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A mean bound financial model and options pricing

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Abstract

Most of financial models, including the famous Black Scholes-Merton options pricing model, rely upon the assumption that asset returns follow a normal distribution. However, this assumption is not justified by empirical data. To be more concrete, the empirical observations exhibit fat tails or heavy tails and implied volatilities against the strike prices demonstrate U-shaped curve resembling a smile, which is the famous volatility smile. In this paper we present a mean bound financial model and show that asset returns per time unit are Pareto distributed and assets are log Gamma distributed under this model. Based on this we study the sensitivity of the options prices to a change in underlying parameters, which are commonly called the Greeks, and derive options pricing formulas. Finally, we reveal the relation between correct volatility and implied volatility in Black Scholes model and provide a mathematical explanation of volatility smile.

Keywords. heavy-tailed distribution, power law, Pareto distribution, volatility smile, maximal entropy, options pricing, the greeks, Black Scholes model

1 Introduction

Black Scholes model was a major breakthrough in the pricing of European stock options in the early 1970s. The model had a huge influence on options trading and legitimized scientifically the activities of the Chicago Board Options Exchange and other options markets around the world. Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work [7, 19].

In Black Scholes model, financial asset returns are considered as the cumulative outcome of a vast number of pieces of information and individual decisions arriving almost continuously in time. As such they have been modeled by the Gaussian distribution based on the central limit theorem [7, 19].

However, this basic assumption is not justified by empirical data. Rather, the empirical observations exhibit fat tails or heavy tails and implied volatilities against the strike prices demonstrate U-shaped curve resembling a smile [4, 5, 16, 29].

Studies of heavy tailed distributions in finance are empirically established [6]. A large amount of literature in mathematical finance deals with the problem of finding rigorous models which reproduce heavy-tails returns.

Student distribution was proposed as symmetric stable distribution which counts for observed “fat tails”. However, it cannot capture skew feature of observed data [2, 3].

Stable distributions can accommodate the fat tails and asymmetry, they were proposed as alternative models [11, 23]. However, a common criticism of the stable distribution is that their tails are too heavy, that is, all non-Gaussian stable distributions have infinite variance [6].

The class of generalized hyperbolic distributions and normal inverse Gaussian distributions possess these semi heavy tails [25, 26]. Exponentially truncated stable distributions are some of the parametric models which have been proposed [13]. However, they are all difficult to reproduce.

In Black Scholes model, volatility is assumed to be constant overall. However, implied volatility surface such as volatility smile and skew has been long observed, which indicates implied volatility tends to vary with respect to strike price and expiry. To resolve this shortcoming of the BlackScholes model, stochastic volatility models were presented. The most commonly used stochastic volatility models are Heston model, constant elasticity of variance model and SABR model [12, 17, 18, 29].

2 Stylized empirical facts of financial data

The random variations of asset prices seem to share some quite nontrivial statistical properties. Such properties across different markets, asset types, and time periods, including a high kurtosis and a rather low-starting and slowly decaying auto-correlation function of the squared or absolute-valued observations, are called stylized empirical facts [9, 21, 22, 27]. In order to introduce some new insights provided by methods based on statistical techniques recently applied in empirical finance, we present stylized empirical facts emerging from the statistical analysis of price variations in various types of financial markets.

Let us list the most relevant stylized facts.

1. **Heavy tail and conditional heavy tail:** the (unconditional) returns distribution seems to display a power-law or Pareto-like tail, with a tail index α with $2 < \alpha < 5$ for most data sets studied, which excludes stable laws with infinite variance and the normal distribution. Even conditional distributions exhibit heavy tail, however, with less heavier tail than the unconditional distribution of returns. Mathematically speaking, a random variable X has a heavy tail if there exists a positive parameter α such that

$$P[X > x] = x^{-\alpha} L(x) \quad (1)$$

where α is called tail index and L denotes a slowly varying function, i.e,

$$\lim_{x \rightarrow +\infty} \frac{L(tx)}{L(x)} = 1, \forall t > 0$$

2. **Gain/loss asymmetry:** Larger downward movement in stock prices and stock index values than upward movements are observed in financial market.

3. **Aggregational gaussianity:** returns distribution looks more and more like a normal distribution as time scale Δt increases.
4. **Intermittency:** returns display, at any time scale, a high degree of variability. This is quantified by the presence of irregular bursts in time series of a wide variety of volatility estimators.
5. **Absence and slow decay of auto-correlations:** (linear) auto-correlations of asset returns are often insignificant, except for very small time scales. The auto-correlation function of absolute returns decays slowly as a function of the time lag, roughly as a power law with an exponent β with $0.2 < \beta < 0.5$.
6. **Volatility clustering:** different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high-volatility events tend to cluster in time.
7. **Long range dependence:** There is a phenomenon, called Long-range dependence arising in the analysis of financial data. It relates to the slower dependence decays than an exponential decay, typically a power-like decay.

3 Mean bound model

In this section we establish a model based on a mean assumption and a bound assumption. So this model can be called “mean-bound” model.

3.1 Central limit theorem and lower bound

The central limit theorem (CLT) is the most important theorem in probability. It states the sum of a large number of independent, identically distributed variables from a finite-variance distribution will tend to be normally distributed. The mean of all samples from the same population will be approximately equal to the mean of the population.

Theorem 3.1 (Central limit theorem). *Let $\{X_k\}_k$ be a sequence of independent and identically distributed random variables, then the random variables $\frac{\sum_{k=1}^n X_k}{\sqrt{n}}$ converge in distribution to a normal distribution*

$$\frac{\sum_{k=1}^n X_k}{\sqrt{n}} \xrightarrow{d} N(\mu, \sigma)$$

However, the central limit theorem “erases” the trace of lower bound. To see this, we consider a sequence of independent and identically distributed random variables $\{X_n\}_n$, which are all bounded from below $X_n \geq b$. Due to the central limit theorem, the random variable

$$\frac{\sum_{k=1}^n X_k}{\sqrt{n}}$$

is asymptotically normally distributed and is not bounded from below even if all components X_k are lower bounded by b . Nevertheless, the bound b still leaves a trace even if n is arbitrarily large

$$\frac{\sum_{k=1}^n X_k}{n} \geq b, \forall n > 0$$

In order to rediscover this lost lower bound and present a more rigorous financial model, we devote some time to the studying of lower bounded random variables, $b \leq X \leq +\infty$. Using this random variable, we build a mean-bound financial model.

3.2 Maximal entropy principle and lower bounded random variables

The Principle of Maximum Entropy states the rather obvious point that probability distribution with largest uncertainty remained should be selected. In other words, subject to precisely testable information, the probability distribution which best represents the current state of knowledge is the one with largest entropy and the least informative default. The actual mathematical procedure is called the “method of Lagrange multipliers” [14].

