Stochastic Advection by Lie Transport and Location Uncertainty: a common ground for uncertainty quantification in fluid dynamics

Baylor Fox-Kemper, Darryl D Holm, Wei Pan, Valentin Resseguier

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Motivations

- Rigorously identified sudgrid dynamics effects

- Injecting likely small-scale dynamics

- Studying bifurcations and attractors

  → Climate projections

- Quantification of modeling errors

  → Ensemble forecasts and data assimilation

Contents

- Stochastic transport
- Stochastic Navier-Stokes : SALT vs LU
- Unresolved velocity parametrisation
- Unresolved velocity non-stationary heterogeneity

Part I
Stochastic transport

Usual random CFD

- Random parameters, boundary conditions, forcing → Other (complementary) issues
- Random initial conditions → Underdispersive + need large ensemble
- Arbitrary Gaussian forcing → Adding energy + wrong phase
- Averaging, homogenization → Assumptions and energy issues
- Mori-Zwanzig, MSM → Increase dimension

Usual random CFD

• Random parameters, boundary conditions, forcing	Other (complementary) issues
• Random initial conditions	Overdispersive + need large ensemble
• Arbitrary Gaussian forcing	Adding energy + wrong phase
• Averaging, homogenization	Assumptions and energy issues
• Mori-Zwanzig, MSM	Increase dimension

SALT & LU :
Adding random velocity

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma} \dot{\boldsymbol{B}}$$

SALT & LU :

Adding random velocity

Resolved
large scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma} \dot{\boldsymbol{B}}$$

SALT & LU :

Adding random velocity

Resolved large scales

White-in-time small scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

SALT & LU :

Adding random velocity

Resolved large scales

White-in-time small scales

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{\sigma}\dot{\boldsymbol{B}}$$

Large scales:

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$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} \boldsymbol{d}\boldsymbol{B} \, (\boldsymbol{\sigma} \boldsymbol{d}\boldsymbol{B})^T\}}{dt}$

SALT & LU :

Adding random velocity

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{w}^*(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t) \circ \mathrm{d}\boldsymbol{B}_t$$

$$= \boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma} \dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} \boldsymbol{d} \boldsymbol{B} \, (\boldsymbol{\sigma} \boldsymbol{d} \boldsymbol{B})^T\}}{dt}$

SALT & LU :

Adding random velocity

Stratonovich

$$\mathrm{d}\boldsymbol{X}_t = \boxed{\boldsymbol{w}^*(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t) \circ \mathrm{d}\boldsymbol{B}_t}$$

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Large scales:

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Small scales:

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$$= \boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t$$

Ito

Large scales:

$\boldsymbol{w}$

Small scales:

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Ito

References :

Mikulevicius and Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz & Holm 2017
Cotter and al 2018 a, b

Large scales:

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Small scales:

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Ito

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Flandoli, 2011

LU

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Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

SALT

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017
Cotter and al 2017
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Large scales:

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Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \left(\boldsymbol{\sigma} d\boldsymbol{B}\right)^T\}}{dt}$$

Advection of tracer $\boldsymbol{\Theta}$

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Advection of tracer $\boldsymbol{\Theta}$

Ito-Wentzell formula (Kunita 1990)

$$\frac{D\Theta}{Dt} = 0$$

Large scales:

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Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

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Advection of tracer Θ

Ito-Wentzell formula (Kunita 1990)

Advection of tracer Θ

Ito-Wentzell formula (Kunita 1990)

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Small scales:

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Variance tensor:

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Stratonovich notations:

Advection of tracer $\boldsymbol{\Theta}$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Ito-Wentzell formula (Kunita 1990)

Stratonovich notations:

$$\partial_t \Theta + (w^* + \sigma \circ \dot{B}) \cdot \nabla \Theta = 0$$

Advection of tracer Θ

Ito-Wentzell formula (Kunita 1990)

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Ito notations:

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Ito-Wentzell formula (Kunita 1990)

Ito notations:

$$\partial_t \Theta + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla}\Theta \right)$$

Advection of tracer $\boldsymbol{\Theta}$

Large scales:

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Small scales:

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Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

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Ito notations:

Advection

$$\partial_t \Theta + \boxed{\boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta} = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \Theta \right)$$

Advection of tracer Θ

Large scales: $\boldsymbol{w}$

Small scales: $\boldsymbol{\sigma}\dot{B}$

Variance tensor: $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB\,(\boldsymbol{\sigma} dB)^T\}}{dt}$

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Ito notations:

Advection

Diffusion

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Small scales: $\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

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Stratonovich drift : « Drift correction » in Ito notations

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \dfrac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Advection of tracer Θ

Ito-Wentzell formula (Kunita 1990)

Multiplicative random forcing

Ito notations:

Advection

Diffusion

$$\partial_t \Theta + \boldsymbol{w}^{\star} \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)$$

Stratonovich drift : « Drift correction » in Ito notations

Advection of tracer Θ

Ito-Wentzell formula (Kunita 1990)

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d B\, (\boldsymbol{\sigma} d B)^T\}}{dt}$$

Multiplicative random forcing

Ito notations:

Advection

Diffusion

$$\partial_t \Theta + \boldsymbol{w}^{\star} \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{B} \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \Theta \right)$$

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Large scales:

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Small scales:

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Ito-Wentzell formula (Kunita 1990)

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Ito notations:

Advection

Diffusion

$$\partial_t \Theta + \boldsymbol{w}^{\star} \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla}\Theta \right)$$

Stratonovich drift : « Drift correction » in Ito notations

Advection of tracer Θ

Large scales:

$\boldsymbol{w}$

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Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Ito-Wentzell formula (Kunita 1990)

Ito notations:

Multiplicative random forcing

Balanced energy exchanges

Advection

Diffusion

$$\partial_t \Theta + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta + \boldsymbol{\sigma}\dot{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)$$

Stratonovich drift : « Drift correction » in Ito notations

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp\!\!\!\perp \boldsymbol{w}(\bullet, t), \forall t' \leq t$$

Advection | Diffusion | Source term

$$\partial_t \mathbb{E}(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \mathbb{E}(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta) \right)$$

$$\partial_t Var(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} Var(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} Var(\Theta) \right) + (\boldsymbol{\nabla} \mathbb{E}(\Theta))^T \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta)$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \dfrac{\mathbb{E}\{\boldsymbol{\sigma} dB\,(\boldsymbol{\sigma} dB)^T\}}{dt}$

Moments of a passive tracer $\boldsymbol{\Theta}$

$$\Theta(\bullet, t') \perp\!\!\!\perp \boldsymbol{w}(\bullet, t), \forall t' \leq t$$

Advection

Diffusion

Source term

$$\partial_t \mathbb{E}(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \mathbb{E}(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta) \right)$$

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Large scales:

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Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma}d\boldsymbol{B}\,(\boldsymbol{\sigma}d\boldsymbol{B})^T\}}{dt}$

Moments of a passive tracer Θ

$$\Theta(\bullet,t') \perp\!\!\!\perp \boldsymbol{w}(\bullet,t), \forall t' \leq t$$

Conversion from deterministic to random energy

Advection

Diffusion

Source term

$$\partial_t \mathbb{E}(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\mathbb{E}(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\mathbb{E}(\Theta)\right)$$

$$\partial_t Var(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla}Var(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}Var(\Theta)\right) + (\boldsymbol{\nabla}\mathbb{E}(\Theta))^T\boldsymbol{a}\boldsymbol{\nabla}\mathbb{E}(\Theta)$$

Large scales:

$\boldsymbol{w}$

Small scales:

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Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB\, (\boldsymbol{\sigma} dB)^T\}}{dt}$$

Moments of a passive tracer Θ

$$\Theta(\bullet, t') \perp\!\!\!\perp \boldsymbol{w}(\bullet, t), \forall t' \leq t$$

Conservation of the energy $\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \Theta^2 = 0$

Conversion from deterministic to random energy

Advection

Diffusion

Source term

$$\partial_t \mathbb{E}(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} \mathbb{E}(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta) \right)$$

$$\partial_t Var(\Theta) + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla} Var(\Theta) = \boldsymbol{\nabla} \cdot \left(\frac{1}{2} \boldsymbol{a} \boldsymbol{\nabla} Var(\Theta) \right) + (\boldsymbol{\nabla} \mathbb{E}(\Theta))^T \boldsymbol{a} \boldsymbol{\nabla} \mathbb{E}(\Theta)$$

Part II
Stochastic Navier-Stokes
SALT vs LU

Comparison

	LU	SALT
Scalar (e.g. SQG)	Identical	
Navier-Stokes	2 differences	
Vorticity	2 differences	
Kinetic energy conservation	✓	✗
Helicity (& 2D enstrophy) conservation	✗	✓

Navier-Stokes LU

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\nabla p$$

Navier-Stokes LU

Pressure

$$\frac{Dw}{Dt} = \boxed{-\frac{1}{\rho}\nabla p}$$

Navier-Stokes LU

Stochastic transport of **Ito** drift

Pressure

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\nabla p$$

Navier-Stokes LU

Stochastic transport of **Ito** drift

Pressure

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\nabla p$$

Conserve (Ito) kinetic energy

Navier-Stokes SALT

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho}\nabla p$$

Navier-Stokes SALT

Pressure

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = \boxed{-\frac{1}{\rho}\nabla p}$$

Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

Pressure

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho}\nabla p$$

Navier-Stokes SALT

Stochastic transport of **Stratonovich** drift

Additional term

Pressure

$$\frac{Dw^*}{Dt} + \nabla(\sigma \circ \dot{B})^T w^* = -\frac{1}{\rho}\nabla p$$

Conserve (Stratonovich) helicity

Part III

Unresolved velocity parametrisation

Code available online

Large scales:

$\boldsymbol{w}$

Small scales:

$\sigma \dot{B}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

Parameter-free models for $\boldsymbol{\sigma}$

	Cotter et al. 2018b	Resseguier et al. 2017b
Applicable to :	LU & SALT	LU & SALT
Homogeneous	No	Yes
Stationary	Yes	Not anymore
Self-similar assumption	No	Yes
Data-driven	Yes	No

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

Cotter et al. 2018b:

Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^{n} \xi_i(x) \; dW_i(t)$$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma \dot{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma dB} (\boldsymbol{\sigma dB})^T\}}{dt}$

Cotter et al. 2018b:

Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^{n} \xi_i(x) \boxed{dW_i(t)}$$

iid Brownian motion

Large scales:

$\boldsymbol{w}$

Small scales:

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$$\sigma dB_t = \sum_{i=1}^{n} \boxed{\xi_i(x)} \, \boxed{dW_i(t)}$$

iid Brownian motion

weighted EOFs learned on data

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

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Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^{n} \boxed{\xi_i(x)}\ \boxed{dW_i(t)}$$

iid Brownian motion

weighted EOFs learned on data

PCA on : $\Delta X_k(x) = X^{HR}_{t_k,t_k+\Delta t}(x) - X^{LR}_{t_k,t_k+\Delta t}(x)$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB\,(\boldsymbol{\sigma} dB)^T\}}{dt}$

Cotter et al. 2018b:
Karhunen–Loève decomposition (EOF)

$$\sigma dB_t = \sum_{i=1}^{n} \boxed{\xi_i(x)}\, \boxed{dW_i(t)}$$

iid Brownian motion

weighted EOFs learned on data

PCA on : $\Delta X_k(x) = \boxed{X^{HR}_{t_k,t_k+\Delta t}(x)} - \boxed{X^{LR}_{t_k,t_k+\Delta t}(x)}$

Flow from **high**-resolution velocity

Flow from **low**-resolution velocity

Large scales:

$\boldsymbol{w}$

Small scales:

