{-j}{n}\right) + C(0) \]

For polynomial with non-vanishing odd terms, there is no simple numerical formula to calculate the eigenvalue of \( H \). But as \( H \) is a compact operator, we can use Fredholm determinant \[8\].

**Theorem 4.5.** Let \( C \) be a polynomial of the form

\[ C(t) = r - \frac{1}{2} |t| + \sum_{i=2}^{n+1} b_i |t|^i, \quad -1 \leq t \leq 1 \]

Then \( C \) is a covariance function if and only if

1. the following equation admits only \( \mu < 1 \)

\[ \sum_{n \geq 0} \frac{1}{\mu^n} \text{tr} \left( \Lambda^n \left( \frac{H}{\mu} \right) \right) = 0 \]

2. the inequality holds

\[ C(0) \geq r_n, \quad \forall n \in \mathbb{N} \]

where

\[ r_n = -\frac{\det(\beta_{i,j})_{i,j=0}}{\det(\alpha_{i,j})_{i,j=1}} \]

with

\[ \beta_{i,j} = C\left(\frac{i-j}{n}\right) - C(0) \]
\[ \alpha_{i,j} = C\left(\frac{i-j}{n}\right) - C\left(\frac{i}{n}\right) - C\left(\frac{-j}{n}\right) + C(0) \]
Proof. \( \mathcal{H} \) is a finite rank operator by definition, then it is trace class operator. The Fredholm determinant is well-defined. \(-\mu\) is an eigenvalue of \( \mathcal{H} \), then the function \( \varphi(\lambda) = \det(I + \lambda \mathcal{H}) \) has a zero of order \( n \) at \( \lambda = \frac{1}{\mu} \), where \( n \) is the algebraic multiplicity of \(-\mu\). Then the Fredholm determinant can be expressed
\[
\det(I + \frac{\mathcal{H}}{\mu}) = \sum_{n \geq 0} \frac{1}{\mu^n} \text{tr} \left( \Lambda^n \left( \frac{\mathcal{H}}{\mu} \right) \right)
\]
where \( \text{tr} \) denotes operator trace and \( \Lambda \) denotes exterior product.

A Maple Program for calculation of eigenvalues of \( \mathcal{H} \)

```maple
restart;
with(LinearAlgebra);
with(CodeGeneration);

n := ;
expr := {};

CL := t -> eval(-1/2 + sum(a[k]*t^(k-1)/(k-1)!, k=2..n+1), expr);

b := (l, j) -> piecewise(j < l, eval(-sum(a[k]/(k-l+j-1)!, k=1+l..n+1), expr), j >= l, eval(mu/(j-l)! - sum(a[k]/(k-l+j-1)!, k=1+l..n+1), expr));

B := Matrix(n, n, (x, y) -> b(x, y));

factor(eval(Determinant(B))) = 0;
```

B Maple Program for calculation of inner product

```maple
restart;
with(LinearAlgebra);
with(CodeGeneration);

n := ;
```

CL:=t->eval(-1/2+sum(a[k]*t^(k-1)/(k-1)!,k=2..n+1),expr);
a:=(l,j)->piecewise(j<l,eval(-sum(a[k]/(k-l+j-1)!,k=1..n+1),expr),
j>=l,eval(1/(j-1)!-sum(a[k]/(k-l+j-1)!,k=1..n+1),expr));
f:=l->eval(-sum(sum(a[k]*a[j]/(k-1-l+j)!,k=1+l..n+1),j=1..n+1),expr);
A:=Matrix(n,n,(x,y)->a(x,y));

f:=Matrix(n,1,x->f(x));
p:=t->Determinant(Transpose(Matrix(n,1,j->t^(j-1)/(j-1)!)).A^(-1).F);
factor(eval(int((CL(t)-p(t))*CL(t),t=0..1)));

C Maple Programm for calculation of $r_m$

restart;
with(LinearAlgebra);
with(CodeGeneration);

n:=;
m:=;

expr:={};
c:=t->piecewise(t>=0,eval(r-1/2*t+sum(a[k]*t^k/k!,k=2..n+1),expr),t<0,eval(r+1/2*t+sum(a[k]*(-t)^k/k!,k=2..n+1),expr));
A:=Matrix(m,m,(x,y)->c((x-y)/m)+c(0)-c(x/m)-c(-y/m));
B:=Matrix(m+1,m+1,(x,y)->c((x-y)/m)-r);
-evalf[6](Det(B)/Det(A));

References


