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CONTROLLING ITERATED JUMPS OF SOLUTIONS TO COMBINATORIAL PROBLEMS

LUDOVIC PATEY

ABSTRACT. Among the Ramsey-type hierarchies, namely, Ramsey's theorem, the free set, the thin set and the rainbow Ramsey theorem, only Ramsey's theorem is known to collapse in reverse mathematics. A promising approach to show the strictness of the hierarchies would be to prove that every computable instance at level n has a low_n solution. In particular, this requires to control effectively iterations of the Turing jump.

In this paper, we design some variants of Mathias forcing to construct solutions to cohesiveness, the Erdős-Moser theorem and stable Ramsey's theorem for pairs, while controlling their iterated jumps. For this, we define forcing relations which, unlike Mathias forcing, have the same definitional complexity as the formulas they force. This analysis enables us to answer two questions of Wei Wang, namely, whether cohesiveness and the Erdős-Moser theorem admit preservation of the arithmetic hierarchy, and can be seen as a step towards the resolution of the strictness of the Ramsey-type hierarchies.

1. Introduction

The effective forcing is a very powerful tool in the computational analysis of mathematical statements. In this framework, lowness is achieved by deciding formulas during the forcing argument, while ensuring that the whole construction remains effective. Thus, the definitional strength of the forcing relation is very sensitive in effective forcing. We present a new forcing argument enabling one to control iterated jumps of solutions to Ramsey-type theorems. Our main motivation is reverse mathematics.

1.1. Reverse mathematics

Reverse mathematics is a vast mathematical program whose goal is to classify ordinary theorems in terms of their provability strength. It uses the framework of subsystems of second order arithmetic, which is sufficiently rich to express in a natural way many theorems. The base system, RCA_0 standing for Recursive Comprehension Axiom, contains the basic first order Peano arithmetic together with the Δ^0_1 comprehension scheme and the Σ^0_1 induction scheme. Thanks to the equivalence between Δ^0_1-definable sets and computable sets, RCA_0 can be considered as capturing "computational mathematics". The proof-theoretic analysis of the theorems in reverse mathematics is therefore closely related to their computational analysis. See Simpson [19] for a formal introduction to reverse mathematics.

Early reverse mathematics have led to two main empirical observations: First, many ordinary (i.e. non set-theoretic) theorems require very weak set existence axioms. Second, most of those theorems are in fact *equivalent* to one of five main subsystems, known as the "Big Five". However, among the theorems studied in reverse mathematics, a notable class of theorems fails to support those observations, namely, Ramsey-type theorems. This article focuses on consequences of Ramsey's theorem below the arithmetic comprehension axiom (ACA_0). See Hirschfeldt [7] for a gentle introduction to the reverse mathematics below ACA_0.

1.2. Controlling iterated jumps

Among the hierarchies of combinatorial principles, namely, Ramsey's theorem [9, 18, 4], the rainbow Ramsey theorem [6, 20, 16], and the free sets and thin set theorems [3, 22] – only Ramsey's theorem is known to collapse within the framework of reverse mathematics. The above mentioned hierarchies satisfy the lower bounds of Jockusch [9], that is, there exists a computable instance at every level $n \geq 2$ with no Σ^0_n solution. Thus, a possible strategy for proving that a hierarchy is strict consists of showing the existence, for every computable instance at level n, of a low$_n$ solution.

The solutions to combinatorial principles are often built by Mathias forcing, whose forcing relation is known to be of higher definitional strength than the formula it forces [2]. Therefore there is a need for new notions of forcing with a better-behaving forcing relation. In this paper, we design three notions of forcing to construct solutions to cohesiveness, the Erdős-Moser theorem and stable Ramsey's theorem for pairs, respectively. We define a forcing relation with the expected properties, and which formalises the first and the second jump control of Cholak, Jockusch and Slaman [4]. This can be seen as a step toward the resolution the strictness of the Ramsey-type hierarchies. We take advantage of this new analysis of Ramsey-type statements to prove two conjectures of Wang about the preservation of the arithmetic hierarchy.

1.3. Preservation of the arithmetic hierarchy

The notion of preservation of the arithmetic hierarchy has been introduced by Wang in [21], in the context of a new analysis of principles in reverse mathematics in terms of their definitional strength.

Definition 1.1 (Preservation of definitions)

1. A set Y *preserves* Ξ*-definitions* (relative to X) for Ξ among $\Delta^0_{n+1}, \Pi^0_n, \Sigma^0_n$ where $n > 0$, if every properly Ξ (relative to X) set is properly Ξ relative to Y ($X \oplus Y$). Y *preserves the arithmetic hierarchy* (relative to X) if Y preserves Ξ-definitions (relative to X) for all Ξ among $\Delta^0_{n+1}, \Pi^0_n, \Sigma^0_n$ where $n > 0$.
2. Suppose that $\Phi = (\forall X)(\exists Y)\varphi(X, Y)$ and φ is arithmetic. Φ *admits preservation of* Ξ*-definitions* if for each Z and $X \leq_T Z$ there exists Y such that Y preserves Ξ-definitions relative to Z and $\varphi(X, Y)$ holds. Φ *admits preservation of the arithmetic hierarchy* if for each Z and $X \leq_T Z$ there exists Y such that Y preserves the arithmetic hierarchy relative to Z and $\varphi(X, Y)$ holds.

The preservation of the arithmetic hierarchy seems closely related to the problem of controlling iterated jumps of solutions to combinatorial problems. Indeed, a proof of such a preservation usually consists of noticing that the forcing relation has the same strength as the formula it forces, and then deriving a diagonalization from it. See Lemma 2.16 for a case-in-point. Wang proved in [21] that weak König's lemma ($\mathsf{WKL_0}$), the rainbow Ramsey theorem for pairs (RRT^2_2) and the atomic model theorem (AMT) admit preservation of the arithmetic hierarchy. He conjectured that this is also the case for cohesiveness and the Erdős Moser theorem. We prove the two conjectures through the following theorem, where COH stands for cohesiveness and EM for the Erdős-Moser theorem.

Theorem 1.2 COH and EM admit preservation of the arithmetic hierarchy.

1.4. Definitions and notation

Fix an integer $k \in \omega$. A *string* (over k) is an ordered tuple of integers $a_0, \dots, a_{n-1}$ (such that $a_i < k$ for every $i < n$). The empty string is written ϵ. A *sequence* (over k) is an infinite listing of integers $a_0, a_1, \dots$ (such that $a_i < k$ for every $i \in \omega$). Given $s \in \omega$, k^s is the set of strings of length s over k and $k^{<s}$ is the set of strings of length $< s$ over k. As well, $k^{<\omega}$ is the set of finite strings over k and k^ω is the set of sequences (i.e. infinite strings) over k. Given a string $\sigma \in k^{<\omega}$, we denote by $|\sigma|$ its length. Given two strings $\sigma, \tau \in k^{<\omega}$, σ is a *prefix* of τ (written $\sigma \preceq \tau$) if there exists a string $\rho \in k^{<\omega}$ such that $\sigma\rho = \tau$. Given a sequence X, we write $\sigma \prec X$ if $\sigma = X{\upharpoonright}n$ for some $n \in \omega$. A *binary string* (resp. real) is a *string* (resp. sequence) over 2. We may identify a real with a set of integers by considering that the real is its characteristic function.

A tree $T \subseteq k^{<\omega}$ is a set downward-closed under the prefix relation. A *binary* tree is a set $T \subseteq 2^{<\omega}$. A set $P \subseteq \omega$ is a *path* through T if for every $\sigma \prec P$, $\sigma \in T$. A string $\sigma \in k^{<\omega}$ is a *stem* of a tree T if every $\tau \in T$ is comparable with σ. Given a tree T and a string $\sigma \in T$, we denote by $T^{[\sigma]}$ the subtree $\{\tau \in T : \tau \preceq \sigma \vee \tau \succeq \sigma\}$.

Given two sets A and B, we denote by $A < B$ the formula $(\forall x \in A)(\forall y \in B)[x < y]$ and by $A \subseteq^* B$ the formula $(\forall^\infty x \in A)[x \in B]$, intuitively meaning that A is included into B *up to finite changes.* A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition (F_1, X_1) *extends* (F, X) (written $(F_1, X_1) \leq (F, X)$) if $F \subseteq F_1$, $X_1 \subseteq X$ and $F_1 \setminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \setminus F \subseteq X$.

2. Cohesiveness preserves the arithmetic hierarchy

Cohesiveness plays a central role in reverse mathematics. It appears naturally in the standard proof of Ramsey's theorem, as a preliminary step to reduce an instance of Ramsey's theorem over $(n+1)$-tuples into a non-effective instance over n-tuples.

Definition 2.1 (Cohesiveness) An infinite set C is $\vec{R}$-cohesive for a sequence of sets $R_0, R_1, \dots$ if for each $i \in \omega$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$. A set C is *cohesive* (resp. *p-cohesive*, *r-cohesive*) if it is $\vec{R}$-cohesive where $\vec{R}$ is the sequence of all the c.e. sets (resp. primitive recursive sets, computable sets). COH is the statement "Every uniform sequence of sets $\vec{R}$ admits an infinite $\vec{R}$-cohesive set."

Mileti [13] and Jockusch & Lempp [unpublished] proved that COH is a consequence of Ramsey's theorem for pairs over RCA_0. The computational power of COH is relatively well understood. Jockusch and Stephan characterized in [8] the degrees bounding COH as the degrees whose jump is PA relative to $\emptyset'$. The author [17] extended this characterization to an instance-wise correspondance between cohesiveness and the statement "For every Δ^0_2 tree T, there is a set whose jump computes a path through T". Wang [21] conjectured that COH admits preservation of the arithmetic hierarchy. We prove his conjecture by using a new forcing argument.

Theorem 2.2 COH admits preservation of the arithmetic hierarchy.

Before proving Theorem 2.2, we state an immediate corollary.

Corollary 2.3 There exists a cohesive set preserving the arithmetic hierarchy.

Proof. Jockusch [10] proved that every PA degree computes a sequence of sets containing, among others, all the computable sets. Wang proved in [21] that WKL_0 preserves the arithmetic hierarchy. Therefore there exists a uniform sequence of sets $\vec{R}$ containing all the computable sets and preserving the arithmetic hierarchy. By Theorem 2.2 relativized to $\vec{R}$, there exists an infinite $\vec{R}$-cohesive set C preserving the arithmetic hierarchy relative to $\vec{R}$. In particular C is r-cohesive and preserves the arithmetic hierarchy. By [8], the degrees of r-cohesive and cohesive sets coincide. Therefore C computes a cohesive set which preserves the arithmetic hierarchy. $\square$

Given a uniformly computable sequence of sets $R_0, R_1, \dots$, the construction of an $\vec{R}$-cohesive set is usually done with computable Mathias forcing, that is, using conditions (F, X) in which X is computable. The construction starts with $(\emptyset, \omega)$ and interleaves two kinds of steps. Given some condition (F, X),

- (S1) the *extension* step consists of taking an element x from X and adding it to F, therefore forming the extension $(F \cup \{x\}, X \setminus [0, x])$;
- (S2) the *cohesiveness* step consists of deciding which one of $X \cap R_i$ and $X \cap \overline{R}_i$ is infinite, and taking the chosen one as the new reservoir.

Cholak, Dzhafarov, Hirst and Slaman [2] studied the definitional complexity of the forcing relation for computable Mathias forcing. They proved that it has the good definitional properties to decide the first jump, but not iterated jumps. Indeed, given a computable Mathias condition $c = (F, X)$ and a Σ^0_1 formula $(\exists x)\varphi(G, x)$, one can $\emptyset'$-effectively decide whether there is an extension d forcing $(\exists x)\varphi(G, x)$ by asking the following question:

Is there an extension $d = (E, Y) \leq c$ and some $n \in \omega$ such that $\varphi(E, n)$ holds?

If there is such an extension, then we can choose it to be a *finite extension*, that is, such that $Y =^* X$. Therefore, the question is $\Sigma^{0,X}_1$. Consider now a Π^0_2 formula $(\forall x)(\exists y)\varphi(G, x, y)$. The question becomes

For every extension $d \leq c$ and every $m \in \omega$, is there some extension $e = (E, Y) \leq d$ and some $n \in \omega$ such that $\varphi(E, m, n)$ holds?

In this case, the extension d is not usually a finite extension and therefore the question cannot be presented in a Π^0_2 way. In particular, the formula "Y is an infinite subset of X" is definitionally complex. In general, deciding iterated jumps of a generic set requires to be able to talk about the future of a given condition, and in particular to describe by simple means the formula "d is a valid condition" and the formula "d is an extension of c".

Thankfully, in the case of cohesiveness, we do not need the full generality of the computable Mathias forcing. Indeed, the reservoirs have a very special shape. After the first application of stage (S2), the set X is, up to finite changes, of the form $\omega \cap R_0$ or $\omega \cap \overline{R_0}$. After the second application of (S2), it is in one of the following forms: $\omega \cap R_0 \cap R_1$, $\omega \cap R_0 \cap \overline{R}_1$, $\omega \cap \overline{R}_0 \cap R_1$, $\omega \cap \overline{R}_0 \cap \overline{R}_1$, and so on. More generally, after n applications of (S2), a condition $c = (F, X)$ is characterized by a pair (F, σ) where σ is a string of length n representing the choices made during (S2).

Even within this restricted partial order, the decision of the Π^0_2 formula remains too complicated sinces it requires to decide whether R_σ is infinite. However, notice that the σ's such that R_σ is infinite are exactly the initial segments of the $\Pi^{0,\emptyset'}_1$ class $\mathcal{C}(\vec{R})$ defined as the collection of the reals X such that R_σ has more than $|\sigma|$ elements for every $\sigma \prec X$. We can therefore use a compactness argument at the second level to decrease the definitional strength of the forcing relation, as did Wang [21] for weak König's lemma.

2.1. **The forcing notion**

We let $\mathbb{T}$ denote the collection of all the infinite $\emptyset'$-primitive recursive trees T such that $[T] \subseteq \mathcal{C}(\vec{R})$. Note that $\mathbb{T}$ is a computable set. We are now ready to defined our partial order.

Definition 2.4 Let $\mathbb{P}$ be the partial order whose conditions are tuples (F, σ, T) where $F \subseteq \omega$ is a finite set, $\sigma \in 2^{<\omega}$, R_σ is infinite and $T \in \mathbb{T}$ with stem σ. A condition $d = (E, \tau, S)$ *extends* $c = (F, \sigma, T)$ (written $d \leq c$) if $E \leq_\sigma F$, $\tau \succeq \sigma$ and $S \subseteq T$.

Given a condition $c = (F, \sigma, T)$, the string σ imposes a finite restriction on the possible extensions of the set F. The condition c intuitively denotes the Mathias condition $(F, R_\sigma \cap (max(F), +\infty))$ with some additional constraints on the extensions of σ represented by the tree T. Accordingly, set G *satisfies* (F, σ, T) if it satisfies the induced Mathias condition, that is, if $F \subseteq G \subseteq F \cup (R_\sigma \cap (max(F), +\infty))$. We let $\text{Ext}(c)$ be the collection of all the extensions of c.

Note that although we did not explicitely require R_σ to be infinite, this property holds for every condition $(F, \sigma, T) \in \mathbb{P}$. Indeed, since $[T] \subseteq \mathcal{C}(\vec{R})$, then R_τ is infinite for every extensible node $\tau \in T$. Since σ is a stem of T, it is extensible and therefore R_σ is infinite.

2.2. **Preconditions and forcing Σ^0_1 (Π^0_1) formulas**

When forcing complex formulas, we need to be able to consider all possible extensions of some condition c. Checking that some $d = (E, \tau, S)$ is a valid condition extending c requires to decide whether the tree $\emptyset'$-p.r. S is infinite, which is a Π^0_2 question. At some point, we will need to decide a Σ^0_1 formula without having enough computational power to check that the tree part is infinite (see clause (ii) of Definition 2.10). As the tree part of a condition is not accurate for such formulas, we may define the corresponding forcing relation over a weaker notion of condition where the tree is not anymore required to be infinite.

Definition 2.5 (Precondition) A *precondition* is a condition (F, σ, T) without the assumption that T is infinite.

In particular, R_σ may be a finite set. The notion of condition extension can be generalized to the preconditions. The set of all preconditions is computable, contrary to the set $\mathbb{P}$. Given a precondition c, we denote by $\mathrm{Ext}_1(c)$ the set of all preconditions (E, τ, S) extending c such that $\tau = \sigma$ and $T = S$. Here, $T = S$ in a strong sense, that is, the Turing indices of T and S are the same. This fact is used in clause a) of Lemma 2.14. We let $\mathbb{A}$ denote the collection of all the finite sets of integers. The set $\mathbb{A}$ represents the set of finite approximations of the generic set G. We also fix a uniformly computable enumeration $\mathbb{A}_0 \subseteq \mathbb{A}_1 \subseteq \dots$ of finite subsets of $\mathbb{A}$ cofinal in $\mathbb{A}$, that is, such that $\bigcup_s \mathbb{A}_s = \mathbb{A}$. We denote by $\mathrm{Apx}(c)$ the set $\{E \in \mathbb{A} : (E, \sigma, T) \in \mathrm{Ext}_1(c)\}$. In particular, $\mathrm{Apx}(c)$ is collection of all finite sets E satisfying c, that is, $\mathrm{Apx}(c) = \{E \in \mathbb{A} : E \leq_\sigma F\}$. Last, we let $\mathrm{Apx}_s(c) = \mathrm{Apx}(c) \cap \mathbb{A}_s$. We start by proving a few trivial statements.

Lemma 2.6 Fix a precondition $c = (F, \sigma, T)$.

1) If c is a condition then $\mathrm{Ext}_1(c) \subseteq \mathrm{Ext}(c)$.
2) If c is a condition then $\mathrm{Apx}(c) = \{E : (E, \tau, S) \in \mathrm{Ext}(c)\}$.
3) If d is a precondition extending c then $\mathrm{Apx}(d) \subseteq \mathrm{Apx}(c)$ and $\mathrm{Apx}_s(d) \subseteq \mathrm{Apx}_s(c)$.

Proof.

