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Dynamics under location uncertainty: model derivation, modified transport and uncertainty quantification

Valentin Resseguier, Baylor Fox-Kemper
Etienne Memin, Bertrand Chapron
Motivations

- Rigorously identified subgrid dynamics effects
- Injecting likely small-scale dynamics
- Predict extreme events
- Quantification of modeling errors
- Studying different likely scenarios and attractors

Motivations:
- Ensemble forecasts and data assimilation
- Climate projections
Contents

I. Location uncertainty
II. SQG under moderate uncertainty
Part I

Location uncertainty
Adding random velocity

\[ v = w + \sigma \dot{B} \]
Adding random velocity

\[ v = \omega + \sigma \dot{B} \]
Adding random velocity

\[ \mathbf{v} = \mathbf{w} + \sigma \dot{\mathbf{B}} \]

- Resolved large scales
- White-in-time small scales
Adding random velocity

Resolved large scales

White-in-time small scales

\[ \nu = w + \sigma \dot{B} \]
Adding random velocity

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor: $a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt$

\[ \nu = w + \sigma \dot{B} \]

Resolved large scales
White-in-time small scales

References:
Mikulevicius and Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz and Holm 2017
Advection of tracer $\Theta$

Large scales:
$w$

Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \frac{E\{\sigma dB (\sigma dB)^T\}}{dt}$

$$\frac{D\Theta}{Dt} = 0$$
Advection of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}$$
Advection of tracer $\Theta$

\[
\partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{\mathbf{B}} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)
\]
Advection of tracer $\Theta$

$$\partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$

Large scales:
- $\mathbf{w}$
- $\sigma \dot{B}$

Small scales:
- Variance tensor:
  $$a = a(x, x) = \mathbb{E} \{ \sigma dB (\sigma dB)^T \}$$

$$\frac{\partial \Theta}{\partial t} + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$
Advection of tracer $\Theta$

Large scales:

$w$

Small scales:

$\sigma \dot{B}$

Variance tensor:

$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} / dt$

\[
\partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)
\]
Advection of tracer $\Theta$

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:

$$\sigma = a(x, x) = E\{\sigma dB (\sigma dB)^T\}$$

$$\frac{\partial \Theta}{\partial t} + \omega^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \left( \frac{1}{2} a \nabla \Theta \right)$$

Drift correction
Large scales: 
\( w \)

Small scales: 
\( \sigma \dot{B} \)

Variance tensor:
\[
a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} \frac{dt}{dt}
\]
Advection of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:
$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}$$

$$\partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$
Advection of tracer $\Theta$

\[ \partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \mathbf{\dot{B}} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right) \]

- Large scales: $w$
- Small scales: $\sigma \mathbf{\dot{B}}$
- Variance tensor: $a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt$

Drift correction

Multiplicative random forcing
Advection of tracer $\Theta$

\[ \partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{\mathbf{B}} \cdot \nabla \Theta = \mathbf{\nabla} \cdot \left( \frac{1}{2} a \mathbf{\nabla} \Theta \right) \]

- **Advection**
- **Diffusion**
- **Drift correction**
- **Multiplicative random forcing**
- **Balanced energy exchanges**

Large scales: $\mathbf{w}$
Small scales: $\sigma \dot{\mathbf{B}}$

Variance tensor:

\[ a = a(x, x) = \mathbb{E} \{ \sigma dB (\sigma dB)^T \} \]
Derived random models

Large scales: \( w \)
Small scales: \( \sigma \dot{B} \)
Variance tensor:
\[
a = a(x, x) = E\{\sigma dB (\sigma dB)^T\} dt
\]

Conservations (mass, linear momentum, …)

\[ \frac{D}{Dt} \]

Navier-Stokes
Derived random models

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

\[ a = a(x, x) = \frac{1}{dt} E\{ \sigma dB \sigma dB^T \} \]

Conservations
(mass, linear momentum, …)

\[ \frac{D}{Dt} \]

Navier-Stokes

Simplifications
Derived random models

Large scales:
\[ w \]
Small scales:
\[ \sigma \dot{B} \]
Variance tensor:
\[ a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt} \]

Conservations (mass, linear momentum, …)

\[ \frac{D}{Dt} \]

Navier-Stokes

Simplifications
Derived random models

Large scales: \( w \)
Small scales: \( \sigma \dot{B} \)

Variance tensor:
\[
a = a(x, x) = E\left\{ \sigma dB (\sigma dB)^T \right\} dt
\]

Conservations (mass, linear momentum, ...)

\[
\frac{D}{Dt}
\]

Reduced Order Model

Derived random models

Boussinesq

Navier-Stokes

Conservations (mass, linear momentum, ...)

Reduced Order Model

Data

Simplifications
Derived random models

Large scales: \( w \)
Small scales: \( \sigma \dot{B} \)
Variance tensor:
\[
a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} / dt
\]

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt}
\]

Reduced Order Model

Lorenz 63

Boussinesq

Navier-Stokes

Simplifications
Derived random models

Large scales: \( w \)
Small scales: \( \sigma \dot{B} \)
Variance tensor:
\[
a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt
\]

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt}
\]

\[
\text{Data} \rightarrow \text{Reduced Order Model} \rightarrow \text{Lorenz 63} \rightarrow \text{Boussinesq} \rightarrow \text{Navier-Stokes}
\]

Simplifications
Derived random models

Conservations (mass, linear momentum, ...)

