Dynamics under location uncertainty: model derivation, modified transport and uncertainty quantification
Valentin Resseguier, Baylor Fox-Kemper, Etienne Mémin, Bertrand Chapron

To cite this version:
Valentin Resseguier, Baylor Fox-Kemper, Etienne Mémin, Bertrand Chapron. Dynamics under location uncertainty: model derivation, modified transport and uncertainty quantification. AGU 2017 - American Geophysical Union, Dec 2017, New Orleans, United States. pp.1-47. <hal-01891163>
Dynamics under location uncertainty: model derivation, modified transport and uncertainty quantification

Valentin Resseguier, Baylor Fox-Kemper
Etienne Memin, Bertrand Chapron
Motivations

• Rigorously identified subgrid dynamics effects

• Injecting likely small-scale dynamics

• Predict extreme events

• Quantification of modeling errors → Ensemble forecasts and data assimilation

• Studying different likely scenarios and attractors → Climate projections
Contents

I. Location uncertainty
II. SQG under moderate uncertainty
Part I
Location uncertainty
Adding random velocity

\[ v = w + \sigma \dot{B} \]
Adding random velocity

\[ \nu = w + \sigma \dot{B} \]

Resolved large scales
Adding random velocity

\[ \nu = \omega + \sigma \dot{B} \]

Resolved large scales

White-in-time small scales
Adding random velocity

\[ \nu = w + \sigma \dot{B} \]

- Large scales:
  \( w \)

- Small scales:
  \( \sigma \dot{B} \)

Variance tensor:

\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt \]

Resolved large scales

White-in-time small scales
Adding random velocity

Large scales: \( w \)
Small scales: \( \sigma \dot{B} \)
Variance tensor: 

\[
a = a(x, x) = \frac{1}{dt} \mathbb{E} \{ \sigma dB (\sigma dB)^T \}
\]

\( \nu = w + \sigma \dot{B} \)

Resolved large scales

White-in-time small scales

References:

Mikulevicius and Rozovskii, 2004
Flandoli, 2011

Memin, 2014
Resseguier et al. 2017 a, b, c
Chapron et al. 2017
Cai et al. 2017

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al., 2017

Cotter and al 2017
Crisan et al., 2017
Gay-Balmaz and Holm 2017
Advection of tracer $\Theta$

\[
\frac{D\Theta}{Dt} = 0
\]

Large scales: $w$
Small scales: $\sigma \dot{B}$
Variance tensor:

\[
a = a(x, x) = \mathbb{E}\{\sigma dB \sigma dB^T\} \frac{dt}{dt}
\]
Advection of tracer $\Theta$

Large scales:

$w$

Small scales:

$\dot{B}$

Variance tensor:

$a = a(x, x) = \frac{E\{\sigma dB (\sigma dB)^T\}}{dt}$
Advection of tracer $\Theta$

\[
\partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{\mathbf{B}} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)
\]

Large scales:
- $\mathbf{w}$
- $\sigma \dot{\mathbf{B}}$

Small scales:
- $a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt$

Variance tensor:
- $\sigma \dot{\mathbf{B}}$
Advection of tracer $\Theta$

\[ \partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \mathbf{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right) \]

Large scales:
- $\mathbf{w}$
- $a = a(x, x) = \mathbb{E}\{\sigma \mathbf{B} (\sigma \mathbf{B})^T\}$

Small scales:
- $\sigma \mathbf{B}$
- Variance tensor:

\[ \dot{a} = \dot{a}(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} \]
Advection of tracer $\Theta$

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt$$

$$\frac{\partial t}{\partial t} \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)$$
Advection of tracer $\Theta$

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E}\{\sigma dB(\sigma dB)^T\}$

$\frac{\partial t}{\theta} + \nabla \cdot (w^* \nabla \theta + \sigma \dot{B} \cdot \nabla \theta) = \nabla \cdot \left( \frac{1}{2} a \nabla \theta \right)$

Drift correction
Advection of tracer $\Theta$

Large scales:
- $w$

Small scales:
- $\sigma \dot{B}$

Variance tensor:
- $a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt$

\[
\partial_t \Theta + \underbrace{w^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta}_{\text{Advection}} = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)_{\text{Diffusion}}
\]