We consider a random variable X bounded from below $+\infty \geq X \geq b$ with probability density $f(x)$. This implies that there exists a real number b such that

$$P[X \leq b] = 0 \tag{2}$$

Now we calculate its distribution with maximum entropy principle. The differential entropy of X is defined as

$$H(f) = - \int_{\mathbb{R}} f(x) \ln f(x) dx$$

The following optimization solves for a maximum entropy distribution that satisfies some constraints:

$$\begin{aligned} \min_f & -H(f) \\ \text{s.t.} & f(x) \geq 0 \\ & \int f(x) dx = 1 \\ & \int x f(x) dx = m_1 \end{aligned} \tag{3}$$

We define the Lagrangian functional

$$L(f, \lambda) = -H(f) + \lambda_0 \left(\int f(x) dx - 1 \right) + \lambda_1 \left(\int x f(x) dx - m_1 \right)$$

take functional derivative and set $\frac{\partial L}{\partial f} = 0$,

$$\frac{\partial L}{\partial f} = 1 + \ln f + \lambda_0 + \lambda_1 x = 0 \tag{4}$$

Then the only distribution that solves this optimization problem (3) is the shifted exponential distribution

$$P[X < x] = 1 - e^{-\lambda(x-b)} \quad (5)$$

with mean and standard derivation

$$\begin{aligned} \mu &= \frac{1}{\lambda} + b \\ \sigma &= \frac{1}{\lambda} \end{aligned} \quad (6)$$

where b is the largest lower bound of X . For convenience, we use (b, λ) pair instead of mean-variance (μ, σ) pair.

3.3 A discrete time model for finance

In this section we present a time discrete model to describe the asset evolution and show that asset single-period returns are Pareto distributed and gross returns are logarithm Erlang, or more general, logarithm Gamma distributed in this model.

We fix a time unit and let $S = \{S_t\}_t$ denote a process of assets and let r_t denote single period simple returns at time t . It is convenient to consider the single-period simple logarithm returns $h_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$ [28]. We assume

- $h = \{h_t\}_t$ are independent and identically distributed with finite mean,
- $h = \{h_t\}_t$ are all bounded from below but free from above, $b \leq h_t \leq +\infty$.

The boundary assumption is not very restrictive as the lower bound b can be arbitrarily small. The only maximal entropy distribution satisfying these two assumptions, due to principle of maximum entropy (5), is exponential distribution of the form

$$P[h_t < x] = 1 - e^{-\lambda(x-b)} \quad (7)$$

Here b is the lower bound of h_t , $\frac{1}{\lambda}$ is the standard deviation of h_t and e^b serves as the lower bound of single-period simple return $r_t = e^{h_t}$. Hence, the asset returns per unit time obey power law and are Pareto distributed

$$P[r_t < x] = P[h_t < \ln x] = 1 - x^{-\lambda} e^{\lambda b} \quad (8)$$

with density function

$$f(x) = \lambda e^{\lambda b} x^{-\lambda-1}$$

Now we are in the state to calculate the mean and variance of asset returns per time unit

$$\begin{aligned} E[r_t] &= \lambda e^{\lambda b} \int_{e^b}^{\infty} x^{-\lambda} dx = \frac{\lambda e^b}{\lambda - 1} \\ \sigma^2(r_t) &= \frac{\lambda e^{2b}}{(\lambda - 2)(\lambda - 1)^2}, \lambda > 2 \end{aligned} \quad (9)$$

Recall that sum of independent exponential distributed random variables is Erlang distributed, so $\sum_{i=1}^t h_i$ is Erlang distributed from (7) with a shift term bt

$$P \left[\sum_{i=1}^t h_i < x \right] = 1 - e^{-\lambda(x-bt)} \sum_{i=0}^{t-1} \frac{\lambda^i (x-bt)^i}{i!}$$

Then the tail of the underlying asset is

$$\begin{aligned} P[S_t > x] &= P \left[S_0 e^{(\sum_{i=1}^t h_i)} > x \right] = P \left[\left(\sum_{i=1}^t h_i \right) > \ln \frac{x}{S_0} \right] \\ &= \underbrace{x^{-\lambda} e^{\lambda bt} \sum_{i=0}^{t-1} \frac{\lambda^i (\ln \frac{x}{S_0} - bt)^i}{i!}}_{L(x)} \end{aligned} \quad (10)$$

It is easy to check this $L(x)$ is a slow varying function satisfying (1) and hence, S_t has a heavy tail. Since Erlang distributions can be expressed in terms of Gamma functions (see appendix C), formula (10) is equivalent to

$$P[S_t < x] = \frac{\gamma \left(t, \lambda (\ln \frac{x}{S_0} - bt) \right)}{\Gamma(t)} \quad (11)$$

where γ is an incomplete Gamma function

$$\gamma(k, x) = \int_0^x t^{k-1} e^{-t} dt$$

and Γ a complete Gamma function

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$$

So the underlying asset S_t is log Gamma distributed with mean and standard deviation

$$\begin{aligned} E[S_t] &= S_0 E \left[\prod_{i=1}^t r_i \right] = S_0 \left(\frac{\lambda e^b}{\lambda - 1} \right)^t \\ E(S_t^2) &= S_0^2 \left[E \left[\prod_{i=1}^t r_i \right]^2 \right] = X_0^2 e^{2bt} \left(\frac{\lambda}{\lambda - 2} \right)^t, \lambda > 2 \end{aligned} \quad (12)$$

or equivalently (from equations (6))

$$\begin{aligned} E[S_t] &= S_0 \left(\frac{e^{\mu-\sigma}}{1-\sigma} \right)^t \\ E(S_t^2) &= S_0^2 e^{2(\mu-\sigma)t} \left(\frac{1}{1-2\sigma} \right)^t, \sigma < \frac{1}{2} \end{aligned} \quad (13)$$

Notice that $\left\{ \left(\frac{\lambda e^b}{\lambda - 1} \right)^{-t} S_t \right\}_t$ is a martingale

$$E[S_t | \mathcal{F}_{t-1}] = E[S_{t-1} h_t | \mathcal{F}_{t-1}] = S_{t-1} \frac{\lambda e^b}{\lambda - 1} \quad (14)$$

where $\mathcal{F}_t$ denote generated σ algebra, $\mathcal{F}_t = \sigma(h_1, \dots, h_t)$ and $\left\{ \left(\frac{\lambda e^b}{\lambda - 1} \right)^{-t} \right\}_t$ serves as a discount process. Then the risk free interest rate r is

$$r = -\frac{D'(t)}{D(t)} = \ln \left(\frac{\lambda e^b}{\lambda - 1} \right) \quad (15)$$

Power-law distributions occur in many situations of finance and play an important role in financial modeling. We provide an alternative interpretation of power laws. If a random variable $\ln X$ is a (shifted) exponentially distributed random variable

$$P[\ln X \geq x] = e^{-\lambda(x-b)}$$

then X obeys a power law

$$P[X \geq x] = e^{\lambda b} x^{-\lambda}$$

For this reason, many power laws can be understood as “logarithmically transformed” exponential distributions. More generally, many power laws can be understood as “logarithmically transformed” Erlang or Gamma distributions. Such distributions can be called log Erlang or log Gamma distributions. We give a formal definition in the following.