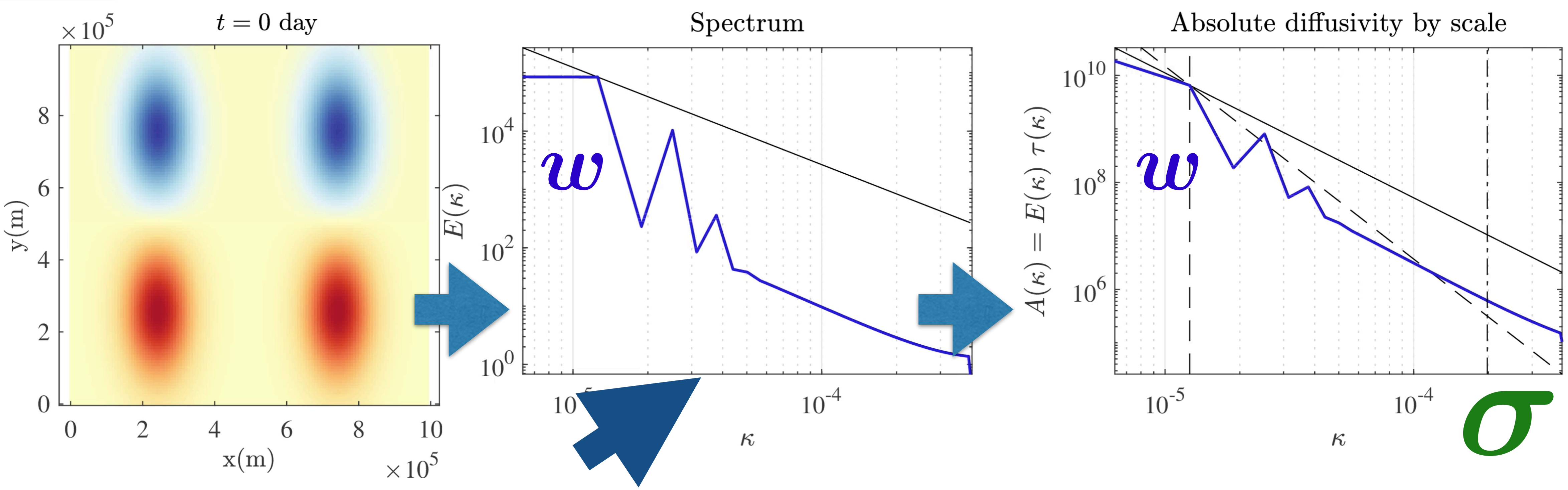
$\boldsymbol{\sigma} \dot{\boldsymbol{B}}$

Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$

(improvement of) Resseguier et al. 2017b:

Self-similar model



Kinetic energy Spectrum $E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \left\| \widehat{v}(k) \right\|^2$

Large scales:

$\boldsymbol{w}$

Small scales:

$\boldsymbol{\sigma}\dot{\boldsymbol{B}}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B}\,(\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

(improvement of) Resseguier et al. 2017b:

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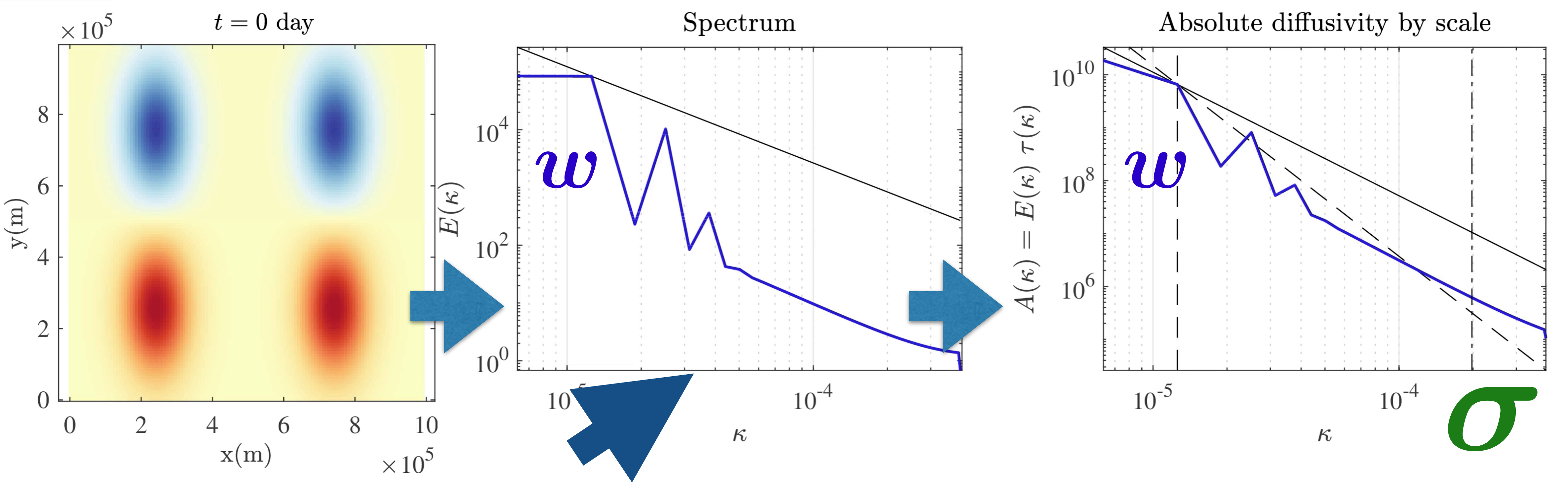

Kinetic energy Spectrum $$E(\kappa) = \frac{1}{\mu(\Omega)}\mathbb{E}\oint_{\|k\|=\kappa} d\theta_k \kappa \left\|\widehat{v}(k)\right\|^2$$

Large scales:

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Small scales:

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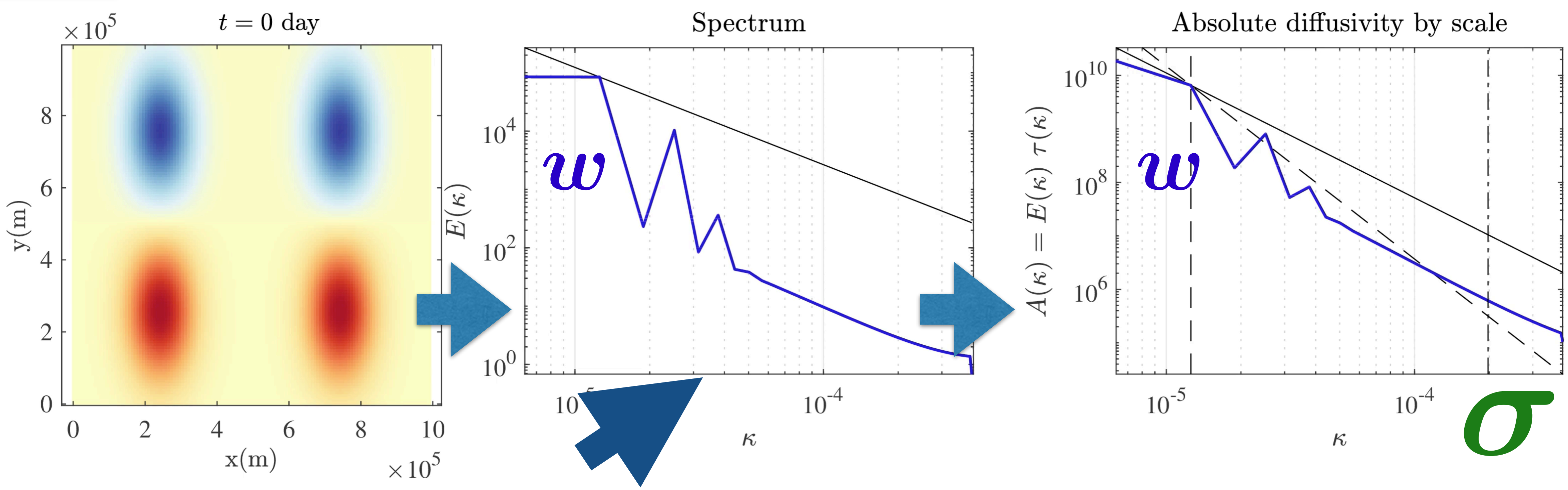
Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} d\boldsymbol{B} \, (\boldsymbol{\sigma} d\boldsymbol{B})^T\}}{dt}$$

(improvement of)
Resseguier et al. 2017b:
Self-similar model

Absolute Diffusivity Spectral Density

$$A(\kappa) = E(\kappa)\tau(\kappa) = \kappa^{-3/2} E^{1/2}(\kappa)$$



Kinetic energy Spectrum

$$E(\kappa) = \frac{1}{\mu(\Omega)} \mathbb{E} \oint_{\|k\|=\kappa} d\theta_k \kappa \, \|\widehat{v}(k)\|^2$$

SQG

$$\frac{Db}{Dt} = \boxed{-\alpha_{HV}\Delta^4 b} \text{ Hyper-viscosity}$$

$$\boxed{-\frac{1}{\tau}(b - F)} \text{ Forcing}$$

$$u = \left(\text{cst}.\nabla^\perp \Delta^{-1/2}\right) b$$

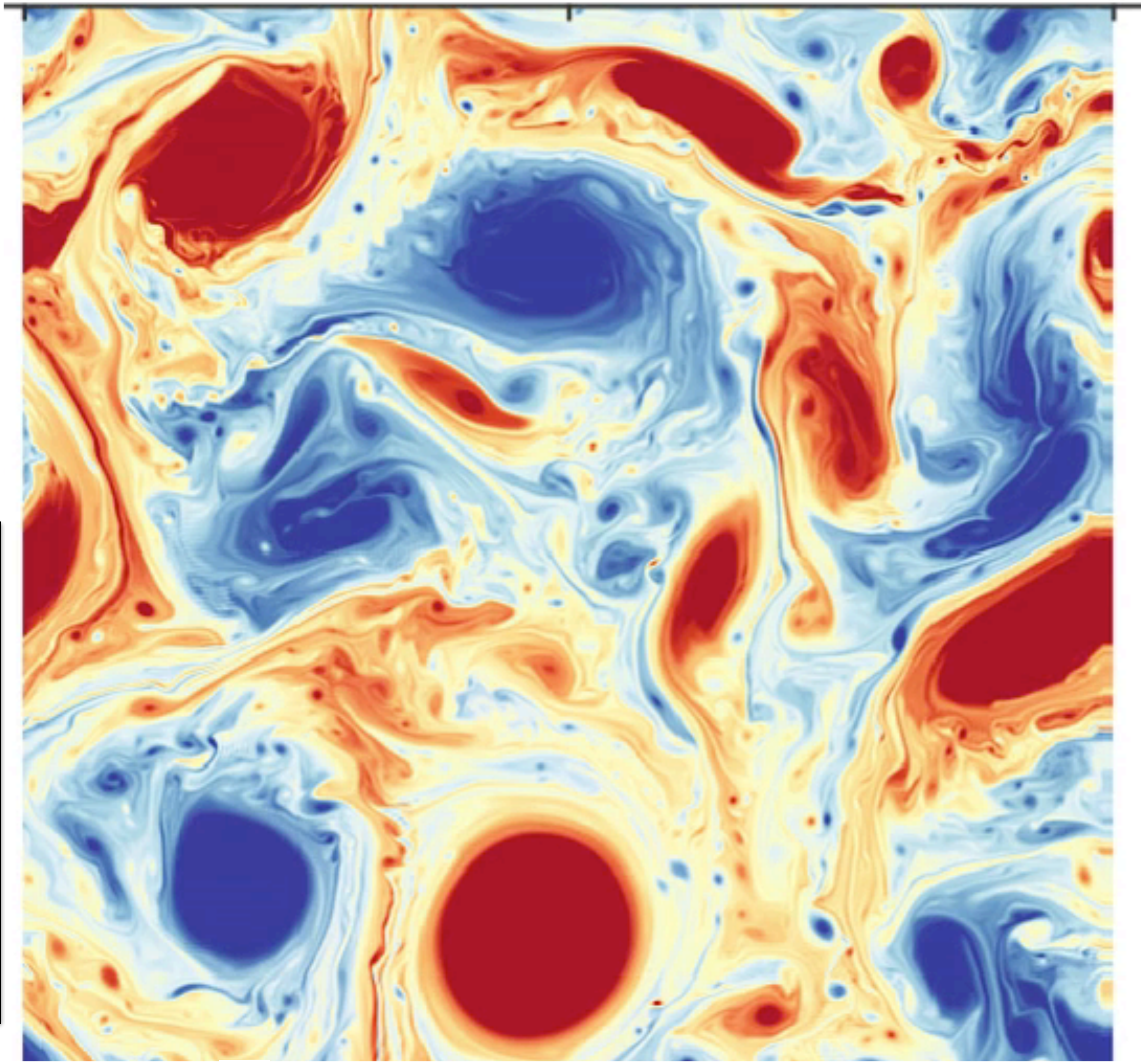
Reference flow:

deterministic

SQG

512 x 512

$t = 100$ day



SQG

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deterministic

SQG

512 x 512

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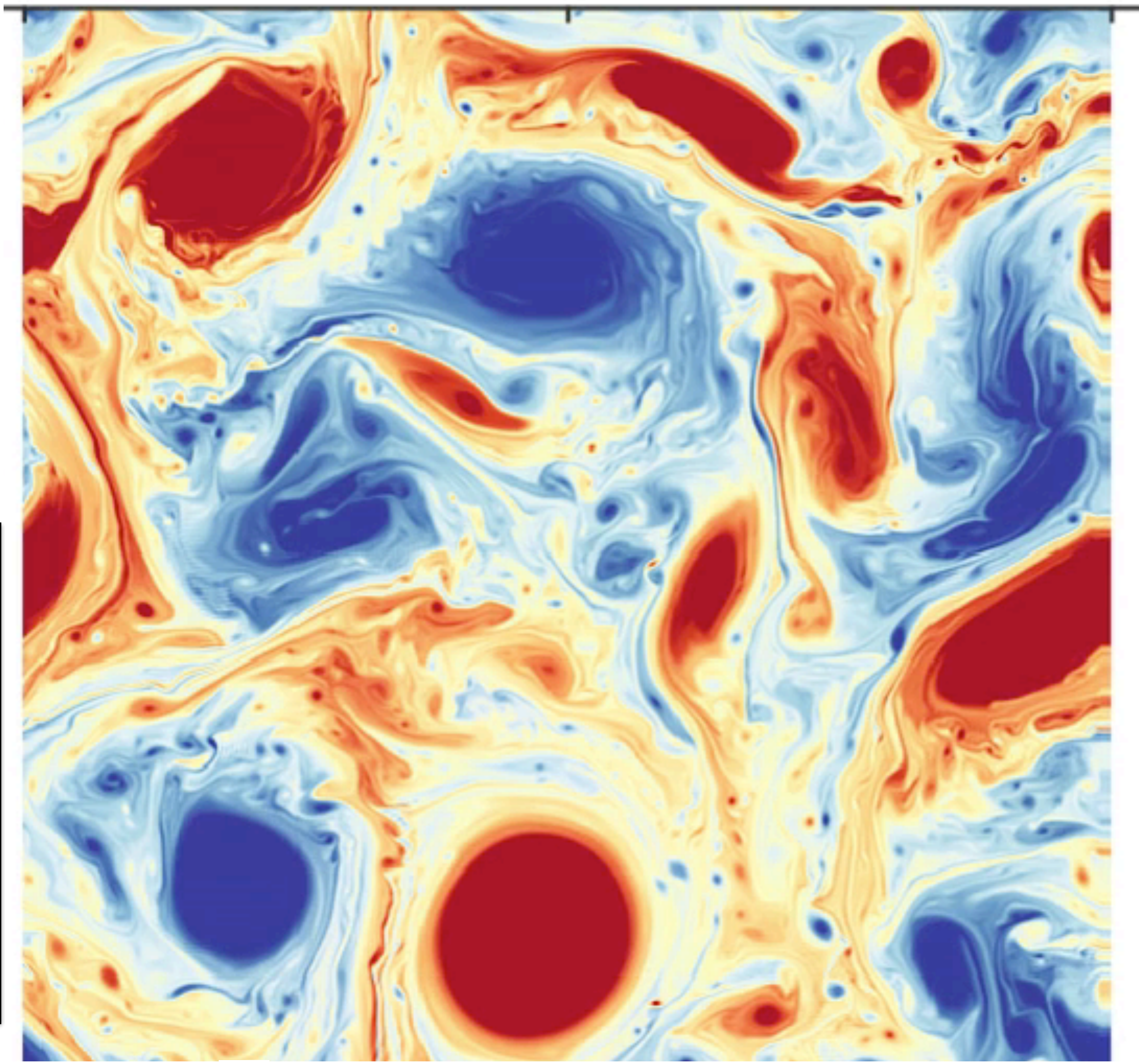
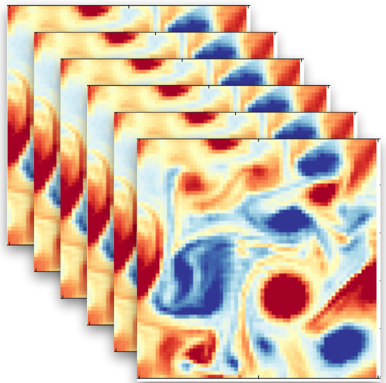
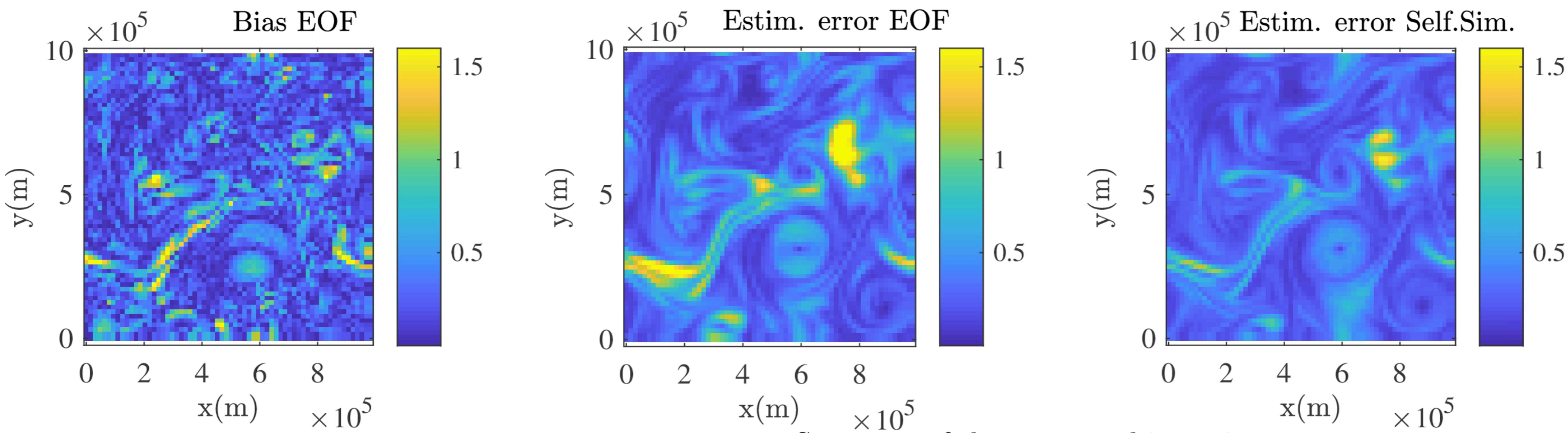
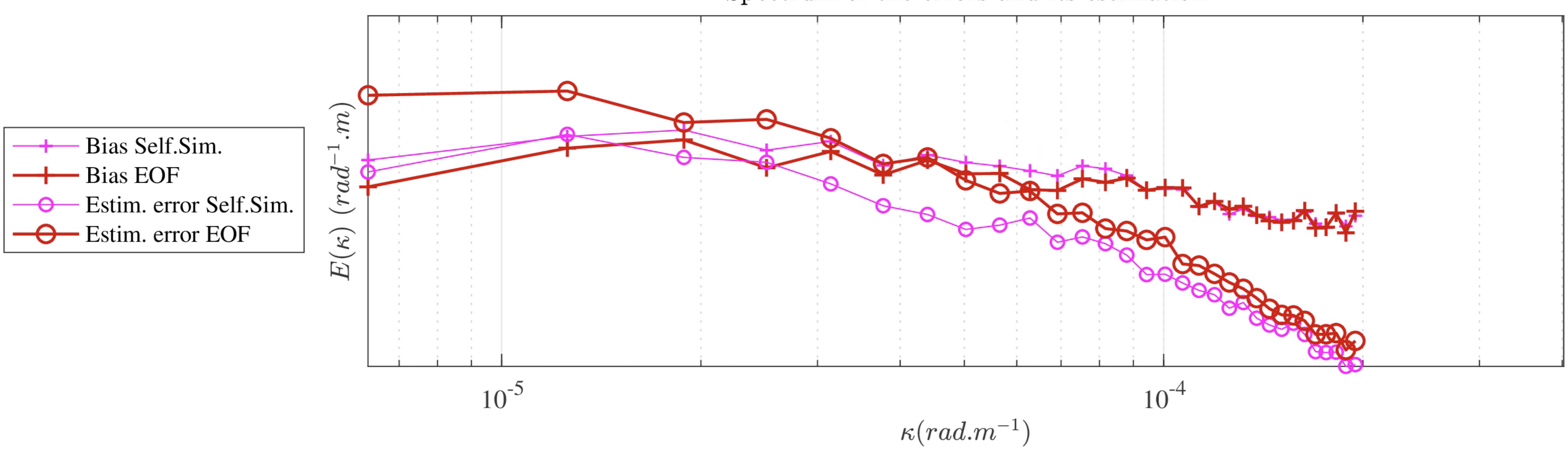
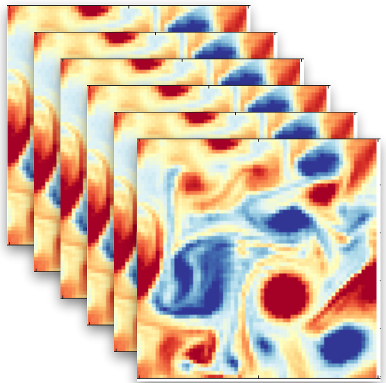