Large scales: $w$
Small scales: $\sigma\dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E}\{\sigma dB(\sigma dB)^T\} / dt$

Data ⟷ Reduced Order Model

Reduced random models

Lorenz 63

$\frac{a}{2UL}$

Uncertainty

Boussinesq

Navier-Stokes

$\frac{D}{Dt}$

Simplifications
Derived random models

Large scales: \( \mathbf{w} \)
Small scales: \( \sigma \dot{B} \)
Variance tensor:
\[
a = a(x, x) = \frac{1}{dt} \mathbb{E}\left\{\sigma dB (\sigma dB)^T\right\}
\]

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt} \rightarrow \text{Navier-Stokes}
\]

\[
\text{Reduced Order Model} \rightarrow \text{Lorenz 63}
\]

\[
\frac{a/2}{UL} \rightarrow \text{QG MU}
\]

\[
\text{QG} \rightarrow \text{Uncertainty}
\]

Simplifications
Derived random models

Large scales: \( \mathbf{w} \)
Small scales: \( \sigma \dot{\mathbf{B}} \)

Variance tensor:
\[
\mathbf{a} = \mathbf{a}(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt
\]

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt}
\]

Data

Reduced Order Model

Lorenz 63

\[
\frac{\dot{a}}{UL}
\]

SQG MU

QG MU

Boussinesq

Navier-Stokes

Uncertainty

Simplifications
Derived random models

Large scales:
\( \mathbf{w} \)
Small scales:
\( \sigma \dot{\mathbf{B}} \)

Variance tensor:
\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt \]

Conservations (mass, linear momentum, …)

\[ \frac{D}{Dt} \]

\[ \frac{a/2}{UL} \]

Uncertainty

SQG MU

SQG SU

QG MU

Lorenz 63

Boussinesq

Navier-Stokes

Derived random models
Derived random models

Large scales: $\mathbf{w}$
Small scales: $\sigma \mathbf{B}$

Variance tensor:
\[ a = a(x, x) = \mathbb{E}\{\sigma dB \ (\sigma dB)^T \} / dt \]

\[ \dot{b}^2 \]

Conservations (mass, linear momentum, …)

\[ \frac{D}{Dt} \]

Reduced Order Model

Lorenz 63

\[ \frac{a}{2L} \]

Uncertainty

SQG MU

SQG SU

Boussinesq

Navier-Stokes

Derived random models
Part II
SQG under Moderate Uncertainty

SQG MU

Code available online
\[
\frac{Db}{Dt} = -\alpha_{HV} \Delta^4 b \quad \text{Hyper-viscosity}
\]

\[u = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b\]

Reference flow:
- deterministic
- SQG
- 1024 x 1024

\(t = 17\ \text{day}\)
\[ \frac{D b}{D t} = -\alpha_{HV} \Delta^4 b \quad \text{Hyperviscosity} \]

\[ u = \left( \text{cst.} \nabla \perp \Delta^{-\frac{1}{2}} \right) b \]

Reference flow:
- deterministic
- SQG
- 1024 x 1024

\[ t = 17 \text{ day} \]
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128

$t = 17$ days
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128
One realization : Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x128

Spectrum
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128

$t = 17$ days

$x (m) \times 10^5$

$y (m)$

$\hat{b}(\kappa)$

$\kappa (\text{rad.m})$
One realization: Stochastic destabilization

![Deterministic 128 x 128](image1)

![Deterministic 1024 x 1024](image2)

![Location Uncertainty 128 x 128](image3)
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128
One realization: Stochastic destabilization

- Deterministic 128 x 128
- Deterministic 1024 x 1024
- Location Uncertainty 128 x 128

$t = 17$ days
Ensemble: random coherent structures
Ensemble: random coherent structures
Ensemble: uncertainty quantification

Spectrum of the errors and its estimation

- Bias LU
- Estim. error LU
- Bias RanIC
- Estim. error RanIC
Ensemble : uncertainty quantification

Spectrum of the errors and its estimation

<table>
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<th>Graph</th>
<th>Description</th>
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<td>Bias error</td>
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<td>Estim. error LU</td>
<td>Estimation of the error</td>
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<tr>
<td>Estim. error RanIC</td>
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Legend:
- Bias RanIC
- Bias LU
- Estim. error RanIC
- Estim. error LU
Conclusion

Models under location uncertainty blindly describe unresolved triades

• Conserve energy

• Model derivation

• Instabilities triggered, possibly followed by extreme events

• Uncertainty quantification to address filter divergence
Related works and outlooks

• Bifurcations (SQG) and attractor (Lorenz 63) exploration

• Stabilization / destabilization of Reduced Order Model

• Comparisons with data-driven $\sigma$ and Stochastic Lie Derivative approaches (Holm and coauthors)

• Parametrization and tests based on higher-order statistics (curvature, energy flux through scales, bispectrum, …)

• Mimic barotropization

• Girsanov theorem for MLE and Bayesian estimations with satellite images

• Learning $\sigma$ on SWOT data

• Filtering / smoothing with (reduced) models under location uncertainty