1) By definition, if c is a condition, then T is infinite. If $d \in \mathrm{Ext}_1(c)$ then $d = (E, \sigma, T)$ for some $E \in \mathrm{Apx}(c)$. As d is a precondition and T is infinite, d is a condition.
2) By definition, $\mathrm{Apx}(c) = \{E : (E, \sigma, T) \in \mathrm{Ext}_1(c)\} \subseteq \{E : (E, \tau, S) \in \mathrm{Ext}(c)\}$. On the other direction, fix an extension $(E, \tau, S) \in \mathrm{Ext}(c)$. By definition of an extension, $E \leq_\tau F$, so $E \leq_\sigma F$. Therefore $(E, \sigma, T) \in \mathrm{Ext}_1(c)$ and by definition of $\mathrm{Apx}(c)$, $E \in \mathrm{Apx}(c)$.
3) Fix some $(E, \tau, S) \in \mathrm{Ext}_1(d)$. As d extends c, $\tau \succeq \sigma$. By definition of an extension, $E \leq_\tau F$, so $E \leq_\sigma F$, hence $(E, \sigma, T) \in \mathrm{Ext}_1(c)$. Therefore $\mathrm{Apx}(d) = \{E : (E, \tau, S) \in \mathrm{Ext}_1(d)\} \subseteq \{E : (E, \sigma, T) \in \mathrm{Ext}_1(c)\} = \mathrm{Apx}(c)$. For any $s \in \omega$, $\mathrm{Apx}_s(d) = \mathrm{Apx}(d) \cap \mathbb{A}_s \subseteq \mathrm{Apx}(c) \cap \mathbb{A}_s = \mathrm{Apx}_s(c)$.

□

Note that altough the extension relation has been generalized to preconditions, $\mathrm{Ext}(c)$ is defined to be the set of all the *conditions* extending c. In particular, if c is a precondition which is not a condition, $\mathrm{Ext}(c) = \emptyset$, whereas at least $c \in \mathrm{Ext}_1(c)$. This is why clause 1 of Lemma 2.6 gives the useful information that whenever c is a true condition, so are the members of $\mathrm{Ext}_1(c)$.

Definition 2.7 Fix a precondition $c = (F, \sigma, T)$ and a Σ^0_0 formula $\varphi(G, x)$.

(i) $c \Vdash (\exists x)\varphi(G, x)$ iff $\varphi(F, w)$ holds for some $w \in \omega$
(ii) $c \Vdash (\forall x)\varphi(G, x)$ iff $\varphi(E, w)$ holds for every $w \in \omega$ and every set $E \in \mathrm{Apx}(c)$.

As explained, σ restricts the possible extensions of the set F (see clause 3 of Lemma 2.6), so this forcing notion is stable by condition extension. The tree T itself restricts the possible extensions of σ, but has no effect of the decision of a Σ^0_1 formula (Lemma 2.8).

The following trivial lemma expresses the fact that the tree part of a precondition has no effect in the forcing relation for a Σ^0_1 or Π^0_1 formula.

Lemma 2.8 Fix two preconditions $c = (F, \sigma, T)$ and $d = (F, \sigma, S)$, and some Σ^0_1 or Π^0_1 formula $\varphi(G)$.

$$c \Vdash \varphi(G) \quad \text{if and only if} \quad d \Vdash \varphi(G)$$

Proof. Simply notice that the tree part of the condition does not occur in the definition of the forcing relation, and that $\mathrm{Apx}(c) = \mathrm{Apx}(d)$. □

As one may expect, the forcing relation for a precondition is closed under extension.

Lemma 2.9 Fix a precondition c and a Σ^0_1 or Π^0_1 formula $\varphi(G)$. If $c \Vdash \varphi(G)$ then for every precondition $d \leq c$, $d \Vdash \varphi(G)$.

Proof. Fix a precondition $c = (F, \sigma, T)$ such that $c \Vdash \varphi(G)$ and an extension $d = (E, \tau, S) \leq c$.

- If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. As $c \Vdash \varphi(G)$, then by clause (i) of Definition 2.7, there exists a $w \in \omega$ such that $\psi(F, w)$ holds. By definition of $d \leq c$, $E \leq_\sigma F$, so $\psi(E, w)$ holds, hence $d \Vdash \varphi(G)$.
- If $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. As $c \Vdash \varphi(G)$, then by clause (ii) of Definition 2.7, for every $w \in \omega$ and every $H \in \text{Apx}(c)$, $\varphi(H, w)$ holds. By clause 3 of Lemma 2.6, $\text{Apx}(d) \subseteq \text{Apx}(c)$ so $d \Vdash \varphi(G)$.

□

2.3. **Forcing higher formulas**

We are now able to define the forcing relation for any arithmetic formula. The forcing relation for arbitrary arithmetic formulas is induced by the forcing relation for Σ^0_1 formulas. However, the definitional strength of the resulting relation is too high with respect to the formula it forces. We therefore design a custom relation with better definitional properties, and which still preserve the expected properties of a forcing relation, that is, the density of the set of conditions forcing a formula or its negation, and the preservation of the forced formulas under condition extension.

Definition 2.10 Let $c = (F, \sigma, T)$ be a condition and $\varphi(G)$ be an arithmetic formula.

(i) If $\varphi(G) = (\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+1}$ then $c \Vdash \varphi(G)$ iff there is a $w < |\sigma|$ such that $c \Vdash \psi(G, w)$

(ii) If $\varphi(G) = (\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$ then $c \Vdash \varphi(G)$ iff for every $\tau \in T$, every $E \in \text{Apx}_{|\tau|}(c)$ and every $w < |\tau|$, $(E, \tau, T^{[\tau]}) \nVdash \neg\psi(G, w)$

(iii) If $\varphi(G) = \neg\psi(G, x)$ where $\psi \in \Sigma^0_{n+3}$ then $c \Vdash \varphi(G)$ iff $d \nVdash \psi(G)$ for every $d \leq c$.

Note that in clause (ii) of Definition 2.10, there may be some $\tau \in T$ such that $T^{[\tau]}$ is finite, hence $(E, \tau, T^{[\tau]})$ is not necessarily a condition. This is where we use the generalization of forcing of Σ^0_1 and Π^0_1 formulas to preconditions. We now prove that this relation enjoys the main properties of a forcing relation.

Lemma 2.11 Fix a condition c and an arithmetic formula $\varphi(G)$. If $c \Vdash \varphi(G)$ then for every condition $d \leq c$, $d \Vdash \varphi(G)$.

Proof. We prove by induction over the complexity of the formula $\varphi(G)$ that for every condition c, if $c \Vdash \varphi(G)$ then for every condition $d \leq c$, $d \Vdash \varphi(G)$. Fix a condition $c = (F, \sigma, T)$ such that $c \Vdash \varphi(G)$ and an extension $d = (E, \tau, S)$.

- If $\varphi \in \Sigma^0_1 \cup \Pi^0_1$ then it follows from Lemma 2.9.
- If $\varphi \in \Sigma^0_{n+2}$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+1}$. By clause (i) of Definition 2.10, there exists a $w \in \omega$ such that $c \Vdash \psi(G, w)$. By induction hypothesis, $d \Vdash \psi(G, w)$ so by clause (i) of Definition 2.10, $d \Vdash \varphi(G)$.
- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause (ii) of Definition 2.10, for every $\rho \in T$, every $w < |\rho|$, and every $H \in \text{Apx}_{|\rho|}(c)$, $(H, \rho, T^{[\rho]}) \nVdash \neg\psi(G, w)$. As $S \subseteq T$ and $\text{Apx}(d) \subseteq \text{Apx}(c)$, for every $\rho \in S$, every $w < |\rho|$, and every $H \in \text{Apx}_{|\rho|}(d)$, $(H, \rho, T^{[\rho]}) \nVdash \neg\psi(G, w)$. By Lemma 2.8, $(H, \rho, S^{[\rho]}) \nVdash \neg\psi(G, w)$ hence by clause (ii) of Definition 2.10, $d \Vdash \varphi(G)$.
- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause (iii) of Definition 2.10, for every $e \in \text{Ext}(c)$, $e \nVdash \psi(G)$. As $\text{Ext}(d) \subseteq \text{Ext}(c)$, for every $e \in \text{Ext}(d)$, $e \nVdash \psi(G)$, so by clause (iii) of Definition 2.10, $d \Vdash \varphi(G)$.

□

Lemma 2.12 For every arithmetic formula φ, the following set is dense

$$\{c \in \mathbb{P} : c \Vdash \varphi(G) \text{ or } c \Vdash \neg\varphi(G)\}$$

Proof. We prove by induction over $n > 0$ that if φ is a Σ^0_n (Π^0_n) formula then the following set is dense

$$\{c \in \mathbb{P} : c \Vdash \varphi(G) \text{ or } c \Vdash \neg\varphi(G)\}$$

It suffices to prove it for the case where φ is a Σ^0_n formula, as the case where φ is a Π^0_n formula is symmetric. Fix a condition $c = (F, \sigma, T)$.

- In case $n = 1$, the formula φ is of the form $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. Suppose there exists a $w \in \omega$ and a set $E \in \text{Apx}(c)$ such that $\psi(E, w)$ holds. The precondition $d = (E, \sigma, T)$ is a condition extending c by clause 1 of Lemma 2.6 and by definition of $\text{Apx}(c)$. Moreover $d \Vdash (\exists x)\psi(G, x)$ by clause (i) of Definition 2.7 hence $d \Vdash \varphi(G)$. Suppose now that for every $w \in \omega$ and every $E \in \text{Apx}(c)$, $\psi(E, w)$ does not hold. By clause (ii) of Definition 2.7, $c \Vdash (\forall x)\neg\psi(G, x)$, hence $c \Vdash \neg\varphi(G)$.
- In case $n = 2$, the formula φ is of the form $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. Let

$$S = \{\tau \in T : (\forall w < |\tau|)(\forall E \in \text{Apx}_{|\tau|}(c)(E, \tau, T^{[\tau]}) \nVdash \psi(G, w)\}$$

  The set S is obviously $\emptyset'$-p.r. We prove that it is a subtree of T. Suppose that $\tau \in S$ and $\rho \preceq \tau$. Fix a $w < |\rho|$ and $E \in \text{Apx}_{|\rho|}(c)$. In particular $w < |\tau|$ and $E \in \text{Apx}_{|\tau|}(c)$ so $(E, \tau, T^{[\tau]}) \nVdash \psi(G, w)$. Note that $(E, \tau, T^{[\tau]})$ is a precondition extending $(E, \rho, T^{[\rho]})$, so by the contrapositive of Lemma 2.9, $(E, \rho, T^{[\rho]}) \nVdash \psi(G, w)$. Therefore $\rho \in S$. Hence S is a tree, and as $S \subseteq T$, it is a subtree of T.

  If S is infinite, then $d = (F, \sigma, S)$ is an extension of c such that for every $\tau \in S$, every $w < |\tau|$ and every $E \in \text{Apx}_{|\tau|}(c)$, $(E, \tau, T^{[\tau]}) \nVdash \psi(G, w)$. By Lemma 2.8, for every $E \in \text{Apx}_{|\tau|}(c)$, $(E, \tau, S^{[\tau]}) \nVdash \psi(G, w)$ and by clause 3 of Lemma 2.6, $\text{Apx}_{|\tau|}(d) \subseteq \text{Apx}_{|\tau|}(c)$. Therefore, by clause (ii) of Definition 2.10, $d \Vdash (\forall x)\neg\psi(G, x)$ so $d \Vdash \neg\varphi(G)$. If S is finite, then pick some $\tau \in T \setminus S$ such that $T^{[\tau]}$ is infinite. By choice of $\tau \in T \setminus S$, there exists a $w < |\tau|$ and an $E \in \text{Apx}_{|\tau|}(c)$ such that $(E, \tau, T^{[\tau]}) \Vdash \psi(G, w)$. $d = (E, \tau, T^{[\tau]})$ is a valid condition extending c and by clause (i) of Definition 2.10 $d \Vdash \varphi(G)$.
- In case $n > 2$, density follows from clause (iii) of Definition 2.10.

□

Any sufficiently generic filter $\mathcal{F}$ induces a unique generic real G defined by

$$G = \bigcup\{F \in \mathbb{A} : (F, \sigma, T) \in \mathcal{F}\}$$

The following lemma informally asserts that the forcing relation is *sound* and *complete*. Sound because whenever it forces a property, then this property actually holds over the generic real G. The forcing is also complete in that every property which holds over G is forced at some point whenever the filter is sufficiently generic.

Lemma 2.13 Suppose that $\mathcal{F}$ is a sufficiently generic filter and let G be the corresponding generic real. Then for each arithmetic formula $\varphi(G)$, $\varphi(G)$ holds iff $c \Vdash \varphi(G)$ for some $c \in \mathcal{F}$.

Proof. We prove by induction over the complexity of the arithmetic formula $\varphi(G)$ that $\varphi(G)$ holds iff $c \Vdash \varphi(G)$ for some $c \in \mathcal{F}$. Note that thanks to Lemma 2.12, it suffices to prove that if $c \Vdash \varphi(G)$ for some $c \in \mathcal{F}$ then $\varphi(G)$ holds. Indeed, conversely if $\varphi(G)$ holds, then by genericity of G either $c \Vdash \varphi(G)$ or $c \Vdash \neg\varphi(G)$ for some $c \in \mathcal{F}$, but if $c \Vdash \neg\varphi(G)$ then $\neg\varphi(G)$ holds, contradicting the hypothesis. So $c \Vdash \varphi(G)$.

We proceed by case analysis on the formula φ. Note that in the above argument, the converse of the Σ case is proved assuming the Π case. However, in our proof, we use the converse of the Σ^0_{n+3} case to prove the Π^0_{n+3} case. We need therefore prove to the converse of the Σ^0_{n+3} case without Lemma 2.12. Fix a condition $c = (F, \sigma, T) \in \mathcal{F}$ such that $c \Vdash \varphi(G)$.

- If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause (i) of Definition 2.7, there exists a $w \in \omega$ such that $\psi(F, w)$ holds. As $F \subseteq G$ and $G \setminus F \subseteq (max(F), +\infty)$, then by continuity $\psi(G, w)$ holds, hence $\varphi(G)$ holds.

- If $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause (ii) of Definition 2.7, for every $w \in \omega$ and every $E \in \text{Apx}(c)$, $\psi(E, w)$ holds. As $\{E \subset_{fin} G : E \supseteq F\} \subseteq \text{Apx}(c)$, then for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.
- If $\varphi \in \Sigma^0_{n+2}$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+1}$. By clause (i) of Definition 2.10, there exists a $w \in \omega$ such that $c \Vdash \psi(G, w)$. By induction hypothesis, $\psi(G, w)$ holds, hence $\varphi(G)$ holds.

  Conversely, suppose that $\varphi(G)$ holds. Then there exists a $w \in \omega$ such that $\psi(G, w)$ holds, so by induction hypothesis $c \Vdash \psi(G, w)$ for some $c \in \mathcal{F}$, so by clause (i) of Definition 2.10, $c \Vdash \varphi(G)$.
- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause (ii) of Definition 2.10, for every $\tau \in T$, every $w < |\tau|$, and every $E \in \text{Apx}_{|\tau|}(c)$, $(E, \tau, T^{[\tau]}) \nVdash \neg\psi(G, w)$. Suppose for the sake of absurd that $\psi(G, w)$ does not hold for some $w \in \omega$. Then by induction hypothesis, there exists a $d \in \mathcal{F}$ such that $d \Vdash \neg\psi(G, w)$. Let $e = (E, \tau, S) \in \mathcal{F}$ be such that $e \Vdash \neg\psi(G, w)$, $|\tau| > w$ and e extends both c and d. The condition e exists by Lemma 2.9. We can furthermore require that $E \in \text{Apx}_{|\tau|}(c)$, so $e \nVdash \neg\psi(G, w)$ and $e \Vdash \neg\psi(G, w)$. Contradiction. Hence for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.
- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause (iii) of Definition 2.10, for every $d \in \text{Ext}(c)$, $d \nVdash \psi(G)$. By Lemma 2.11, $d \nVdash \psi(G)$ for every $d \in \mathcal{F}$, and by a previous case, $\psi(G)$ does not hold, so $\varphi(G)$ holds.

□

We now prove that the forcing relation enjoys the desired definitional properties, that is, the complexity of the forcing relation is the same as the complexity of the formula it forces. We start by analysing the complexity of some components of this notion of forcing.

Lemma 2.14

a) For every precondition c, $\text{Apx}(c)$ and $\text{Ext}_1(c)$ are Δ^0_1 uniformly in c.
b) For every condition c, $\text{Ext}(c)$ is Π^0_2 uniformly in c.

Proof.

a) Fix a precondition $c = (F, \sigma, T)$. A set $E \in \text{Apx}(c)$ iff the following Δ^0_1 predicate holds:

$$(F \subseteq E) \wedge (\forall x \in E \setminus F)[x > max(F) \wedge x \in R_\sigma]$$

Moreover, $(E, \tau, S) \in \text{Ext}_1(c)$ iff the Δ^0_1 predicate $E \in \text{Apx}(c) \wedge \tau = \sigma \wedge S = T$ holds. As already mentioned, the equality $S = T$ is translated into "the indices of S and T coincide" which is a Σ^0_0 statement.

b) Fix a condition $c = (F, \sigma, T)$. By clause 2) of Lemma 2.6, $(E, \tau, S) \in \text{Ext}(c)$ iff the following Π^0_2 formula holds

$$\begin{array}{ll} E \in \text{Apx}(c) \wedge \sigma \preceq \tau & \\ \wedge(\forall \rho \in S)(\forall \xi)[\xi \preceq \rho \rightarrow \xi \in S] & (S \text{ is a tree}) \\ \wedge(\forall n)(\exists \rho \in 2^n)\rho \in S) & (S \text{ is infinite}) \\ \wedge(\forall \rho \in S)(\sigma \prec \rho \vee \rho \preceq \sigma) & (S \text{ has stem } \sigma) \\ \wedge(\forall \rho \in S)(\rho \in T) & (S \text{ is a subset of } T) \end{array}$$

□

Lemma 2.15 Fix an arithmetic formula $\varphi(G)$.

a) Given a precondition c, if $\varphi(G)$ is a Σ^0_1 (Π^0_1) formula then so is the predicate $c \Vdash \varphi(G)$.
b) Given a condition c, if $\varphi(G)$ is a Σ^0_{n+2} (Π^0_{n+2}) formula then so is the predicate $c \Vdash \varphi(G)$.

Proof. We prove our lemma by induction over the complexity of the formula $\varphi(G)$. Fix a (pre)condition $c = (F, \sigma, T)$.

