Quantification d'incertitude en mécaniques des fluides : l'apport de la dynamique sous incertitude de position et de l'advection stochastic par transport de Lie

Baylor Fox-Kemper, Darryl D Holm, Wei Pan, Valentin Resseguier

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Motivations

- Rigorously identified sudgrid dynamics effects
- Injecting likely small-scale dynamics
- Studying bifurcations and attractors

  → Climate projections
- Quantification of modeling errors

  → Ensemble forecasts and data assimilation

Contents

- Stochastic transport
- Stochastic Navier-Stokes : SALT vs LU
- Unresolved velocity parametrisation
- Unresolved velocity non-stationary heterogeneity

Part I
Stochastic transport

Usual random CFD

• Random parameters, boundary conditions, forcing	Other (complementary) issues
• Random initial conditions	Underdispersive + need large ensemble
• Arbitrary Gaussian forcing	Adding energy + wrong phase
• Averaging, homogenization	Assumptions and fitting issues
• Mori-Zwanzig, MSM	Increase dimension
• SPPT, SKEBS + inflation	Not physically-based, ad hoc tuning needed, increase dimension, stability issues

Usual random CFD

- Random parameters, boundary conditions, forcing → Other (complementary) issues
- Random initial conditions → Und[illegible]ve [illegible] large ensemble
- Arbitrary Gaussian forcing → Adding energy + wrong phase
- Averaging, homogenization → Assumptions and fitting issues
- Mori-Zwanzig, MSM → [illegible]se dimension
- SPPT, SKEBS + inflation → Not physically [illegible] ad hoc tuning needed, increase dimension, stability issues

SALT & LU :
Adding random velocity

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

SALT & LU :

Adding random velocity

Resolved
large scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma} \dot{\boldsymbol{B}}$$

SALT & LU :
Adding random velocity

Resolved
large scales

White-in-time
small scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

SALT & LU :

Adding random velocity

Resolved large scales

White-in-time small scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

Large scales:

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$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} \boldsymbol{d} \boldsymbol{B}\, (\boldsymbol{\sigma} \boldsymbol{d} \boldsymbol{B})^T\}}{dt}$

SALT & LU :

Adding random velocity

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{w}^*(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t) \circ \mathrm{d}\boldsymbol{B}_t$$

$$= \boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma dB}\,(\boldsymbol{\sigma dB})^T\}}{dt}$

SALT & LU :

Adding random velocity

Stratonovich

$$\mathrm{d}\boldsymbol{X}_t = \boxed{\boldsymbol{w}^*(\boldsymbol{X}_t,t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t,t)\circ\mathrm{d}\boldsymbol{B}_t}$$

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$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{w}^*(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t) \circ \mathrm{d}\boldsymbol{B}_t$$

$$= \boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t$$

Ito

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma} \dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \dfrac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

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Ito

References :

Mikulevicius and Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al 2018 a, b

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Stratonovich

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Ito

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SALT

Holm, 2015
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Advection of tracer $\boldsymbol{\Theta}$

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Advection of tracer $\boldsymbol{\Theta}$

Ito-Wentzell formula (Kunita 1990)

$$\frac{D\Theta}{Dt} = 0$$

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Small scales:

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Variance tensor:

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Advection of tracer Θ

Ito-Wentzell formula (Kunita 1990)

Advection of tracer Θ

Large scales:

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Small scales:

$\boldsymbol{\sigma} \dot{B}$

Variance tensor:

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Stratonovich notations:

Advection of tracer $\boldsymbol{\Theta}$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

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$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Ito-Wentzell formula (Kunita 1990)

Stratonovich notations:

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Advection of tracer $\boldsymbol{\Theta}$

Ito-Wentzell formula (Kunita 1990)

Large scales:

$\boldsymbol{w}$

Small scales:

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Ito notations:

$$\partial_t \Theta + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \Theta + \boldsymbol{\sigma}\dot{B} \cdot \boldsymbol{\nabla} \Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \Theta \right)$$

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Ito notations:

Advection

Diffusion

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Stratonovich drift : « Drift correction » in Ito notations

Advection of tracer Θ

Ito-Wentzell formula (Kunita 1990)

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Small scales:

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Ito notations:

Multiplicative random forcing

Advection

Diffusion

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Advection of tracer $\boldsymbol{\Theta}$

Large scales:

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Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

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Ito-Wentzell formula (Kunita 1990)

Multiplicative random forcing

Ito notations:

Advection

Diffusion

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Advection of tracer $\boldsymbol{\Theta}$

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Ito-Wentzell formula (Kunita 1990)

Multiplicative random forcing

Balanced energy exchanges

Ito notations:

Advection

Diffusion

$$\partial_t \Theta + \boldsymbol{w}^{\star} \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla}\Theta \right)$$

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Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp\!\!\!\perp \boldsymbol{w}(\bullet, t), \forall t' \leq t$$

$$\partial_t \mathbb{E}(\Theta) + \underbrace{\boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \mathbb{E}(\Theta)}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta) \right)}_{\text{Diffusion}}$$

$$\partial_t Var(\Theta) + \underbrace{\boldsymbol{w}^\star \cdot \boldsymbol{\nabla} Var(\Theta)}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} Var(\Theta) \right)}_{\text{Diffusion}} + \underbrace{(\boldsymbol{\nabla} \mathbb{E}(\Theta))^T \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta)}_{\text{Source term}}$$

Large scales:

$\boldsymbol{w}$

Small scales:

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Variance tensor:

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$$\Theta(\bullet, t') \perp\!\!\!\perp \boldsymbol{w}(\bullet, t), \forall t' \leq t$$

Advection

Diffusion

Source term

$$\partial_t \mathbb{E}(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \mathbb{E}(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta) \right)$$

$$\partial_t Var(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} Var(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} Var(\Theta) \right) + (\boldsymbol{\nabla} \mathbb{E}(\Theta))^T \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta)$$

Large scales:

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Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB \, (\boldsymbol{\sigma} dB)^T\}}{dt}$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp\!\!\!\perp \boldsymbol{w}(\bullet, t), \forall t' \leq t$$

Conversion from deterministic to random energy

$$\partial_t \mathbb{E}(\Theta) + \underbrace{\boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \mathbb{E}(\Theta)}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta) \right)}_{\text{Diffusion}}$$

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Moments of a passive tracer Θ

Large scales:

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Small scales:

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Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB\,(\boldsymbol{\sigma} dB)^T\}}{dt}$

$$\Theta(\bullet, t') \perp\!\!\!\perp \boldsymbol{w}(\bullet, t), \forall t' \leq t$$

Conservation of the energy $\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \Theta^2 = 0$

Conversion from deterministic to random energy

Advection

Diffusion

Source term

$$\partial_t \mathbb{E}(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \mathbb{E}(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta) \right)$$

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Part II

Stochastic Navier-Stokes

SALT vs LU

Comparison

	LU	SALT
Scalar (e.g. SQG)	Identical	
Navier-Stokes	2 differences	
Vorticity	2 differences	
Kinetic energy conservation	✓	✗
Helicity (& 2D enstrophy) conservation	✗	✓

Navier-Stokes LU

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\nabla p$$

Navier-Stokes LU

Pressure

$$\frac{Dw}{Dt} = \boxed{-\frac{1}{\rho}\nabla p}$$

Navier-Stokes LU

Stochastic transport of **<u>Ito</u>** drift

Pressure

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\nabla p$$

Navier-Stokes LU

Stochastic transport of **<u>Ito</u>** drift

Pressure

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\nabla p$$

Conserve (Ito) kinetic energy

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho}\nabla p$$

Navier-Stokes SALT

Pressure

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = \boxed{-\frac{1}{\rho}\nabla p}$$

Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

Pressure

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho}\nabla p$$

Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

Additional term

Pressure

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho}\nabla p$$

Conserve (Stratonovich) helicity

A word about reduced order models LU

- Very fast simulation of very complex system (e.g. for industrial application)
- Physical model (PDE) simplified using observations

Forecast of reduced models LU

- Rigorous and low-cost estimators
- Stabilization of the unstable modes
- Maintain variability of stable modes
- Uncertainty quantification

State of art

- Possible parametrization with eddy viscosity
- Impossible to parametrize with additional dissipation
  Need ad hoc closure like MQG
  (Sapsis and Majda, 2013a,b,c)

Part III

Unresolved velocity parametrisation

Code available online

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma} \dot{B}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB \, (\boldsymbol{\sigma} dB)^T\}}{dt}$$

Parameter-free models for $\boldsymbol{\sigma}$

	Cotter et al. 2018b	Resseguier et al. 2017b
Applicable to :	LU & SALT	LU & SALT
Homogeneous	No	Yes
Stationary	Yes	Not anymore
Self-similar assumption	No	Yes
Data-driven	Yes	No

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Cotter et al. 2018b:

Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^{n} \xi_i(x)\ dW_i(t)$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma \dot{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma dB}\,(\boldsymbol{\sigma dB})^T\}}{dt}$

Cotter et al. 2018b:

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$$\sigma dB_t = \sum_{i=1}^{n} \xi_i(x) \boxed{dW_i(t)}$$

iid Brownian motion

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

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iid Brownian motion

weighted EOFs learned on data

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$$\sigma dB_t = \sum_{i=1}^{n} \boxed{\xi_i(x)}\, \boxed{dW_i(t)}$$

iid Brownian motion

weighted EOFs learned on data

PCA on : $\Delta X_k(x) = X^{HR}_{t_k, t_k+\Delta t}(x) - X^{LR}_{t_k, t_k+\Delta t}(x)$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

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iid Brownian motion

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PCA on : $\Delta X_k(x) = \boxed{X^{HR}_{t_k, t_k+\Delta t}(x)} - \boxed{X^{LR}_{t_k, t_k+\Delta t}(x)}$

Flow from **high**-resolution velocity

Flow from **low**-resolution velocity

Large scales:

$\boldsymbol{w}$

Small scales:

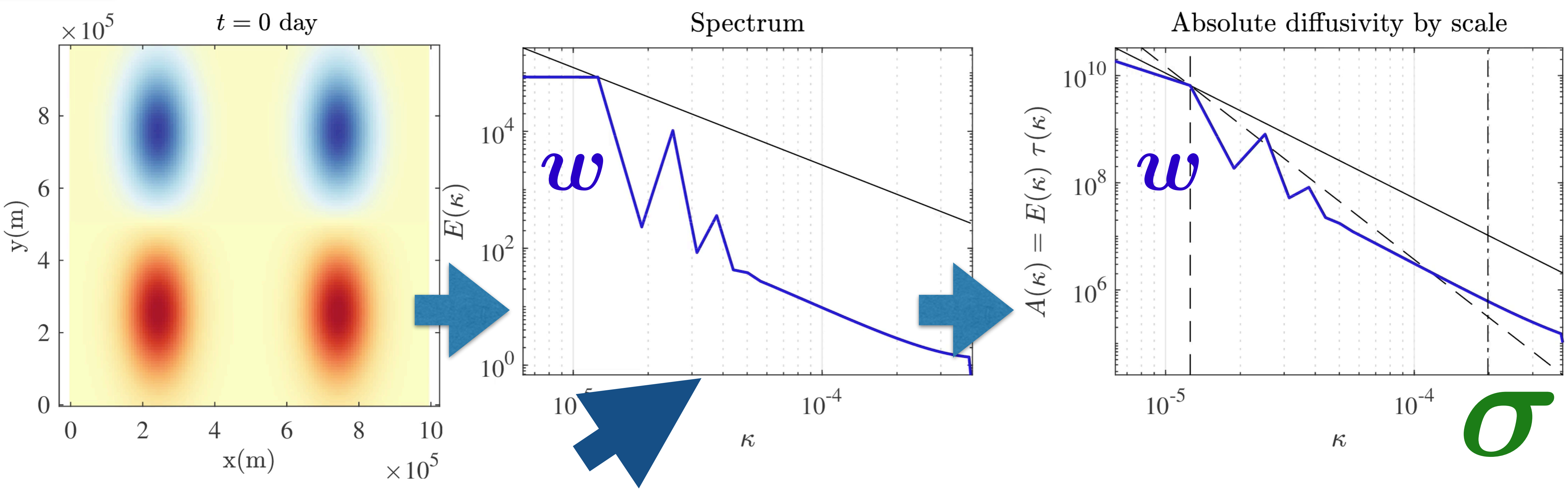
$\boldsymbol{\sigma} \dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

(improvement of) Resseguier et al. 2017b:

Self-similar model



Kinetic energy Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \, \|\widehat{v}(k)\|^2$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