Drift correction

Multiplicative random forcing
Advection of tracer Θ

\[ \partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \mathbf{\dot{B}} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right) \]

Large scales:
- \( w \)
- \( \sigma \mathbf{\dot{B}} \)

Small scales:
- Variance tensor:
  \[ a = a(x, x) = \mathbb{E} \{ \sigma dB (\sigma dB)^T \} \]

Drift correction

Multiplicative random forcing
Advection of tracer $\Theta$

\[
\frac{\partial_t \Theta}{\partial t} + \mathbf{w}^* \cdot \nabla \Theta + \sigma \mathbf{\dot{B}} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)
\]

Large scales: $\mathbf{w}$
Small scales: $\sigma \mathbf{\dot{B}}$
Variance tensor: $a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt$

Drift correction
Multiplicative random forcing
Advection of tracer $\Theta$

Large scales:
$w$
Small scales:
$\sigma \dot{B}$

Variance tensor:
$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} / dt$

\[
\partial_t \Theta + \mathbf{w}^* \cdot \nabla \Theta + \sigma \dot{B} \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)
\]

Drift correction
Multiplicative random forcing
Balanced energy exchanges
Derived random models

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:

$$a = a(x, x) = E\{\sigma dB (\sigma dB)^T\} dt$$

Conservations (mass, linear momentum, …)
Derived random models

Large scales:
\( \mathbf{w} \)
Small scales:
\( \sigma \mathbf{B} \)
Variance tensor:
\[
a = a(x, x) = \mathbb{E}\left\{ \sigma dB \,(\sigma dB)^T \right\} \, dt
\]

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt}
\]

Navier-Stokes

Simplifications
Derived random models

Large scales:
\[ w \]
Small scales:
\[ \sigma \dot{B} \]
Variance tensor:
\[ a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt} \]

\[ D \]

Conservations (mass, linear momentum, …)

\[ \frac{D}{Dt} \]

Navier-Stokes

Simplifications
Derived random models

Conservations
(mass, linear momentum, …)

\[ D \]

\[ \frac{D}{Dt} \]

Large scales:
\( w \)

Small scales:
\( \sigma \dot{B} \)

Variance tensor:
\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} \]

Large scales:

Small scales:

Variance tensor:

Conservations
(mass, linear momentum, …)

Boussinesq

Navier-Stokes

Simplifications
Derived random models

Large scales: \( w \)
Small scales: \( \sigma B \)

Variance tensor:
\[ a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} \]

\[ \frac{D}{Dt} \]

Conservations (mass, linear momentum, …)

\[ \dot{\mathcal{B}} = a(x, x) \mathbb{E}\{\sigma dB (\sigma dB)^T\} \]

Data

Reduced Order Model

Lorenz 63

Boussinesq

Navier-Stokes

Simplifications
Derived random models

Large scales: $w$
Small scales: $\sigma \hat{B}$

Variance tensor:

$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\} dt$$

Conservations
(mass, linear momentum, …)

Conservations
(mass, linear momentum, …)

$$\frac{D}{Dt}$$

Data

Reduced Order Model

Lorenz 63

QG

Boussinesq

Navier-Stokes
Derived random models

Large scales: $\mathbf{w}$
Small scales: $\sigma \mathbf{B}$

Variance tensor:
$$ a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt} $$

Data
Reduced Order Model

Lorenz 63

Boussinesq

Conservations (mass, linear momentum, …)

Navier-Stokes

Simplifications

Uncertainty

$$ \frac{a}{2UL} $$
Derived random models

Large scales: $w$
Small scales: $\sigma \dot{B}$

Variance tensor:
$$a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}$$

Conservations (mass, linear momentum, …)

$$\frac{D}{Dt}$$

Reduced Order Model

Lorenz 63

Boussinesq

Navier-Stokes

QG

QG MU

$\frac{a/2}{UL}$

Uncertainty

Simplifications
Reduced Order Models

Data

Reduced Order Model

Lorenz 63

Boussinesq

Navier-Stokes

Derived random models

Large scales: \( \mathbf{w} \)
Small scales: \( \sigma \frac{\partial \mathbf{B}}{\partial t} \)