- If $\varphi(G) \in \Sigma^0_1$ then it can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Sigma^0_0$. By clause (i) of Definition 2.7, $c \Vdash \varphi(G)$ if and only if the formula $(\exists w \in \omega)\psi(F,w)$ holds. This is a Σ^0_1 predicate.
- If $\varphi(G) \in \Pi^0_1$ then it can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_0$. By clause (ii) of Definition 2.7, $c \Vdash \varphi(G)$ if and only if the formula $(\forall w \in \omega)(\forall E \in \operatorname{Apx}(c))\psi(E,w)$ holds. By clause a) of Lemma 2.14, this is a Π^0_1 predicate.
- If $\varphi(G) \in \Sigma^0_{n+2}$ then it can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Pi^0_{n+1}$. By clause (i) of Definition 2.10, $c \Vdash \varphi(G)$ if and only if the formula $(\exists w < |\sigma|)c \Vdash \psi(G,w)$ holds. This is a Σ^0_{n+2} predicate by induction hypothesis.
- If $\varphi(G) \in \Pi^0_2$ then it can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_1$. By clause (ii) of Definition 2.10, $c \Vdash \varphi(G)$ if and only if the formula $(\forall \tau \in T)(\forall w < |\tau|)(\forall E \in \operatorname{Apx}_{|\tau|}(c))(E,\tau,T^{[\tau]}) \not\Vdash \neg\psi(G,w)$ holds. By induction hypothesis, $(E,\tau,T^{[\tau]}) \not\Vdash \neg\psi(G,w)$ is a Σ^0_1 predicate, hence by clause a) of Lemma 2.14, $c \Vdash \varphi(G)$ is a Π^0_2 predicate.
- If $\varphi(G) \in \Pi^0_{n+3}$ then it can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause (iii) of Definition 2.10, $c \Vdash \varphi(G)$ if and only if the formula $(\forall d)(d \not\in \operatorname{Ext}(c) \vee d \not\Vdash \psi(G))$ holds. By induction hypothesis, $d \not\Vdash \psi(G)$ is a Π^0_{n+3} predicate. Hence by clause b) of Lemma 2.14, $c \Vdash \varphi(G)$ is a Π^0_{n+3} predicate.

□

2.4. **Preserving the arithmetic hierarchy**

The following lemma asserts that every sufficiently generic real for this notion of forcing preserves the arithmetic hierarchy. The argument deeply relies on the fact that this notion of forcing admits a forcing relation with good definitional properties.

Lemma 2.16 If $A \not\in \Sigma^0_{n+1}$ and $\varphi(G,x)$ is Σ^0_{n+1}, then the set of $c \in \mathbb{P}$ satisfying the following property is dense:

$$[(\exists w \in A)c \Vdash \neg\varphi(G,w)] \vee [(\exists w \not\in A)c \Vdash \varphi(G,w)]$$

Proof. Fix a condition $c = (F,\sigma,T)$.

- In case $n = 0$, $\varphi(G,w)$ can be expressed as $(\exists x)\psi(G,w,x)$ where $\psi \in \Sigma^0_0$. Let $U = \{w \in \omega : (\exists E \in \operatorname{Apx}(c))(\exists u)\psi(E,w,u)\}$. By clause a) of Lemma 2.14, $U \in \Sigma^0_1$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of U, there exists an $E \in \operatorname{Apx}(c)$ and a $u \in \omega$ such that $\psi(E,w,u)$ holds. By definition of $\operatorname{Apx}(c)$ and clause 1) of Lemma 2.6, $d = (E,\sigma,T)$ is a condition extending c. By clause (i) of Definition 2.7, $d \Vdash \varphi(G,w)$. If $w \in A \setminus U$, then for every $E \in \operatorname{Apx}(c)$ and every $u \in \omega$, $\psi(E,w,u)$ does not hold, so by clause (ii) of Definition 2.7, $c \Vdash (\forall x)\neg\psi(G,w,x)$, hence $c \Vdash \neg\varphi(G,w)$.
- In case $n = 1$, $\varphi(G,w)$ can be expressed as $(\exists x)\psi(G,w,x)$ where $\psi \in \Pi^0_1$. Let $U = \{w \in \omega : (\exists s)(\forall \tau \in 2^s \cap T)(\exists u < s)(\exists E \in \operatorname{Apx}_s(c))(E,\tau,T^{[\tau]}) \Vdash \psi(G,w,u)\}$. By Lemma 2.15 and clause a) of Lemma 2.14, $U \in \Sigma^0_2$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of U, there exists an $s \in \omega$, a $\tau \in 2^s \cap T$, a $u < s$ and an $E \in \operatorname{Apx}_s(c)$ such that $T^{[\tau]}$ is infinite and $(E,\tau,T^{[\tau]}) \Vdash \psi(G,w,u)$. Thus $d = (E,\tau,T^{[\tau]})$ is a condition extending c and by clause (i) of Definition 2.10, $d \Vdash \varphi(G,w)$. If $w \in A \setminus U$, then let $S = \{\tau \in T : (\forall u < |\tau|)(\forall E \in \operatorname{Apx}_{|\tau|}(c)(E,\tau,T^{[\tau]}) \not\Vdash \psi(G,w,u)\}$. As proven in Lemma 2.12, S is a $\emptyset'$-p.r. subtree of T and by $w \not\in U$, S is infinite. Thus $d = (F,\sigma,S)$ is a condition extending c. By clause 3) of Lemma 2.6, $\operatorname{Apx}(d) \subseteq \operatorname{Apx}(c)$, so for every $\tau \in S$, every $u < |\tau|$, and every $E \in \operatorname{Apx}_{|\tau|}(d)$, $(E,\tau,T^{[\tau]}) \not\Vdash \psi(G,w,u)$. By Lemma 2.8, $(E,\tau,S^{[\tau]}) \not\Vdash \psi(G,w,u)$, so by clause (ii) of Definition 2.10, $d \Vdash (\forall x)\neg\psi(G,w,u)$ hence $d \Vdash \neg\varphi(G,w)$.
- In case $n > 1$, let $U = \{w \in \omega : (\exists d \in \operatorname{Ext}(c))d \Vdash \varphi(G,w)\}$. By clause b) of Lemma 2.14 and Lemma 2.15, $U \in \Sigma^0_n$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of U, there exists a condition d extending c such that $d \Vdash \varphi(G,w)$. If $w \in A \setminus U$, then for every $d \in \operatorname{Ext}(c)d \not\Vdash \varphi(G,w)$ so by clause (iii) of Definition 2.10, $c \Vdash \neg\varphi(G,w)$.

□

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let C be a set and $R_0, R_1, \dots$ be a uniformly C-computable sequence of sets. Let T_0 be a C'-primitive recursive tree such that $[T_0] \subseteq \mathcal{C}(\vec{R})$. Let $\mathcal{F}$ be a sufficiently generic filter containing $c_0 = (\emptyset, \epsilon, T_0)$. and let G be the corresponding generic real. By genericity, the set G is an infinite $\vec{R}$-cohesive set. By Lemma 2.16 and Lemma 2.15, G preserves non-Σ^0_{n+1} definitions relative to C for every $n \in \omega$. Therefore, by Proposition 2.2 of [21], G preserves the arithmetic hierarchy relative to C. □

3. The Erdős Moser theorem preserves the arithmetic hierarchy

We now extend the previous result to the Erdős-Moser theorem. The Erdős-Moser theorem is a statement coming from graph theory. It provides together with the ascending descending principle (ADS) an alternative proof of Ramsey's theorem for pairs (RT^2_2). Indeed, every coloring $f : [\omega]^2 \to 2$ can be seen as a tournament R such that $R(x, y)$ holds if $x < y$ and $f(x, y) = 1$, or $x > y$ and $f(y, x) = 0$. Every infinite transitive subtournament induces a linear order whose infinite ascending or descending sequences are homogeneous for f.

Definition 3.1 (Erdős-Moser theorem) A tournament T on a domain $D \subseteq \omega$ is an irreflexive binary relation on D such that for all $x, y \in D$ with $x \neq y$, exactly one of $T(x, y)$ or $T(y, x)$ holds. A tournament T is *transitive* if the corresponding relation T is transitive in the usual sense. A tournament T is *stable* if $(\forall x \in D)[(\forall^\infty s)T(x, s) \vee (\forall^\infty s)T(s, x)]$. EM is the statement "Every infinite tournament T has an infinite transitive subtournament." SEM is the restriction of EM to stable tournaments.

Bovykin and Weiermann proved in [1] that $\mathsf{EM} + \mathsf{ADS}$ is equivalent to RT^2_2 over RCA_0, equivalence still holding between their stable versions. Lerman et al. [12] proceeded to a combinatorial and effective analysis of the Erdős-Moser theorem, and proved in particular that there is an ω-model of EM which is not a model of SRT^2_2. The author simplified their proof in [15] and showed in [16] that $\mathsf{RCA}_0 \vdash \mathsf{EM} \to [\mathsf{STS}^2 \vee \mathsf{COH}]$, where STS^2 stands for the stable thin set theorem for pairs. In particular, since Wang [21] proved that STS^2 does not admit preservation of the arithmetic hierarchy, Theorem 2.2 follows from Theorem 3.2. On a definitional point of view, Wang proved in [21] that EM admits preservation of Δ^0_2 definitions and preservation of higher definitions. He conjectured that EM admits preservation of the arithmetic hierarchy. We prove his conjecture.

Theorem 3.2 EM admits preservation of the arithmetic hierarchy.

Again, the core of the proof consists of finding the good forcing notion whose generics will preserve the arithmetic hierarchy. For the sake of simplicity, we will restrict ourselves to stable tournaments eventhough it is clear that the forcing notion can be adapted to arbitrary tournament. The proof of Theorem 3.2 will be obtained by composing the proof that cohesiveness and the stable Erdős-Moser theorem admit preservation of the arithmetic hierarchy.

The following notion of *minimal interval* plays a fundamental role in the analysis of EM. See [12] for a background analysis of EM.

Definition 3.3 (Minimal interval) Let T be an infinite tournament and $a, b \in T$ be such that $T(a, b)$ holds. The *interval* (a, b) is the set of all $x \in T$ such that $T(a, x)$ and $T(x, b)$ hold. Let $F \subseteq T$ be a finite transitive subtournament of T. For $a, b \in F$ such that $T(a, b)$ holds, we say that (a, b) is a *minimal interval of* F if there is no $c \in F \cap (a, b)$, i.e., no $c \in F$ such that $T(a, c)$ and $T(c, b)$ both hold.

We must introduce an preliminary variant of Mathias forcing which is more suited to the Erdős-Moser theorem.

3.1. Erdős Moser forcing

The following notion of Erdős-Moser forcing has been first implicitly used by Lerman, Solomon and Towsner [12] to separate the Erdős-Moser theorem from stable Ramsey's theorem for pairs. The author formalized this notion of forcing in [14] to construct a low$_2$ degree bounding the Erdős-Moser theorem.

Definition 3.4 An *Erdős Moser condition* (EM condition) for an infinite tournament R is a Mathias condition (F, X) where

(a) $F \cup \{x\}$ is R-transitive for each $x \in X$
(b) X is included in a minimal R-interval of F.

The Erdős-Moser extension is the usual Mathias extension. EM conditions have good properties for tournaments as state following lemmas. Given a tournament R and two sets E and F, we denote by $E \to_R F$ the formula $(\forall x \in E)(\forall y \in F)R(x, y)$ holds.

Lemma 3.5 (Patey [14]) Fix an EM condition (F, X) for a tournament R. For every $x \in F$, $\{x\} \to_R X$ or $X \to_R \{x\}$.

Lemma 3.6 (Patey [14]) Fix an EM condition $c = (F, X)$ for a tournament R, an infinite subset $Y \subseteq X$ and a finite R-transitive set $F_1 \subset X$ such that $F_1 < Y$ and $[F_1 \to_R Y \vee Y \to_R F_1]$. Then $d = (F \cup F_1, Y)$ is a valid extension of c.

3.2. Partition trees

Given a string $\sigma \in k^{<\omega}$, we denote by $\text{set}_\nu(\sigma)$ the set $\{x < |\sigma| : \sigma(x) = \nu\}$ where $\nu < k$. The notion can be extended to sequences $P \in k^\omega$ where $\text{set}_\nu(P) = \{x \in \omega : P(x) = \nu\}$.

Definition 3.7 (Partition tree) A *k-partition tree of* $[t, +\infty)$ for some $k, t \in \omega$ is a tuple (k, t, T) such that T is a subtree of $k^{<\omega}$. A *partition tree* is a k-partition tree of $[t, +\infty)$ for some $k, t \in \omega$.

For the simplicity of notations, we may use the same letter T to denote both a partition tree (k, t, T) and the actual tree $T \subseteq k^{<\omega}$. We then write $\text{dom}(T)$ for $[t, +\infty)$ and $\text{parts}(T)$ for k. Given a p.r. partition tree T, we write $\#T$ for its Turing index, and may refer to it as its *code*.

Definition 3.8 (Refinement) Given a function $f : \ell \to k$, a string $\sigma \in \ell^{<\omega}$ *f-refines* a string $\tau \in k^{<\omega}$ if $|\sigma| = |\tau|$ and for every $\nu < \ell$, $\text{set}_\nu(\sigma) \subseteq \text{set}_{f(\nu)}(\tau)$. A p.r. ℓ-partition tree S of $[u, +\infty)$ *f-refines* a p.r. k-partition tree T of $[t, +\infty)$ (written $S \leq_f T$) if $\#S \geq \#T$, $\ell \geq k$, $u \geq t$ and for every $\sigma \in S$, σ f-refines some $\tau \in T$.

The collection of partition trees is equipped with a partial order $\leq$ such that $(\ell, u, S) \leq (k, t, T)$ if there exists a function $f : \ell \to k$ such that $S \leq_f T$. Given a k-partition tree of $[t, +\infty)$ T, we say that part ν of T is *acceptable* if there exists a path P through T such that $\text{set}_\nu(P)$ is infinite. Moreover, we say that part ν of T is *empty* if $(\forall \sigma \in T)[dom(T) \cap \text{set}_\nu(\sigma) = \emptyset]$. Note that each partition tree has at least one acceptable part since for every path P through T, $\text{set}_\nu(P)$ is infinite for some $\nu < k$. It can also be the case that part ν of T is non-empty, while for every path P through T, $\text{set}_\nu(P) \cap \text{dom}(T) = \emptyset$. However, in this case, we can choose the infinite computable subtree $S = \{\sigma \in T : \text{set}_\nu(\sigma) \cap \text{dom}(T) = \emptyset\}$ of T which has the same collection of infinite paths and such that part ν of S is empty.

Given a k-partition tree T, a finite set $F \subseteq \omega$ and a part $\nu < k$, define

$$T^{[\nu, F]} = \{\sigma \in T : F \subseteq \text{set}_\nu(\sigma) \vee |\sigma| < max(F)\}$$

The set $T^{[\nu,F]}$ is a (possibly finite) subtree of T which id-refines T and such that $F \subseteq \text{set}_\nu(P)$ for every path P through $T^{[\nu,F]}$.

We denote by $\mathbb{U}$ the set of all ordered pairs (ν, T) such that T is an infinite, primitive recursive k-partition tree of $[t, +\infty)$ for some $t, k \in \omega$ and $\nu < k$. The set $\mathbb{U}$ is equipped with a partial ordering $\leq$ such that $(\mu, S) \leq (\nu, T)$ if S f-refines T and $f(\mu) = \nu$. In this case we say that *part* μ *of* S *refines part* ν *of* T. Note that the domain of $\mathbb{U}$ and the relation $\leq$ are co-c.e. We denote by $\mathbb{U}[T]$ the set of all $(\nu, S) \leq (\mu, T)$ for some $(\mu, T) \in \mathbb{U}$.

Definition 3.9 (Promise for a partition tree) Fix a p.r. k-partition tree of $[t, +\infty)$ T. A class $\mathcal{C} \subseteq \mathbb{U}[T]$ is a *promise for* T if

a) $\mathcal{C}$ is upward-closed under the $\leq$ relation restricted to $\mathbb{U}[T]$
b) for every infinite p.r. partition tree $S \leq T$, $(\mu, S) \in \mathcal{C}$ for some non-empty part μ of S.

A promise for T can be seen as a two-dimensional tree with at first level the acyclic digraph of refinement of partition trees. Given an infinite path in this digraph, the parts of the members of this path form an infinite, finitely branching tree.

Lemma 3.10 Let T and S be p.r. partition trees such that $S \leq_f T$ for some function f : $\text{parts}(S) \to \text{parts}(T)$ and let $\mathcal{C}$ be a $\emptyset'$-p.r. promise for T.

a) The predicate "T is an infinite k-partition tree of $[t, +\infty)$" is Π^0_1 uniformly in T, k and t.
b) The relations "S f-refines T" and "part ν of S f-refines part μ of T" are Π^0_1 uniformly in S, T and f.
c) The predicate "$\mathcal{C}$ is a promise for T" is Π^0_2 uniformly in an index for $\mathcal{C}$ and T.

Proof.

a) T is an infinite k-partition tree of $[t, +\infty)$ if and only if the Π^0_1 formula $[(\forall \sigma \in T)(\forall \tau \preceq \sigma)\tau \in T \cap k^{<\infty}] \wedge [(\forall n)(\exists \tau \in k^n)\tau \in T]$ holds.
b) Suppose that T is a k-partition tree of $[t, +\infty)$ and S is an ℓ-partition tree of $[u, +\infty)$. S f-refines T if and only if the Π^0_1 formula $u \geq t \wedge \ell \geq k \wedge [(\forall \sigma \in S)(\exists \tau \in k^{|\sigma|})(\forall \nu < u) set_\nu(\sigma) \subseteq \text{set}_{f(\nu)}(\tau)]$ holds. Part ν of S f-refines part μ of T if and only if $\mu = f(\nu)$ and S f-refines T.
c) Given $k, t \in \omega$, let $PartTree(k, t)$ denote the Π^0_1 set of all the infinite p.r. k-partition trees of $[t, +\infty)$. Given a k-partition tree S and a part ν of S, let $Empty(S, \nu)$ denote the Π^0_1 formula "part ν of S is empty", that is the formula $(\forall \sigma \in S)\text{set}_\nu(\sigma) \cap \text{dom}(S) = \emptyset$. $\mathcal{C}$ is a promise for T if and only if the following Π^0_2 formula holds:

$$(\forall \ell, u)(\forall S \in PartTree(\ell, u))[S \leq T \to (\exists \nu < \ell)\neg Empty(S, \nu) \wedge (\nu, S) \in \mathcal{C})]$$
$$\wedge(\forall \ell', u')(\forall V \in PartTree(\ell', u'))(\forall g : \ell \to \ell')[S \leq_g V \leq T \to$$
$$(\forall \nu < \ell)((\nu, S) \in \mathcal{C} \to (g(\nu), V) \in \mathcal{C})]$$

□

Given a promise $\mathcal{C}$ for T and some infinite p.r. partition tree S refining T, we denote by $\mathcal{C}[S]$ the set of all $(\nu, S') \in \mathcal{C}$ below some $(\mu, S) \in \mathcal{C}$, that is, $\mathcal{C}[S] = \mathcal{C} \cap \mathbb{U}[S]$. Note that by clause b) of Lemma 3.10, if $\mathcal{C}$ is $\emptyset'$-p.r. promise for T then $\mathcal{C}[S]$ is a $\emptyset'$-p.r. promise for S.