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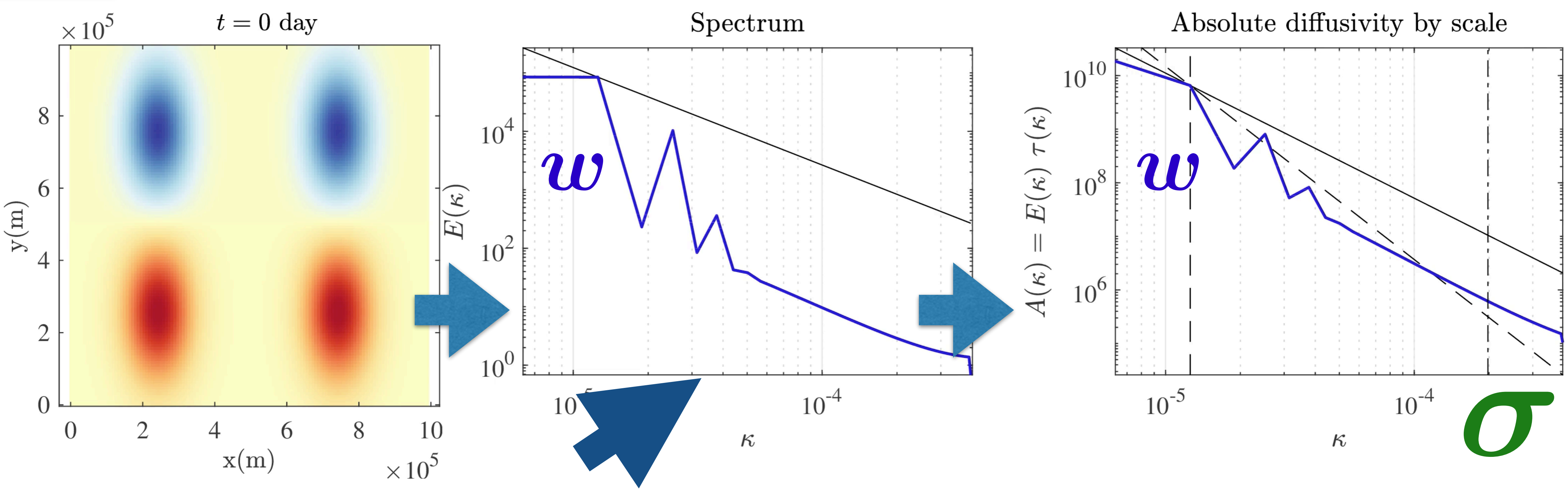
(improvement of) Resseguier et al. 2017b:

Self-similar model



Kinetic energy Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \left\| \widehat{v}(k) \right\|^2$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma} \dot{\boldsymbol{B}}$

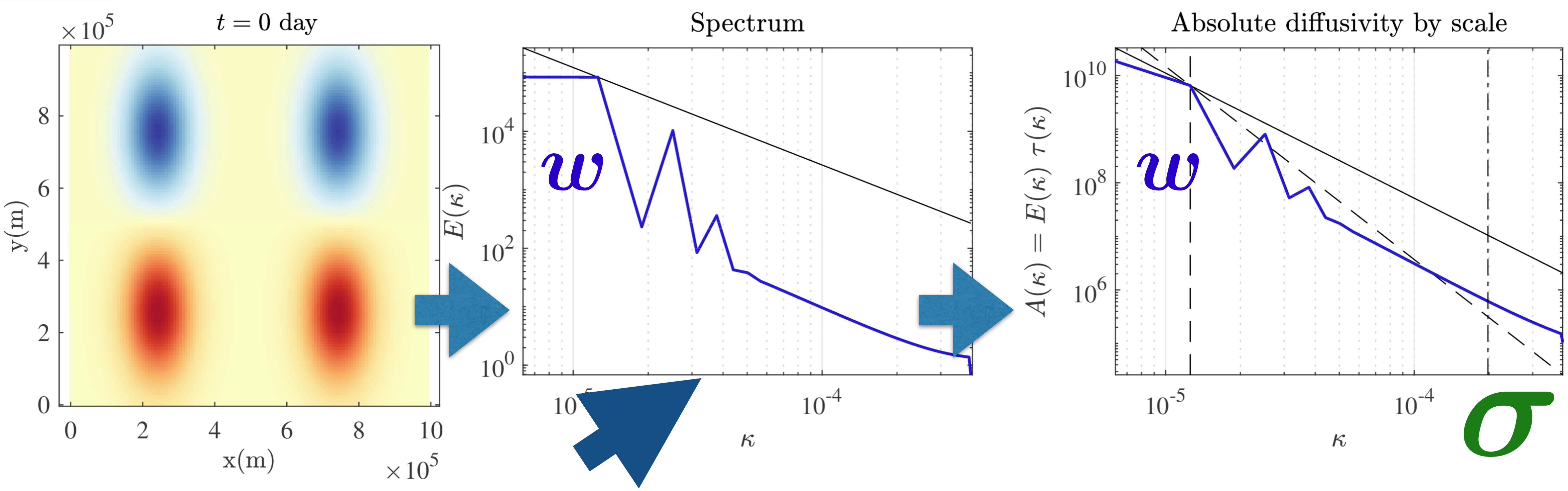
Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

(improvement of) Resseguier et al. 2017b:
Self-similar model

Absolute Diffusivity Spectral Density

$$A(\kappa) = E(\kappa)\tau(\kappa) = \kappa^{-3/2} E^{1/2}(\kappa)$$



Kinetic energy Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \left\| \widehat{v}(k) \right\|^2$$

SQG

$$\frac{Db}{Dt} = -\alpha_{HV}\Delta^4 b \quad \text{Hyper-viscosity}$$
$$-\frac{1}{\tau}(b-F) \quad \text{Forcing}$$

$$u = \left(\text{cst}.\nabla^{\perp}\Delta^{-1/2}\right) b$$

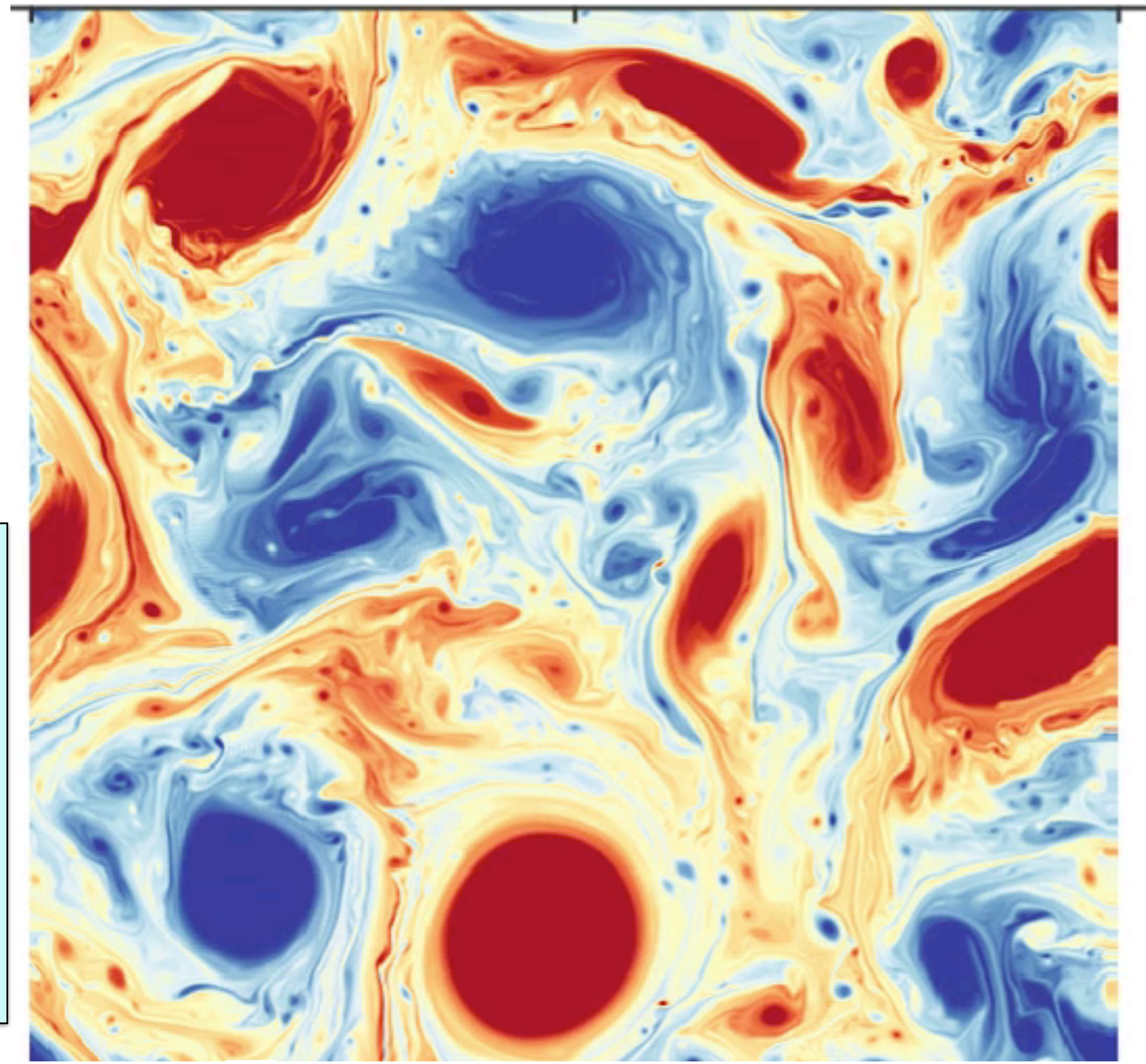
Reference flow:

deterministic

SQG

512 x 512

$t = 100$ day



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$$u = \left(\text{cst}.\nabla^{\perp}\Delta^{-1/2}\right) b$$