Definition 3.1. A random variable X is called log Erlang (or log Gamma) distributed if its logarithm $\ln X$ has an Erlang (or Gamma) distribution.

All above discussion can be formulated as the following theorem.

Theorem 3.2. *In the discrete time financial model, the single period return r_t is Pareto distributed*

$$P[r_t < x] = 1 - x^{-\frac{1}{\sigma}} e^{\frac{(\mu - \sigma)}{\sigma}} \quad (16)$$

the return R_t over time t is log Gamma distributed

$$P[R_t < x] = \frac{\gamma \left(t, \frac{1}{\sigma} (\ln x - (\mu - \sigma)t) \right)}{\Gamma(t)} \quad (17)$$

the mean and second moment of asset prices S_t

$$\begin{aligned} E[S_t] &= S_0 e^{(\mu - \sigma)t} (1 - \sigma)^{-t} \\ E[S_t^2] &= S_0^2 e^{2(\mu - \sigma)t} (1 - 2\sigma)^{-t} \end{aligned} \quad (18)$$

the risk free interest rate r is the root of the equation

$$e^r = (1 - \sigma)^{-1} e^{\mu - \sigma}$$

3.4 A continuous-time model for finance

The Black-Scholes model assumes log returns of the stock price are normal distributed. However, there is much empirical evidence for that these log-increments are not Gaussian. This leads to consider an alternative asset price model based on Gamma process instead of Wiener process.

We start with a random variable E and work on mean bound assumptions

- E has the mean 1,
- E is lower bounded by 0.

As we showed in section 3, the only maximal entropy distribution satisfying mean and bound assumptions is exponential distribution

$$P[E < x] = 1 - e^{-x}$$

As exponential distribution is infinitely divisible, there exists uniquely a Levy process with independent increments $\{E_t\}_{t>0}$ such that [1, 8, 20]

$$E_1 = E$$

Proposition 3.3. *The stochastic process $\{E_t\}_{t>0}$ is a Gamma process with distributions*

$$P[E_t < x] = \frac{\gamma(t, x)}{\Gamma(t)}$$

Proof. Directly from the one-one correspondence between infinitely divisible distribution and Levy process. □

Furthermore, we define a stochastic process $\{G_t\}_{t>0}$

$$G_t = E_t - t$$

with properties

1. $G_0 = 0$ is almost surely continuous
2. $\{G_t\}_t$ is almost surely continuous
3. $\{G_t\}_t$ has independent increments
4. $G_{t+s} - G_s$ is Gamma distributed

$$P[G_{t+s} - G_s < x] = \frac{\gamma(t, x - t)}{\Gamma(t)}$$

with mean and variance

$$\begin{aligned} E[G_{t+s} - G_s] &= 0 \\ \text{Var}[G_{t+s} - G_s] &= t \end{aligned}$$

This Gamma process is analogue to Wiener process except that it has a largest lower bound $b_t = -t$ and we model the risky asset price process H_t using the Gamma process

$$H_t = \mu t + \sigma G_t \quad (19)$$

Then

$$S_t = S_0 e^{H_t} \quad (20)$$

In the following we present some simple but useful lemmas.

Lemma 3.4. *For a random variable X , it holds*

$$E[e^{a+bX}] = e^a M_X(b) \quad (21)$$

where M_X is the moment generating function of X .

Proof.

$$E[e^{a+bX}] = e^a E[e^{bX}] = e^a M_X(b)$$

□

Example 3.5. *If S_t is a geometric Brownian motion in Black Scholes model, then*

$$E[S_t] = E[S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}] = S_0 e^{(r - \frac{1}{2}\sigma^2)t} M_{W_t}(\sigma) = e^{rt}$$

Lemma 3.6. *For any positive number σ , it holds*

$$E[e^{\sigma(G_t+t)}] = (1 - \sigma)^{-t} \quad (22)$$

Proof. $(G_t + t)$ is Gamma distributed with starting point 0 and $\lambda = 1$ and then we come to the conclusion by using moment generating function (see appendix C). □

Theorem 3.7. *In the continuous time financial model, the return R_t over time t is log Gamma distributed*

$$P[R_t < x] = \frac{\gamma\left(t, \frac{1}{\sigma}(\ln x - (\mu - \sigma)t)\right)}{\Gamma(t)} \quad (23)$$

the mean and second moment of asset prices S_t

$$\begin{aligned} E[S_t] &= S_0 e^{(\mu - \sigma)t} (1 - \sigma)^{-t} \\ E[S_t^2] &= S_0^2 e^{2(\mu - \sigma)t} (1 - 2\sigma)^{-t} \end{aligned} \quad (24)$$

the risk free interest rate r is the root of the equation

$$e^r = (1 - \sigma)^{-1} e^{\mu - \sigma} \quad (25)$$

Proof.

$$\begin{aligned} P[R_t < x] &= P[e^{\mu t + \sigma G_t} < x] = P\left[G_t + t < \frac{\ln x - \mu t + \sigma t}{\sigma}\right] \\ &= \frac{\gamma\left(t, \frac{1}{\sigma}(\ln x - (\mu - \sigma)t)\right)}{\Gamma(t)} \end{aligned}$$

$$E[S_t] = S_0 e^{(\mu - \sigma)t} E[e^{\sigma(G_t + t)}] = S_0 e^{(\mu - \sigma)t} (1 - \sigma)^{-t}$$

and

$$E[S_t]^2 = S_0^2 e^{2(\mu - \sigma)t} E[e^{2\sigma(G_t + t)}] = S_0^2 e^{2(\mu - \sigma)t} (1 - 2\sigma)^{-t}$$

The risk free interest rate r satisfies

$$e^r = (1 - \sigma)^{-1} e^{\mu - \sigma}$$

□

4 Applications

Empirical research in finance mainly focuses on modeling the distribution of returns. Mandelbrot pointed out that the normal distribution is insufficient for modeling the log asset returns and since then non-Gaussian features have been repeatedly observed in various market data [9, 23]. In this section we will show with formulas that the log Gamma model is a good fit for empirical stylized financial data.

Heavy tail of returns Empirical studies show that the distribution of asset return is not a Gaussian and the complementary cumulative distribution function $\bar{F}(x)$ of real returns is found to be approximately a power law $\bar{F}(x) \sim x^{-\alpha}$ with exponent in the range $2 - 4$ [9, 21, 22, 27]. In our model we provide an explicit formula of tail function (10) and show that returns are log Gamma distributed. Volatility is the inverse of exponent $\frac{1}{\alpha}$ and takes the value between $0.25 - 0.5$.