Establishing a distinction between the acceptable parts and the non-acceptable ones requires a lot of definitional power. However, we prove that we can always find an extension where the distinction is Δ^0_2. We say that an infinite p.r. partition tree T *witnesses its acceptable parts* if its parts are either acceptable or empty.

Lemma 3.11 For every infinite p.r. k-partition tree T of $[t, +\infty)$, there exists an infinite p.r. k-partition tree S of $[u, +\infty)$ refining T with the identity function and such that S witnesses its acceptable parts.

Proof. Given a partition tree T, we let $I(T)$ be the set of its empty parts. Fix an infinite p.r. k-partition tree of $[t, +\infty)$ T, It suffices to prove that if ν is a non-empty and non-acceptable part of T, then there exists an infinite p.r. k-partition tree S refining T with the identity function, such that $\nu \in I(S) \setminus I(T)$. As $I(T) \subseteq I(S)$ and $|I(S)| \leq k$, it suffices to iterate the process at most k times to obtain a refinement witnessing its acceptable parts.

So fix a non-empty and non-acceptable part ν of T. By definition of being non-acceptable, there exists a path P through T and an integer $u > max(t, \text{set}_\nu(P))$. Let $S = \{\sigma \in T :$

$\mathrm{set}_\nu(\sigma) \cap [u, +\infty) = \emptyset\}$. The set S is a p.r. k-partition tree of $[u, +\infty)$ refining T with the identity function and such that part ν of S is empty. Moreover, S is infinite since $P \in [S]$. ☐

The following lemma strengthen clause b) of Definition 3.9.

Lemma 3.12 Let T be a p.r. partition tree and $\mathcal{C}$ be a promise for T. For every infinite p.r. partition tree $S \leq T$, $(\mu, S) \in \mathcal{C}$ for some acceptable part μ of S.

Proof. Fix an infinite p.r. ℓ-partition tree $S \leq T$. By Lemma 3.11, there exists an infinite p.r. ℓ-partition tree $S' \leq_{id} S$ witnessing its acceptable parts. As $\mathcal{C}$ is a promise for T and $S' \leq T$, there exists a non-empty (hence acceptable) part ν of S' such that $(\nu, S') \in \mathcal{C}$. As $\mathcal{C}$ is upward-closed, $(\nu, S) \in \mathcal{C}$. ☐

3.3. **Forcing conditions**

We now describe the forcing notion for the Erdős-Moser theorem. Recall that an EM condition for an infinite tournament R is a Mathias condition (F, X) where $F \cup \{x\}$ is R-transitive for each $x \in X$ and X is included in a minimal R-interval of F.

Definition 3.13 We denote by $\mathbb{P}$ the forcing notion whose conditions are tuples $(\vec{F}, T, \mathcal{C})$ where

(a) T is an infinite p.r. partition tree
(b) $\mathcal{C}$ is a $\emptyset'$-p.r. promise for T
(c) $(F_\nu, \mathrm{dom}(T))$ is an EM condition for R and each $\nu < \mathrm{parts}(T)$

A condition $d = (\vec{E}, S, \mathcal{D})$ *extends* $c = (\vec{F}, T, \mathcal{C})$ (written $d \leq c$) if there exists a function $f : \ell \to k$ such that $\mathcal{D} \subseteq \mathcal{C}$ and the following holds:

(i) $(E_\nu, \mathrm{dom}(S))$ EM extends $(F_{f(\nu)}, \mathrm{dom}(T))$ for each $\nu < \mathrm{parts}(S)$
(ii) S f-refines $\bigcap_{\nu < \mathrm{parts}(S)} T^{[f(\nu), E_\nu]}$

We may think of a condition $c = (\vec{F}, T, \mathcal{C})$ as a collection of EM conditions (F_ν, H_ν) for R, where $H_\nu = \mathrm{dom}(T) \cap \mathrm{set}_\nu(P)$ for some path P through T. H_ν must be infinite for at least one of the parts $\nu < \mathrm{parts}(T)$. At a higher level, $\mathcal{D}$ restricts the possible subtrees S and parts μ refining some part of T in the condition c. Given a condition $c = (\vec{F}, T, \mathcal{C})$, we write parts$(c)$ for parts(T).

Lemma 3.14 For every condition $c = (\vec{F}, T, \mathcal{C})$ and every $n \in \omega$, there exists an extension $d = (\vec{E}, S, \mathcal{D})$ such that $|E_\nu| \geq n$ on each acceptable part ν of S.

Proof. It suffices to prove that for every condition $c = (\vec{F}, T, \mathcal{C})$ and every acceptable part ν of T, there exists an extension $d = (\vec{E}, S, \mathcal{D})$ such that $S \leq_{id} T$ and $|E_\nu| \geq n$. Iterating the process at most parts(T) times enables us to conclude. Fix an acceptable part ν of T and a path P trough T such that $\mathrm{set}_\nu(P)$ is infinite. Let F' be an R-transitive subset of $\mathrm{set}_\nu(P) \cap \mathrm{dom}(T)$ of size n. Such a set exists by the classical Erdős-Moser theorem. Let $\vec{E}$ be defined by $E_\mu = F_\mu$ if $\mu \neq \nu$ and $E_\nu = F_\nu \cup F'$ otherwise. As the tournament R is stable, there exists some $u \geq t$ such that $(E_\nu, [u, +\infty))$ is an EM condition and therefore EM extends $(F_\nu, \mathrm{dom}(T))$. Let S be the p.r. partition tree $T^{[\nu, E_\nu]}$ of $[u, +\infty)$. The condition $(\vec{E}, S, \mathcal{C}[S])$ is the desired extension. ☐

Given a condition $c \in \mathbb{P}$, we denote by Ext(c) the set of all its extensions.

3.4. **The forcing relation**

The forcing relation at the first level, namely, for Σ^0_1 and Π^0_1 formulas, is parameterized by some part of the tree of the considered condition. Thanks to the forcing relation we will define, we can build an infinite decreasing sequence of conditions which decide Σ^0_1 and Π^0_1 formulas effectively in $\emptyset'$. The sequence however yields a $\emptyset'$-computably bounded $\emptyset'$-computable tree of (possibly empty) parts. Therefore, any PA degree relative to $\emptyset'$ is sufficient to control the first jump of an infinite transitive subtournament of a stable infinite computable tournament.

We cannot do better since Kreuzer proved in [11] the existence of an infinite, stable, computable tournament with no low infinite transitive subtournament. If we ignore the promise part of a condition, the careful reader will recognize the construction of Cholak, Jockusch and Slaman [4] of a low_2 infinite subset of a Δ^0_2 set or its complement by the first jump control. The difference, which at first look seems only presentational, is in fact one of the key features of this notion of forcing. Indeed, forcing iterated jumps require to have a definitionally weak description of the set of the extensions of a condition, and it requires much less computational power to describe a primitive recursive tree than an infinite reservoir of a Mathias condition.

Definition 3.15 Fix a condition $c = (\vec{F}, T, \mathcal{C})$, a Σ^0_0 formula $\varphi(G, x)$ and a part $\nu < \text{parts}(T)$.

1. $c \Vdash_\nu (\exists x)\varphi(G, x)$ iff there exists a $w \in \omega$ such that $\varphi(F_\nu, w)$ holds.
2. $c \Vdash_\nu (\forall x)\varphi(G, x)$ iff for every $\sigma \in T$, every $w < |\sigma|$ and every R-transitive set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$, $\varphi(F_\nu \cup F', w)$ holds.

We start by proving some basic properties of the forcing relation over Σ^0_1 and Π^0_1 formulas. As one may expect, the forcing relation at first level is closed under the refinement relation.

Lemma 3.16 Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and a Σ^0_1 (Π^0_1) formula $\varphi(G)$. If $c \Vdash_\nu \varphi(G)$ for some $\nu < \text{parts}(T)$, then for every $d = (\vec{E}, S, \mathcal{D}) \leq c$ and every part μ of S refining part ν of T, $d \Vdash_\mu \varphi(G)$.

Proof. We have two cases.

- If $\varphi \in \Sigma^0_1$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 3.15, there exists a $w \in \omega$ such that $\psi(F_\nu, w)$ holds. By property (i) of the definition of an extension, $E_\mu \supseteq F_\nu$ and $(E_\mu \setminus F_\nu) \subset \text{dom}(T)$, therefore $\psi(E_\mu, w)$ holds by continuity, so by clause 1 of Definition 3.15, $d \Vdash_\mu (\exists x)\psi(G, x)$.
- If $\varphi \in \Pi^0_1$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. Fix a $\tau \in S$, a $w < |\tau|$ and an R-transitive set $F' \subseteq \text{dom}(S) \cap \text{set}_\mu(\tau)$. It suffices to prove that $\varphi(E_\mu \cup F')$ holds to conclude that $d \Vdash_\mu (\forall x)\psi(G, x)$ by clause 2 of Definition 3.15. By property (ii) of the definition of an extension, there exists a $\sigma \in T^{[\nu, E_\mu]}$ such that $|\sigma| = |\tau|$ and $\text{set}_\mu(\tau) \subseteq \text{set}_\nu(\sigma)$. As $\text{dom}(S) \subseteq \text{dom}(T)$, $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$. As $\sigma \in T^{[\nu, E_\mu]}$, $E_\mu \subseteq \text{set}_\nu(\sigma)$ and by property (i) of the definition of an extension, $E_\mu \subseteq \text{dom}(T)$. So $E_\mu \cup F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$. As $w < |\tau| = |\sigma|$ and $E_\mu \cup F'$ is an R-transitive subset of $\text{dom}(T) \cap \text{set}_\nu(\sigma)$, then by clause 2 of Definition 3.15 applied to $c \Vdash_\nu (\forall x)\psi(G, x)$, $\varphi(F_\nu \cup (E_\mu \setminus F_\nu) \cup F', w)$ holds, hence $\varphi(E_\mu \cup F')$ holds.

□

Before defining forcing relation at the higher levels, we prove a density lemma for Σ^0_1 and Π^0_1 formulas. It enables us in particular to reprove that every PA relative to $\emptyset'$ computes the jump of an infinite R-transitive set.

Lemma 3.17 For every Σ^0_1 (Π^0_1) formula φ, the following set is dense

$$\{c = (\vec{F}, T, \mathcal{C}) \in \mathbb{P} : (\forall \nu < \text{parts}(T))[c \Vdash_\nu \varphi(G) \vee c \Vdash_\nu \neg\varphi(G)]\}$$

Proof. It suffices to prove the statement for the case where φ is a Σ^0_1 formula, as the case where φ is a Π^0_1 formula is symmetric. Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and let $I(c)$ be the set of the parts $\nu < \text{parts}(T)$ such that $c \nVdash_\nu \varphi(G)$ and $c \nVdash_\nu \neg\varphi(G)$. If $I(c) = \emptyset$ then we are done, so suppose $I(c) \neq \emptyset$ and fix some $\nu \in I(c)$. We will construct an extension d of c such that $I(d) \subseteq I(c) \setminus \{\nu\}$. Applying iteratively the operation enables us to conclude.

The formula φ is of the form $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. Define $f : k+1 \to k$ as $f(\mu) = \mu$ if $\mu < k$ and $f(k) = \nu$ otherwise. Let S be the set of all $\sigma \in (k+1)^{<\omega}$ which f-refine some $\tau \in T \cap k^{|\sigma|}$ and such that for every $w < |\sigma|$, every part $\mu \in \{\nu, k\}$ and every finite R-transitive set $F' \subseteq \text{dom}(T) \cap \text{set}_\mu(\sigma)$, $\varphi(F_\nu \cup F', w)$ does not hold.

Note that S is a p.r. partition tree of $[t, +\infty)$ refining T with witness function f. Suppose that S is infinite. Let $\vec{E}$ be defined by $E_\mu = F_\mu$ if $\mu < k$ and $E_k = F_\nu$ and consider the extension $d = (\vec{E}, S, \mathcal{C}[S])$. We claim that $\nu, k \notin I(d)$. Fix a part $\mu \in \{\nu, k\}$ of S. By definition of S, for every $\sigma \in S$, every $w < |\sigma|$ and every R-transitive set $F' \subseteq \mathrm{dom}(S) \cap \mathrm{set}_\mu(\sigma)$, $\varphi(E_\mu \cup F', w)$ does not hold. Therefore, by clause 2 of Definition 3.15, $d \Vdash_\mu (\forall x)\neg\psi(G, x)$, hence $d \Vdash_\mu \neg\varphi(G)$. Note that $I(d) \subseteq I(c) \setminus \{\nu\}$.

Suppose now that S is finite. Fix a threshold $\ell \in \omega$ such that $(\forall \sigma \in S)|\sigma| < \ell$ and a $\tau \in T \cap k^\ell$ such that $T^{[\tau]}$ is infinite. Consider the 2-partition $E_0 \sqcup E_1$ of $\mathrm{set}_\nu(\tau) \cap \mathrm{dom}(T)$ defined by $E_0 = \{i \geq t : \tau(i) = \nu \wedge (\forall^\infty s)R(i, s) \text{ holds}\}$ and $E_1 = \{i \geq t : \tau(i) = \nu \wedge (\forall^\infty s)R(s, i) \text{ holds}\}$. This is a 2-partition since the tournament R is stable. As there exists no $\sigma \in S$ which f-refines τ, there exists a $w < \ell$ and an R-transitive set $F' \subseteq E_0$ or $F' \subseteq E_1$ such that $\varphi(F_\nu \cup F', w)$ holds. By choice of the partition, there exists a $t' > t$ such that $F' \to_R [t', +\infty)$ or $[t', +\infty) \to_R F'$. By Lemma 3.6, $(F_\nu \cup F', [t', +\infty))$ is a valid EM extension for $(F_\nu, [t, +\infty))$. As $T^{[\tau]}$ is infinite, $T^{[\nu, F']}$ is also infinite. Let $\vec{E}$ be defined by $E_\mu = F_\mu$ if $\mu \neq \nu$ and $E_\mu = F_\nu \cup F'$ otherwise. Let S be the k-partition tree $(k, t', T^{[\nu, F']})$. The condition $d = (\vec{E}, S, \mathcal{C}[S])$ is a valid extension of c. By clause 1 of Definition 3.15, $d \Vdash_\mu \varphi(G)$. Therefore $I(d) \subseteq I(c) \setminus \{\nu\}$. ☐

As in the previous notion of forcing, the following trivial lemma expresses the fact that the promise part of a condition has no effect in the forcing relation for a Σ^0_1 or Π^0_1 formula.

Lemma 3.18 Fix two conditions $c = (\vec{F}, T, \mathcal{C})$ and $d = (\vec{F}, T, \mathcal{D})$, and a Σ^0_1 (Π^0_1) formula. For every part ν of T, $c \Vdash_\nu \varphi(G)$ if and only if $d \Vdash_\nu \varphi(G)$.

Proof. If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 3.15, $c \Vdash_\nu \varphi(G)$ iff there exists a $w \in \omega$ such that $\psi(F_\nu, w)$ holds, iff $d \Vdash_\nu \varphi(G)$. Similarily, if $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 2 of Definition 3.15, $c \Vdash_\nu \varphi(G)$ iff for every $\sigma \in T$, every $w < |\sigma|$ and every R-transitive set $F' \subseteq \mathrm{dom}(T) \cap \mathrm{set}_\nu(\sigma)$, $\varphi(F_\nu \cup F', w)$ holds, iff $d \Vdash_\nu \varphi(G)$. ☐

We are now ready to define the forcing relation for an arbitrary arithmetic formula. Again, the natural forcing relation induced by the forcing of Σ^0_0 formulas is too complex, so we design a more effective relation which still enjoys the main properties of a forcing relation.

Definition 3.19 Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and an arithmetic formula $\varphi(G)$.

1. If $\varphi(G) = (\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$ then $c \Vdash \varphi(G)$ iff for every part $\nu < \mathrm{parts}(T)$ such that $(\nu, T) \in \mathcal{C}$ there exists a $w < \mathrm{dom}(T)$ such that $c \Vdash_\nu \psi(G, w)$
2. If $\varphi(G) = (\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$ then $c \Vdash \varphi(G)$ iff for every infinite p.r. k'-partition tree S, every function $f : \mathrm{parts}(S) \to \mathrm{parts}(T)$, every w and $\vec{E}$ smaller than $\#S$ such that the following hold
   i) $(E_\nu, \mathrm{dom}(S))$ EM extends $(F_{f(\nu)}, \mathrm{dom}(T))$ for each $\nu < \mathrm{parts}(S)$
   ii) S f-refines $\bigcap_{\nu < \mathrm{parts}(S)} T^{[f(\nu), E_\nu]}$

   for every $(\mu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \nVdash_\mu \neg\psi(G, w)$
3. If $\varphi(G) = (\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+2}$ then $c \Vdash \varphi(G)$ iff there exists a $w \in \omega$ such that $c \Vdash \psi(G, w)$
4. If $\varphi(G) = \neg\psi(G, x)$ where $\psi \in \Sigma^0_{n+3}$ then $c \Vdash \varphi(G)$ iff $d \nVdash \psi(G)$ for every $d \in \mathrm{Ext}(c)$.

Notice that, unlike the forcing relation for Σ^0_1 and Π^0_1 formulas, the relation over higher formuals does not depend on the part of the relation. The careful reader will have recognized the combinatorics of the second jump control introduced by Cholak, Jockusch and Slaman in [4]. We now prove the main properties of this forcing relation.

Lemma 3.20 Fix a condition c and a Σ^0_{n+2} (Π^0_{n+2}) formula $\varphi(G)$. If $c \Vdash \varphi(G)$ then for every $d \leq c$, $d \Vdash \varphi(G)$.

Proof. We prove the statement by induction over the complexity of the formula $\varphi(G)$. Fix a condition $c = (\vec{F}, T, \mathcal{C})$ such that $c \Vdash \varphi(G)$ and an extension $d = (\vec{E}, S, \mathcal{D})$ of c.