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512 x 512

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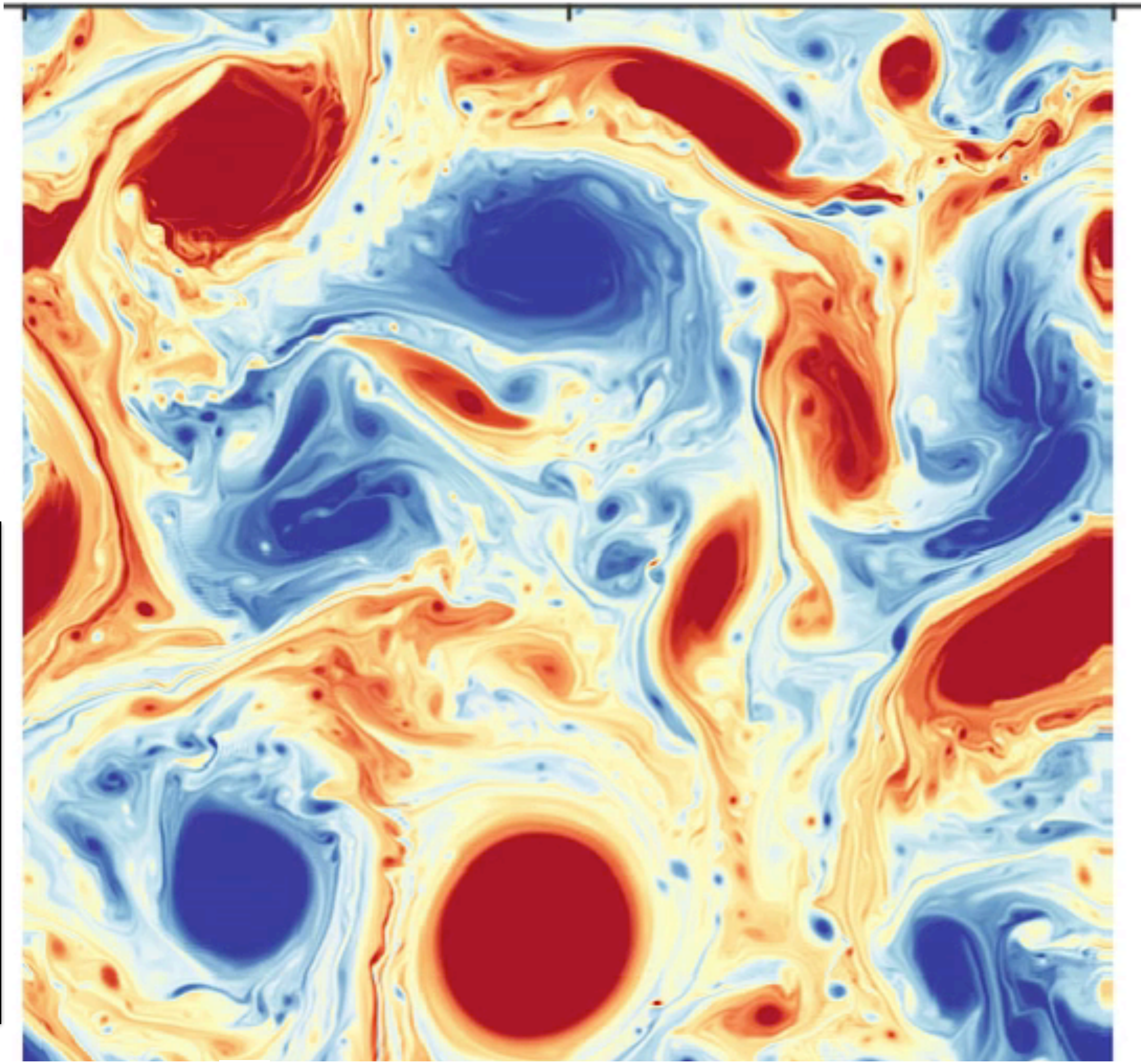
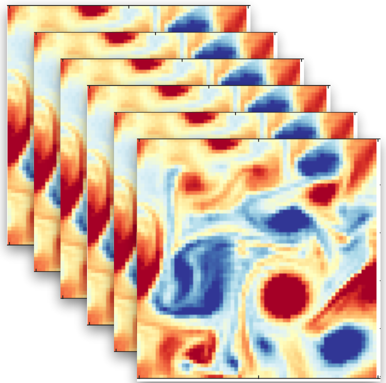
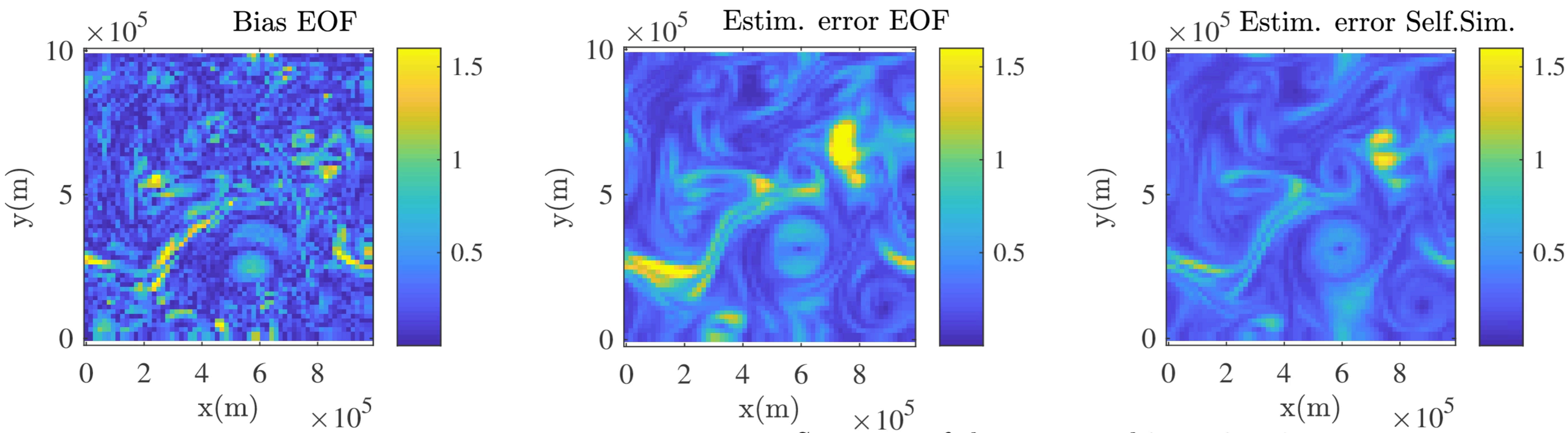
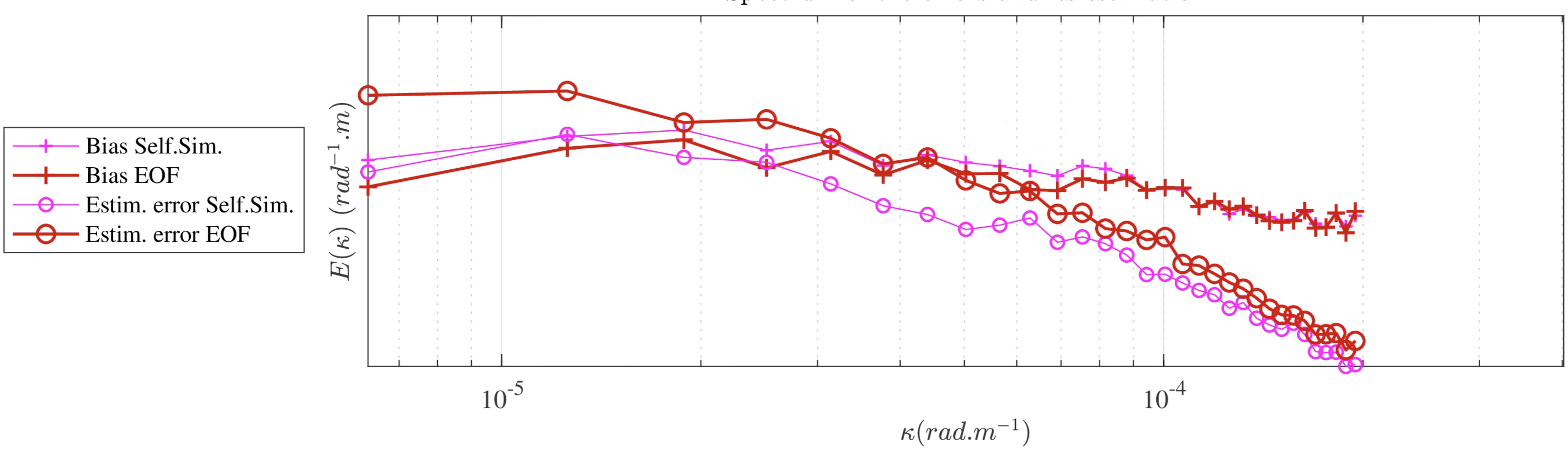
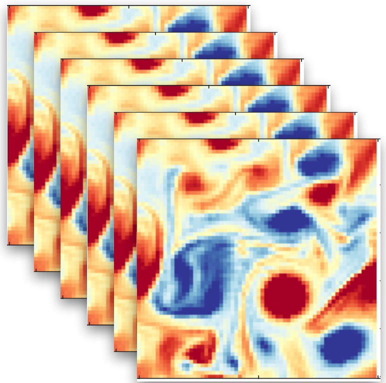