Gain/loss asymmetry There is large drawdowns in stock prices and stock index values but not equally large upward movements. In other words, the distribution is skewed [9, 21, 22, 27]. In our model, return per time unit r_t is proved to have a Pareto tail in (16) by applying maximal entropy principle with skewness $\frac{2(1+\lambda)}{\lambda-3} \sqrt{\frac{\lambda-2}{\lambda}}$ for $\lambda > 3$ and log return per time unit h_t is shifted exponential distributed. For exponentially distributed h_t , loss is more probable than gain

$$P[x - \Delta x < h_t < x] > P[x < h_t < x + \Delta x]$$

Aggregational gaussianity Distribution of returns looks more and more like a normal distribution as time scale increases. In our model, log return over time t denoted

by $H_t = \sum_{i=1}^t h_i$, is asymptotically normally distributed due to central limit theorem. Hence, return over large time scale appears Gaussian

$$R_t = e^{H_t} \approx 1 + H_t$$

Volatility clustering As noted as Mandelbrot (1963), that "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes" [23]. In other words, there is a tendency of large changes in prices of financial assets to cluster together, which results in the persistence of these magnitudes of price changes [21, 22, 27]. In log Gamma model there is a relation derived from (13) that variance of asset prices is asymptotically a power function of time t

$$\sigma^2(S_t) \sim S_0^2 e^{2(\mu-\sigma)t} (1-2\sigma)^{-t}, \sigma < \frac{1}{2}$$

5 Options pricing

In finance, an option is a contract which provides the holder the right, but not the obligation, to buy or sell an underlying asset or instruments at a fixed price, called a strike price or an exercise price at the expiration date (European options) or before the expiration date of the option (American options). Options where the payoff is calculated differently are categorized as "exotic options".

In this section we present options pricing formulas under mean-bound model.

5.1 Valuation of European options

The European calls and puts are sometimes called vanilla options. Their payoffs depend only on the final value of the underlying asset.

European futures options can be valued based on the log Gamma models that we have produced.

Theorem 5.1. *The prices of European call and put options on a stock S can be expressed as*

$$\begin{aligned} C &= \frac{S\Gamma(T, l_1)}{\Gamma(T)} - \frac{K\Gamma(T, l_2)e^{-rT}}{\Gamma(T)} \\ P &= \frac{e^{-rT}K\gamma(T, l_2)}{\Gamma(T)} - \frac{S\gamma(T, l_1)}{\Gamma(T)} \end{aligned} \quad (26)$$

where

$$\begin{aligned} l_1 &= \left(\frac{1}{\sigma} - 1\right) \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma)) \right] \\ l_2 &= \frac{1}{\sigma} \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma)) \right] \end{aligned}$$

Proof. In this model the free interest rate r satisfies (from equation (24))

$$e^{-r} = (1 - \sigma)e^{-(\mu - \sigma)}$$

then

$$\begin{aligned} C &= e^{-rT} E[(Se^{H_T} - K)^+] = e^{-rT} E[(Se^{(\mu - \sigma)T + \sigma(G_T + T)} - K)^+] \\ &= e^{-rT} \int_{x > \frac{\ln \frac{K}{S} - (\mu - \sigma)T}{\sigma}} \left(Se^{(\mu - \sigma)T + \sigma x} - K \right) \left(\frac{x^{T-1} e^{-x}}{\Gamma(T)} \right) dx \\ &= \frac{S_0 \Gamma(T, (\frac{1}{\sigma} - 1) [\ln \frac{K}{S} - (\mu - \sigma)T])}{\Gamma(T)} - \frac{K e^{-rT} \Gamma(T, \frac{1}{\sigma} [\ln \frac{K}{S} - (\mu - \sigma)T])}{\Gamma(T)} \\ &= \frac{S \Gamma(T, l_1)}{\Gamma(T)} - \frac{K \Gamma(T, l_2) e^{-rT}}{\Gamma(T)} \end{aligned}$$

To calculate the first integral we use a parameter transform $x' = x(1 - \sigma)$.

Then we can prove analogically

$$P = \frac{e^{-rT} K \gamma(T, l_2)}{\Gamma(T)} - \frac{S \gamma(T, l_1)}{\Gamma(T)}$$

□

Notice: In Black Scholes model $N(d)$ is a monotone increasing function with $d_1 > d_2$. In mean bound model Gamma function $\Gamma(T, x)$ is a monotone decreasing function with respect to x with $l_1 < l_2$.

5.2 Valuation of American put options without expiry date

Perpetual American options, also called American options without expiry date, are contracts that can be exercised at any time. It is essential to determine the optimal exercise boundary and the optimal stopping time.

We define stopping time τ_L via

$$\tau_L = \inf \{t : S_t \leq L\}$$

The optimal exercise boundary of a perpetual American put option is a constant. The option is exercised as soon as the stock price falls to the level L for the first time [19, 24].

Theorem 5.2. *The pricing formula for American put options without expiry date can be expressed*

$$V_L(S_0) = \begin{cases} K - S_0, & 0 \leq S_0 \leq L \\ (K - L) \left(\frac{S_0}{L} \right)^\beta, & S_0 \geq L \end{cases}$$

where β is the negative root of the equation

$$\frac{e^{-r(1-\beta)}(1-\sigma)^\beta}{1-\sigma\beta} = 1 \quad (27)$$

Proof. We search a number β such that the stochastic process $\{e^{-rt+\beta(\mu t+\sigma G_t)}\}_t$ is a martingale with respect to risk-neutral measure, i.e.

$$1 = E[e^{-rt+\beta(\mu t+\sigma G_t)}] = e^{-rt+\beta(\mu-\sigma)t} E[e^{\beta\sigma(G_t+t)}] = e^{-rt+\beta(\mu-\sigma)t} (1-\sigma\beta)^{-t}$$

or after combination with (25) and simplification

$$\frac{e^{-r(1-\beta)}(1-\sigma)^\beta}{1-\sigma\beta} = 1$$

The equation (27) has a positive solution $\beta_1 = 1$ and a negative solution β_2 (see figure 1). By the optional sampling theorem, we have

$$1 = E[e^{-r\tau_L+\beta_2(\mu\tau_L+\sigma G_{\tau_L})}] = E[e^{-r\tau_L}] \left(\frac{L}{S_0}\right)^{\beta_2}$$

or

$$E[e^{-r\tau_L}] = \left(\frac{L}{S_0}\right)^{-\beta_2}$$

Finally we derive the pricing formula for American options

$$V_L(S_0) = \begin{cases} K - S_0, & 0 \leq S_0 \leq L \\ (K - L) \left(\frac{S_0}{L}\right)^\beta, & S_0 \geq L \end{cases}$$

where β is the negative solution of equation

$$\frac{e^{-r(1-\beta)}(1-\sigma)^\beta}{1-\sigma\beta} = 1$$

□

Under the Black Scholes model, American option prices are given by

$$V_L(S_0) = \begin{cases} K - S_0, & 0 \leq S_0 \leq L \\ (K - L) \left(\frac{S_0}{L}\right)^\beta, & S_0 \geq L \end{cases}$$

where β is the negative root of the equation

$$\frac{1}{2}\sigma^2\beta^2 + (r - \frac{1}{2}\sigma^2)\beta - r = 0$$

In comparison between figure 1 and figure 2 we see that the value of β in Black Scholes model is larger than that in mean bound model. This implies that Black Scholes model overestimates the option prices of underlying assets. However, for larger σ , two values are very closed and Black Scholes model provides a better approximation of option price.