- If $\varphi \in \Sigma^0_2$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 3.19, for every part ν of T such that $(\nu, T) \in \mathcal{C}$, there exists a $w < \text{dom}(T)$ such that $c \Vdash_\nu \psi(G, w)$. Fix a part μ of S such that $(\mu, S) \in \mathcal{D}$. As $\mathcal{D} \subseteq \mathcal{C}$, $(\mu, S) \in \mathcal{C}$. By upward-closure of $\mathcal{C}$, part μ of S refines some part ν of $\mathcal{C}$ such that $(\nu, T) \in \mathcal{C}$. Therefore by Lemma 3.16, $d \Vdash_\mu \psi(G, w)$, with $w < \text{dom}(T) \leq \text{dom}(S)$. Applying again clause 1 of Definition 3.19, we deduce that $d \Vdash (\forall x)\psi(G, x)$, hence $d \Vdash \varphi(G)$.
- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. Suppose for the sake of absurd that $d \nVdash (\forall x)\psi(G, x)$. Let $f : \text{parts}(S) \to \text{parts}(T)$ witness the refinement $S \leq T$. By clause 2 of Definition 3.19, there exists an infinite p.r. k'-partition tree S', a function $g : \text{parts}(S') \to \text{parts}(S)$, a $w \in \omega$, and $\vec{H}$ smaller than the code of S' such that
  i) $(H_\nu, \text{dom}(S'))$ EM extends $(E_{g(\nu)}, \text{dom}(S))$ for each $\nu < \text{parts}(S')$
  ii) S' g-refines $\bigcap_{\nu < \text{parts}(S')} S^{[g(\nu), H_\nu]}$
  iii) there exists a $(\mu, S') \in \mathcal{D}$ such that $(\vec{H}, S', \mathcal{D}[S']) \Vdash_\mu \neg\psi(G, w)$.

  To deduce by clause 2 of Definition 3.19 that $c \nVdash (\forall x)\psi(G, x)$ and derive a contradiction, it suffices to prove that the same properties hold w.r.t. T.
  i) As by property (i) of the definition of an extension, $(E_{g(\nu)}, \text{dom}(S))$ EM extends $(F_{f(g(\nu))}, \text{dom}(T))$ and $(H_\nu, \text{dom}(S')$ EM extends $(E_{g(\nu)}, \text{dom}(S))$, then $(H_\nu, \text{dom}(S'))$ EM extends $(F_{f(g(\nu))}, \text{dom}(T))$.
  ii) As by property (ii) of the definition of an extension, S f-refines $\bigcap_{\nu < \text{parts}(S')} T^{[f(g(\nu)), E_{g(\nu)}]}$ and S' g-refines $\bigcap_{\nu < \text{parts}(S')} S^{[g(\nu), H_\nu]}$, then S' $(g \circ f)$-refines $\bigcap_{\nu < \text{parts}(S')} T^{[(g \circ f)(\nu), H_\nu]}$.
  iii) As $\mathcal{D} \subseteq \mathcal{C}$, there exists a part $(\mu, S') \in \mathcal{C}$ such that $(\vec{H}, S', \mathcal{D}[S']) \Vdash_\mu \neg\psi(G, w)$. By Lemma 3.18, $(\vec{H}, S', \mathcal{C}[S']) \Vdash_\mu \neg\psi(G, w)$.
- If $\varphi \in \Sigma^0_{n+3}$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+2}$. By clause 3 of Definition 3.19, there exists a $w \in \omega$ such that $c \Vdash \psi(G, w)$. By induction hypothesis, $d \Vdash \psi(G, w)$ so by clause 3 of Definition 3.19, $d \Vdash \varphi(G)$.
- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause 4 of Definition 3.19, for every $e \in \text{Ext}(c)$, $e \nVdash \psi(G)$. As $\text{Ext}(d) \subseteq \text{Ext}(c)$, for every $e \in \text{Ext}(d)$, $e \nVdash \psi(G)$, so by clause 4 of Definition 3.19, $d \Vdash \varphi(G)$.

□

Lemma 3.21 For every Σ^0_{n+2} (Π^0_{n+2}) formula φ, the following set is dense

$$\{c \in \mathbb{P} : c \Vdash \varphi(G) \text{ or } c \Vdash \neg\varphi(G)\}$$

Proof. We prove the statement by induction over n. It suffices to treat the case where φ is a Σ^0_{n+2} formula, as the case where φ is a Π^0_{n+2} formula is symmetric. Fix a condition $c = (\vec{F}, T, \mathcal{C})$.

- In case $n = 0$, the formula φ is of the form $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. Suppose there exists an infinite p.r. k'-partition tree S for some $k' \in \omega$, a function $f : \text{parts}(S) \to \text{parts}(T)$ and a k'-tuple of finite sets $\vec{E}$ such that
  i) $(E_\nu, [\ell, +\infty))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$.
  ii) S f-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E_\nu]}$
  iii) for each non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \Vdash_\nu \psi(G, w)$ for some $w < \#S$

  We can choose $\text{dom}(S)$ so that $(E_\nu, \text{dom}(S))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$. Properties i-ii) remain trivially true. By Lemma 3.16 and Lemma 3.18, property iii) remains true too. Let $\mathcal{D} = \mathcal{C}[S] \setminus \{(\nu, S') \in \mathcal{C} : \text{ part } \nu \text{ of } S' \text{ is empty}\}$. As $\mathcal{C}$ is an $\emptyset'$-p.r. promise for T, $\mathcal{C}[S]$ is an $\emptyset'$-p.r. promise for S. As $\mathcal{D}$ is obtained from $\mathcal{C}[S]$ by removing only empty parts, $\mathcal{D}$ is also an $\emptyset'$-p.r. promise for S. By clause 1 of Definition 3.19, $d = (\vec{E}, S, \mathcal{D}) \Vdash (\exists x)\psi(G, x)$ hence $d \Vdash \varphi(G)$.

We may choose a coding of the p.r. trees such that such that the code of S is sufficiently large to witness ℓ and $\vec{E}$. So suppose now that for every infinite p.r. k'-partition tree S, every function $f : \text{parts}(S) \to \text{parts}(T)$ and $\vec{E}$ smaller than the code of S such that properties i-ii) hold, there exists a non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$ and $(\vec{E}, S, \mathcal{C}) \not\Vdash_\nu \psi(G, w)$ for every $w < \ell$. Let $\mathcal{D}$ be the collection of all such (ν, S). The set $\mathcal{D}$ is $\emptyset'$-p.r. since by Lemma 3.25, both $(\vec{E}, S, \mathcal{C}) \not\Vdash_\nu \psi(G, w)$ and "part ν of S is non-empty" are Σ^0_1. By Lemma 3.16 and since we require that $\#S \geq \#T$ in the definition of $S \leq T$, $\mathcal{D}$ is upward-closed under the refinement relation, hence is a promise for T. By clause 2 of Definition 3.19, $d = (\vec{F}, T, \mathcal{D}) \Vdash (\forall x)\neg\psi(G, x)$, hence $d \Vdash \neg\varphi(G)$.
- In case $n > 0$, density follows from clause 4 of Definition 3.19.

□

Given any filter $\mathcal{F} = \{c_0, c_1, \dots\}$ with $c_s = (\vec{F}_s, T_s, \mathcal{C}_s)$, the set of the acceptable parts ν of T_s such that $(\nu, T_s) \in \mathcal{C}_s$ forms an infinite, directed acyclic graph $\mathcal{G}(\mathcal{F})$. Whenever $\mathcal{F}$ is sufficiently generic, the graph $\mathcal{G}(\mathcal{F})$ has a unique infinite path P. The path P induces an infinite set $G = \bigcup_s F_{P(s),s}$. We call P the *generic path* and G the *generic real.*

Lemma 3.22 Suppose that $\mathcal{F}$ is sufficiently generic and let P and G be the generic path and the generic real, respectively. For any Σ^0_1 (Π^0_1) formula $\varphi(G)$, $\varphi(G)$ holds iff $c_s \Vdash_{P(s)} \varphi(G)$ for some $c_s \in \mathcal{F}$.

Proof. Fix a condition $c_s = (\vec{F}, T, \mathcal{C}) \in \mathcal{F}$ such that $c \Vdash_{P(s)} \varphi(G)$, and let $\nu = P(s)$.
- If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 3.15, there exists a $w \in \omega$ such that $\psi(F_\nu, w)$ holds. As $\nu = P(s)$, $F_\nu = F_{P(s)} \subseteq G$ and $G \setminus F_\nu \subseteq (max(F_\nu), +\infty)$, so $\psi(G, w)$ holds by continuity, hence $\varphi(G)$ holds.
- If $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 2 of Definition 3.15, for every $\sigma \in T$, every $w < |\sigma|$ and every R-transitive set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$, $\psi(F_\nu \cup F', w)$ holds. For every $F' \subseteq G \setminus F_\nu$, and $w \in \omega$ there exists a $\sigma \in T$ such that $w < |\sigma|$ and $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$. Hence $\psi(F_\nu \cup F', w)$ holds. Therefore, for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.

The other direction holds by Lemma 3.17. □

Lemma 3.23 Suppose that $\mathcal{F}$ is sufficiently generic and let P and G be the generic path and the generic real, respectively. For any Σ^0_{n+2} (Π^0_{n+2}) formula $\varphi(G)$, $\varphi(G)$ holds iff $c_s \Vdash \varphi(G)$ for some $c_s \in \mathcal{F}$.

Proof. We prove the statement by induction over the complexity of the formula $\varphi(G)$. As previously noted in Lemma 2.13, it suffices to prove that if $c_s \Vdash \varphi(G)$ for some $c_s \in \mathcal{F}$ then $\varphi(G)$ holds. Indeed, conversely if $\varphi(G)$ holds, then by Lemma 3.21 and by genericity of $\mathcal{F}$ either $c_s \Vdash \varphi(G)$ or $c_s \Vdash \neg\varphi(G)$, but if $c \Vdash \neg\varphi(G)$ then $\neg\varphi(G)$ holds, contradicting the hypothesis. So $c_s \Vdash \varphi(G)$. Fix a condition $c_s = (\vec{F}, T, \mathcal{C}) \in \mathcal{F}$ such that $c_s \Vdash \varphi(G)$. We proceed by case analysis on φ.
- If $\varphi \in \Sigma^0_2$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 3.19, for every part ν of T such that $(\nu, T) \in \mathcal{C}$, there exists a $w < \text{dom}(T)$ such that $c_s \Vdash_\nu \psi(G, w)$. In particular $(P(s), T) \in \mathcal{C}$, so $c_s \Vdash_{P(s)} \psi(G, w)$. By Lemma 3.22, $\psi(G, w)$ holds, hence $\varphi(G)$ holds.
- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 3.19, for every infinite k'-partition tree S, every function $f : \text{parts}(S) \to \text{parts}(T)$, every w and $\vec{E}$ smaller than the code of S such that the following hold
  i) $(E_\nu, \text{dom}(S))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$
  ii) S f-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E_\nu]}$

for every $(\mu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \not\Vdash_\mu \neg\psi(G, w)$. Suppose for the sake of absurd that $\psi(G, w)$ does not hold for some $w \in \omega$. Then by Lemma 3.22, there exists a $d_t \in \mathcal{F}$ such that $d_t \Vdash_{P(t)} \neg\psi(G, w)$. Since $\mathcal{F}$ is a filter, there is a condition $e_r = (\vec{E}, S, \mathcal{D}) \in \mathcal{F}$ extending both c_s and d_t. Let $\mu = P(r)$. By choice of P, $(\mu, S) \in \mathcal{C}$, so by clause ii), $(\vec{E}, S, \mathcal{C}[S]) \not\Vdash_\mu \psi(G, w)$, hence by Lemma 3.18, $e_r \not\Vdash_\mu \neg\psi(G, w)$. However, since part μ of S refines part $P(t)$ of d_t, then by Lemma 3.16, $e_r \Vdash_\mu \neg\psi(G, w)$. Contradiction. Hence for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.

- If $\varphi \in \Sigma^0_{n+3}$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+2}$. By clause 3 of Definition 3.19, there exists a $w \in \omega$ such that $c_s \Vdash \psi(G, w)$. By induction hypothesis, $\psi(G, w)$ holds, hence $\varphi(G)$ holds.

  Conversely, if $\varphi(G)$ holds, then there exists a $w \in \omega$ such that $\psi(G, w)$ holds, so by induction hypothesis $c_s \Vdash \psi(G, w)$ for some $c_s \in \mathcal{F}$, so by clause 3 of Definition 3.19, $c_s \Vdash \varphi(G)$.
- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause 4 of Definition 3.19, for every $d \in \operatorname{Ext}(c_s)$, $d \not\Vdash \psi(G)$. By Lemma 3.20, $d \not\Vdash \psi(G)$ for every $d \in \mathcal{F}$ and by a previous case, $\psi(G)$ does hold, so $\varphi(G)$ holds.

□

We now prove that the forcing relation has good definitional properties as we did with the notion of forcing for cohesiveness.

Lemma 3.24 For every condition c, $\operatorname{Ext}(c)$ is Π^0_2 uniformly in c.

Proof. Recall from Lemma 3.10 that given $k, t \in \omega$, $PartTree(k, t)$ denotes the Π^0_1 set of all the infinite p.r. k-partition trees of $[t, +\infty)$, and given a k-partition tree S and a part ν of S, the predicate $Empty(S, \nu)$ denotes the Π^0_1 formula "part ν of S is empty", that is, the formula $(\forall \sigma \in S)[\operatorname{set}_\nu(\sigma) \cap \operatorname{dom}(S) = \emptyset]$. If T is p.r. then so is $T^{[\nu, H]}$ for some finite set H.

Fix a condition $c = (\vec{F}, (k, t, T), \mathcal{C})$. By definition, $(\vec{H}, (k', t', S), \mathcal{D}) \in \operatorname{Ext}(c)$ iff the following formula holds:

$$\begin{array}{ll}
(\exists f : k' \to k) & \\
(\forall \nu < k')(H_\nu, [t', +\infty)) \text{ EM extends } (F_{f(\nu)}, [t, +\infty)) & (\Pi^0_1) \\
\wedge S \in PartTree(k', t') \wedge S \leq_f \bigwedge_{\nu < k'} T^{[f(\nu), H_\nu]} & (\Pi^0_1) \\
\wedge \mathcal{D} \text{ is a promise for } S \wedge \mathcal{D} \subseteq \mathcal{C} & (\Pi^0_2)
\end{array}$$

By Lemma 3.10 and the fact that $\bigwedge_{\nu < k'} T^{[f(\nu), H_\nu]}$ is p.r. uniformly in T, f, $\vec{H}$ and k', the above formula is Π^0_2. □

Lemma 3.25 Fix an arithmetic formula $\varphi(G)$, a condition $c = (\vec{F}, T, \mathcal{C})$ and a part ν of T.

a) If $\varphi(G)$ is a Σ^0_1 (Π^0_1) formula then so is the predicate $c \Vdash_\nu \varphi(G)$.
b) If $\varphi(G)$ is a Σ^0_{n+2} (Π^0_{n+2}) formula then so is the predicate $c \Vdash \varphi(G)$.

Proof. We prove our lemma by induction over the complexity of the formula $\varphi(G)$.

- If $\varphi(G) \in \Sigma^0_1$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 3.15, $c \Vdash_\nu \varphi(G)$ if and only if the formula $(\exists w \in \omega)\psi(F_\nu, w)$ holds. This is a Σ^0_1 predicate.
- If $\varphi(G) \in \Pi^0_1$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 2 of Definition 3.15, $c \Vdash_\nu \varphi(G)$ if and only if the formula $(\forall \sigma \in T)(\forall w < |\sigma|)(\forall F' \subseteq \operatorname{dom}(T) \cap \operatorname{set}_\nu(\sigma))[F' \ R\text{-transitive} \to \psi(F_\nu \cup F', w)]$ holds. This is a Π^0_1 predicate.
- If $\varphi(G) \in \Sigma^0_2$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 3.19, $c \Vdash \varphi(G)$ if and only if the formula $(\forall \nu < \operatorname{parts}(T))(\exists w < \operatorname{dom}(T))[(\nu, T) \in \mathcal{C} \to c \Vdash_\nu \psi(G, w)]$ holds. This is a Σ^0_2 predicate by induction hypothesis and the fact that $\mathcal{C}$ is $\emptyset'$-computable.
- If $\varphi(G) \in \Pi^0_2$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 3.19, $c \Vdash \varphi(G)$ if and only if for every infinite k'-partition tree S, every

function $f : \text{parts}(S) \to \text{parts}(T)$, every w and $\vec{E}$ smaller than the code of S such that the following hold

i) $(E_\nu, \text{dom}(S))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$

ii) S f-refines $\bigcap_{\nu<\text{parts}(S)} T^{[f(\nu),E_\nu]}$

for every $(\mu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \not\Vdash_\mu \neg\psi(G, w)$. By Lemma 3.10, Properties i-ii) are Δ^0_2. Moreover, the predicate $(\mu, S) \in \mathcal{C}$ is Δ^0_2. By induction hypothesis, $(\vec{E}, S, \mathcal{C}) \not\Vdash_\mu \neg\psi(G, w)$ is Σ^0_1. Therefore $c \Vdash \varphi(G)$ is a Π^0_2 predicate.

- If $\varphi(G) \in \Sigma^0_{n+3}$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+2}$. By clause 3 of Definition 3.19, $c \Vdash \varphi(G)$ if and only if the formula $(\exists w \in \omega)c \Vdash \psi(G, w)$ holds. This is a Σ^0_{n+3} predicate by induction hypothesis.
- If $\varphi(G) \in \Pi^0_{n+3}$ then it can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause 4 of Definition 3.19, $c \Vdash \varphi(G)$ if and only if the formula $(\forall d)(d \notin \text{Ext}(c) \vee d \not\Vdash \psi(G))$ holds. By induction hypothesis, $d \not\Vdash \psi(G)$ is a Π^0_{n+3} predicate. By Lemma 3.24, the set $\text{Ext}(c)$ is Π^0_2-computable uniformly in c, thus $c \Vdash \varphi(G)$ is a Π^0_{n+3} predicate.

□

3.5. **Preserving the arithmetic hierarchy**

We now prove the core lemmas showing that every sufficiently generic real preserves the arithmetic hierarchy. The proof is split into two lemmas since the forcing relation for Σ^0_1 and Π^0_1 formulas depends on the part of the condition, and therefore has to be treated separately.

Lemma 3.26 If $A \notin \Sigma^0_1$ and $\varphi(G, x)$ is Σ^0_1, then the set of $c = (\vec{F}, T, \mathcal{C}) \in \mathbb{P}$ satisfying the following property is dense:

$$(\forall \nu < \text{parts}(T))[(\exists w \in A)c_s \Vdash_\nu \neg\varphi(G, w)] \vee [(\exists w \notin A)c_s \Vdash_\nu \varphi(G, w)]$$

Proof. The formula $\varphi(G, w)$ can be expressed as $(\exists x)\psi(G, w, x)$ where $\psi \in \Sigma^0_0$. Given a condition $c = (\vec{F}, T, \mathcal{C})$, let $I(c)$ be the set of the parts ν of T such that for every $w \in A$, $c \not\Vdash_\nu \neg\varphi(G, w)$ and for every $w \in \overline{A}$, $c \not\Vdash_\nu \varphi(G, w)$. If $I(c) = \emptyset$ then we are done, so suppose $I(c) \neq \emptyset$ and fix some $\nu \in I(c)$. We will construct an extension d such that $I(d) \subseteq I(c) \setminus \{\nu\}$. Applying iteratively the operation enables us to conclude.