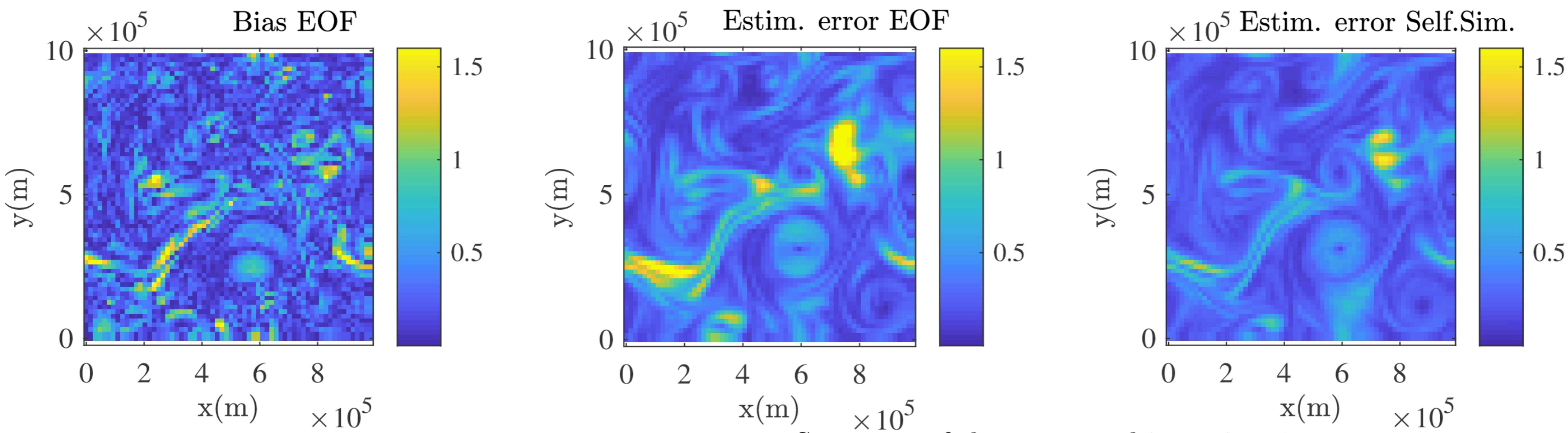
Ensemble : uncertainty quantification

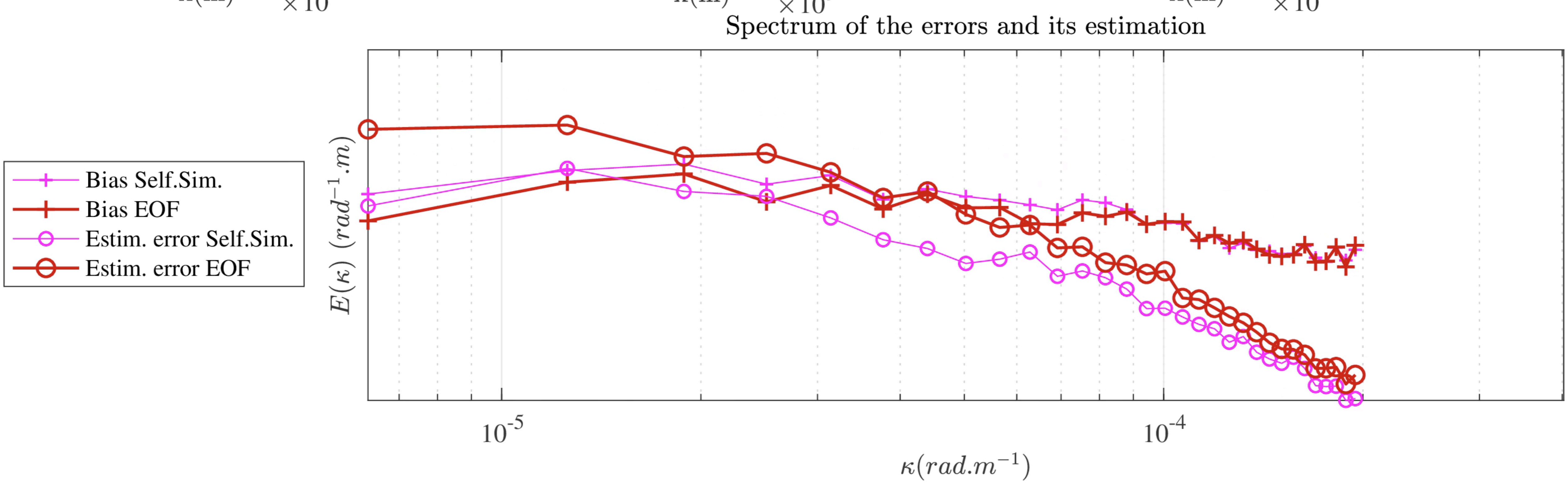


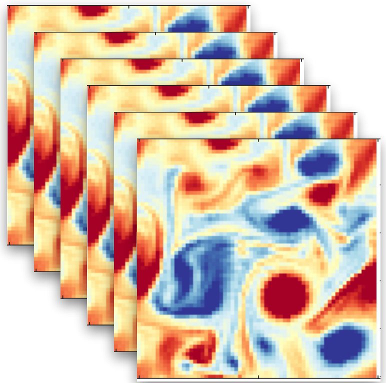




Ensemble : uncertainty quantification



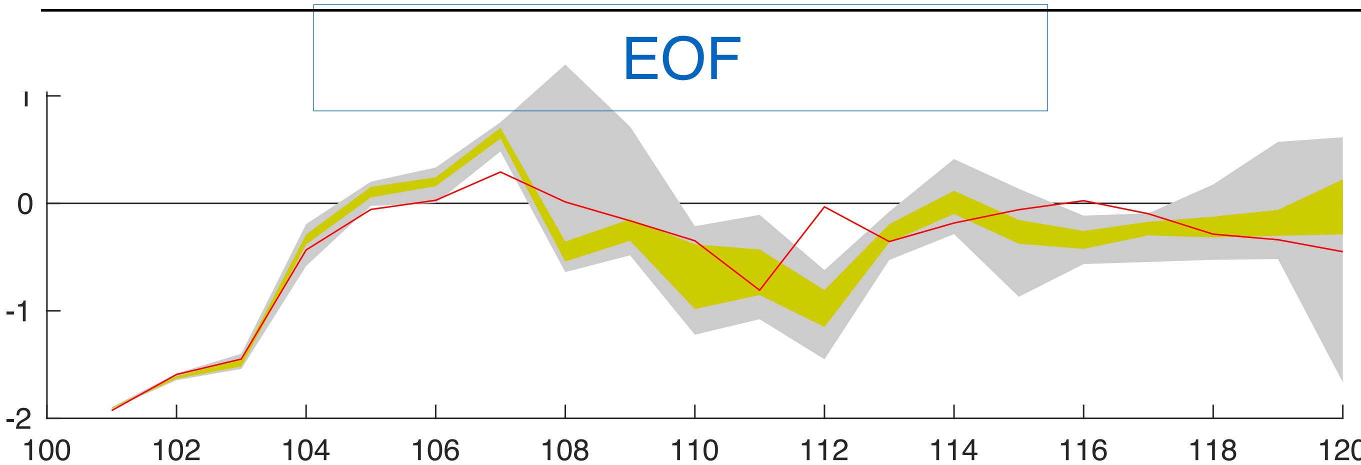


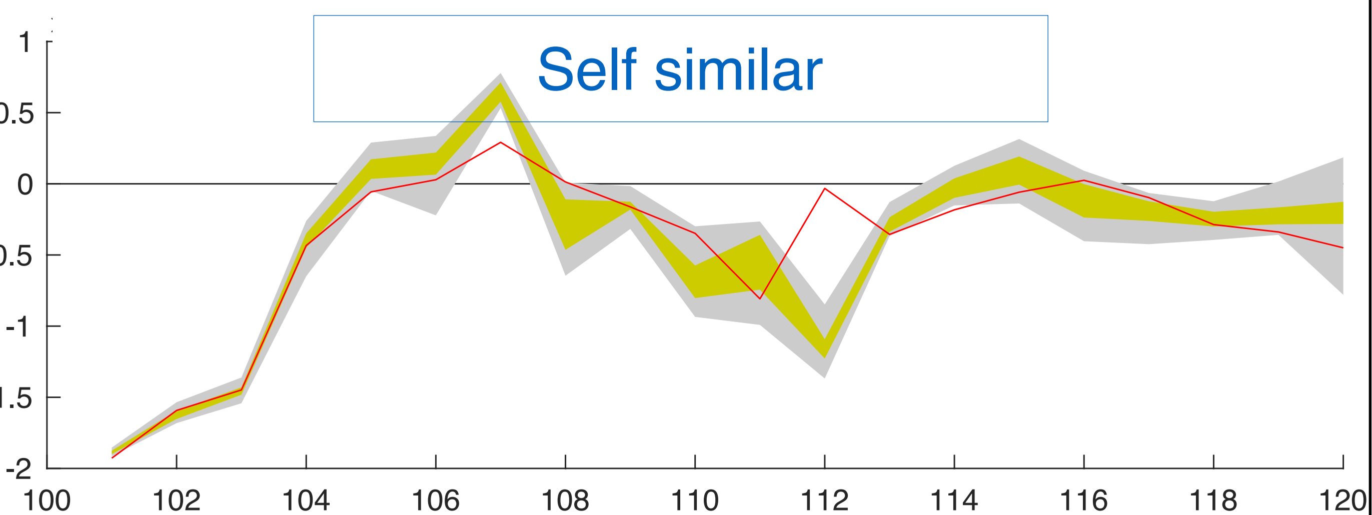


20 realisations

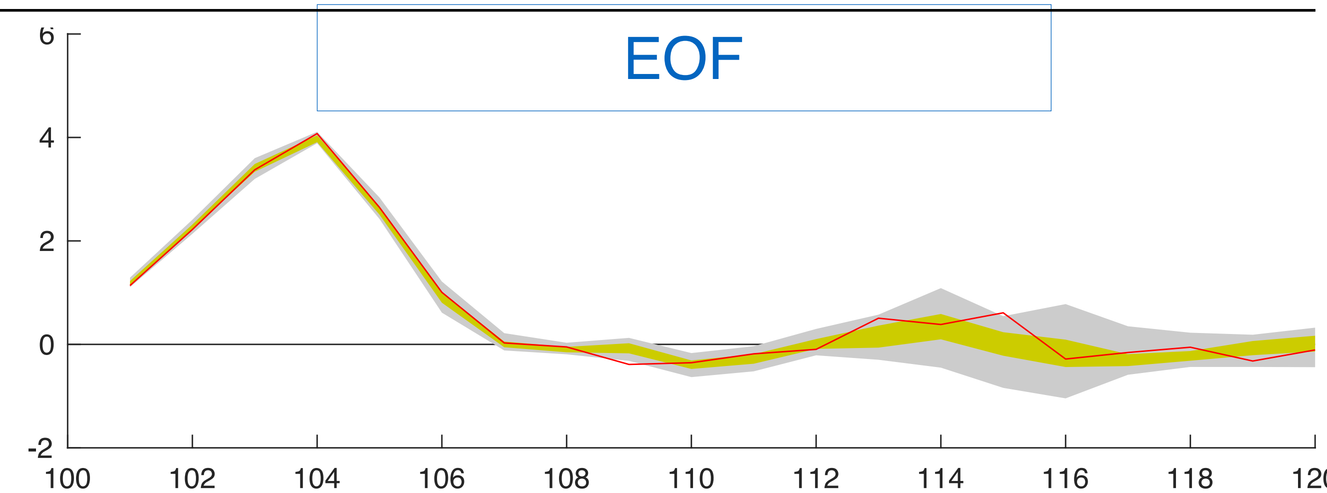
Ensemble : uncertainty quantification

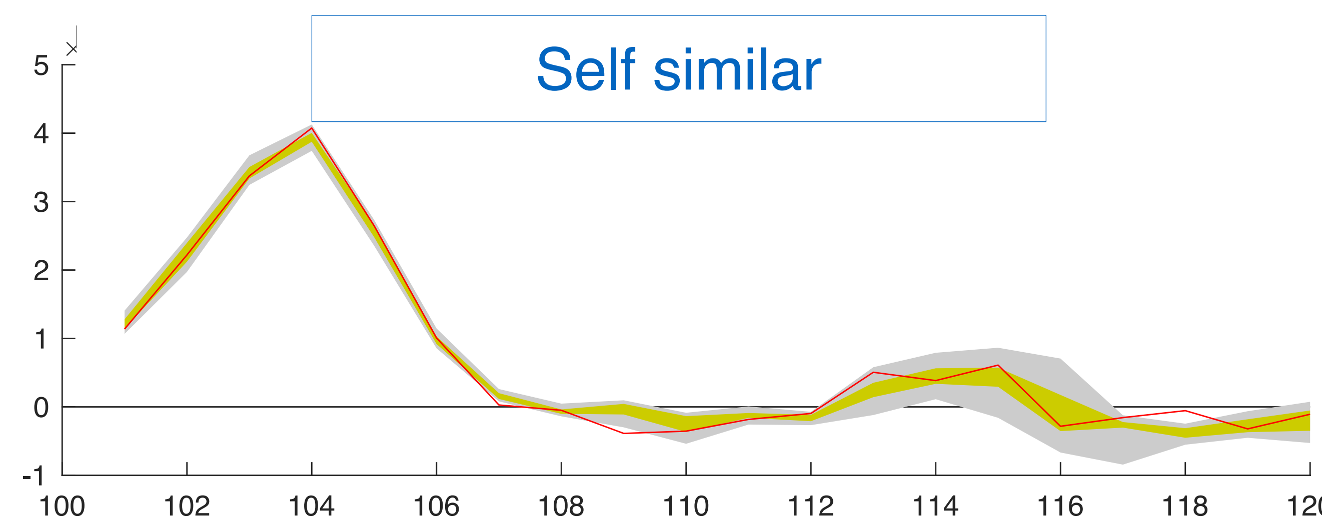
At point (250km,250km)

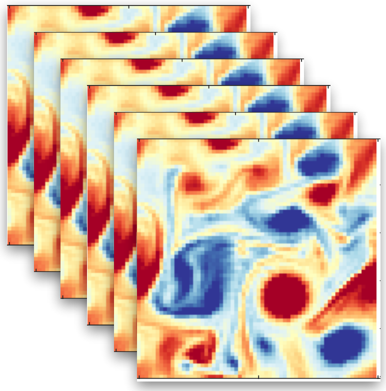




At point (500km,250km)





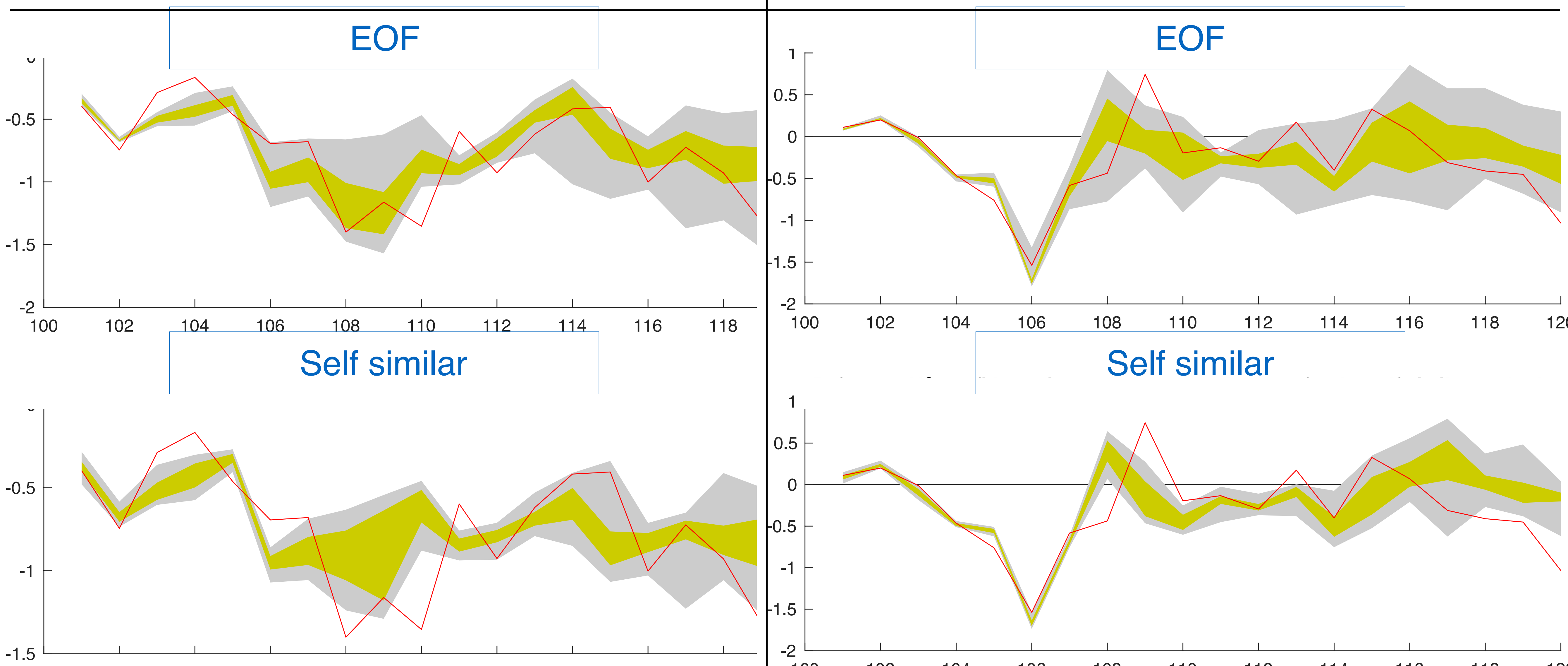


20 realisations

Ensemble : uncertainty quantification

At point (250km,500km)

At point (500km,500km)