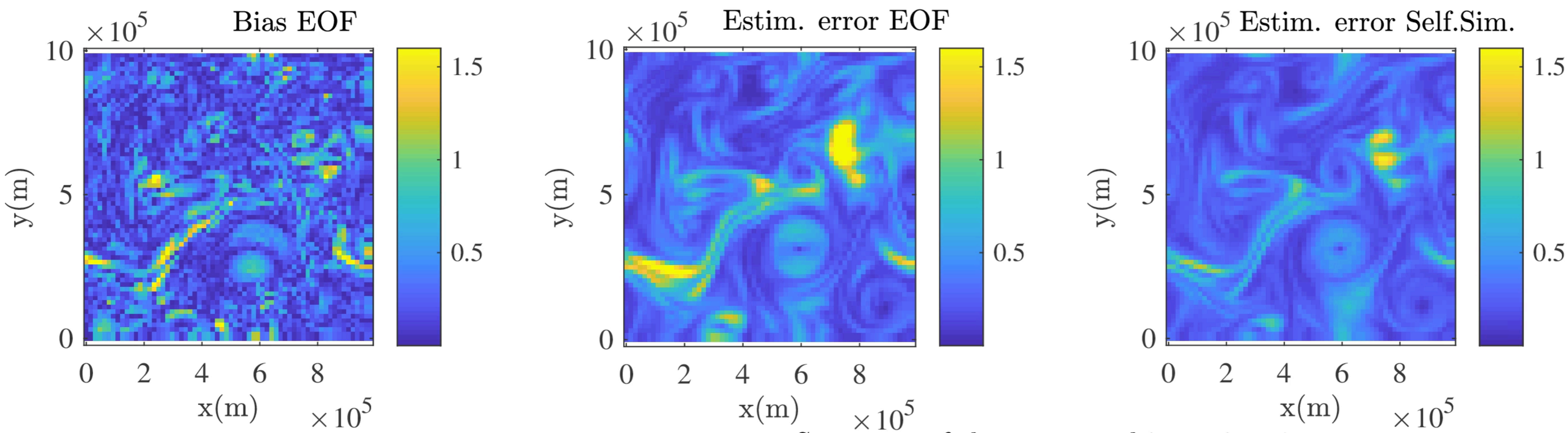
Ensemble : uncertainty quantification

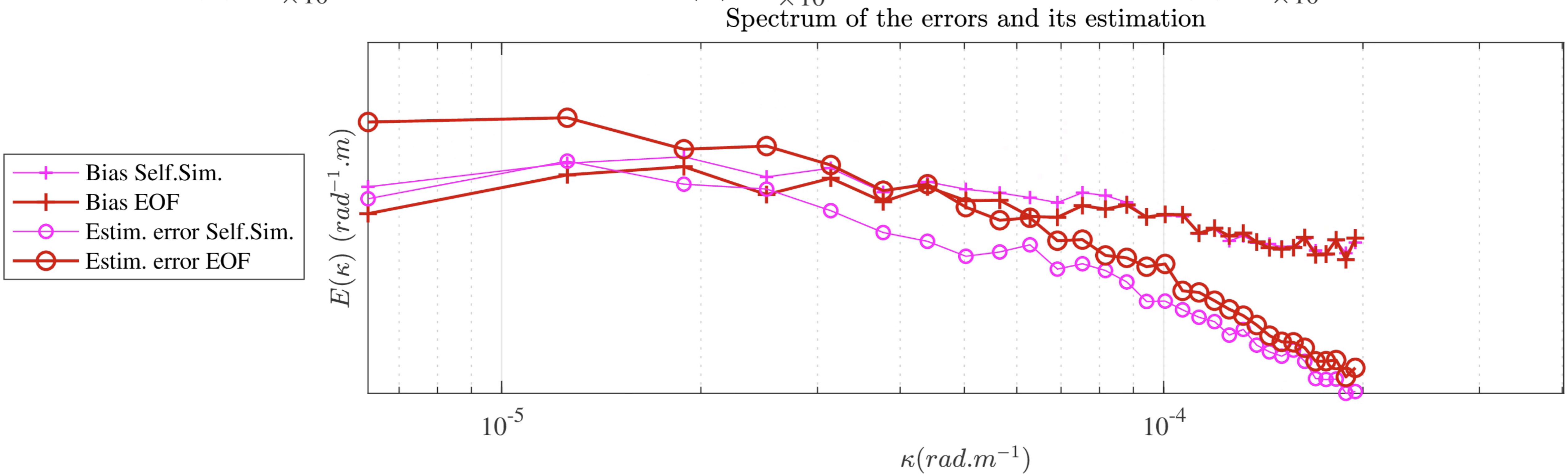






Ensemble : uncertainty quantification





Part IV

Unresolved velocity non-stationary heterogeneity

SQG

$$\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}$$

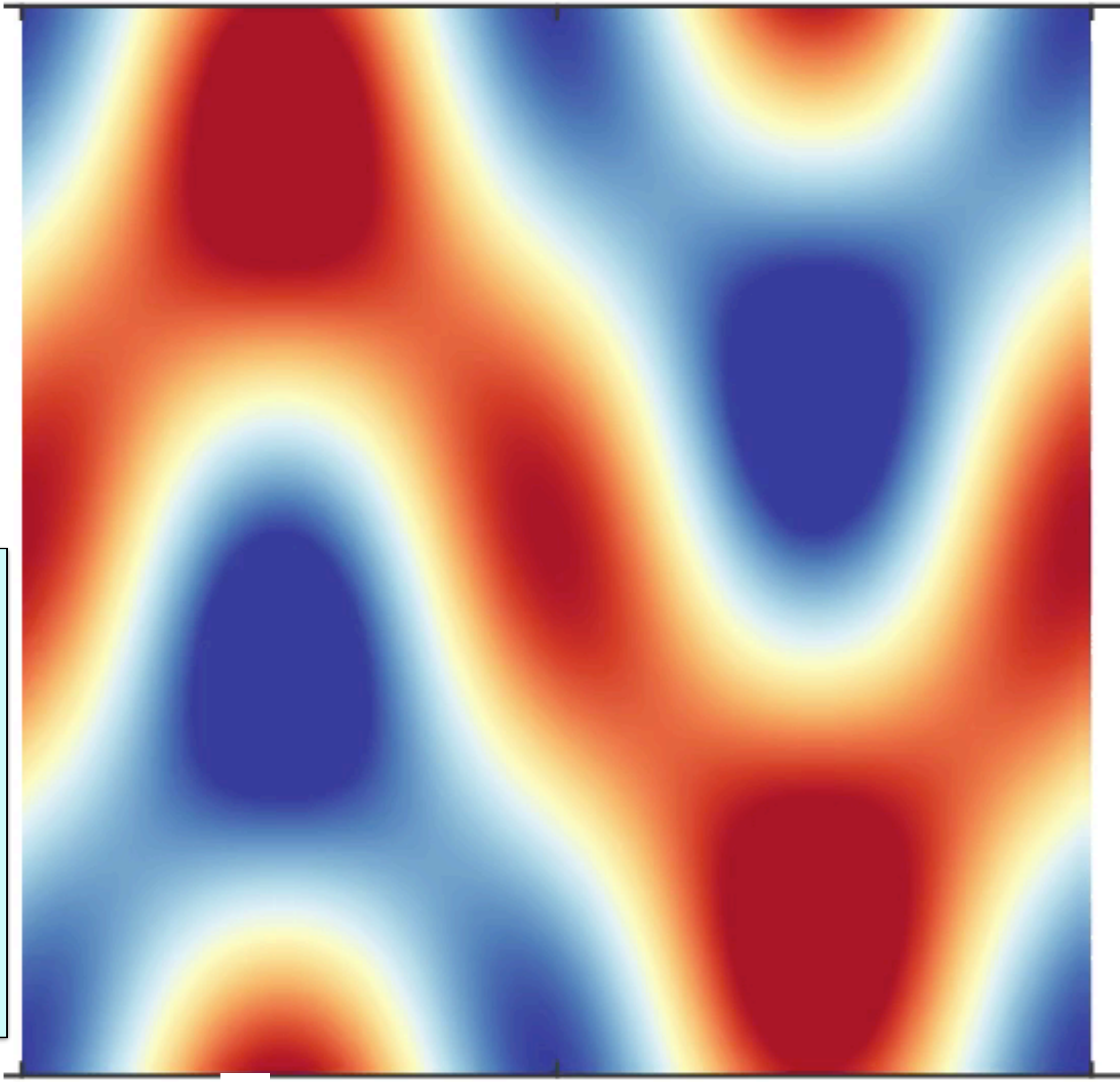
$$\boldsymbol{u} = \left(\text{cst.} \boldsymbol{\nabla}^{\perp} \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)

$t = 0$ day



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$$\frac{Db}{Dt} = \boxed{-\alpha_{HV}\Delta^4 b \quad \text{Hyper-viscosity}}$$

$$\boldsymbol{u} = \left(\text{cst.}\boldsymbol{\nabla}^{\perp}\Delta^{-\frac{1}{2}}\right) b$$

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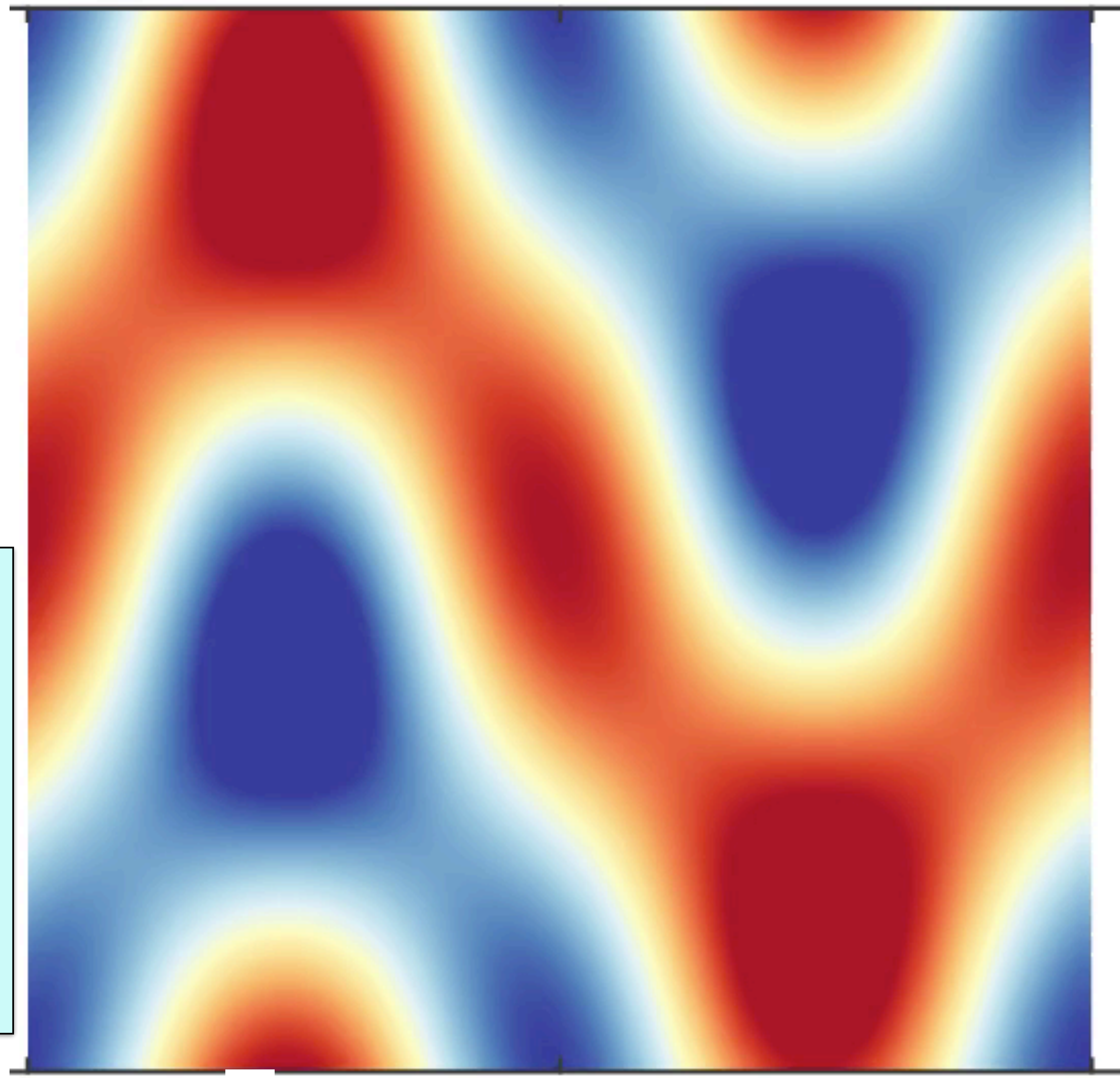
deterministic SQG
1024 x 1024