Figure 1: Roots of equation $\frac{e^{-r(1-\beta)}(1-\sigma)^\beta}{1-\sigma\beta} - 1 = 0$

Figure 2: Roots of equation $\frac{1}{2}\sigma^2\beta^2 + (r - \frac{1}{2}\sigma^2)\beta - r = 0$

5.3 Valuation of exotic options

Exotic Options, also called path-dependent options are options whose payoffs depend on the path of the underlying asset. The most important exotic options are barrier options, lookback options and Asian options. Here we present only Barrier options.

Barrier options are option contracts whose payoff depends on whether or not the underlying asset has reached or exceeded a certain level during the option's lifetime [15]. Here we consider a Up-and-Out Call European call, expiring at time T , with strike price K and up-and-out barrier B .

Theorem 5.3. *The payoff of the option is*

$$C = \frac{S[\Gamma(T, l_1) - \Gamma(T, l_3)]}{\Gamma(T)} - \frac{e^{-rT} K[\Gamma(T, l_2) - \Gamma(T, l_4)]}{\Gamma(T)}$$

where

$$\begin{aligned} l_1 &= \left(\frac{1}{\sigma} - 1\right) \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma)) \right] \\ l_2 &= \frac{1}{\sigma} \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma)) \right] \\ l_3 &= \left(\frac{1}{\sigma} - 1\right) \left[\ln \frac{B}{S} - T(r + \ln(1 - \sigma)) \right] \\ l_4 &= \frac{1}{\sigma} \left[\ln \frac{B}{S} - T(r + \ln(1 - \sigma)) \right] \end{aligned}$$

Proof. We calculate the call with upper bound B as in (26). The conclusion is immediate. $\square$

6 The greeks

In finance, the Greeks measure the sensitivity of the price of derivatives such as options to a small change in underlying parameters of the formula [15]. This section provides simple derivations of Greek letters for European call and put options within the mean-bound model framework.

To repeat, European options on a stock S in mean bound model can be expressed as

$$\begin{aligned} C &= \frac{S\Gamma(T, l_1)}{\Gamma(T)} - \frac{K\Gamma(T, l_2)e^{-rT}}{\Gamma(T)} \\ P &= \frac{e^{-rT} K\gamma(T, l_2)}{\Gamma(T)} - \frac{S\gamma(T, l_1)}{\Gamma(T)} \end{aligned}$$

where

$$\begin{aligned} l_1 &= \left(\frac{1}{\sigma} - 1\right) \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma)) \right] \\ l_2 &= \frac{1}{\sigma} \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma)) \right] \end{aligned}$$

We define differential operator

$$\partial_{i,j}\Gamma(t,x) = \frac{\partial^{i+j}\Gamma(t,x)}{\partial t^i \partial x^j}$$

Lemma 6.1. *For l_1 and l_2 it holds*

$$\begin{aligned} \frac{\partial l_1}{\partial S} &= (1-\sigma) \frac{\partial l_2}{\partial S} \\ \frac{\partial l_1}{\partial T} &= (1-\sigma) \frac{\partial l_2}{\partial T} \\ \frac{\partial l_1}{\partial r} &= (1-\sigma) \frac{\partial l_2}{\partial r} \\ \frac{\partial l_1}{\partial K} &= (1-\sigma) \frac{\partial l_2}{\partial K} \\ \frac{\partial l_1}{\partial \sigma} &= (1-\sigma) \frac{\partial l_2}{\partial \sigma} - l_2 \\ \frac{\partial l_1}{\partial \sigma} &= \frac{\ln \frac{S}{K} + T(r + \ln(1-\sigma) + \sigma - \sigma^2)}{\sigma^2} \end{aligned}$$

Proof. From the given relationship

$$l_1 = l_2(1-\sigma)$$

and notice

$$\frac{\partial r}{\partial \sigma} = -\frac{\sigma}{1-\sigma}$$

These equations are immediate. □

Lemma 6.2. *Two relations hold for l_1 and l_2*

$$\begin{aligned} (1-\sigma)S\partial_{0,1}\Gamma(T, l_1) &= Ke^{-rT}\partial_{0,1}\Gamma(T, l_2) \\ (1-\sigma)S\partial_{0,1}\gamma(T, l_1) &= Ke^{-rT}\partial_{0,1}\gamma(T, l_2) \end{aligned}$$

Proof. We calculate

$$\frac{\partial_{0,1}\Gamma(T, l_1)}{\partial_{0,1}\Gamma(T, l_2)} = \frac{l_1^{T-1}e^{-l_1}}{l_2^{T-1}e^{-l_2}} = (1-\sigma)^{-1} \frac{Ke^{-rT}}{S}$$

and

$$\frac{\partial_{0,1}\gamma(T, l_1)}{\partial_{0,1}\gamma(T, l_2)} = \frac{l_1^{T-1}e^{-l_1}}{l_2^{T-1}e^{-l_2}} = (1-\sigma)^{-1} \frac{Ke^{-rT}}{S}$$

The lemma is proved. □

Lemma 6.3. *Two equations hold*

$$\frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial x} \right) - \frac{Ke^{-rT}\partial_{0,1}\Gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial x} \right) = 0$$

$$\frac{S\partial_{0,1}\gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial x} \right) - \frac{Ke^{-rT}\partial_{0,1}\gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial x} \right) = 0$$

where $x = S, r, T, K$.