Part IV

Unresolved velocity non-stationary heterogeneity

SQG

$$\frac{Db}{Dt} = -\alpha_{HV}\Delta^4 b \quad \text{Hyper-viscosity}$$

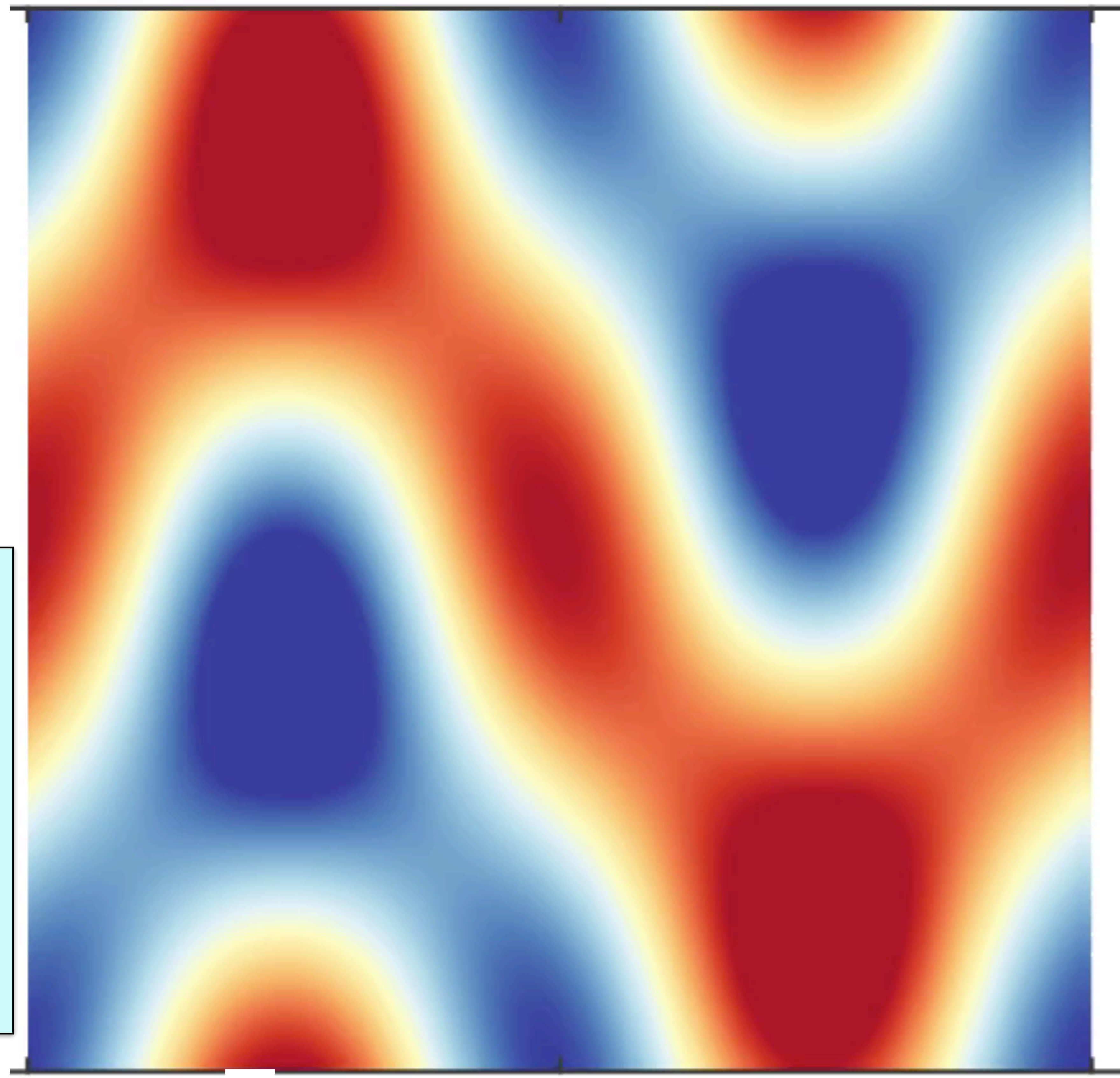
$$\boldsymbol{u} = \left(\text{cst.}\boldsymbol{\nabla}^{\perp}\Delta^{-\frac{1}{2}}\right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)

$t = 0$ day



SQG

$$\frac{Db}{Dt} = -\alpha_{HV}\Delta^4 b$$ Hyper-viscosity

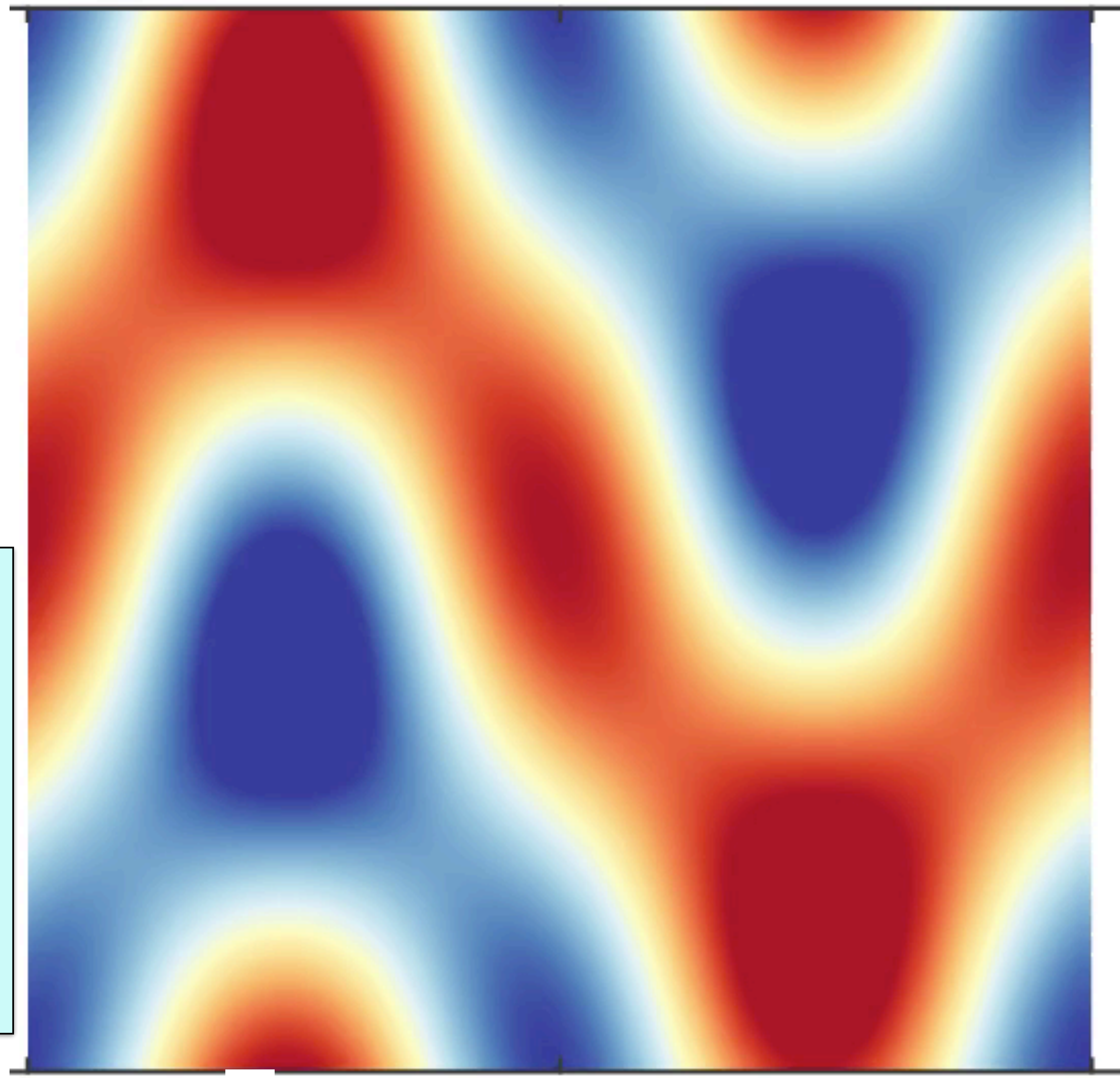
$$\boldsymbol{u} = \left(\text{cst.}\boldsymbol{\nabla}^{\perp}\Delta^{-\frac{1}{2}}\right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)

$t = 0$ day



$$t = 15 \text{ day}$$

SQG

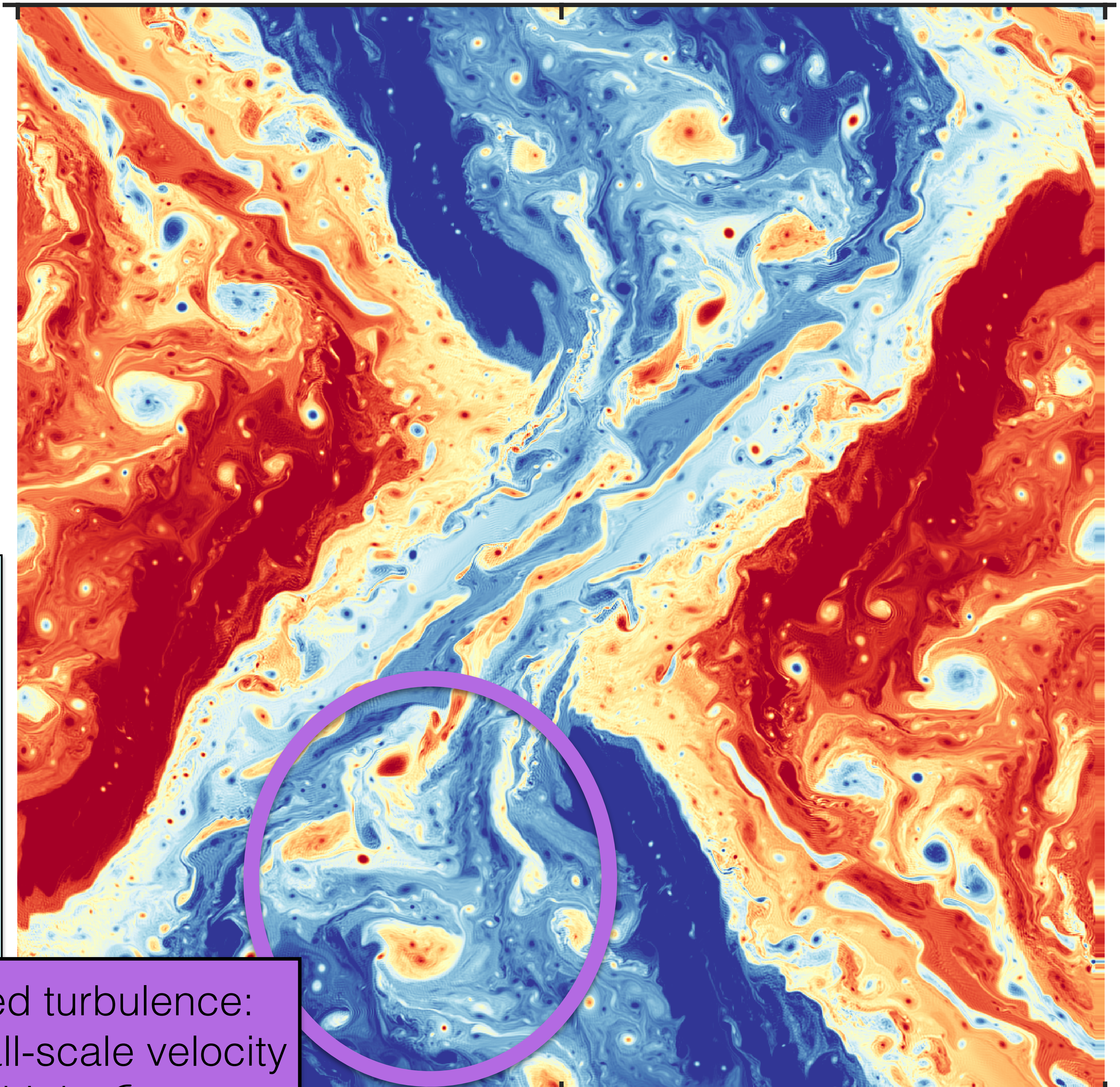
$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b$$ Hyper-viscosity

$$\boldsymbol{u} = \left(\text{cst.} \boldsymbol{\nabla}^{\perp} \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 199



Developed turbulence:
strong small-scale velocity
high ϵ

$$t = 15 \text{ day}$$

SQG

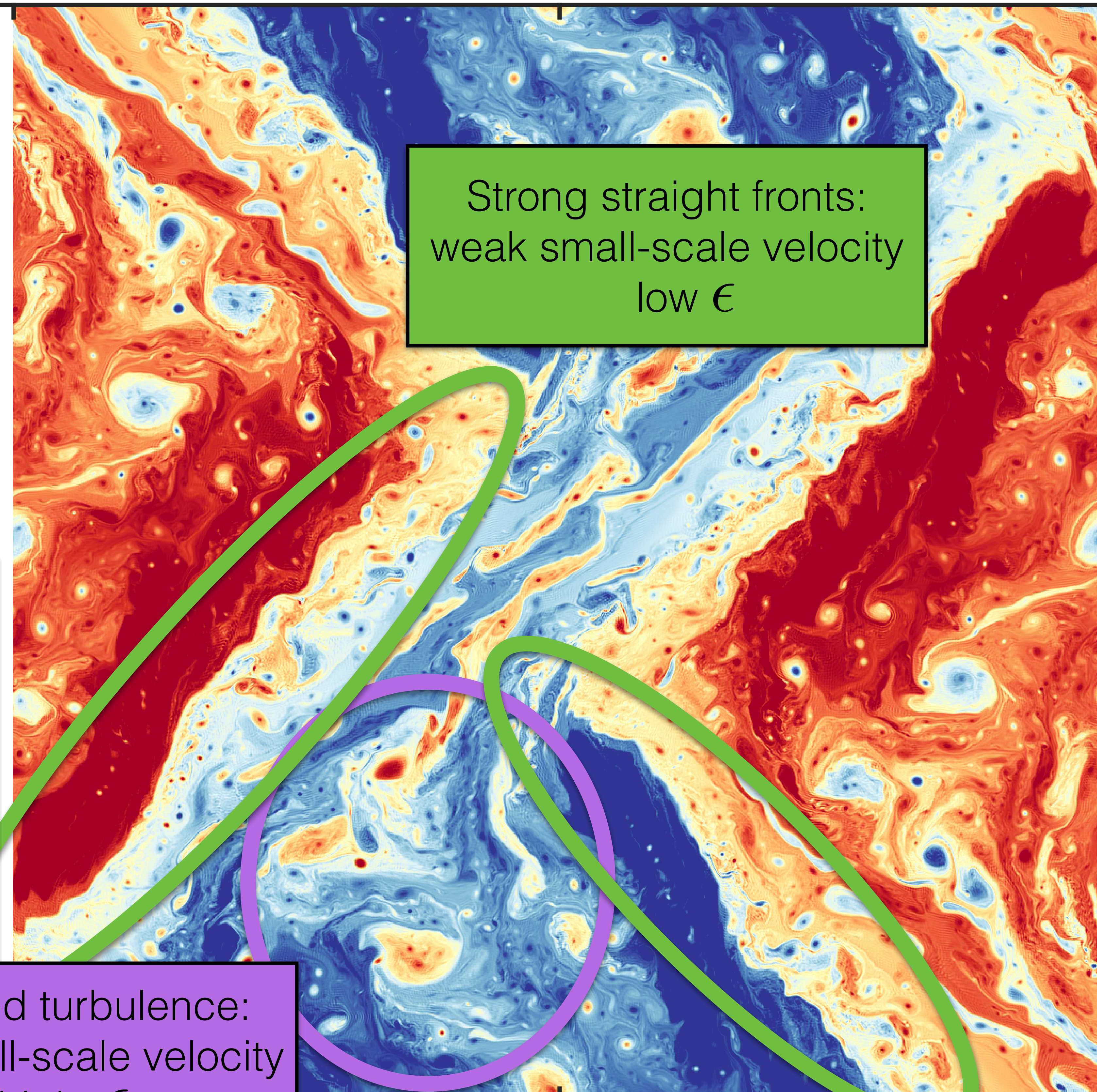
$$\frac{Db}{Dt} = \boxed{-\alpha_{HV}\Delta^4 b \quad \text{Hyper-viscosity}}$$

$$\boldsymbol{u} = \left(\text{cst.}\boldsymbol{\nabla}^{\perp}\Delta^{-\frac{1}{2}}\right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 199