Say that T is a k-partition tree of $[t, +\infty)$ for some $k, t \in \omega$. Define $f : k+1 \to k$ as $f(\mu) = \mu$ if $\mu < k$ and $f(k) = \nu$ otherwise. Given an integer $w \in \omega$, let S_w be the set of all $\sigma \in (k+1)^{<\omega}$ which f-refine some $\tau \in T \cap k^{|\sigma|}$ and such that for every $u < |\sigma|$, every part $\mu \in \{\nu, k\}$ and every finite R-transitive set $F' \subseteq \text{dom}(T) \cap \text{set}_\mu(\sigma)$, $\varphi(F_\nu \cup F', w, u)$ does not hold.

The set S_w is a p.r. (uniformly in w) partition tree of $[t, +\infty)$ refining T with witness function f. Let $U = \{w \in \omega : S_w \text{ is finite }\}$. $U \in \Sigma^0_1$, thus $U \neq A$. Fix some $w \in U \Delta A$. Suppose first that $w \in A \setminus U$. By definition of U, S_w is infinite. Let $\vec{E}$ be defined by $E_\mu = F_\mu$ if $\mu < k$ and $E_k = F_\nu$, and consider the extension $d = (\vec{E}, S_w, \mathcal{C}[S_w])$. We claim that $I(d) \subseteq I(c) \setminus \{\nu\}$. Fix a part $\mu \in \{\nu, k\}$ of S_w. By definition of S_w, for every $\sigma \in S_w$, every $u < |\sigma|$ and every R-transitive set $F' \subseteq \text{dom}(S_w) \cap \text{set}_\mu(\sigma)$, $\varphi(E_\mu \cup F', w, u)$ does not hold. Therefore, by clause 2 of Definition 3.15, $d \Vdash_\mu (\forall x)\neg\psi(G, w, x)$, hence $d \Vdash_\mu \neg\varphi(G, w)$, and this for some $w \in A$. Thus $I(d) \subseteq I(c) \setminus \{\nu\}$.

Suppose now that $w \in U \setminus A$, so S_w is finite. Fix an $\ell \in \omega$ such that $(\forall \sigma \in S)|\sigma| < \ell$ and a $\tau \in T \cap k^\ell$ such that $T^{[\tau]}$ is infinite. Consider the 2-partition $E_0 \cup E_1$ of $\text{set}_\nu(\tau) \cap \text{dom}(T)$ defined by $E_0 = \{i \geq t : \tau(i) = \nu \wedge (\forall^\infty s)R(i, s) \text{ holds}\}$ and $E_0 = \{i \geq t : \tau(i) = \nu \wedge (\forall^\infty s)R(s, i) \text{ holds}\}$. As there exists no $\sigma \in S_w$ which f-refines τ, there exists a $u < \ell$ and an R-transitive set $F' \subseteq E_0$ or $F' \subseteq E_1$ such that $\varphi(F_\nu \cup F', w, u)$ holds. By choice of the partition, there exists a $t' > t$ such that $F' \to_R [t', +\infty)$ or $[t', +\infty) \to_R F'$. By Lemma 3.6, $(F_\nu \cup F', [t', +\infty))$ is a valid EM extension of $(F_\nu, [t, +\infty))$. As $T^{[\tau]}$ is infinite, $T^{[\nu,F']}$ is also infinite. Let $\vec{E}$ be defined by $E_\mu = F_\mu$ if $\mu \neq \nu$ and $E_\mu = F_\nu \cup F'$ otherwise. Let S be the k-partition tree $(k, t', T^{[\nu,F']})$. The condition $d = (\vec{E}, S, \mathcal{C}[S])$ is a valid extension of c. By clause 1 of Definition 3.15, $d \Vdash_\mu \varphi(G, w)$ with $w \notin A$. . Therefore $I(d) \subseteq I(c) \setminus \{\nu\}$. □

Lemma 3.27 If $A \not\in \Sigma^0_{n+2}$ and $\varphi(G,x)$ is Σ^0_{n+2}, then the set of $c \in \mathbb{P}$ satisfying the following property is dense:

$$[(\exists w \in A) c \Vdash \neg\varphi(G,w)] \vee [(\exists w \not\in A) c \Vdash \varphi(G,w)]$$

Proof. Fix a condition $c = (\vec{F}, T, \mathcal{C})$.

- In case $n = 0$, $\varphi(G,w)$ can be expressed as $(\exists x)\psi(G,w,x)$ where $\psi \in \Pi^0_1$. Let U be the set of integers w such that there exists an infinite p.r. k'-partition tree S for some $k' \in \omega$, a function $f : \text{parts}(S) \to \text{parts}(T)$ and a k'-tuple of finite sets $\vec{E}$ such that
  i) $(E_\nu, [\ell, +\infty))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$.
  ii) S f-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E_\nu]}$
  iii) for each non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \Vdash_\nu \psi(G,w,u)$ for some $u < \#S$

  By Lemma 3.25 and Lemma 3.10, $U \in \Sigma^0_2$, thus $U \neq A$. Let $w \in U \Delta A$. Suppose that $w \in U \setminus A$. We can choose $\text{dom}(S)$ so that $(E_\nu, \text{dom}(S))$ EM extends $(F_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$. By Lemma 3.16 and Lemma 3.18, properties i-ii) remain true. Let $\mathcal{D} = \mathcal{C}[S] \setminus \{(\nu, S') \in \mathcal{C} : \text{ part } \nu \text{ of } S' \text{ is empty}\}$. As $\mathcal{C}$ is an $\emptyset'$-p.r. promise for T, $\mathcal{C}[S]$ is an $\emptyset'$-p.r. promise for S. As $\mathcal{D}$ is obtained from $\mathcal{C}[S]$ by removing only empty parts, $\mathcal{D}$ is also an $\emptyset'$-p.r. promise for S. By clause 1 of Definition 3.19, $d = (\vec{E}, S, \mathcal{D}) \Vdash (\exists x)\psi(G,w,x)$ hence $d \Vdash \varphi(G,w)$ for some $w \not\in A$.

  We may choose a coding of the p.r. trees such that such that the code of S is sufficiently large to witness u and $\vec{E}$. So suppose now that $w \in A \setminus U$. Then for every infinite p.r. k'-partition tree S, every ℓ and $\vec{E}$ smaller than the code of S such that properties i-ii) hold, there exists a non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$ and $(\vec{E}, S, \mathcal{C}) \not\Vdash_\nu \psi(G,w,u)$ for every $u < \ell$. Let $\mathcal{D}$ be the collection of all such (ν, S). The set $\mathcal{D}$ is $\emptyset'$-p.r. By Lemma 3.16 and since $\#S \geq \#T$ whenever $S \leq_f T$, $\mathcal{D}$ is upward-closed under the refinement relation, hence it is a promise for T. By clause 2. of Definition 3.19, $d = (\vec{F}, T, \mathcal{D}) \Vdash (\forall x)\neg\psi(G,w,x)$, hence $d \Vdash \neg\varphi(G,w)$ for some $w \in A$.
- In case $n > 0$, let $U = \{w \in \omega : (\exists d \in \text{Ext}(c)) d \Vdash \varphi(G,w)\}$. By Lemma 3.24 and Lemma 3.25, $U \in \Sigma^0_{n+2}$, thus $U \neq A$. Fix some $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of U, there exists a condition d extending c such that $d \Vdash \varphi(G,w)$. If $w \in A \setminus U$, then for every $d \in \text{Ext}(c)$, $d \not\Vdash \varphi(G,w)$ so by clause 4 of Definition 3.19, $c \Vdash \neg\varphi(G,w)$.

□

We are now ready to prove Theorem 3.2. It follows from the preservation of the arithmetic hierarchy for cohesiveness and the stable Erdős-Moser theorem.

Proof of Theorem 3.2. Since $\mathsf{RCA}_0 \vdash \mathsf{COH} \wedge \mathsf{SEM} \to \mathsf{EM}$, then by Theorem 2.2 it suffices to prove that SEM admits preservation of the arithmetic hierarchy. Fix some set C and a C-computable stable infinite tournament R. Let $\mathcal{C}_0$ be the C'-p.r. set of all $(\nu, T) \in \mathbb{U}$ such that $(\nu, T) \leq (0, 1^{<\omega})$. Let $\mathcal{F}$ be a sufficiently generic filter containing $c_0 = (\{\emptyset\}, 1^{<\omega}, \mathcal{C}_0)$. Let P and G be the corresponding generic path and generic real, respectively. By definition of a condition, the set G is R-transitive. By Lemma 3.14, G is infinite. By Lemma 3.26 and Lemma 3.25, G preserves non-Σ^0_1 definitions relative to C. By Lemma 3.27 and Lemma 3.25, G preserves non-Σ^0_{n+2} definitions relative to C for every $n \in \omega$. Therefore, by Proposition 2.2 of [21], G preserves the arithmetic hierarchy relative to C. □

4. D^2_2 preserves higher definitions

Among the Ramsey-type hierarchies, the D hierarchy is conceptually the simplest one. It is therefore natural to study it in order to understand better the control of iterated jumps and focus on the core combinatorics without the technicalities specific to another hierarchy.

Definition 4.1 For every $n, k \geq 1$, D^n_k is the statement "Every Δ^0_n k-partition of the integers has an infinite subset of one of its parts".

In particular, D^1_k is nothing but RT^1_k. Cholak et al. [4] proved that D^2_k and stable Ramsey's theorem for pairs and k colors (SRT^2_k) are computably equivalent and that the proof is formalizable over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Later, Chong et al. [5] proved that D^2_2 implies $\mathsf{B}\Sigma^0_2$ over RCA_0, showing therefore that $\mathsf{RCA}_0 \vdash \mathsf{D}^2_k \leftrightarrow \mathsf{SRT}^2_\ell$ for every $k, \ell \geq 2$. Wang [21] studied D^2_2 within his framework of preservation of definitions and proved that D^2_2 admits preservation of Ξ definitions simultaneously for all Ξ in $\{\Sigma^0_{n+2}, \Pi^0_{n+2}, \Delta^0_{n+2} : n \in \omega\}$, but not Δ^0_2 definitions. He used for this a combination of the first jump control of Cholak, Jockusch and Slaman [4] and a relativization of the preservation of the arithmetic hierarchy by WKL_0.

In this section, we design a notion of forcing for D^2_2 with a forcing relation which has the same definitional complexity as the formula it forces. It enables us to reprove that D^2_2 admits preservation of Ξ definitions simultaneously for all Ξ in $\{\Sigma^0_{n+2}, \Pi^0_{n+2}, \Delta^0_{n+2} : n \in \omega\}$. The proof is significantly more involved than the previous proofs of preservation of the arithmetic hierarchy.

4.1. **Sides of a sequence of sets**

A main feature in the construction of an solution to an instance R_0, R_1 of D^2_2 is the parallel construction of a subset of R_0 and a subset of R_1. The intrinsic disjunction in the forcing argument prevents us from applying the same strategy as for the Erdős-Moser theorem and obtain a preservation of the arithmetic hierarchy. Given some $\alpha < 2$, we shall refer to R_α or simply α as a *side* of $\vec{R}$. We also need to define a relative notion of acceptation and emptiness of a part.

Definition 4.2 Fix a k-partition tree T of $[t, +\infty)$ and a set X. We say that part ν of T is *X-acceptable* if there exists a path P through T such that $\text{set}_\nu(P) \cap X$ is infinite. We say that part ν of T is *X-empty* if $(\forall \sigma \in T)[\text{dom}(T) \cap \text{set}_\nu(\sigma) \cap X = \emptyset]$.

The intended uses of those notions will be R_α-acceptation and R_α-emptiness. Every partition tree has an R_α-acceptable part for some $\alpha < 2$. The notion of X-emptiness is $\Pi^{0,X}_1$, and therefore Π^0_2 if X is Δ^0_2, which raises new problems for obtaining a forcing relation of weak definitional complexity. We would like to define a stronger notion of "witnessing its acceptable parts" and prove that for every infinite p.r. partition tree T, there is a p.r. refined tree S such that for each side α and each part ν of S, either ν is R_α-empty in S, or ν is R_α-acceptable. However, the resulting tree S would be $\emptyset'$-p.r. since R_α is $\emptyset'$-computable. Thankfully, we will be able to circumvent this problem in Lemma 4.17.

4.2. **Forcing conditions**

Fix a Δ^0_2 2-partition $R_0 \cup R_1 = \omega$. We now describe the notion of forcing to build an infinite subset of R_0 or of R_1.

Definition 4.3 We denote by $\mathbb{P}$ the forcing notion whose conditions are tuples $(\vec{F}, T, \mathcal{C})$ where

(a) T is an infinite, p.r. k-partition tree for some $k \in \omega$
(b) $\mathcal{C}$ is a $\emptyset'$-p.r. promise for T
(c) $(F^\alpha_\nu, \text{dom}(T))$ is a Mathias condition for each $\nu < k$ and $\alpha < 2$

A condition $d = (\vec{E}, S, \mathcal{D})$ *extends* $c = (\vec{F}, T, \mathcal{C})$ (written $d \leq c$) if there exists a function $f : \text{parts}(S) \to \text{parts}(T)$ such that $\mathcal{D} \subseteq \mathcal{C}$ and the following holds

(i) $(E^\alpha_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\alpha_{f(\nu)}, \text{dom}(T) \cap R_\alpha)$ for each $\nu < \text{parts}(S)$ and $\alpha < 2$
(ii) S f-refines $\bigcap_{\nu < \text{parts}(S), \alpha < 2} T^{[f(\nu), E^\alpha_\nu]}$

In the whole construction, the index α indicates that we are constructing a set cofinitely in R_α. Given a condition $c = (\vec{F}, T, \mathcal{C})$, we write again parts($c$) for parts($T$). The following lemma shows that we can force our constructed set to be infinite if we choose it among the acceptable parts.

Lemma 4.4 For every condition $c = (\vec{F}, T, \mathcal{C})$ and every $n \in \omega$, there exists an extension $d = (\vec{E}, S, \mathcal{D})$ such that $|E^\alpha_\nu| \geq n$ on each R_α-acceptable part ν of S for each $\alpha < 2$.

Proof. It suffices to prove that for every condition $c = (\vec{F}, T, \mathcal{C})$, every side $\alpha < 2$ and every R_α-acceptable part ν of T, there exists an extension $d = (\vec{E}, S, \mathcal{D})$ such that $S \leq_{id} T$ and $|E^\alpha_\nu| \geq n$. Iterating the process at most parts$(T) \times 2$ times enables us to conclude. Fix an R_α-acceptable part ν of T and a path P trough T such that $\text{set}_\nu(P) \cap R_\alpha$ is infinite. Let F' be a subset of $\text{set}_\nu(P) \cap \text{dom}(T) \cap R_\alpha$ of size n. Let $\vec{E}$ be defined by $E^\beta_\mu = F^\beta_\mu$ if $\mu \neq \nu \vee \beta \neq \alpha$ and $E^\alpha_\nu = F^\alpha_\nu \cup F'$ otherwise. Let S be the p.r. partition tree obtained from $T^{[\nu, E^\alpha_\nu]}$ by restricting its domain so that $(E^\alpha_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\alpha_\nu, \text{dom}(T) \cap R_\alpha)$. The condition $(\vec{E}, S, \mathcal{C}[S])$ is the desired extension. □

Given a condition c, we denote by Ext(c) the set of all its extensions.

4.3. **Forcing relation**

We need to define two forcing relations at the first level: the "true" forcing relation, i.e., the one having the good density properties but whose decision requires too much computational power, and a "weak" forcing relation having better computational properties, but which does not behave well with respect to the forcing. We start with the definition of the true forcing relation.

Definition 4.5 (True forcing relation) Fix a condition $c = (\vec{F}, T, \mathcal{C})$, a Σ^0_0 formula $\varphi(G, x)$, a part $\nu <$ parts(T), and a side $\alpha < 2$.

1. $c \Vdash^\alpha_\nu (\exists x)\varphi(G, x)$ iff there exists a $w \in \omega$ such that $\varphi(F^\alpha_\nu, w)$ holds.
2. $c \Vdash^\alpha_\nu (\forall x)\varphi(G, x)$ iff for every $\sigma \in T$ such that $T^{[\sigma]}$ is infinite, every $w < |\sigma|$ and every set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha$, $\varphi(F^\alpha_\nu \cup F', w)$ holds.

Given a condition c, a side $\alpha < 2$, a part ν of c and a Π^0_1 formula φ, the relation $c \Vdash^\alpha_\nu \varphi(G)$ is $\Pi^{0, \emptyset' \oplus R_\alpha}_1$, hence Π^0_2 as R_α is Δ^0_2. This relation enjoys the good properties of a forcing relation, that is, it is downward-closed under the refinement relation (Lemma 4.6), and the set of the conditions forcing either a Σ^0_1 formula or its negation is dense (Lemma 4.7).

Lemma 4.6 Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and a Σ^0_1 (Π^0_1) formula $\varphi(G)$. If $c \Vdash^\alpha_\nu \varphi(G)$ for some $\nu <$ parts(T) and $\alpha < 2$, then for every $d = (\vec{E}, S, \mathcal{D}) \leq c$ and every part μ of S refining part ν of T, $d \Vdash^\alpha_\mu \varphi(G)$.

Proof.