Variance tensor:
\[
\sigma = \sigma(x, x) = \mathbb{E}\left\{\sigma dB (\sigma dB)^T\right\} dt
\]

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt}
\]

\[
\mathbf{w} = \mathbf{w}(x, x)
\]

\[
\sigma^2 \frac{\partial}{\partial t} (\sigma dB (\sigma dB)^T)
\]

\[
E\{\sigma dB (\sigma dB)^T\} dt
\]

QG

SQG MU

QG MU

Uncertainty

Lorenz 63

Boussinesq

Navier-Stokes

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt}
\]
Derived random models

Large scales: \( w \)
Small scales: \( \sigma \dot{B} \)

Variance tensor:
\[
a = a(x, x) = \frac{\mathbb{E}\{\sigma dB (\sigma dB)^T\}}{dt}
\]

\[\dot{B} = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}\]

Conservations (mass, linear momentum, …)

\[\frac{D}{Dt}\]

\[\mathbb{E}\{\sigma dB (\sigma dB)^T\}\]

Uncertainty

Simplifications
Derived random models

Large scales: \( \mathbf{w} \)
Small scales: \( \sigma \mathbf{\dot{B}} \)

Variance tensor:
\[
a = a(x, x) = \mathbb{E}\{\sigma dB (\sigma dB)^T\}/dt
\]

Conservations (mass, linear momentum, …)

\[
\frac{D}{Dt}
\]

Data

Reduced Order Model

Lorenz 63

\[\frac{a/2}{UL}\]

Uncertainty

SQG MU

SQG SU

Boussinesq

Navier-Stokes
Part II
SQG under Moderate Uncertainty

SQG MU

Code available online
\[
\frac{D b}{D t} = -\alpha_H V \Delta^4 b \\
u = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b
\]

Reference flow:
- deterministic
- SQG
- \(1024 \times 1024\)

\(t = 17\) day
\[
\frac{D b}{D t} = -\alpha_{HV} \Delta^{4} b \\
\mathbf{u} = \left( \text{cst.} \nabla^\perp \Delta^{-\frac{1}{2}} \right) b
\]
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128
One realization: Stochastic destabilization

- Deterministic 128 x 128
- Deterministic 1024 x 1024
- Location Uncertainty 128 x 128
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128

Spectrum

$E_{\phi}(\kappa, m, s^{-1}, \text{rad})$

-5/3

$\kappa(\text{rad.m}^{-1})$

$x(m)$

$t = 17$ days

$y(m)$

$10^5$

$10^6$

$10^7$

$10^8$
One realization: Stochastic destabilization

- Deterministic 128 x 128
- Deterministic 1024 x 1024
- Location Uncertainty 128 x 128

$t = 17$ days

Spectrum
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128
One realization: Stochastic destabilization
One realization: Stochastic destabilization

- **Deterministic 128 x 128**
- **Deterministic 1024 x 1024**
- **Location Uncertainty 128 x 128**
One realization: Stochastic destabilization

Deterministic 128 x 128

Deterministic 1024 x 1024

Location Uncertainty 128 x 128
Ensemble: random coherent structures
Ensemble: random coherent structures
Ensemble: uncertainty quantification
Ensemble : uncertainty quantification
Conclusion

Models under location uncertainty blindly describe unresolved triades

- Conserve energy
- Model derivation
- Instabilities triggered, possibly followed by extreme events
- Uncertainty quantification to address filter divergence
Related works and outlooks

- Bifurcations (SQG) and attractor (Lorenz 63) exploration
- Stabilization / destabilization of Reduced Order Model
- Comparisons with data-driven $\sigma$ and Stochastic Lie Derivative approaches (Holm and coauthors)
- Parametrization and tests based on higher-order statistics (curvature, energy flux through scales, bispectrum, …)
- Mimic barotropization
- Girsanov theorem for MLE and Bayesian estimations with satellite images
- Learning $\sigma$ on SWOT data
- Filtering / smoothing with (reduced) models under location uncertainty