Strong straight fronts:
weak small-scale velocity
low ϵ

Developed turbulence:
strong small-scale velocity
high ϵ

Large scales:

$\boldsymbol{w}$

Small scales:

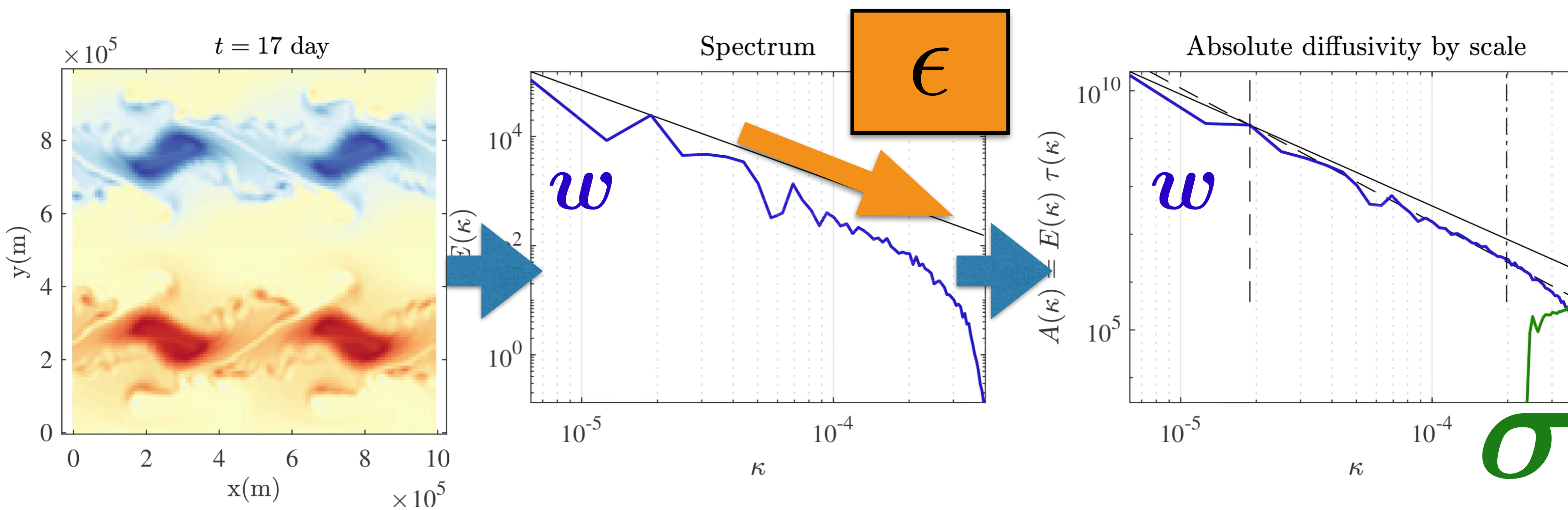
$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Heterogeneous modulation of $\boldsymbol{\sigma}$

(for Resseguier et al. 2017b)



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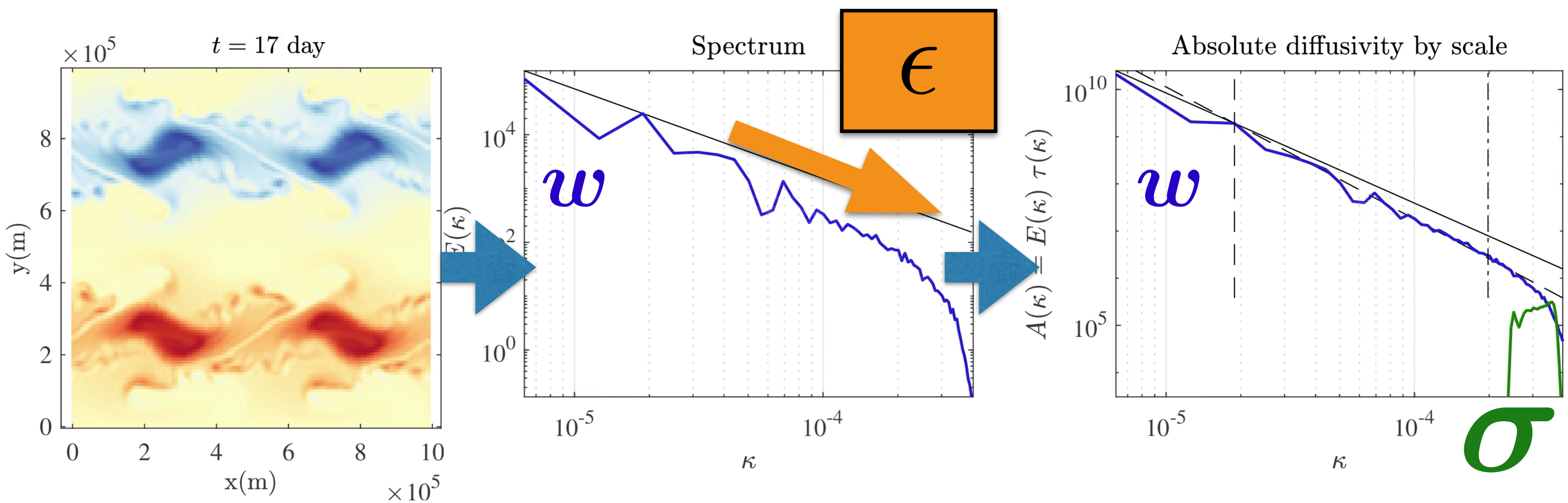

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Small scales:

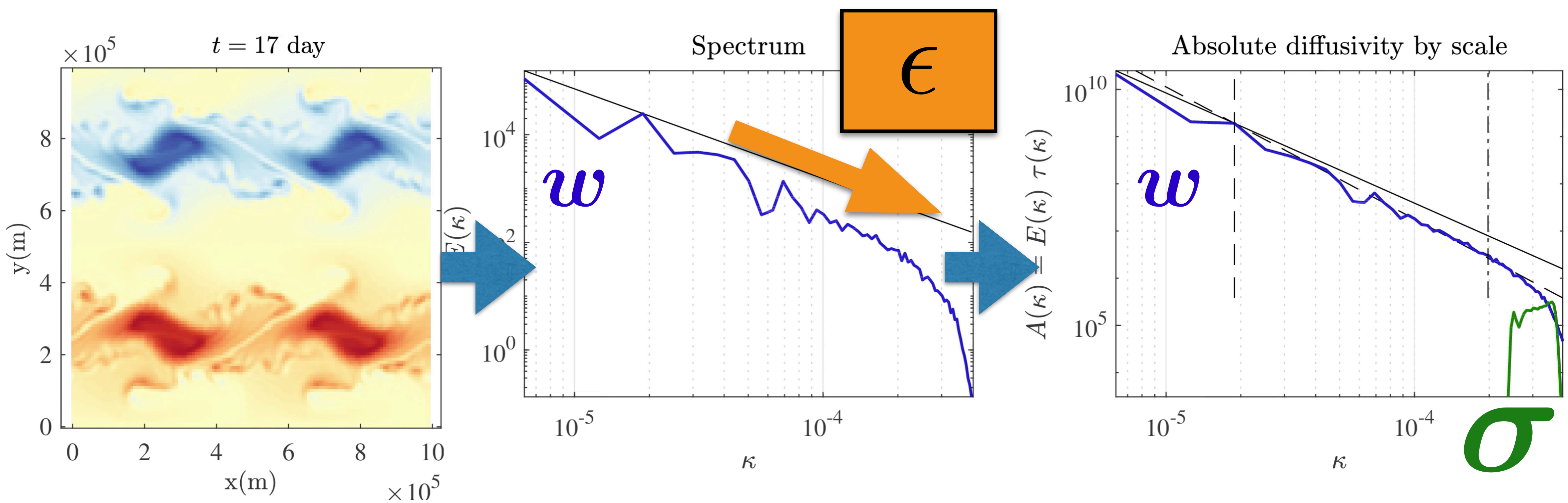
$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB\, (\boldsymbol{\sigma} dB)^T\}}{dt}$

Heterogeneous modulation of $\boldsymbol{\sigma}$ (for Resseguier et al. 2017b)

$$A(\kappa, x) = E(\kappa, x)\tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst. } \epsilon^p(x)\ \kappa^{-q}$$



Heterogeneous modulation of $\boldsymbol{\sigma}$
(for Resseguier et al. 2017b)

Large scales:

$\boldsymbol{w}$

Small scales:

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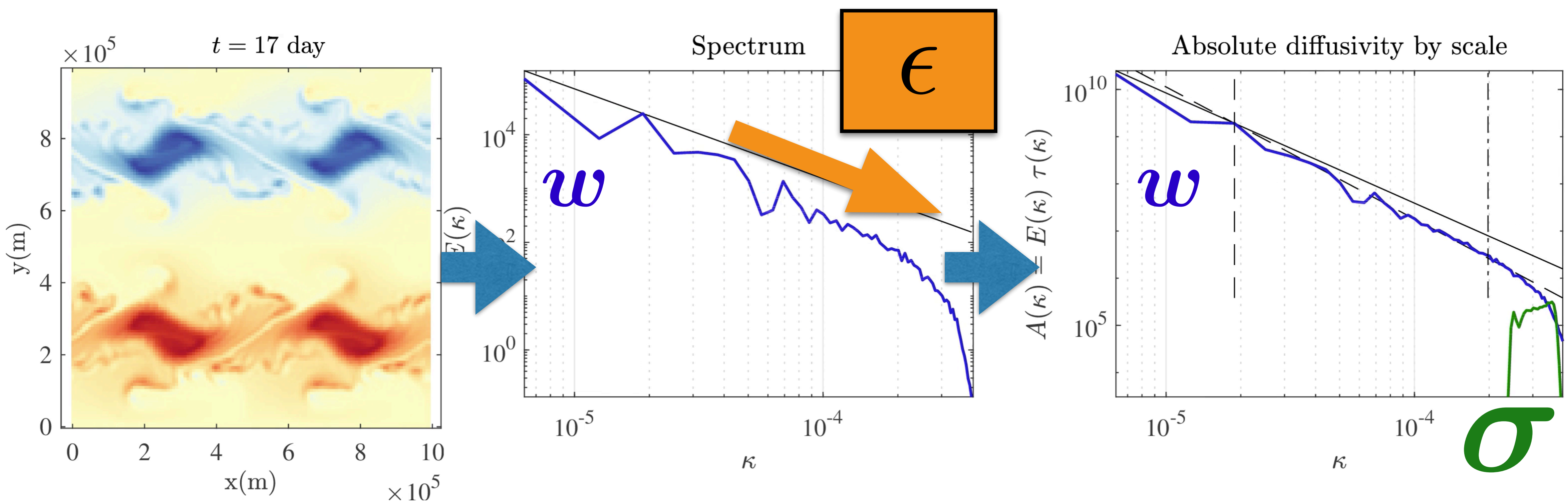
Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Dynamics
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x)\tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst. } \epsilon^{p}(x)\ \kappa^{-q}$$



Large scales:

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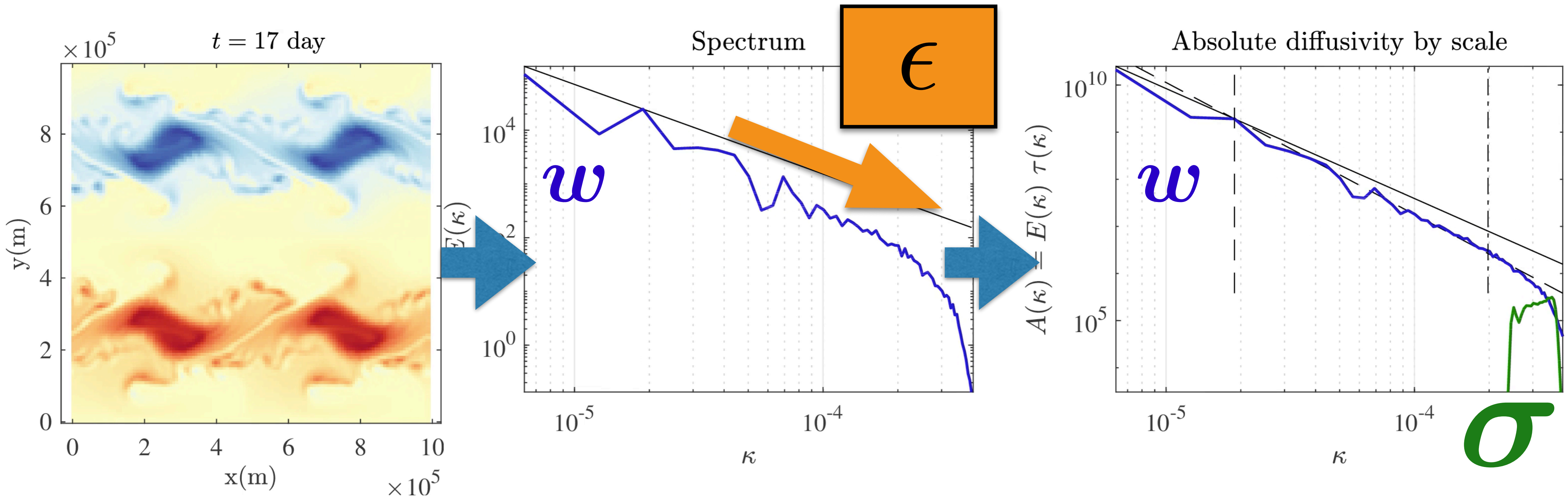
Heterogeneous modulation of $\boldsymbol{\sigma}$

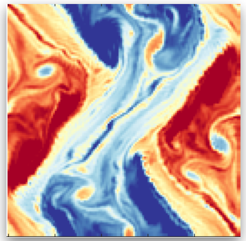
(for Resseguier et al. 2017b)

Dynamics (1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x)\tau(\kappa, x) = \kappa^{-3/2} E^{1/2}(\kappa, x) = \text{cst}\; \epsilon^{p}(x)\; \kappa^{-q}$$

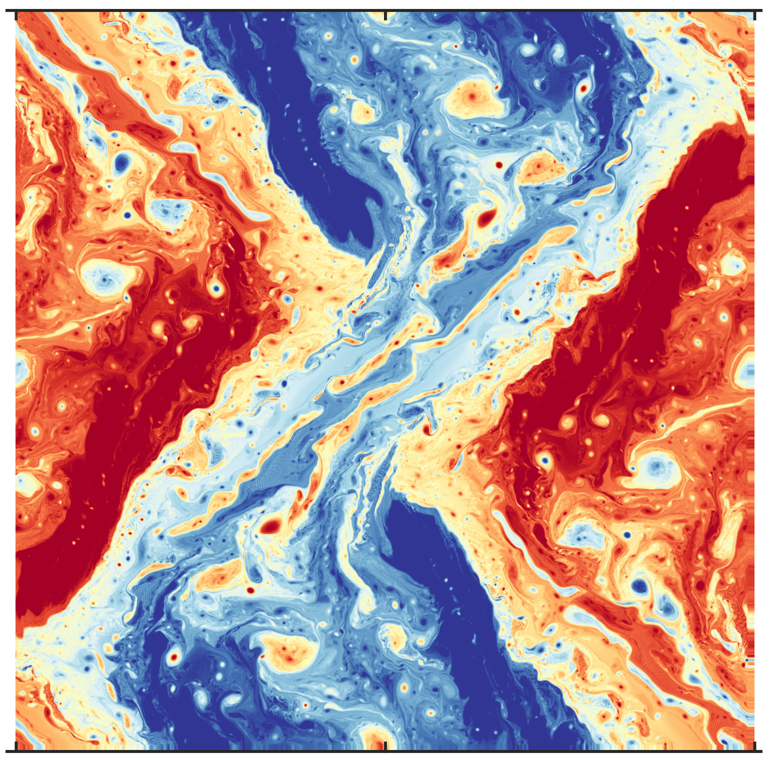
Learned on large scales at each time t



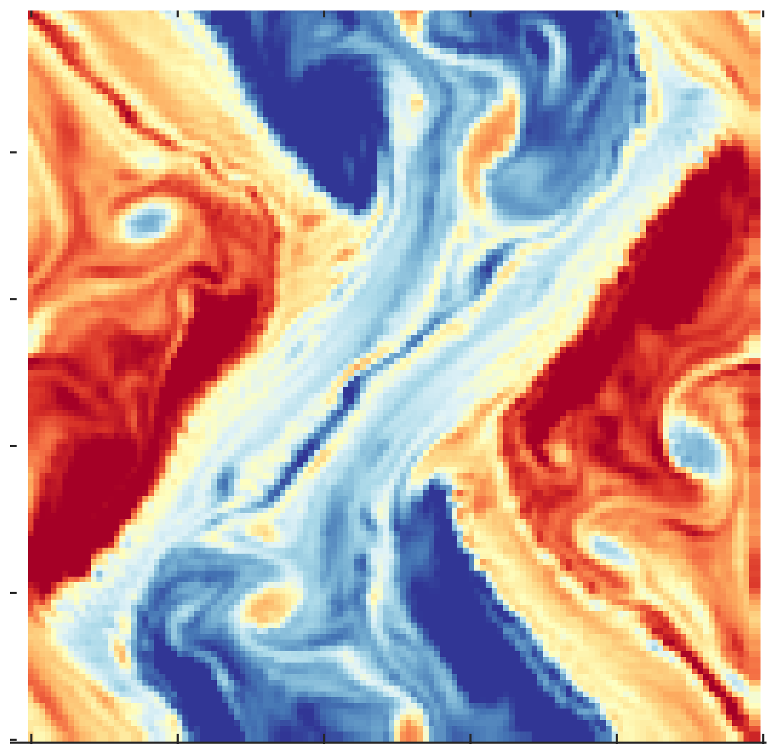


Heterogenous modulation

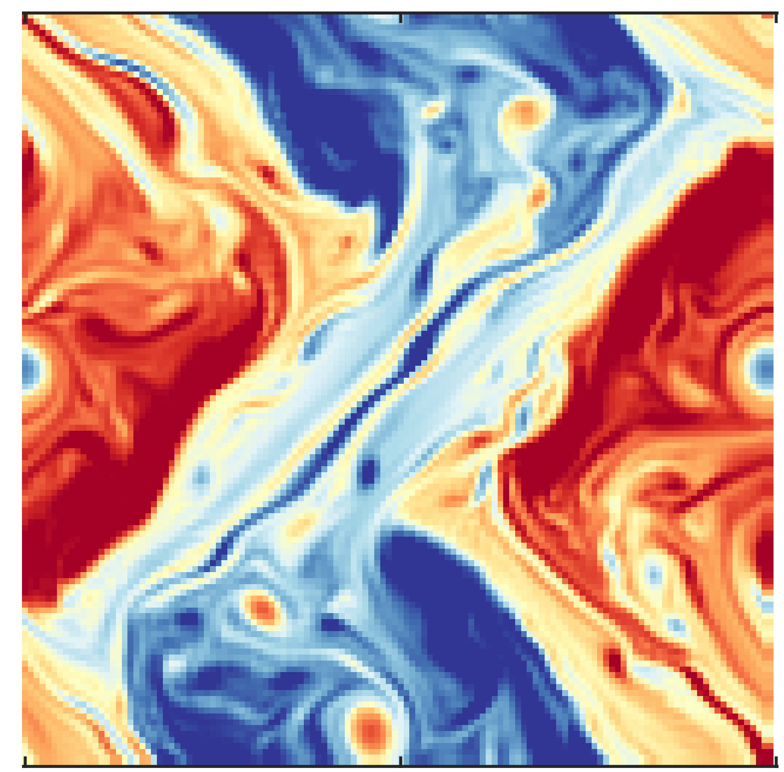
Deterministic 1024 x 1024

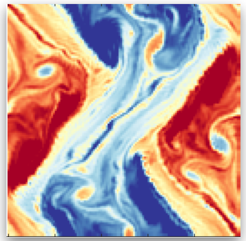


Stochastic 128 x128
with Homogeneous
small-scale velocity



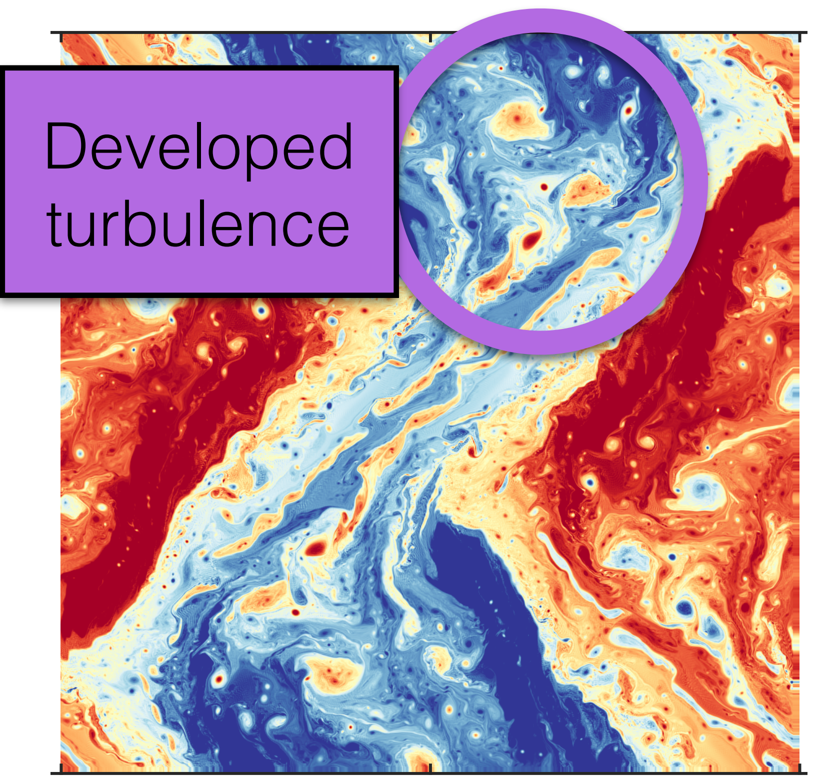
Stochastic 128 x128
with Energy-flux-based
modulation