Constantin et al.
(1994, 1999, 2012)

$t = 0$ day



$$t = 15 \text{ day}$$

SQG

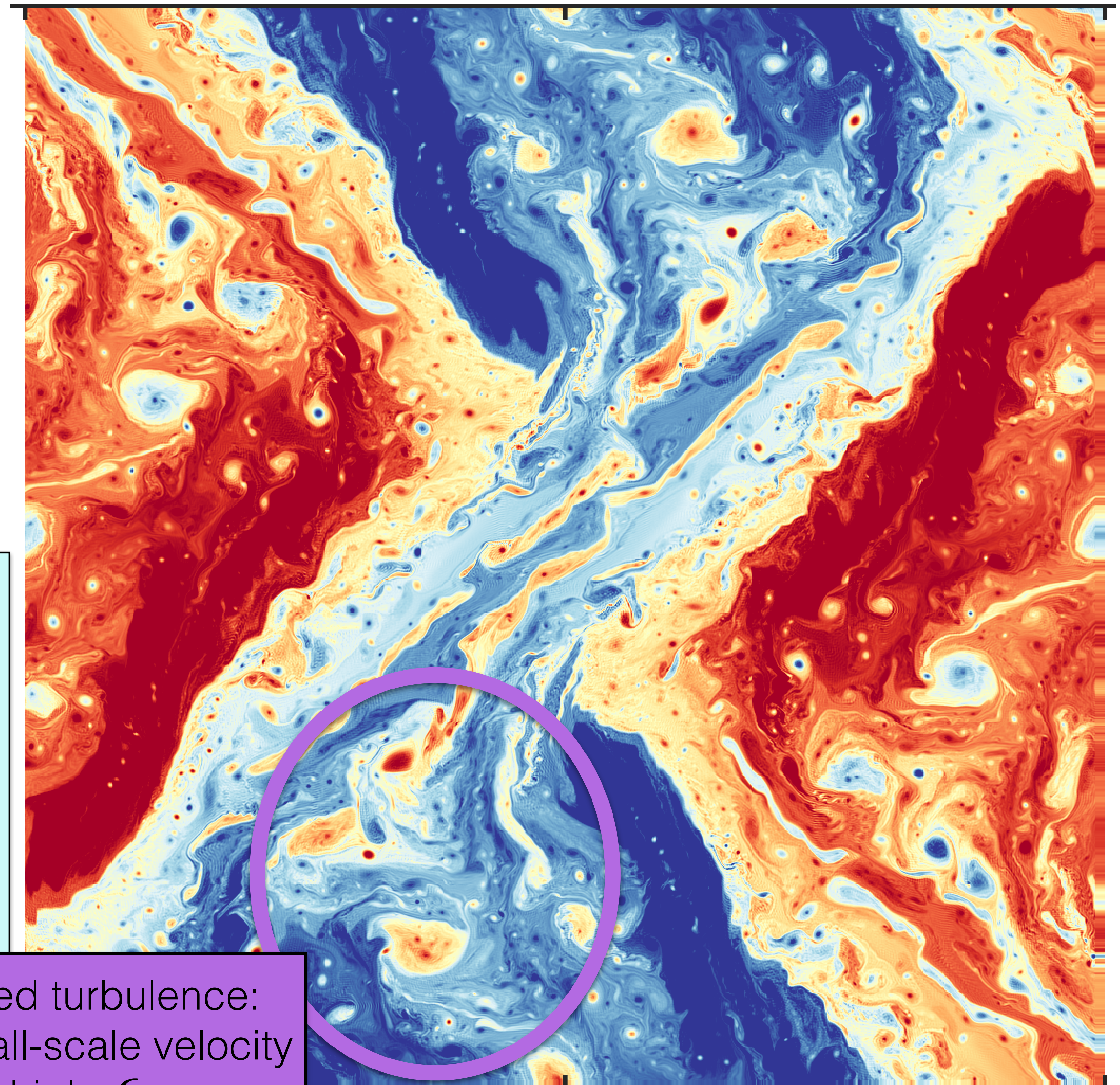
$$\frac{Db}{Dt} = \boxed{-\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}}$$

$$\boldsymbol{u} = \left(\text{cst.} \boldsymbol{\nabla}^{\perp} \Delta^{-\frac{1}{2}} \right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 199



Developed turbulence:
strong small-scale velocity
high ϵ

$$t = 15 \text{ day}$$

SQG

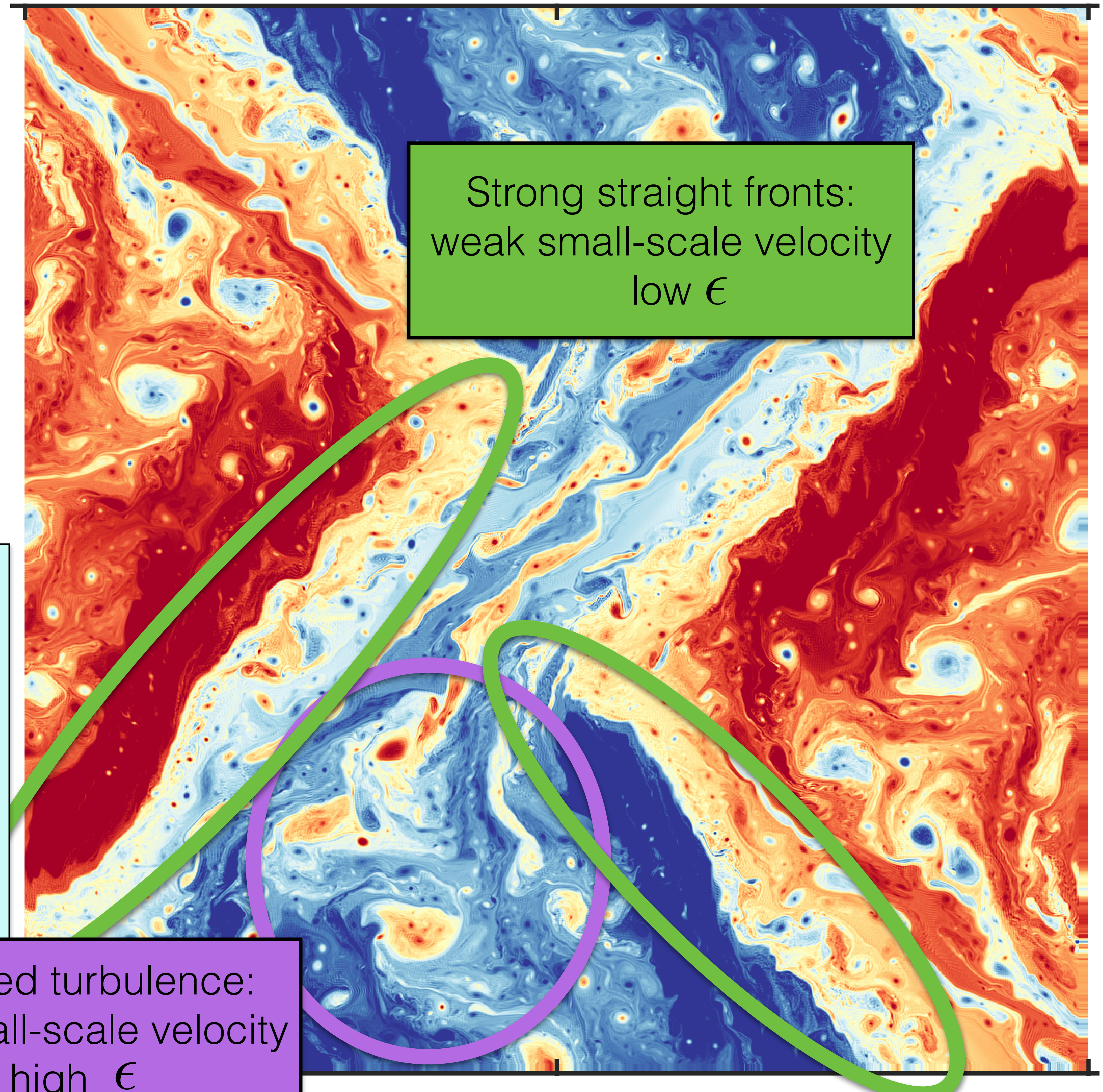
$$\frac{Db}{Dt} = \boxed{-\alpha_{HV}\Delta^4 b \quad \text{Hyper-viscosity}}$$

$$\boldsymbol{u} = \left(\text{cst.}\boldsymbol{\nabla}^{\perp}\Delta^{-\frac{1}{2}}\right) b$$

Reference flow:

deterministic SQG
1024 x 1024

Constantin et al.
(1994, 199



Strong straight fronts:
weak small-scale velocity
low ϵ

Developed turbulence:
strong small-scale velocity
high ϵ

Large scales:

$\boldsymbol{w}$

Small scales:

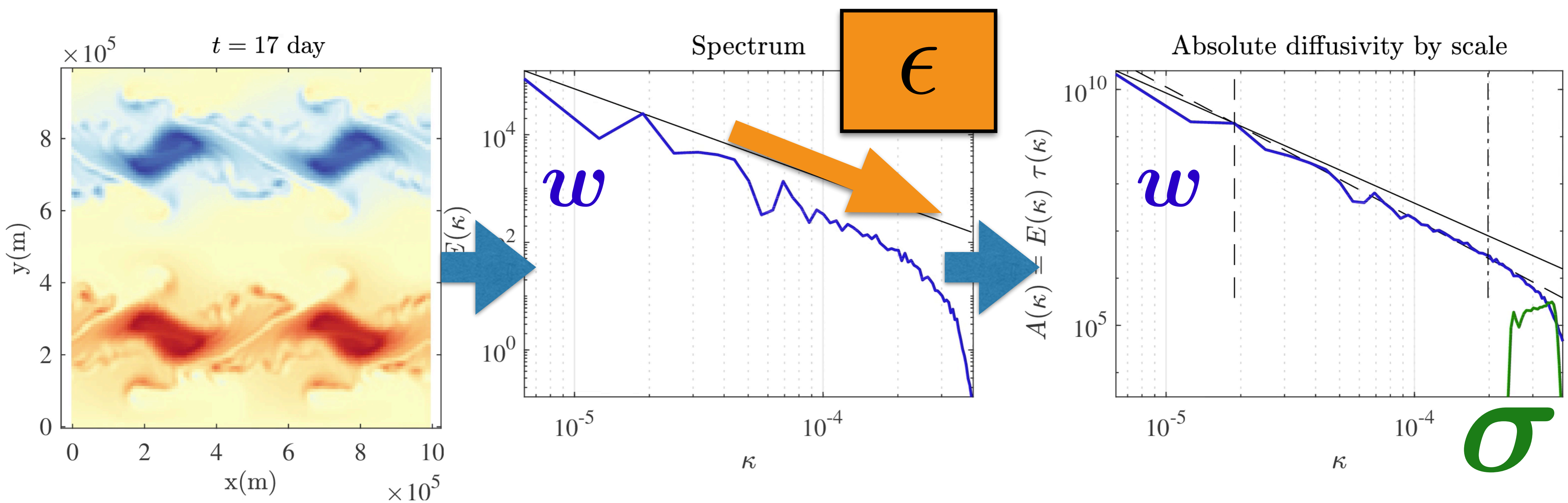
$\boldsymbol{\sigma}\dot{B}$

Variance tensor:

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB\,(\boldsymbol{\sigma} dB)^T\}}{dt}$$