- If $\varphi \in \Sigma^0_1$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 4.5, there exists a $w \in \omega$ such that $\psi(F^\alpha_\nu, w)$ holds. By property (i) of the definition of an extension, $E^\alpha_\mu \supseteq F^\alpha_\nu$ and $(E^\alpha_\mu \setminus F^\alpha_\nu) \subset \text{dom}(T) \cap R_\alpha$, therefore by continuity $\psi(E^\alpha_\mu, w)$ holds, so by clause 1 of Definition 4.5, $d \Vdash^\alpha_\mu (\exists x)\psi(G, x)$.
- If $\varphi \in \Pi^0_1$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. Fix a $\tau \in S$ such that $S^{[\tau]}$ is infinite, a $w < |\tau|$ and a set $F' \subseteq \text{dom}(S) \cap \text{set}_\mu(\tau) \cap R_\alpha$. Let f be the function witnesing $d \leq c$. By property (ii) of the definition of an extension, τ f-refines a $\sigma \in T^{[\nu, E^\alpha_\mu]}$. We claim that we can even choose σ to be extendible in $T^{[\nu, E^\alpha_\mu]}$. Indeed, since τ is extendible in S, let P be a path through S extending τ and let U be the set of σ's in T such that $P \upharpoonright s$ f-refines σ for some s. The set U is an infinite subtree of T. Let σ be a string of length $|\tau|$ and extendible in U, hence in T. By definition of U, τ f-refines σ. By definition of a refinement, such that $|\sigma| = |\tau|$ and $\text{set}_\mu(\tau) \subseteq \text{set}_\nu(\sigma)$. As $w < |\tau|$ and $\text{dom}(S) \subseteq \text{dom}(T)$, $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha$. As $\sigma \in T^{[\nu, E^\alpha_\mu]}$, $E^\alpha_\mu \subseteq \text{set}_\nu(\sigma)$ and by property (i) of the definition of an extension, $E^\alpha_\mu \subseteq \text{dom}(T) \cap R_\alpha$ so $E^\alpha_\mu \subseteq \text{dom}(T) \cap R_\alpha$. Therefore $E^\alpha_\mu \cup F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha$. By clause 2 of

Definition 4.5 applied to $c \Vdash_\nu^\alpha (\forall x)\psi(G,x)$, $\psi(F_\nu^\alpha \cup (E_\mu^\alpha \setminus F_\nu^\alpha) \cup F', w)$ holds, hence $\psi(E_\mu^\alpha \cup F', w)$ holds and still by clause 2 of Definition 4.5, $d \Vdash_\mu (\forall x)\psi(G,x)$.

□

Lemma 4.7 For every Σ_1^0 (Π_1^0) formula φ, the following set is dense in $\mathbb{P}$:

$$\{c \in \mathbb{P} : (\forall \nu < \text{parts}(c))(\forall \alpha < 2)[c \Vdash_\nu^\alpha \varphi(G) \text{ or } c \Vdash_\nu^\alpha \neg\varphi(G)]\}$$

Proof. It suffices to prove the statement for the case where φ is a Σ_1^0 formula, as the case where φ is a Π_1^0 formula is symmetric. Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and let $I(c)$ be the set of pairs $(\nu, \alpha) \in \text{parts}(T) \times 2$ such that $c \nVdash_\nu^\alpha \varphi(G)$ and $c \nVdash_\nu^\alpha \neg\varphi(G)$. If $I(c) = \emptyset$ we are done, so suppose $I(c) \neq \emptyset$. Fix some $(\alpha, \nu) \in I(c)$. We will construct an extension d such that $I(d) \subseteq I(c) \setminus \{(\alpha, \nu)\}$. Applying iteratively the operation enables us to conclude.

The formula φ is of the form $(\exists x)\psi(G,x)$ where $\psi \in \Sigma_0^0$. Suppose there exists a $\sigma \in T$ such that $T^{[\sigma]}$ is infinite, a $w < |\sigma|$ and a set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha$ such that $\psi(F_\nu^\alpha \cup F', w)$ holds. In this case, letting $\vec{E}$ be defined by $E_\mu^\beta = F_\mu^\beta$ if $\mu \neq \nu \vee \beta \neq \alpha$ and $E_\nu^\alpha = F_\nu^\alpha \cup F'$, and letting S be the tree $T^{[\sigma]}$ where the domain is restricted so that $(E_\nu^\alpha, \text{dom}(S))$ Mathias extends $(F_\nu^\alpha, \text{dom}(T))$, by clause 1 of Definition 4.5, the condition $d = (\vec{E}, S, \mathcal{C}[S])$ is a valid extension of c such that $d_\nu^\alpha \Vdash \varphi(G)$.

Suppose now that for every $\sigma \in T$ such that $T^{[\sigma]}$ is infinite, every $w < |\sigma|$ and every set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma) \cap R_\alpha$, $\psi(F_\nu^\alpha \cup F', w)$ does not hold. In this case, by clause 2 of Definition 4.5, $c \Vdash_\nu^\alpha \neg\varphi(G)$. □

We now define the weak forcing relation which is almost the same as the true one, expect that the set F' is not required to be a subset of R_α in the case of a Π_1^0 formula.

Definition 4.8 (Weak forcing relation) Fix a condition $c = (\vec{F}, T, \mathcal{C})$, a Σ_0^0 formula $\varphi(G,x)$, a part $\nu < \text{parts}(T)$ and a side $\alpha < 2$.

1. $c \vdash_\nu^\alpha (\exists x)\varphi(G,x)$ iff there exists a $w \in \omega$ such that $\varphi(F_\nu^\alpha, w)$ holds.
2. $c \vdash_\nu^\alpha (\forall x)\varphi(G,x)$ iff for every $\sigma \in T$, every $w < |\sigma|$ and every set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$, $\varphi(F_\nu^\alpha \cup F', w)$ holds.

As one may expect, the weak forcing relation at the first level is also closed under the refinement relation.

Lemma 4.9 Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and a Σ_1^0 (Π_1^0) formula $\varphi(G)$. If $c \vdash_\nu^\alpha \varphi(G)$ for some $\nu < \text{parts}(T)$ and $\alpha < 2$, then for every $d = (\vec{E}, S, \mathcal{D}) \leq c$ and every part μ of S refining part ν of T, $d \vdash_\mu^\alpha \varphi(G)$.

Proof.

- If $\varphi \in \Sigma_1^0$ then this is exactly clause 1 of Lemma 4.6 since the definition of the weak and the true forcing relations coincide for Σ_1^0 formulas.
- If $\varphi \in \Pi_1^0$ then it can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma_0^0$. Fix a $\tau \in S$, a $w < |\tau|$ and a set $F' \subseteq \text{dom}(S) \cap \text{set}_\mu(\tau)$. By property (ii) of the definition of an extension, there exists a $\sigma \in T^{[\nu, E_\mu^\alpha]}$ such that $|\sigma| = |\tau|$ and $\text{set}_\mu(\tau) \subseteq \text{set}_\nu(\sigma)$. As $w < |\tau|$ and $\text{dom}(S) \subseteq \text{dom}(T)$, $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$. As $\sigma \in T^{[\nu, E_\mu^\alpha]}$, $E_\mu^\alpha \subseteq \text{set}_\nu(\sigma)$ and by property (i) of the definition of an extension, $E_\mu^\alpha \subseteq \text{dom}(T)$. Therefore $E_\mu^\alpha \cup F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$. By clause 2 of Definition 4.8 applied to $c \vdash_\nu^\alpha (\forall x)\psi(G,x)$, $\psi(F_\nu^\alpha \cup (E_\mu^\alpha \setminus F_\nu^\alpha) \cup F', w)$ holds, hence $\psi(E_\mu^\alpha \cup F', w)$ holds and still by clause 2 of Definition 4.8, $d \vdash_\mu^\alpha (\forall x)\psi(G,x)$.

□

The following trivial lemma simply reflects the fact that the promise $\mathcal{C}$ is not part of the definition of the weak forcing relation for Σ_1^0 or Π_1^0 formulas, and therefore has no effect on it.

Lemma 4.10 Fix two conditions $c = (\vec{F}, T, \mathcal{C})$ and $d = (\vec{E}, T, \mathcal{D})$ and a Σ^0_1 (Π^0_1) formula. For every part ν of T such that $F^\alpha_\nu = E^\alpha_\nu$, $c \Vdash^\alpha_\nu \varphi(G)$ if and only if $d \Vdash^\alpha_\nu \varphi(G)$.

Proof. If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 4.8, $c \Vdash^\alpha_\nu \varphi(G)$ iff there exists a $w \in \omega$ such that $\psi(F^\alpha_\nu, w)$ holds. As $F^\alpha_\nu = E^\alpha_\nu$, $c \Vdash^\alpha_\nu \varphi(G)$ iff $d \Vdash^\alpha_\nu \varphi(G)$. Similarily, if $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_0$. By clause 2 of Definition 4.8, $c \Vdash^\alpha_\nu \varphi(G)$ iff for every $\sigma \in T$, every $w < |\sigma|$ and every set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$, $\psi(F^\alpha_\nu \cup F', w)$ holds. As $F^\alpha_\nu = E^\alpha_\nu$, $c \Vdash^\alpha_\nu \varphi(G)$ iff $d \Vdash^\alpha_\nu \varphi(G)$. □

We can now define the forcing relation over higher formulas. It is defined inductively, starting with Σ^0_1 and Π^0_1 formulas. We extend the weak forcing relation instead of the true one for effectiveness purposes. We shall see later that the weak forcing relation behaves like the true one for some parts and some sides of a condition, and therefore that it tell us something about the truth of the formula over some carefully defined generic real G. Note that the forcing relation over higher formulas is still parameterized by the side α of the condition.

Definition 4.11 Fix a condition $c = (\vec{F}, T, \mathcal{C})$, a side $\alpha < 2$ and an arithmetic formula $\varphi(G)$.

1. If $\varphi(G) = (\exists x)\psi(G,x)$ where $\psi \in \Pi^0_1$ then $c \Vdash^\alpha \varphi(G)$ iff for every part ν of T such that $(\nu, T) \in \mathcal{C}$ there exists a $w < \text{dom}(T)$ such that $c \Vdash^\alpha_\nu \psi(G,w)$
2. If $\varphi(G) = (\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_1$ then $c \Vdash^\alpha \varphi(G)$ iff for every infinite p.r. k'-partition tree S, every function $f : \text{parts}(S) \to \text{parts}(T)$, every w and $\vec{E}$ smaller than $\#S$ such that the following holds
   i) $E^\beta_\nu = F^\beta_{f(\nu)}$ for each $\nu < \text{parts}(S)$ and $\beta \neq \alpha$
   ii) $(E^\alpha_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\alpha_{f(\nu)}, \text{dom}(T) \cap R_\alpha)$ for each $\nu < \text{parts}(S)$
   iii) S f-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E^\alpha_\nu]}$

   for every $(\mu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \not\Vdash^\alpha_\mu \neg\psi(G,w)$
3. If $\varphi(G) = (\exists x)\psi(G,x)$ where $\psi \in \Pi^0_{n+2}$ then $c \Vdash^\alpha \varphi(G)$ iff there exists a $w \in \omega$ such that $c \Vdash^\alpha \psi(G,w)$
4. If $\varphi(G) = \neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$ then $c \Vdash^\alpha \varphi(G)$ iff $d \not\Vdash^\alpha \psi(G)$ for every $d \in \text{Ext}(c)$.

Note that clause 2.ii) of Definition 4.11 seems to be Π^0_2 since R_α is Δ^0_2. However, in fact, one just needs to ensure that $\text{dom}(S) \subseteq \text{dom}(T)$ and $E^\alpha_\nu \setminus F^\alpha_{f(\nu)} \subseteq \text{dom}(T) \cap R_\alpha$. This is a Δ^0_2 predicate, and so is its negation, so one can already easily check that the forcing relation over a Π^0_2 formula will be also Π^0_2. Before proving the usual properties about the forcing relation, we need to discuss the role of the sides in the forcing relation. We are now ready to prove that the forcing relation is closed under extension.

Lemma 4.12 Fix a condition c, a side $\alpha < 2$ and a Σ^0_{n+2} (Π^0_{n+2}) formula $\varphi(G)$. If $c \Vdash^\alpha \varphi(G)$ then for every $d \leq c$, $d \Vdash^\alpha \varphi(G)$.

Proof. We prove the statement by induction over the complexity of the formula $\varphi(G)$. Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and a side $\alpha < 2$ such that $c \Vdash^\alpha \varphi(G)$. Fix an extension $d = (\vec{E}, S, \mathcal{D})$ of c.

- If $\varphi \in \Sigma^0_2$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 4.11, for every part ν of T such that $(\nu, T) \in \mathcal{C}$, there exists a $w < \text{dom}(T)$ such that $c \Vdash^\alpha_\nu \psi(G,w)$. Fix a part μ of S such that $(\mu, S) \in \mathcal{D}$. As $\mathcal{D} \subseteq \mathcal{C}$, $(\mu, S) \in \mathcal{C}$. By upward-closure of $\mathcal{C}$, part μ of S refines some part ν of $\mathcal{C}$ such that $(\nu, T) \in \mathcal{C}$. Therefore by Lemma 4.9, $d \Vdash^\alpha_\mu \psi(G,w)$, with $w < \text{dom}(T) \leq \text{dom}(S)$. Applying again clause 1 of Definition 4.11, we deduce that $d \Vdash (\forall x)\psi(G,x)$, hence $d \Vdash^\alpha \varphi(G)$.
- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_1$. Suppose for the sake of absurd that $d \not\Vdash^\alpha (\forall x)\psi(G,x)$. Let $f : \text{parts}(S) \to \text{parts}(T)$ witness the refinement $S \leq T$. By clause 2 of Definition 4.11, there exists an infinite p.r. k'-partition tree S', a

function $g : \text{parts}(S') \to \text{parts}(S)$, a $w \in \omega$, and a $2k'$-tuple of finite sets $\vec{H}$ smaller than the code of S' such that

i) $H^\beta_\nu = E^\beta_{g(\nu)}$ for each $\nu < \text{parts}(S')$ and $\beta \neq \alpha$
ii) $(H^\alpha_\nu, \text{dom}(S') \cap R_\alpha)$ Mathias extends $(E^\alpha_{g(\nu)}, \text{dom}(S) \cap R_\alpha)$ for each $\nu < \text{parts}(S')$
iii) S' g-refines $\bigcap_{\nu<\text{parts}(S')} S^{[g(\nu), H^\alpha_\nu]}$
iv) there exists a $(\mu, S') \in \mathcal{D}$ such that $(\vec{H}, S', \mathcal{D}[S']) \Vdash^\alpha_\mu \neg\psi(G, w)$.

To deduce by clause 2 of Definition 4.11 that $c \nVdash^\alpha (\forall x)\psi(G, x)$ and derive a contradiction, it suffices to prove that the same properties hold with respect to T. Let $\vec{H}'$ be defined by $H'^\beta_\nu = F^\beta_{f(g(\nu))}$ for each $\nu < \text{parts}(S')$ and $\beta \neq \alpha$ and $H'^\alpha_\nu = H^\alpha_\nu$.

i) It trivially holds by choice of $\vec{H}'$.
ii) By property (i) of the definition of an extension, $(E^\alpha_{g(\nu)}, \text{dom}(S))$ Mathias extends $(F^\alpha_{f(g(\nu))}, \text{dom}(T))$. Moreover $(H^\alpha_\nu, \text{dom}(S')$ Mathias extends $(E^\alpha_{g(\nu)}, \text{dom}(S))$, so $(H'^\alpha_\nu, \text{dom}(S')) = (H^\alpha_\nu, \text{dom}(S'))$ Mathias extends $(F_{f(g(\nu))}, \text{dom}(T))$.
iii) As by property (ii) of the definition of an extension,
S f-refines $\bigcap_{\nu<\text{parts}(S')} T^{[f(g(\nu)), E^\alpha_{g(\nu)}]}$, and
S' g-refines $\bigcap_{\nu<\text{parts}(S')} S^{[g(\nu), H^\alpha_\nu]}$ then
S' $(g \circ f)$-refines $\bigcap_{\nu<\text{parts}(S')} T^{[g(\nu), H'^\alpha_\nu]}$.
iv) As $\mathcal{D} \subseteq \mathcal{C}$, there exists a part $(\mu, S') \in \mathcal{C}$ such that $(\vec{H}, S', \mathcal{D}[S']) \Vdash^\alpha_\mu \neg\psi(G, w)$. By Lemma 4.10, $(\vec{H}', S', \mathcal{C}[S']) \Vdash^\alpha_\mu \neg\psi(G, w)$.

– If $\varphi \in \Sigma^0_{n+3}$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+2}$. By clause 3 of Definition 4.11, there exists a $w \in \omega$ such that $c \Vdash^\alpha \psi(G, w)$. By induction hypothesis, $d \Vdash^\alpha \psi(G, w)$ so by clause 3 of Definition 4.11, $d \Vdash^\alpha \varphi(G)$.
– If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. Suppose for the sake of absurd that $d \nVdash^\alpha \varphi(G)$. By clause 4 of Definition 4.11, there exists an $e \in \text{Ext}(d)$ such that $e \Vdash^\alpha \psi(G)$. In particular, $e \in \text{Ext}(c)$, so by clause 4 of Definition 4.11, $e \nVdash^\alpha \psi(G)$ since $c \Vdash^\alpha \varphi(G)$. Contradiction.

□

Although the weak forcing relation does not satisfy the density property, the forcing relation over higher formulas does. The reason is that the extended forcing relation does not involve the weak forcing relation over Σ^0_1 formulas in the clause 2 of Definition 4.11, but uses instead the weaker statement "c does not force the negation of the Σ^0_1 formula". The link between this statement and the statement "c has an extension which forces the Σ^0_1 formula" is used when proving that $\varphi(G)$ holds iff $c \Vdash \varphi(G)$ for some condition belonging to a sufficiently generic filter. We now prove the density of the forcing relation for higher formulas.

Lemma 4.13 For every Σ^0_{n+2} (Π^0_{n+2}) formula φ, the following set is dense in $\mathbb{P}$:

$$\{c \in \mathbb{P} : (\forall \alpha < 2)[c \Vdash^\alpha \varphi(G) \text{ or } c \Vdash^\alpha \neg\varphi(G)]\}$$

Proof. We prove the statement by induction over n. It suffices to treat the case where φ is a Σ^0_{n+2} formula, as the case where φ is a Π^0_{n+2} formula is symmetric. Moreover, it is enough to prove that for every condition c and every $\alpha < 2$, there exists an extension $d \leq c$ such that $d \Vdash^\alpha \varphi(G)$ or $d \Vdash^\alpha \neg\varphi(G)$. Iterating the process at most twice enables us to conclude. Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and a part $\alpha < 2$.

– In case $n = 0$, the formula φ is of the form $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. Suppose there exists an infinite p.r. k'-partition tree S for some $k' \in \omega$, a function $f : \text{parts}(S) \to \text{parts}(T)$, and a $2k'$-tuple of finite sets $\vec{E}$ such that

i) $E^\beta_\nu = F^\beta_{f(\nu)}$ for each $\nu < \text{parts}(S)$ and $\beta \neq \alpha$
ii) $(E^\alpha_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\alpha_{f(\nu)}, \text{dom}(T) \cap R_\alpha)$ for each $\nu < \text{parts}(S)$.
iii) S f-refines $\bigcap_{\nu<\text{parts}(S)} T^{[f(\nu), E^\alpha_\nu]}$

iv) for each non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \Vdash^{\alpha}_{\nu} \psi(G, w)$ for some $w < \#S$

Let $\mathcal{D} = \mathcal{C}[S] \setminus \{(\nu, S') \in \mathcal{C} : \text{ part } \nu \text{ of } S' \text{ is empty}\}$. As $\mathcal{C}$ is an $\emptyset'$-p.r. promise for T, $\mathcal{C}[S]$ is an $\emptyset'$-p.r. promise for S. As $\mathcal{D}$ is obtained from $\mathcal{C}[S]$ by removing only empty parts, $\mathcal{D}$ is also an $\emptyset'$-p.r. promise for S. By clause 1 of Definition 4.11, $d = (\vec{E}, S, \mathcal{D}) \Vdash^{\alpha} (\exists x)\psi(G, x)$ hence $d \Vdash^{\alpha} \varphi(G)$.