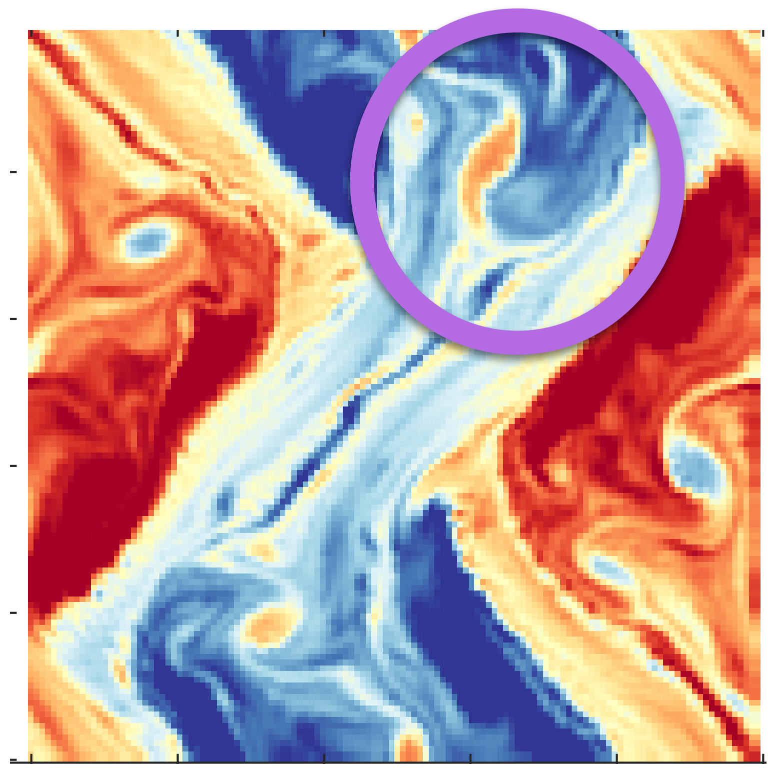


Heterogenous modulation

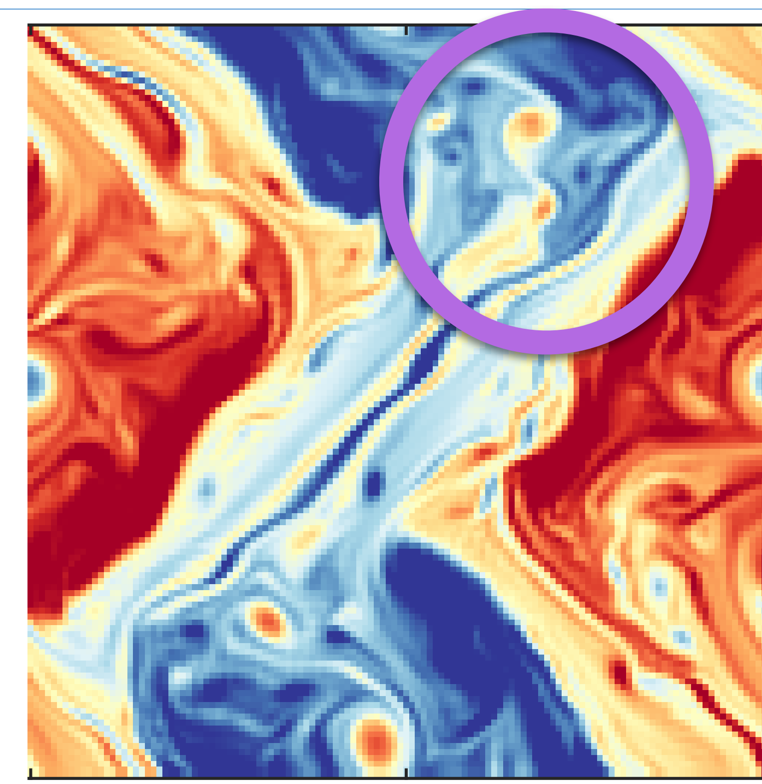
Deterministic 1024 x 1024

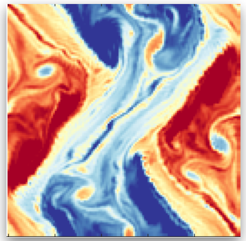


Stochastic 128 x128
with Homogeneous
small-scale velocity



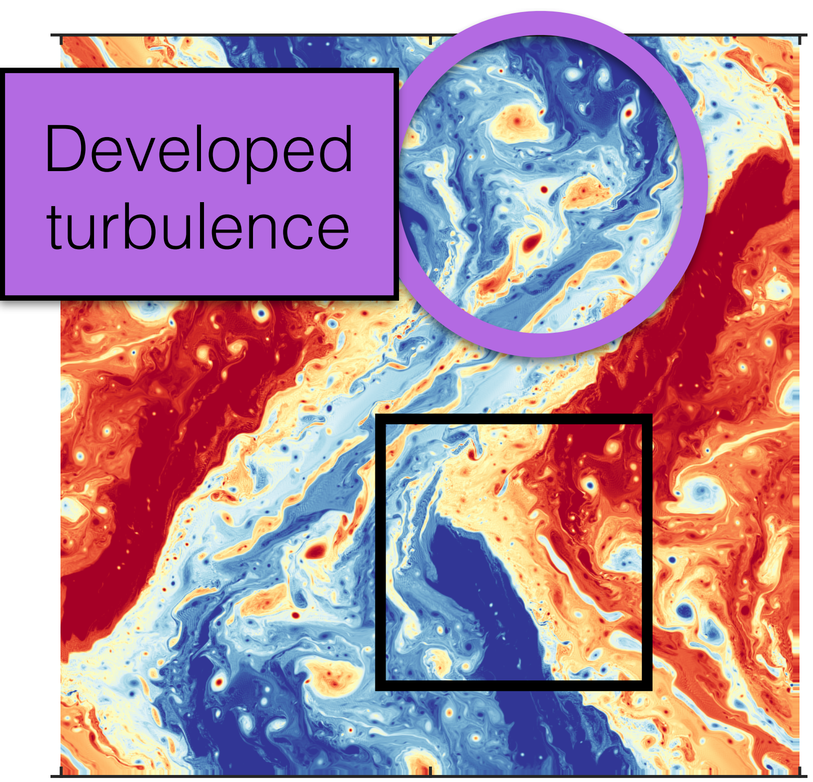
Stochastic 128 x128
with Energy-flux-based
modulation



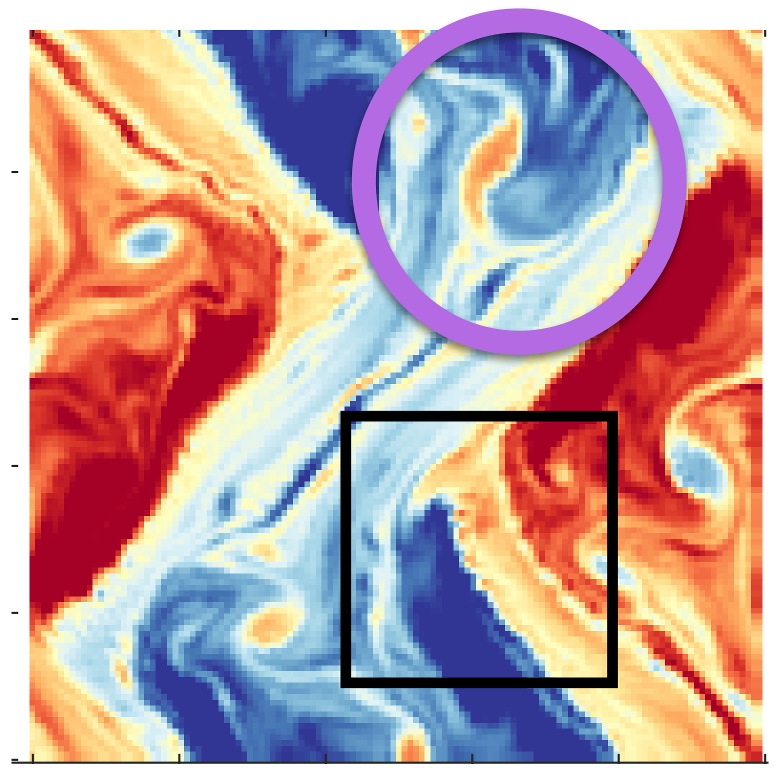


Heterogenous modulation

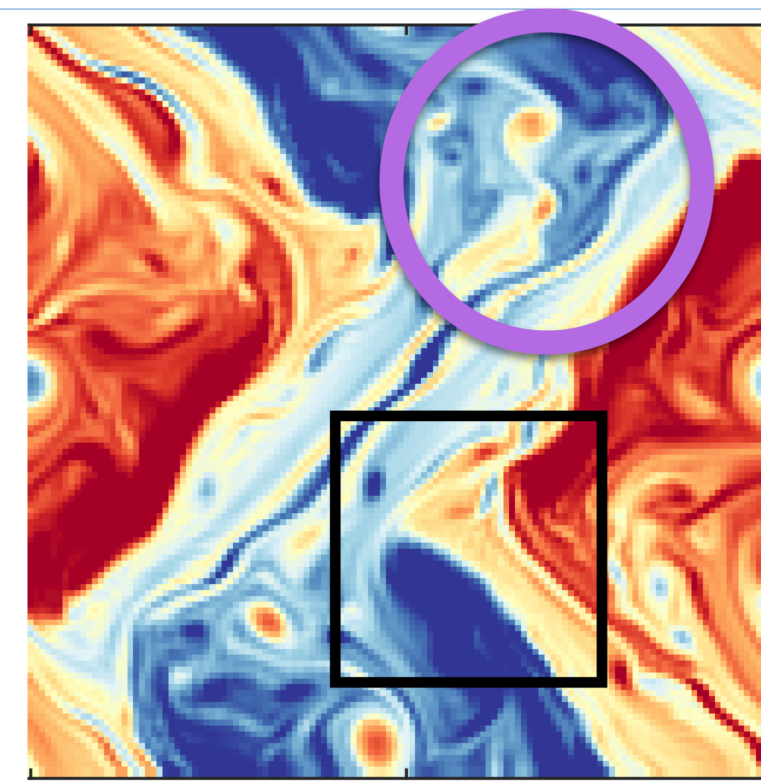
Deterministic 1024 x 1024

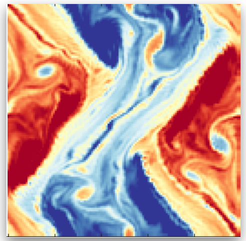


Stochastic 128 x128
with Homogeneous
small-scale velocity

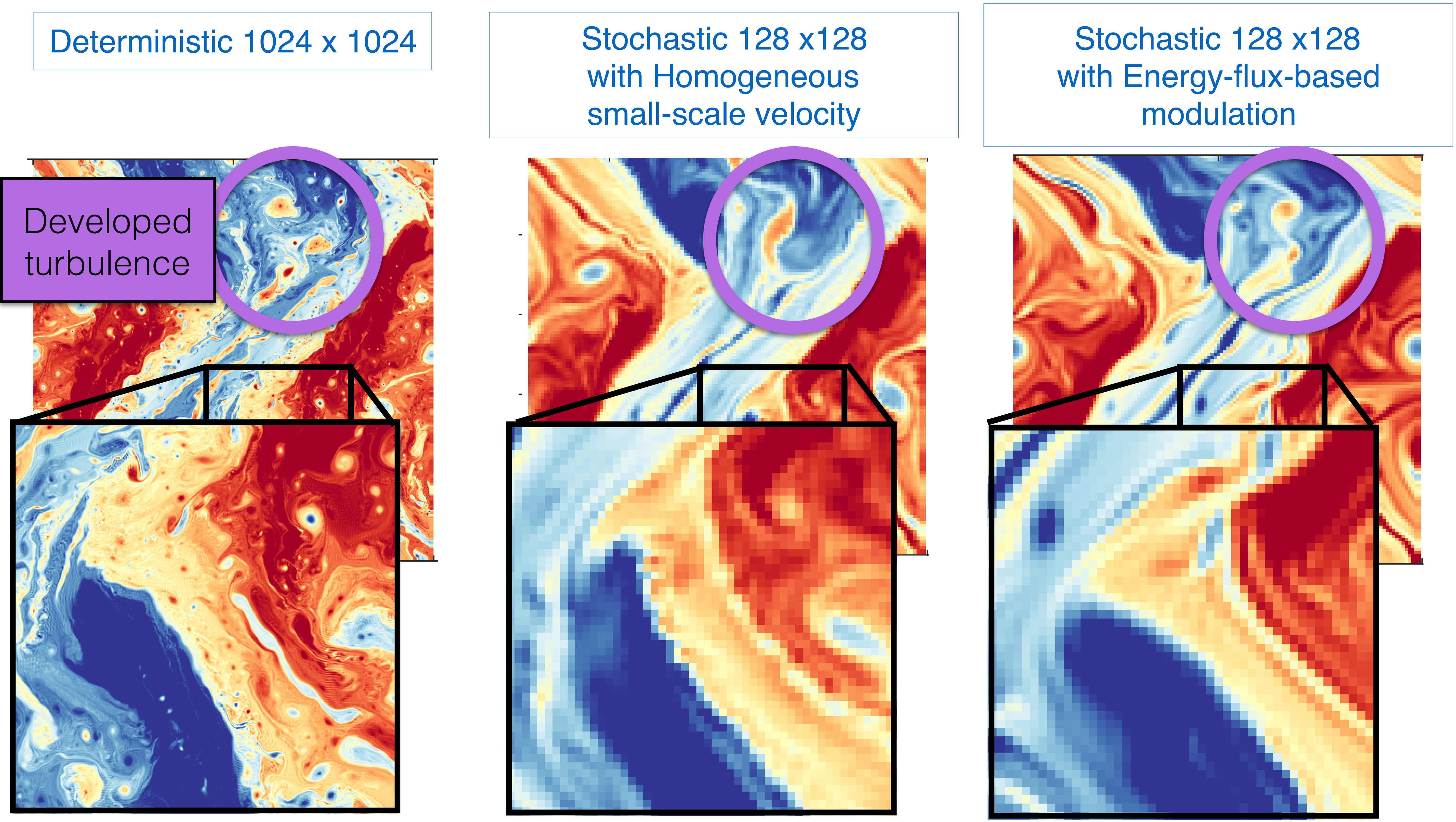


Stochastic 128 x128
with Energy-flux-based
modulation





Heterogenous modulation



Conclusion

Conclusion

- Stochastic transport (in LU & SALT) blindly describes unresolved triades
- LU conserves kinetic energy / SALT conserves helicity
- For the small-scale velocity parametrisation,
  both the data-driven (Cotter et al. 2018b) and the self-similar method (Resseguier et al. 2017b)
  lead to accurate uncertainty quantification
  (to address filter divergence)
- Energy-flux modulation improves the simulations

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- Kunita, H., Stochastic Flows and Stochastic Differential Equations, 1990. (Cambridge: Cambridge University Press).

- Resseguier, V., Mémin, E., & Chapron, B. (2017). Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophysical & Astrophysical Fluid Dynamics*, *111*(3), 177-208.

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