Heterogeneous modulation of $\boldsymbol{\sigma}$

(for Resseguier et al. 2017b)



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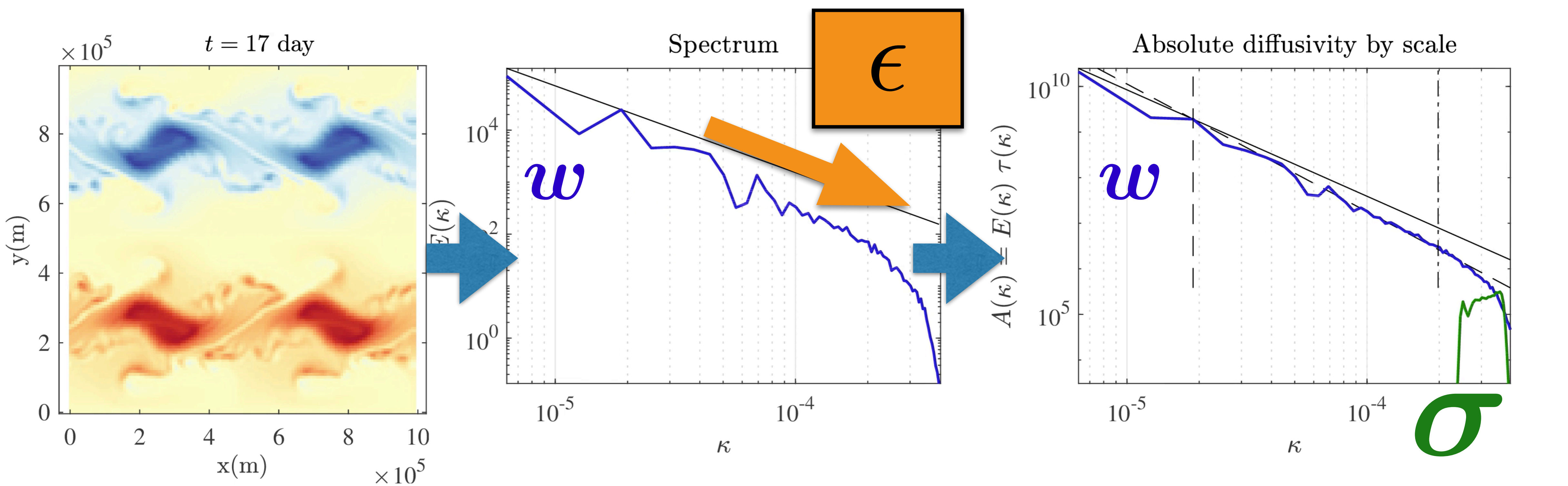

Large scales:

$\boldsymbol{w}$

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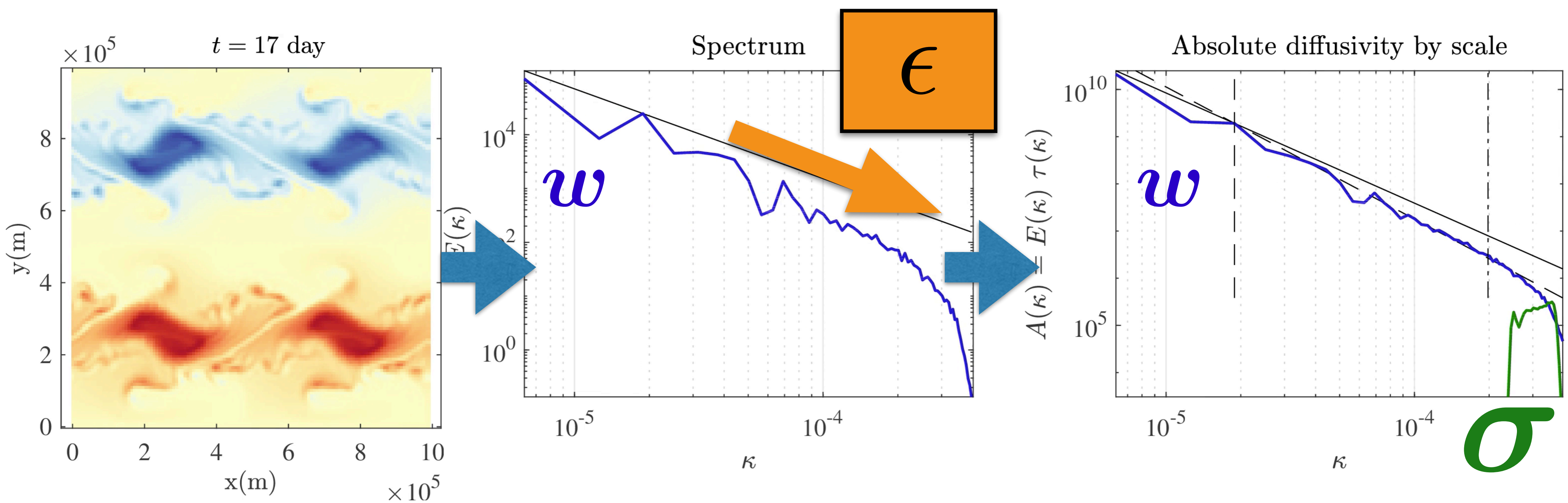
Variance tensor:

$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{\mathbb{E}\{\boldsymbol{\sigma} dB\,(\boldsymbol{\sigma} dB)^T\}}{dt}$

Heterogeneous modulation of $\boldsymbol{\sigma}$ (for Resseguier et al. 2017b)

$$A(\kappa, x) = E(\kappa, x)\tau(\kappa, x) = \kappa^{-3/2}E^{1/2}(\kappa, x) = \text{cst. } \epsilon^p(x)\ \kappa^{-q}$$



Heterogeneous modulation of $\boldsymbol{\sigma}$
(for Resseguier et al. 2017b)

Large scales:

$\boldsymbol{w}$

Small scales:

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Variance tensor:

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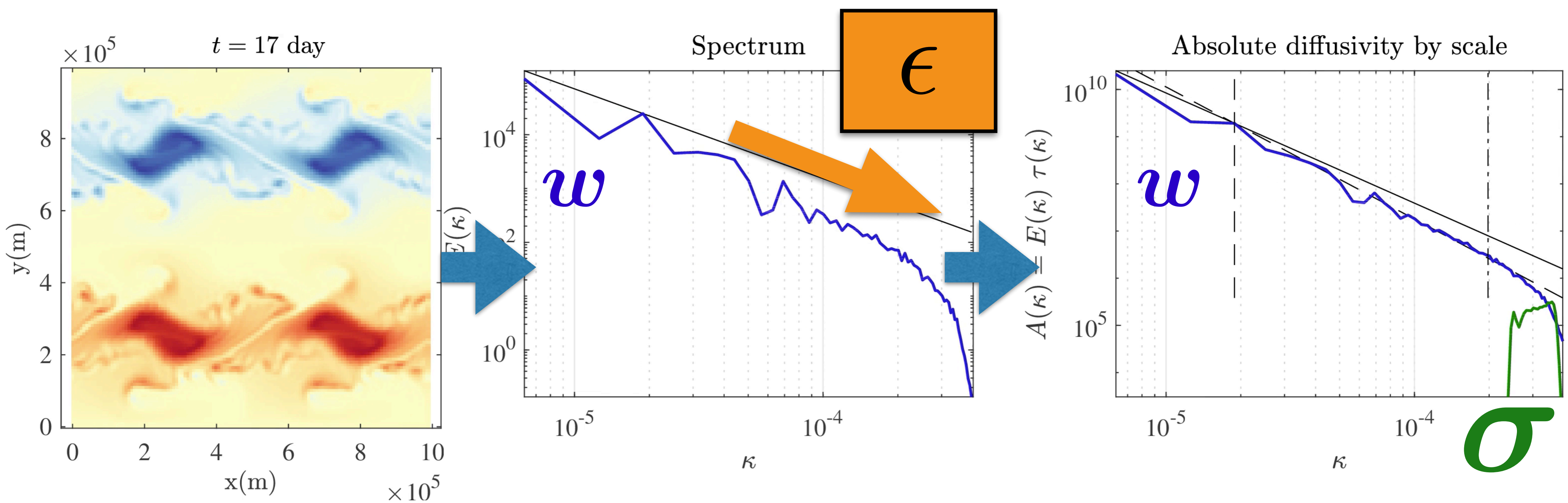
Dynamics
(1/3 for SQG)

$$A(\kappa,x) = E(\kappa,x)\tau(\kappa,x) = \kappa^{-3/2}E^{1/2}(\kappa,x) = \text{cst. } \epsilon^{p}(x)\ \kappa^{-q}$$



Heterogeneous modulation of $\boldsymbol{\sigma}$

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Large scales:

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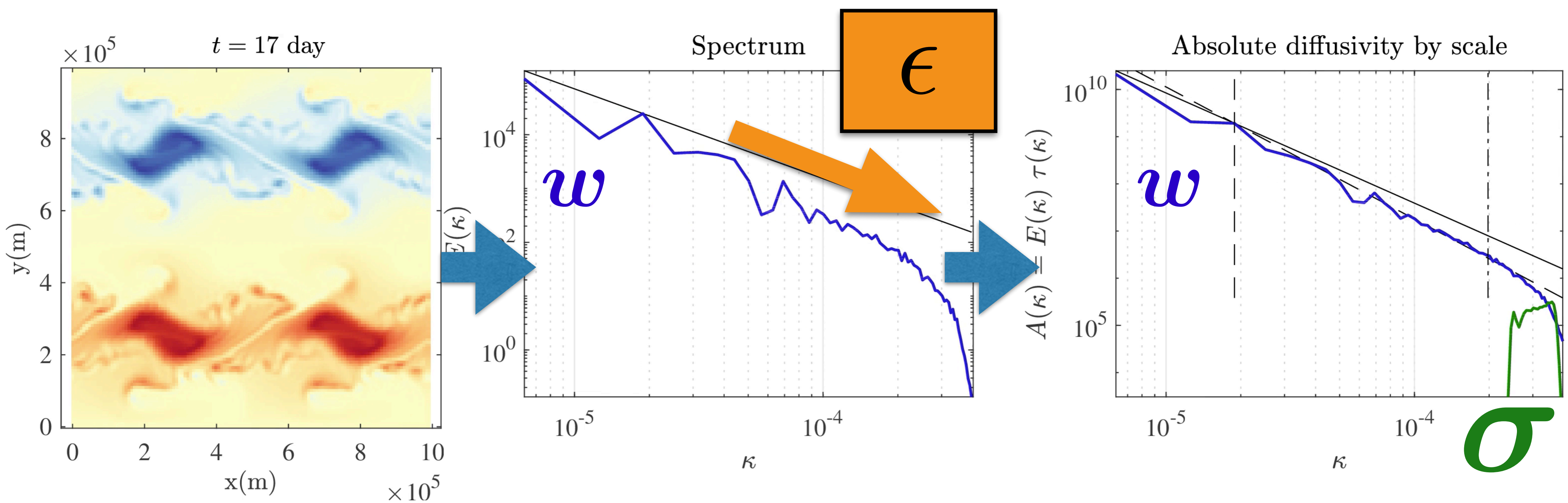
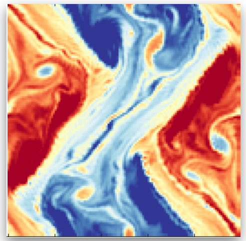
(1/3 for SQG)

$$A(\kappa, x) = E(\kappa, x)\tau(\kappa, x) = \kappa^{-3/2}E^{1/2}(\kappa, x) = \text{cst}\ \epsilon^{p}(x)\ \kappa^{-q}$$