Proof. We calculate

$$\begin{aligned} & \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial x} \right) - \frac{Ke^{-rT}\partial_{0,1}\Gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial x} \right) \\ &= \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial x} \right) (1 - \sigma) - (1 - \sigma) \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial x} \right) \\ &= 0 \end{aligned}$$

The second equation is proved analogously. $\square$

Lemma 6.4. *For σ*

$$\begin{aligned} \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial \sigma} \right) - \frac{Ke^{-rT}\partial_{0,1}\Gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial \sigma} \right) &= \frac{1}{1 - \sigma} \frac{l_1^T e^{-l_1}}{\Gamma(T)} \\ \frac{S\partial_{0,1}\gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial \sigma} \right) - \frac{Ke^{-rT}\partial_{0,1}\gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial \sigma} \right) &= -\frac{1}{1 - \sigma} \frac{l_1^T e^{-l_1}}{\Gamma(T)} \end{aligned}$$

Proof. We calculate

$$\begin{aligned} & \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial x} \right) - \frac{Ke^{-rT}\partial_{0,1}\Gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial x} \right) \\ &= \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left[\left(\frac{\partial l_2}{\partial x} \right) (1 - \sigma) - l_2 \right] - (1 - \sigma) \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial x} \right) \\ &= \frac{1}{1 - \sigma} \frac{l_1^T e^{-l_1}}{\Gamma(T)} \end{aligned}$$

The second equation is proved analogously. $\square$

6.1 First order greeks

Delta

Delta Δ measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. The delta of european options can be expressed

$$\begin{aligned} \Delta_C &= \frac{\Gamma(T, l_1)}{\Gamma(T)} \\ \Delta_P &= -\frac{\gamma(T, l_1)}{\Gamma(T)} \end{aligned}$$

Proof.

$$\begin{aligned}\Delta_C &= \frac{\partial C}{\partial S} = \frac{\Gamma(T, l_1)}{\Gamma(T)} + \frac{S \partial_{0,1} \Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial S} \right) - \frac{K e^{-rT} \partial_{0,1} \Gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial S} \right) \\ &= \frac{\Gamma(T, l_1)}{\Gamma(T)}\end{aligned}$$

and

$$\begin{aligned}\Delta_P &= \frac{\partial P}{\partial S} = -\frac{\gamma(T, l_1)}{\Gamma(T)} - \frac{S \partial_{0,1} \gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial S} \right) + \frac{K e^{-rT} \partial_{0,1} \gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial S} \right) \\ &= -\frac{\gamma(T, l_1)}{\Gamma(T)}\end{aligned}$$

□

There is a relationship between call and put delta

$$\Delta_C - \Delta_P = 1$$

Vega: The volatility factor

Vega measures how sensitive the volatility σ is in an option. It is determined by finding the derivative of the option with respect to volatility.

$$\nu = \frac{1}{1 - \sigma} \frac{l_1^T e^{-l_1}}{\Gamma(T)}$$

Proof.

$$\begin{aligned}\nu &= \frac{\partial C}{\partial \sigma} = \frac{S \partial_{0,1} \Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial \sigma} \right) - \frac{K e^{-rT} \partial_{0,1} \Gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial \sigma} \right) \\ &= \frac{1}{1 - \sigma} \frac{l_1^T e^{-l_1}}{\Gamma(T)}\end{aligned}$$

□

Rho: The interest rate factor

Rho represents the rate of change between an option portfolio's value. It is the derivative of the option value with respect to the risk free interest rate .

$$\begin{aligned}\rho_C &= \frac{T K e^{-rT} \Gamma(T, l_2)}{\Gamma(T)} \\ \rho_P &= -\frac{T K e^{-rT} \gamma(T, l_2)}{\Gamma(T)}\end{aligned}$$

Proof.

$$\begin{aligned}\rho_C &= \frac{\partial C}{\partial r} = \frac{TKe^{-rT}\Gamma(T, l_2)}{\Gamma(T)} + \frac{S\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial r} \right) - \frac{Ke^{-rT}\partial_{0,1}\Gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial r} \right) \\ &= \frac{TKe^{-rT}\Gamma(T, l_2)}{\Gamma(T)}\end{aligned}$$

and

$$\begin{aligned}\rho_P &= \frac{\partial P}{\partial r} = \frac{-TKe^{-rT}\gamma(T, l_2)}{\Gamma(T)} - \frac{S\partial_{0,1}\gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial r} \right) + \frac{Ke^{-rT}\partial_{0,1}\gamma(T, l_2)}{\Gamma(T)} \left(\frac{\partial l_2}{\partial r} \right) \\ &= -\frac{TKe^{-rT}\gamma(T, l_2)}{\Gamma(T)}\end{aligned}$$

□

Theta: The time decay factor

The Theta measures the sensitivity of the value of the derivative to time

$$\Theta = -\frac{\partial C}{\partial T}$$

The derivative of the incomplete gamma function $\Gamma(s, x)$ with respect to the s is related to Meijer G-function, this implies theta has no simple expression. We can only calculate it with numerical methods.

6.2 Second order greeks

Gamma: The convexity factor

The Gamma of a derivative is the sensitivity of Δ with respect to S . In order to distinguish it from Gamma functions, we denote Γ_V .

$$\Gamma_V = \frac{l_1^{T-1}e^{-l_1}}{S\Gamma(T)} \left(\frac{1}{\sigma} - 1 \right)$$

Proof.

$$\Gamma_V = \frac{\partial^2 V}{\partial S^2} = \frac{\partial_{0,1}\Gamma(T, l_1)}{\Gamma(T)} \left(\frac{\partial l_1}{\partial S} \right) = \frac{l_1^{T-1}e^{-l_1}}{S\Gamma(T)} \left(\frac{1}{\sigma} - 1 \right)$$

□

Vanna

Vanna is a second order derivative of the option value with respect underlying spot price and to volatility. It measures the sensitivity of the option delta with respect to change in volatility; or alternatively, the partial of vega with respect to the underlying instrument's price.

$$\text{Vanna} = - \left(\frac{l_1^{T-1} e^{-l_1}}{\Gamma(T)} \right) \left(\frac{\ln \frac{S}{K} + T[r + \ln(1 - \sigma) + \sigma - \sigma^2]}{\sigma^2} \right)$$

Proof.

$$\begin{aligned} \text{Vanna} &= \frac{\partial^2 V}{\partial S \partial \sigma} = - \frac{l_1^{T-1} e^{-l_1}}{\Gamma(T)} \frac{\partial l_1}{\partial \sigma} \\ &= - \left(\frac{l_1^{T-1} e^{-l_1}}{\Gamma(T)} \right) \left(\frac{\ln \frac{S}{K} + T[r + \ln(1 - \sigma) + \sigma - \sigma^2]}{\sigma^2} \right) \end{aligned}$$

□

Vera

Vera measures the rate of change in the vega with respect to the passage of time and is the second derivative of the value function; once to volatility and once to time.

$$\text{Vera} = T^2 K e^{-rT} \left(\frac{l_2^{T-1} e^{-l_2}}{\Gamma(T)} \right) \left(\frac{\ln \frac{S}{K} + T[r + \ln(1 - \sigma) + \sigma - \sigma^2]}{\sigma^2(1 - \sigma)} \right)$$

Proof.

$$\begin{aligned} \text{Vera} &= \frac{\partial^2 V}{\partial \sigma \partial r} = T K e^{-rT} \frac{l_2^{T-1} e^{-l_2}}{\Gamma(T)} \frac{\partial l_2}{\partial \sigma} \\ &= T^2 K e^{-rT} \left(\frac{l_2^{T-1} e^{-l_2}}{\Gamma(T)} \right) \left(\frac{\ln \frac{S}{K} + T[r + \ln(1 - \sigma) + \sigma - \sigma^2]}{\sigma^2(1 - \sigma)} \right) \end{aligned}$$

□

7 Volatility analysis

In the famous Black Scholes model, the volatility of the underlying asset is assumed to be a constant, which should be independent of its strike and expiration. In practice, however, not only is the implied volatility surface not flat but it actually varies. This suggests that the implied volatility is in general not a constant but is dependent on the strike K under the Black Scholes models[16, 19, 24, 29]. To address this issue stochastic

volatility models are proposed. In stochastic volatility models variance of a stochastic process itself is random distributed.

In this section we assume our mean-bound model is correct, i.e., volatility in mean-bound model, denoted by σ_{mb} , is equivalent to correct volatility $\sigma_{\text{mb}} = \sigma_{\text{correct}}$ and σ_{mb} is a constant overall, and we denote Black Scholes volatility with σ_{bs} . Then we derive a relation between correct volatility and Black Scholes implied volatility

$$Ce^{-\frac{1}{2}\sigma_{\text{bs}}^2 t} = (1 - \sigma_{\text{correct}})e^{\sigma_{\text{correct}}} \quad (28)$$

and show that volatility is fake random and fake stochastic.

Historical volatility is the realized volatility of a financial instrument over a given time period and expressed in terms of annualized standard deviation as a percentage of the stock price. In contrast, implied volatility is often interpreted as the markets opinion for the future volatility of a stock, based on sound reasoning and accurate data [10].

In the relationship between historical volatility and implied volatility, the historical volatility serves as the baseline. When the two measures represent similar values, options premiums are generally considered to be fairly valued. However, at times historical volatility and implied volatility can significantly deviate from each other. When implied volatility is significantly higher (lower) than the average historical levels, options premiums are assumed to be overvalued (undervalued) [10, 29].

7.1 Volatility and time-to-maturity

In this section we will show that Black Scholes implied volatility is fake stochastic and explain why the assumption of Black Scholes models is not justified by empirical data while it can still provide right prices.

We present an equation in the following which describes the relation between the correct volatility σ_{correct} and Black Scholes implied volatility σ_{bs} .

Proposition 7.1. *If equation holds*

$$E[e^{\mu_1 t + \sigma_1 W_t}] = E[e^{\mu_2 t + \sigma_2 G_t}]$$

then

$$Ce^{-\frac{1}{2}\sigma_1^2 t} = e^{\sigma_2}(1 - \sigma_2) \quad (29)$$

where C is a positive constant.

Proof. Obviously

$$E[e^{\mu_1 t + \sigma_1 W_t}] = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t}$$

and

$$E[e^{\mu_2 t + \sigma_2 G_t}] = e^{\mu_2 t - \sigma_2 t} E[e^{\sigma_2(G_t + t)}] = e^{\mu_2 t - \sigma_2 t} (1 - \sigma_2)^{-t}$$

Then we come to the conclusion with $C = e^{\mu_2 - \mu_1}$. □

Now we derive a relation between correct volatility and Black Scholes implied volatility

$$Ce^{-\frac{1}{2}\sigma_{\text{bs}}^2 t} = e^{\sigma_{\text{correct}}(1 - \sigma_{\text{correct}})} \quad (30)$$

Without loss of generality let $C = 1$ and we plot these relations over the interval $[0, 0.1]$ in figure 3. The blue line represents the relation $\sigma_{\text{mb}} = \sigma_{\text{correct}}$ and the colorful (red, orange, violet, green) curves represent implied volatility in Black Scholes model with different time-to-maturity.

Over the interval $[0, 0.01]$, colorful curves are irregularly far from the blue straight line and this implies that Black Scholes implied volatilities do not provide useful information about the correct volatility.

Over the interval $[0.01, 0.1]$, colorful curves are in general lower than the blue line but almost parallel to the blue straight line over the interval $[0.02, 0.1]$. This implies that Black Scholes model underestimates the volatility but we can adjust Black Scholes implied volatility to the correct value by adding a term c such that $\sigma_{\text{bs}}/\sigma_{\text{correct}} \approx 1 + c$. For this reason, stochastic volatility models seem to be more effective as $\sigma_{\text{bs}} \approx (1 + c)\sigma_{\text{correct}}$ over the interval $[0.02, 0.1]$.

However, if we plot this relation over the interval $[0.1, 0.4]$ in figure 4, colorful curves are no more parallel to the red line but increasing on the right side. For longer expiration (for example $t = 15$), it appears parallel to the blue line. For this expiration, Black Scholes model provides a fairly good approximation of financial risk for longer time-to-maturity.

Implied volatilities (colorful curves) appear to be random on some interval, but they are completely determined by equation (30). Simply, they are fake random.

This explains why Black Scholes implied volatility is far from being flat but we can still obtain right prices from Black Scholes model.

7.2 Volatility and strike

In the last two sections we have proved that options pricing formulas in mean bound model can be expressed

$$\begin{aligned} C_{\text{mb}} &= \frac{S\Gamma(T, l_1)}{\Gamma(T)} - \frac{K\Gamma(T, l_2)e^{-rT}}{\Gamma(T)} \\ P_{\text{mb}} &= \frac{e^{-rT}K\gamma(T, l_2)}{\Gamma(T)} - \frac{S\gamma(T, l_1)}{\Gamma(T)} \end{aligned}$$

where

$$\begin{aligned} l_1 &= \left(\frac{1}{\sigma_{\text{mb}}} - 1\right) \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma_{\text{mb}}))\right] \\ l_2 &= \frac{1}{\sigma_{\text{mb}}} \left[\ln \frac{K}{S} - T(r + \ln(1 - \sigma_{\text{mb}}))\right] \end{aligned}$$

Figure 3: Implied volatilities

Figure 4: Implied volatilities

and we recall options pricing formulas in Black Scholes model

$$\begin{aligned} C_{\text{bs}} &= SN(d_1) - Ke^{-rT}N(d_2) \\ P_{\text{bs}} &= Ke^{-rT}N(-d_2) - SN(-d_1) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma_{\text{bs}}^2)T}{\sigma_{\text{bs}}\sqrt{T}} \\ d_2 &= \frac{\ln \frac{S}{K} + (r - \frac{1}{2}\sigma_{\text{bs}}^2)T}{\sigma_{\text{bs}}\sqrt{T}} \end{aligned}$$

Proposition 7.2. *It holds*

$$\frac{\partial \sigma_{\text{bs}}}{\partial K} = \frac{e^{-rT}}{S\phi(d_1)\sqrt{T}}[\gamma(T, l_2)/\Gamma(T) - N(-d_2)]$$

Proof. Here we assume that mean bound model is correct and σ_{mb} is a constant overall, so model prices equal to market prices

$$\begin{aligned} C_{\text{market}} &= C_{\text{mb}} \\ P_{\text{market}} &= P_{\text{mb}} \end{aligned}$$

The implied volatility is the value which, when input in the Black Scholes will return a theoretical value equal to the current market prices of the option

$$\begin{aligned} C_{\text{market}} &= C_{\text{bs}} \\ P_{\text{market}} &= P_{\text{bs}} \end{aligned}$$

Then we define two functions R_1, R_2

$$\begin{aligned} R_1 &= C_{\text{bs}} - C_{\text{mb}} = 0 \\ R_2 &= P_{\text{bs}} - P_{\text{mb}} = 0 \end{aligned}$$

and apply implicit differentiation approach (see appendix B).