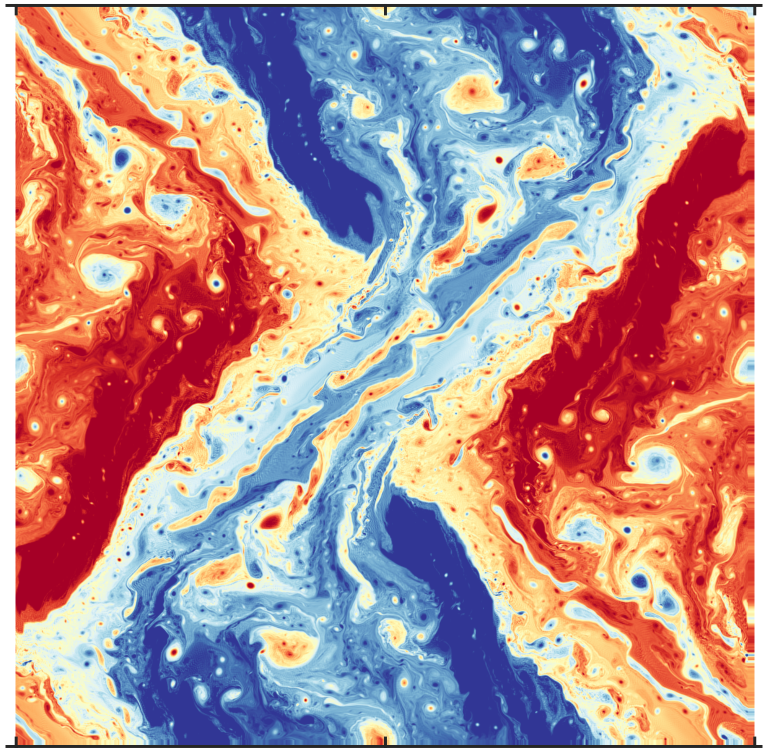
Learned on
large scales at each time t



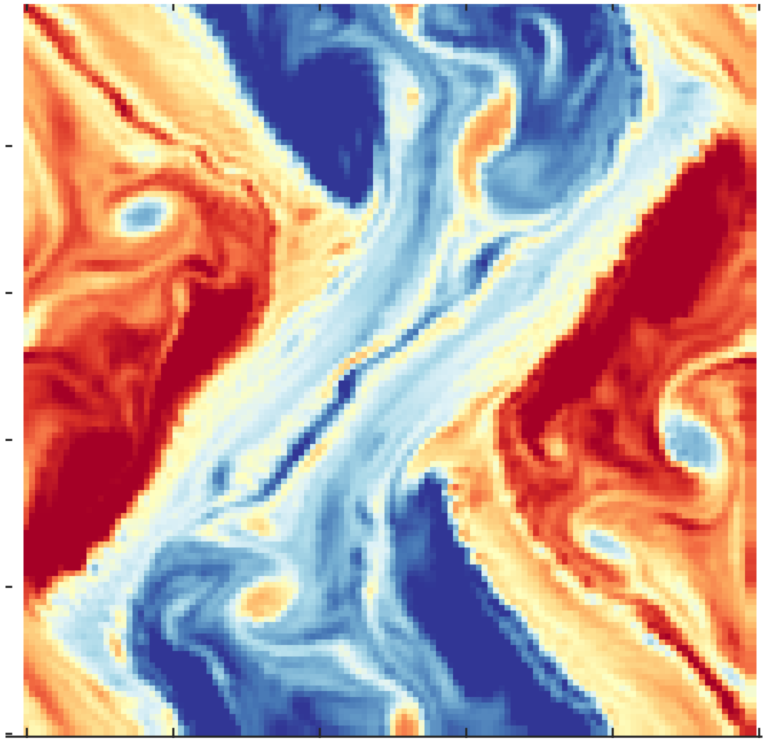


Heterogenous modulation

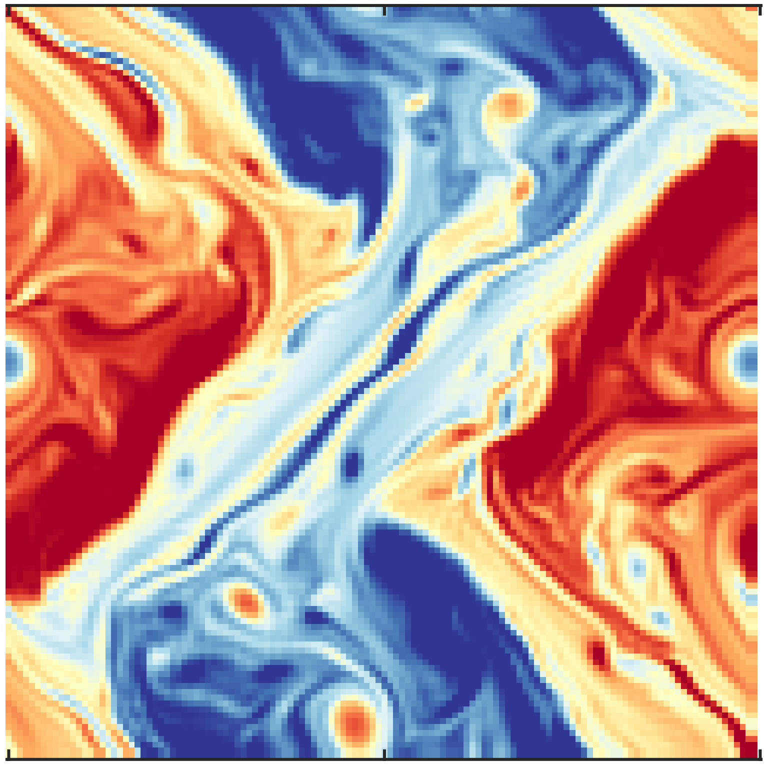
Deterministic 1024 x 1024

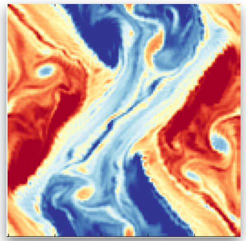


Stochastic 128 x128
with Homogeneous
small-scale velocity



Stochastic 128 x128
with Energy-flux-based
modulation



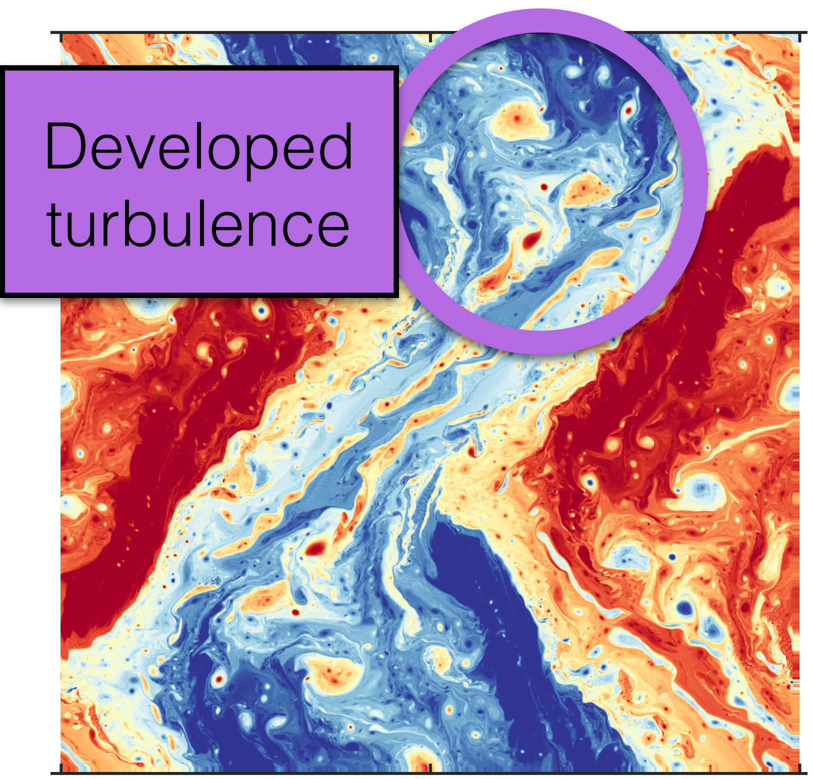


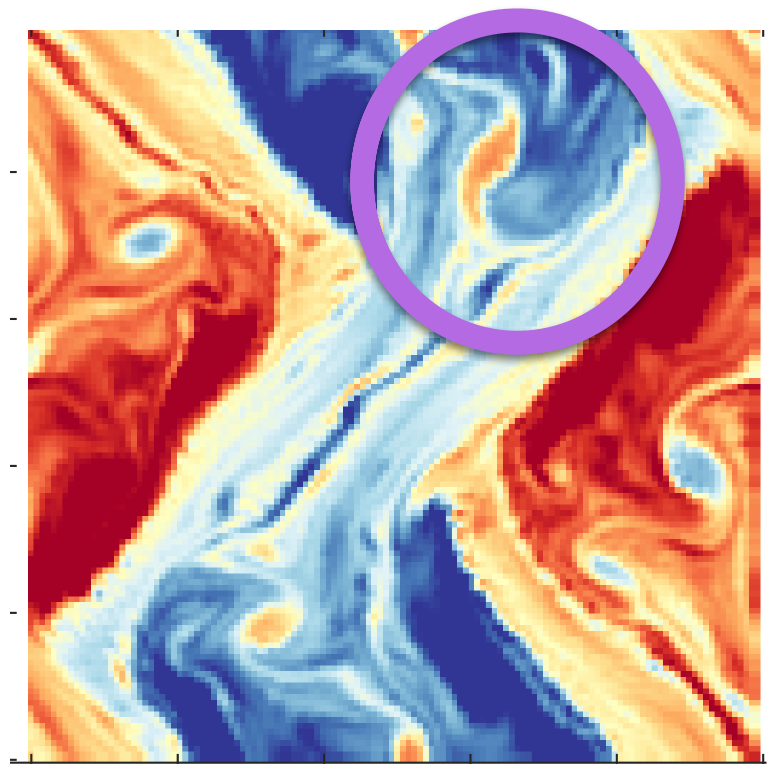
Heterogenous modulation

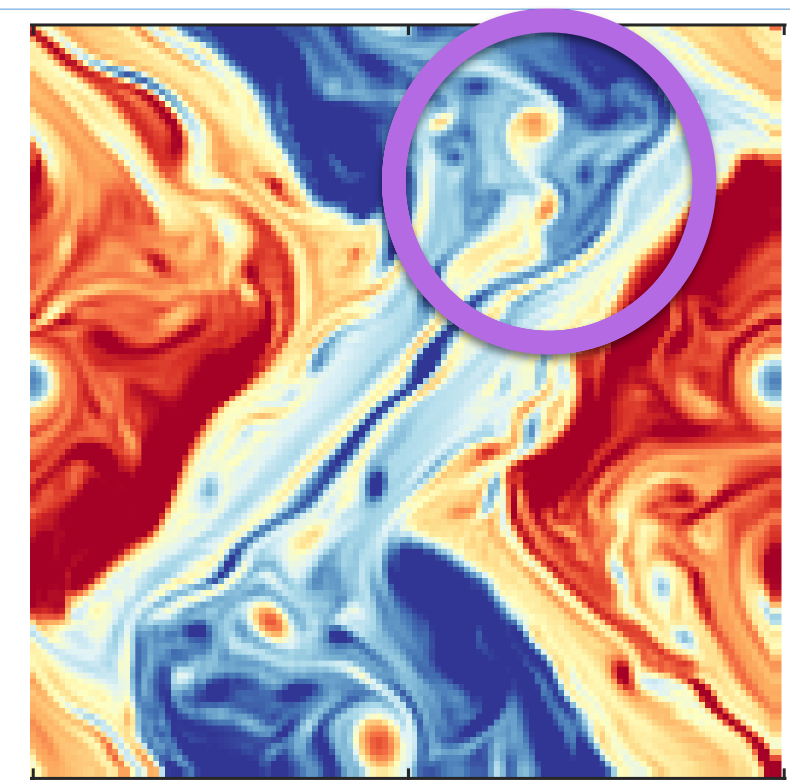
Deterministic 1024 x 1024

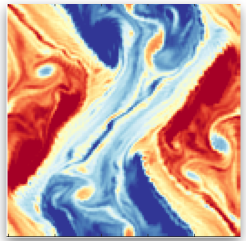
Stochastic 128 x128
with Homogeneous
small-scale velocity