For call in the money

$$\frac{\partial \sigma_{\text{bs}}}{\partial K} = -\frac{\frac{\partial R_1}{\partial K}}{\frac{\partial R_1}{\partial \sigma_{\text{bs}}}} = \frac{-e^{-rT}}{S\phi(d_1)\sqrt{T}}[\Gamma(T, l_2)/\Gamma(T) - N(d_2)]$$

and for put in the money

$$\frac{\partial \sigma_{\text{bs}}}{\partial K} = -\frac{\frac{\partial R_2}{\partial K}}{\frac{\partial R_2}{\partial \sigma_{\text{bs}}}} = \frac{e^{-rT}}{S\phi(d_1)\sqrt{T}}[\gamma(T, l_2)/\Gamma(T) - N(-d_2)]$$

Finally we obtain a unified formula

$$\frac{\partial \sigma_{\text{bs}}}{\partial K} = \frac{e^{-rT}}{S\phi(d_1)\sqrt{T}}[\gamma(T, l_2)/\Gamma(T) - N(-d_2)]$$

□

Figure 5: Normal distribution and Gamma distribution

Black Scholes implied volatility σ_{bs} is not constant but varies with strike K and time-to-maturity T . As we see in the figure 5, normal distribution is unbounded from below while Gamma distribution is bounded from below, so for some small K , the difference between Gamma distribution and normal distribution is negative

$$\gamma(T, l_2)/\Gamma(T) - N(-d_2) < 0$$

and the partial derivative of σ_{bs} with respect to strike K is negative and this implies that σ_{bs} is a monotone decreasing function of K .

Gamma distribution has a heavier right tail than normal distribution and the probability difference is positive for some large enough K ,

$$\gamma(T, l_2)/\Gamma(T) - N(-d_2) > 0$$

The partial derivative of σ_{bs} with respect to strike K is therefore positive and σ_{bs} is a monotone increasing function of K . The implied volatility against the strike price exhibits a smile shape.

8 Conclusion

In this paper we introduce a class of lower bound random variables and present mean-bound financial model. We demonstrate that single period log returns are Pareto distributed and Asset prices are log Gamma distributed. Based on this model we derive options pricing formulas and study sensitivity of derivative prices. Finally, we reveal the relation between correct volatility and Black Scholes volatility and provide a mathematical explanation of the famous volatility smile.

A Black Scholes model

The Black Scholes model is a mathematical model of a financial market containing derivative investment instruments. The model has had a huge influence on the way that traders price and hedge derivatives [8, 19, 28].

There are six factors affecting the price of a stock option:

1. The current stock price S_0
2. The strike price K
3. The time to expiration T
4. The volatility of the stock price σ
5. The risk-free interest rate r
6. The dividends that are expected to be paid.

It is assumed

- no arbitrage
- no dividends
- no transaction costs

In this model, stock price process S_t is assumed to be Geometric Brownian motion and have a log normal distribution

$$S_t = S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \quad (31)$$

The BlackScholes equation is a partial differential equation, which describes the price of the option over time

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0 \quad (32)$$

Solving this equation with boundary conditions

$$C(T, S_T) = (S_T - K)^+$$

$$P(T, S_T) = (K - S_T)^+$$

yields solutions

$$\begin{aligned} C(T, S_T) &= S_0 N(d_1) - K e^{-rT} N(d_2) \\ P(T, S_T) &= K e^{-rT} N(-d_2) - S_0 N(-d_1) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln \frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \end{aligned}$$

Lemma A.1. *For Black Scholes model it holds*

$$S_0 dN'(d_1) \left(\frac{dd_1}{dK} \right) - K e^{-rT} dN'(d_2) \left(\frac{\partial d_2}{\partial K} \right) = S_0 dN'(d_1) \left(\frac{dd_1}{dK} - \frac{dd_2}{dK} \right)$$

Proof. We can verify

$$S_0 dN'(d_1) = K e^{-rT} dN'(d_2)$$

□

B Implicit differentiation

In mathematical analysis, implicit differentiation is an approach that makes use of the chain rule to differentiate functions which are implicitly defined.

For differentiating an implicit function $y(x)$, defined by an equation $R(x, y) = 0$, which is not generally difficult to obtain explicit form for y , one can differentiate $R(x, y)$ with respect to x and y and then solve a linear equation in dy/dx for getting explicitly the derivative in terms of x and y

$$y'(x) = -\frac{\partial_x R}{\partial_y R} \tag{33}$$

C Probability distributions

Pareto distributions

The Pareto distribution, named after the Italian civil engineer, economist, and sociologist Vilfredo Pareto, is a power law probability distribution that is used in description of social, scientific, geophysical, actuarial, and many other types of observable phenomena.

Definition C.1. A random variable has Pareto distribution with parameter α if

$$P[X < x] = 1 - \left(\frac{x_{\min}}{x} \right)^\alpha$$

Gamma functions

Gamma functions are very important in stochastic analysis, economy and engineering and are defined via

$$\begin{aligned}\Gamma(k, x) &= \int_x^\infty t^{k-1} e^{-t} dt \\ \gamma(k, x) &= \int_0^x t^{k-1} e^{-t} dt \\ \Gamma(k) &= \int_0^\infty t^{k-1} e^{-t} dt\end{aligned}$$

Gamma distributions

In probability theory and statistics, the gamma distribution is a two-parameter family of continuous probability distributions. The common exponential distribution and Erlang distribution are special cases of the gamma distribution.

Definition C.2. A random variable X is Gamma distributed if

$$P[X < x] = \frac{\gamma(k, \lambda x)}{\Gamma(k)}$$

or equivalently

$$P[X > x] = \frac{\Gamma(k, \lambda x)}{\Gamma(k)}$$

with density function

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

If k is a positive integer, then the distribution represents an Erlang distribution; i.e., the sum of k independent exponentially distributed random variables.

Proposition C.1. *The moment generating function of Gamma distributed random variable is*

$$M_X(t) = (1 - t/\lambda)^{-k}$$

Infinite divisible distribution

In probability theory, a probability distribution is called to be infinitely divisible if it can be expressed as the convoluted probability distribution of an arbitrary number of independent and identical distributions.

Definition C.3. A distribution is infinitely divisible if for any natural number n , there is a distribution denoted by F_n such that

$$F = F_n^{*n}$$

There is a one-one correspondence between infinitely divisible distribution and Levy process. For each infinitely divisible distribution we can construct a Levy process. On the other hand, each component in Levy process is infinitely divisible.

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