Stochastic 128 x128
with Energy-flux-based
modulation



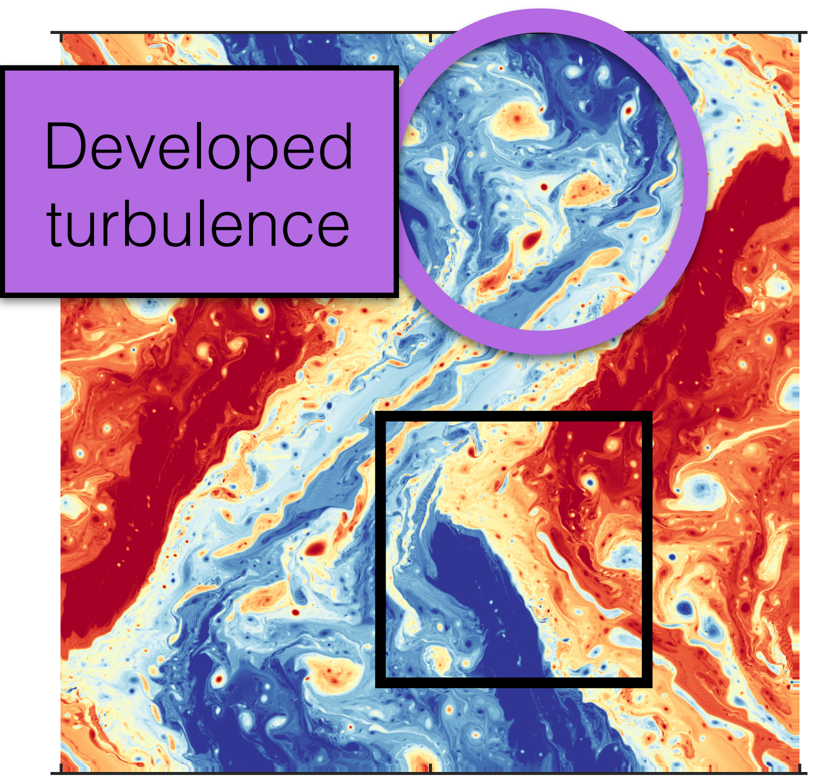




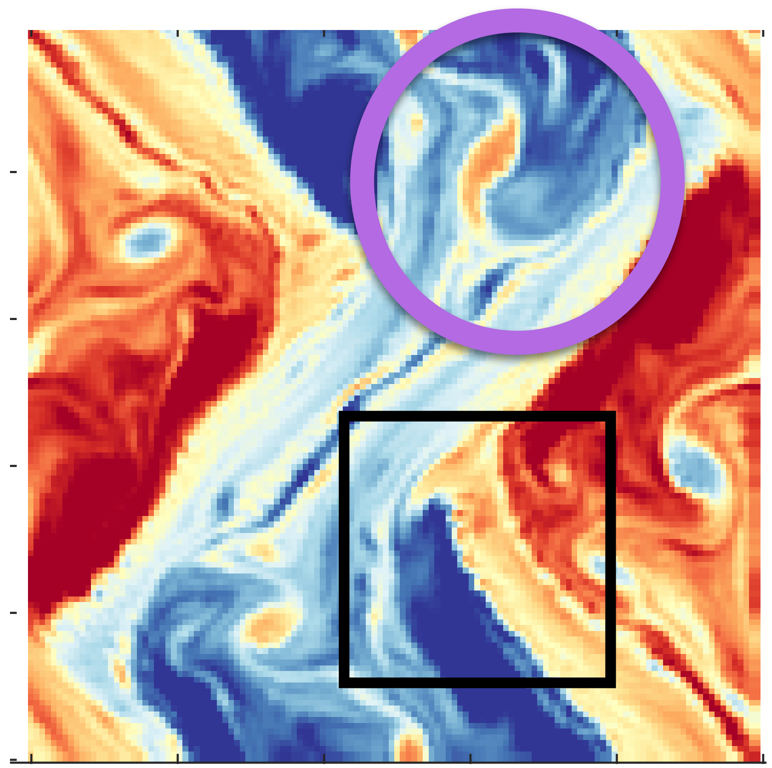


Heterogenous modulation

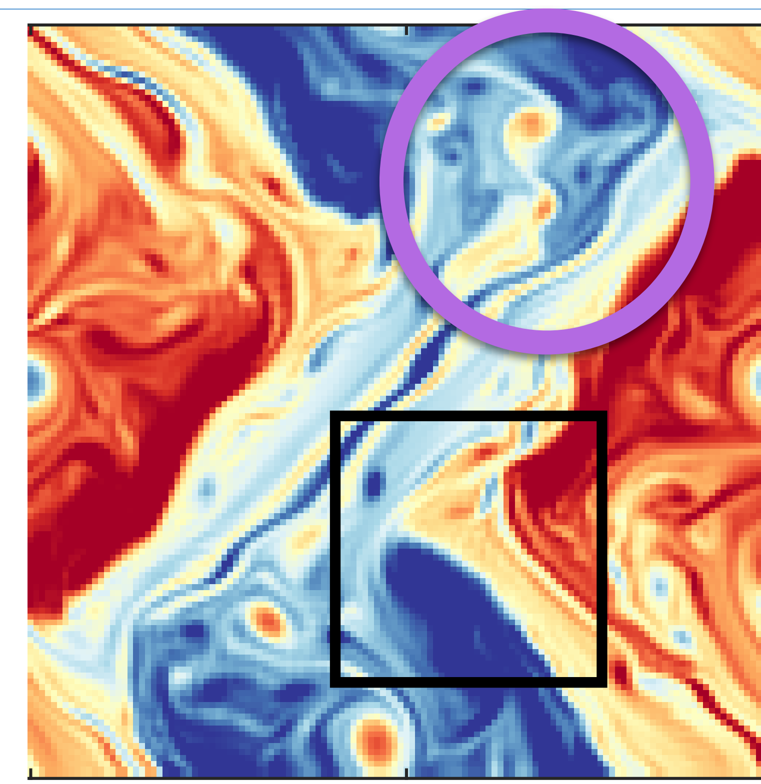
Deterministic 1024 x 1024



Stochastic 128 x128
with Homogeneous
small-scale velocity



Stochastic 128 x128
with Energy-flux-based
modulation

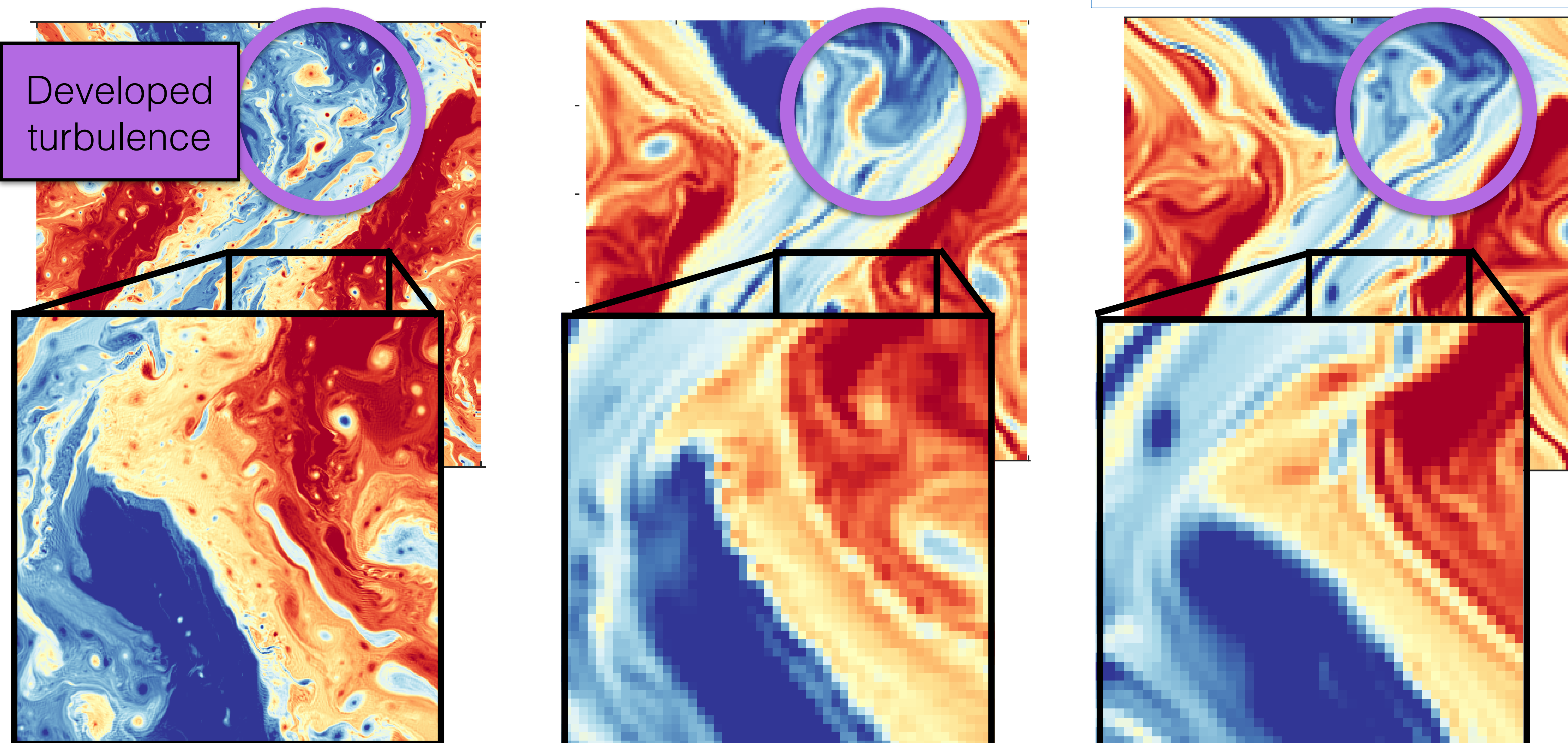


Heterogenous modulation

Deterministic 1024 x 1024

Stochastic 128 x128
with Homogeneous
small-scale velocity

Stochastic 128 x128
with Energy-flux-based
modulation



Conclusion

Conclusion

- Stochastic transport (in LU & SALT) blindly describes unresolved triades

- LU conserves kinetic energy / SALT conserves helicity

- For the small-scale velocity parametrisation,
  both the data-driven (Cotter et al. 2018b) and the self-similar method (Resseguier et al. 2017b)
  lead to accurate uncertainty quantification
  (to address filter divergence)

- Energy-flux modulation improves the simulations

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- Cotter, C. J., Crisan, D., Holm, D. D., Pan, W., & Shevchenko, I. (2018). Numerically modelling stochastic lie transport in fluid dynamics. *arXiv preprint arXiv: 1801.09729*.

- Kunita, H., Stochastic Flows and Stochastic Differential Equations, 1990. (Cambridge: Cambridge University Press).

- Resseguier, V., Mémin, E., & Chapron, B. (2017). Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophysical & Astrophysical Fluid Dynamics*, *111*(3), 177-208.