We may choose a coding of the p.r. trees such that such that the code of S is sufficiently large to witness f, ℓ and $\vec{E}$. So suppose now that for every infinite p.r. k'-partition tree S, every function $f : \text{parts}(S) \to \text{parts}(T)$, $\ell \in \omega$ and $\vec{E}$ smaller than the code of S such that properties i-iii) hold, there exists a non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$ and $(\vec{E}, S, \mathcal{C}[S]) \not\Vdash^{\alpha}_{\nu} \psi(G, w)$ for every $w < \ell$. Let $\mathcal{D}$ be the collection of all such (ν, S). $\mathcal{D}$ is $\emptyset'$-p.r. By Lemma 4.9 and since we require that $\#S \geq \#T$ in the definition of $S \leq T$, $\mathcal{D}$ is upward-closed, hence is a promise for T. By clause 2 of Definition 4.11, $d = (\vec{F}, T, \mathcal{D}) \Vdash^{\alpha} (\forall x)\neg\psi(G, x)$, hence $d \Vdash^{\alpha} \neg\varphi(G)$.

– In case $n > 0$, density follows from clause 4 of Definition 4.11.

□

We now prove that the weak forcing relation extended to any arithmetic formula enjoys the desired definability properties. For this, we start with a lemma showing that the extension relation is Π^0_2. Therefore, only the first two levels have to be treated independently, since the extension relation does not add some extra complexity to the forcing relation for higher formulas.

Lemma 4.14 For every condition c, $\text{Ext}(c)$ is Π^0_2 uniformly in c.

Proof. Recall from Lemma 3.10 that given $k, t \in \omega$, the set $PartTree(k, t)$ denotes the Π^0_1 set of all the infinite p.r. k-partition trees of $[t, +\infty)$, and given a k-partition tree S and a part ν of S, the predicate $Empty(S, \nu)$ denotes the Π^0_1 formula "part ν of S is empty", that is, the formula $(\forall \sigma \in S)[\text{set}_\nu(\sigma) \cap \text{dom}(S) = \emptyset]$. If T is p.r. then so is $T^{[\nu, H]}$ for some finite set H.

Fix a condition $c = (\vec{F}, (k, t, T), \mathcal{C})$. $(\vec{H}, (k', t', S), \mathcal{D}) \in \text{Ext}(c)$ iff the following formula holds:

$$
\begin{aligned}
&(\exists f : k' \to k) \\
&(\forall \nu < k')(\forall \alpha < 2)(H^{\alpha}_{\nu}, [t', +\infty) \cap R_\alpha) \text{ Mathias extends } (F^{\alpha}_{f(\nu)}, [t, +\infty) \cap R_\alpha) && (\Pi^0_2) \\
&\wedge S \in PartTree(k', t') \wedge S \leq_f \textstyle\bigwedge_{\nu < k', \alpha < 2} T^{[f(\nu), H^{\alpha}_{\nu}]} && (\Pi^0_1) \\
&\wedge \mathcal{D} \text{ is a promise for } S \wedge \mathcal{D} \subseteq \mathcal{C} && (\Pi^0_2)
\end{aligned}
$$

The formula $(H^{\alpha}_{\nu}, [t', +\infty) \cap R_\alpha)$ Mathias extends $(F^{\alpha}_{f(\nu)}, [t, +\infty) \cap R_\alpha)$ can be written $(\forall x < t)[x \in H^{\alpha}_{\nu} \leftrightarrow x \in F^{\alpha}_{f(\nu)}] \wedge t' \geq t \wedge (\forall x \in H^{\alpha}_{\nu} \setminus F^{\alpha}_{f(\nu)}) x \in R_\alpha$ and therefore is Π^0_2. By Lemma 3.10 and the fact that $\bigwedge_{\nu < k', \alpha < 2} T^{[f(\nu), H^{\alpha}_{\nu}]}$ is p.r. uniformly in T, f, $\vec{H}$ and k', the above formula is Π^0_2. □

Lemma 4.15 Fix an arithmetic formula $\varphi(G)$, a condition $c = (\vec{F}, T, \mathcal{C})$, a side $\alpha < 2$ and a part ν of T.

a) If $\varphi(G)$ is a Σ^0_1 (Π^0_1) formula then so is the predicate $c \Vdash^{\alpha}_{\nu} \varphi(G)$.
b) If $\varphi(G)$ is a Σ^0_{n+2} (Π^0_{n+2}) formula then so is the predicate $c \Vdash^{\alpha} \varphi(G)$.

Proof. We prove our lemma by induction over the complexity of the formula $\varphi(G)$.

– If $\varphi(G) \in \Sigma^0_1$ then it can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 4.8, $c \Vdash^{\alpha}_{\nu} \varphi(G)$ if and only if the formula $(\exists w \in \omega)\psi(F^{\alpha}_{\nu}, w)$ holds. This is a Σ^0_1 predicate.
– If $\varphi(G) \in \Pi^0_1$ then it can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 2 of Definition 4.8, $c \Vdash^{\alpha}_{\nu} \varphi(G)$ if and only if the formula $(\forall \sigma \in T)(\forall w < |\sigma|)(\forall F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma))\psi(F^{\alpha}_{\nu} \cup F', w)$ holds. This is a Π^0_1 predicate.

– If $\varphi(G) \in \Sigma^0_2$ then it can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 4.11, $c \Vdash^\alpha \varphi(G)$ if and only if the formula $(\forall \nu < \text{parts}(T)(\exists w < \text{dom}(T))[(\nu, T) \in \mathcal{C} \to c \Vdash^\alpha_\nu \psi(G,w)]$ holds. This is a Σ^0_2 predicate by induction hypothesis and the fact that $\mathcal{C}$ is $\emptyset'$-computable.
– If $\varphi(G) \in \Pi^0_2$ then it can be expressed as $(\forall x)\psi(G,x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 4.11, $c \Vdash \varphi(G)$ if and only if for every infinite k'-partition tree S, every function $f : \text{parts}(S) \to \text{parts}(T)$, every w and $\vec{E}$ smaller than the code of S such that the following hold
  i) $(E_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F_{f(\nu)}, \text{dom}(T) \cap R_\alpha)$ for each $\nu < \text{parts}(S)$
  ii) S f-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E_\nu]}$

  for every $(\mu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \nVdash^\alpha_\mu \neg\psi(G,w)$. By Lemma 3.10, Properties i-ii) are Δ^0_2. Moreover the predicate $(\mu, S) \in \mathcal{C}$ is Δ^0_2 since $\mathcal{C}$ is $\emptyset'$-p.r. By induction hypothesis, $(\vec{E}, S, \mathcal{C}[S]) \nVdash^\alpha_\mu \neg\psi(G,w)$ is Σ^0_1. Therefore $c \Vdash^\alpha \varphi(G)$ is a Π^0_2 predicate.
– If $\varphi(G) \in \Sigma^0_{n+3}$ then it can be expressed as $(\exists x)\psi(G,x)$ where $\psi \in \Pi^0_{n+2}$. By clause 3 of Definition 4.11, $c \Vdash^\alpha \varphi(G)$ if and only if the formula $(\exists w \in \omega) c \Vdash^\alpha \psi(G,w)$ holds. This is a Σ^0_{n+3} predicate by induction hypothesis.
– If $\varphi(G) \in \Pi^0_{n+3}$ then it can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause 4 of Definition 4.11, $c \Vdash^\alpha \varphi(G)$ if and only if the formula $(\forall d)(d \notin \text{Ext}(c) \vee d \nVdash^\alpha \psi(G))$ holds. By induction hypothesis, $d \nVdash^\alpha \psi(G)$ is a Π^0_{n+3} predicate. By Lemma 4.14, the set $\text{Ext}(c)$ is Π^0_2-computable uniformly in c, thus $c \Vdash^\alpha \varphi(G)$ is a Π^0_{n+3} predicate.

□

4.4. **Validity**

As we already saw, we have two candidating forcing relations for Σ^0_1 and Π^0_1 formulas:

1. The "true" forcing relation $c \Vvdash^\alpha \varphi(G)$. This relation has been shown to have the expected density properties through Lemma 4.7. However deciding such a relation requires too much computational power.
2. The "weak" forcing relation $c \Vdash^\alpha \varphi(G)$. Deciding such a relation requires the same definitional power as the formula it forces. It provides a sufficient condition for forcing the formula $\varphi(G)$ as $c \Vdash^\alpha \varphi(G)$ implies $c \Vvdash^\alpha \varphi(G)$, but the converse does not hold and we cannot prove the density property in the general case.

Thankfully, there exists some sides and parts of any condition on which those two forcing relations coincide. This leads to the notion of validity.

Definition 4.16 (Validity) Fix an enumeration $\varphi_0(G), \varphi_1(G), \dots$ of all the Π^0_1 formulas. Fix a condition $c = (\vec{F}, T, \mathcal{C})$, a side $\alpha < 2$ and a part ν of T. We say that *side α is valid in part ν of T* if part ν of T is R_α-acceptable and for every $i < \text{dom}(T)$, $c \Vvdash^\alpha_\nu \varphi_i(G)$ iff $c \Vdash^\alpha_\nu \varphi_i(G)$.

Note that the statement "side α is valid in part ν of T" is Π^0_2 as $c \Vvdash^\alpha_\nu \varphi_i(G)$ is Π^0_2 and $c \Vdash^\alpha_\nu \varphi_i(G)$ is Π^0_1. Also note that if side α is valid in part μ of S and part μ of S refines part ν of T, then side α is valid in part ν of T. The following lemma shows in particular that every condition $c = (\vec{F}, T, \mathcal{C})$ has a side which is valid in some of its parts, and that we can furthermore restrict $\mathcal{C}$ so that it "witnesses its valid parts".

Lemma 4.17 The following set is dense in $\mathbb{P}$:

$$\{(\vec{F}, T, \mathcal{C}) \in \mathbb{P} : (\forall \nu)(\exists \alpha < 2)[(\nu, T) \in \mathcal{C} \to \text{side } \alpha \text{ is valid in part } \nu \text{ of } T]\}$$

Proof. Given a condition $c = (\vec{F}, T, \mathcal{C})$, let $I(c)$ be the set of the parts ν of T such that $(\nu, T) \in \mathcal{C}$ and no $\alpha < 2$ is valid in part ν of T. Fix a condition $c = (\vec{F}, T, \mathcal{C}) \in \mathbb{P}$. By iterating the proof of Lemma 4.13, we can assume without loss of generality that for each $i < \text{dom}(T)$,

$$(\forall \alpha < 2)[c \Vdash^\alpha (\exists x)\varphi_i(G) \text{ or } c \Vdash^\alpha (\forall x)\neg\varphi_i(G)]$$

The dummy variable x ensures that the forcing relation for Σ^0_2 and Π^0_2 is applied. It suffices to prove that for every $\nu \in I(c)$, there exists an extension $d = (\vec{E}, S, \mathcal{D})$ such that $I(d) \subseteq I(c) \setminus \{\nu\}$. Iterating the process at most $|\text{parts}(T)|$ times enables us to conclude.

Fix a part $\nu \in I(c)$ and let $\mathcal{D}$ be the set of $(\mu, S) \in \mathcal{C}$ such that part μ of S does not refine part ν of T. The set $\mathcal{D}$ is a $\emptyset'$-p.r. upward-closed subset of $\mathcal{C}$. It suffices to prove that for every infinite p.r. partition tree $S \le T$, there exists a non-empty part μ of S such that $(\mu, S) \in \mathcal{D}$ to deduce that $\mathcal{D}$ is a promise for T and obtain an extension $d = (\vec{E}, T, \mathcal{D})$ of c such that $I(d) \subseteq I(c) \setminus \{\nu\}$.

Fix an infinite p.r. partition tree $S \le T$. By choice of ν, for every $\alpha < 2$, either ν is not R_α-acceptable in S, or there exists an $i_\alpha < \text{dom}(T)$ such that $c \Vdash\!\!\Vdash^\alpha_\nu \varphi_{i_\alpha}(G)$ but $c \nVdash^\alpha_\nu \varphi_{i_\alpha}(G)$. In the latter case, by choice of c, $c \Vdash^\alpha_\nu (\forall x)\neg\varphi_{i_\alpha}(G)$.

Let $f : 2^k + 1 \to 2^k$ be the function such that $f(\mu) = \mu$ for each $\mu \neq \nu$ and $f(\mu_\alpha) = \nu$ for each $\alpha < 2$. In other words, f forks the part ν of S into 2 parts μ_0 and μ_1. Let P be a path through S, and let $t \in \omega$ be large enough to "witness the non R_α-acceptable sides". More formally, let t be such that for every $\alpha < 2$, either $\text{set}_\nu(P) \cap R_\alpha \cap [t, +\infty) = \emptyset$ or there exists an $i_\alpha < \text{dom}(T)$ such that $c \Vdash\!\!\Vdash^\alpha_\nu \varphi_{i_\alpha}(G)$ but $c \nVdash^\alpha_\nu \varphi_{i_\alpha}(G)$.

Let S' be the p.r. tree of all the τ's f-refining some $\sigma \in S$ and such that for each $\alpha < 2$, either $\text{set}_{\mu_\alpha}(\tau) \cap [t, +\infty) = \emptyset$, or $\varphi(E^\alpha_\nu \cup F')$ holds for each $F' \subseteq \text{dom}(S) \cap \text{set}_{\mu_\alpha}(\tau)$. The tree S' is a $(2^k + 1)$-partition tree of $[t, +\infty)$ f-refines S. We claim that S' is infinite. Fix some $s \in \omega$, we will prove that $\tau \in S'$ for some string τ of length s. Let $\sigma = P \upharpoonright s$. In particular, $S^{[\sigma]}$ is infinite, so for every $\alpha < 2$, either $\text{set}_\nu(P) \cap R_\alpha \cap [t, +\infty) = \emptyset$ by definition of t, or, unfolding clause 2 of Definition 4.5 for $c \Vdash\!\!\Vdash^\alpha_\nu \varphi_{i_\alpha}(G)$, for every set $F' \subseteq \text{dom}(S) \cap \text{set}_\nu(\sigma) \cap R_\alpha$, $\varphi(E^\alpha_\nu \cup F')$ holds. Let τ be the string refining σ such that $\text{set}_{\mu_\alpha}(\tau) = \text{set}_\nu(\sigma) \cap R_\alpha$. By definition of S', $\tau \in S'$. Therefore S' is infinite. Moreover, by definition of S', for each $\alpha < 2$, either μ_α is empty in S' or $(\vec{E}, S', \mathcal{C}[S']) \Vdash^\alpha_{\mu_\alpha} \varphi_{i_\alpha}(G)$.

By definition of $c \Vdash^\alpha_\nu (\forall x)\neg\varphi_{i_\alpha}(G)$, $(\vec{E}, S', \mathcal{C}[S']) \nVdash^\alpha_\mu \varphi_{i_\alpha}(G)$ for each part μ of S' such that $(\mu, S') \in \mathcal{C}$. Then, for each $\alpha < 2$, either μ_α is empty in S', or $(\mu_\alpha, S') \notin \mathcal{C}$, as otherwise it would contradict $(\vec{E}, S', \mathcal{C}[S']) \Vdash^\alpha_{\mu_\alpha} \varphi_{i_\alpha}(G)$. So there must exists a non-empty part μ of S' not refining part ν of T such that $(\mu, S') \in \mathcal{C}$, and by upward closure of a promise, there exists a non-empty part μ of S not refining part ν of T such that $(\mu, S) \in \mathcal{C}$. By definition of $\mathcal{D}$, $(\mu, S) \in \mathcal{D}$. Therefore $\mathcal{D}$ is a promise for T and we conclude. ☐

Given any filter $\mathcal{F} = \{c_0, c_1, \dots\}$ with $c_s = (\vec{F}_s, T_s, \mathcal{C}_s)$, the set of pairs (α_s, ν_s) such that $(\nu_s, T_s) \in \mathcal{C}_s$ and the side α_s is valid in the part ν of T_s forms again an infinite, directed acyclic graph $\mathcal{G}(\mathcal{F})$. Lemma 4.17 enables us to say that this graph has at least one infinite directed path. Whenever $\mathcal{F}$ is sufficiently generic, the graph $\mathcal{G}(\mathcal{F})$ yields a pair (α, P) such that for every s, the side α is valid in part $P(s)$ of c_s, and if if c_s refines c_t, then part $P(s)$ of c_s refines part $P(t)$ of c_t. The path P induces an infinite set $G = \bigcup\{F^\alpha_{P(s),s} : s \in \omega\}$. We call α the *generic side*, P the *generic path* and G the *generic real*.

By choice of the generic path to go only through valid sides and parts of the conditions, we recovered the density property for the weak forcing relation and are therefore able to prove that a property holds over the generic real iff it can be forced by some condition belonging to the generic filter.

Lemma 4.18 Suppose that $\mathcal{F}$ is sufficiently generic and let α, P and G be the generic side, the generic path and the generic real, respectively. For every Σ^0_1 (Π^0_1) formula $\varphi(G)$, $\varphi(G)$ holds iff $c_s \Vdash^\alpha_{P(s)} \varphi(G)$ for some $c_s \in \mathcal{F}$.

Proof. Thanks to validity, it suffices to prove that if $c_s \Vdash^\alpha_\nu \varphi(G)$ for some $c_s \in \mathcal{F}$, then $\varphi(G)$ holds. Indeed, if $\varphi(G)$ holds, then by genericity of $\mathcal{F}$, $c_s \Vdash\!\!\Vdash^\alpha_{P(s)} \varphi(G)$ or $c_s \Vdash\!\!\Vdash^\alpha_{P(s)} \neg\varphi(G)$ for some $c_s \in \mathcal{F}$. By validity of side α in part $P(s)$ of c_s, $c_s \Vdash^\alpha_{P(s)} \varphi(G)$ or $c_s \Vdash^\alpha_{P(s)} \neg\varphi(G)$. If $c_s \Vdash^\alpha_{P(s)} \neg\varphi(G)$ then $\neg\varphi(G)$ holds, contradicting the hypothesis. So $c_s \Vdash^\alpha_{P(s)} \varphi(G)$. Fix a condition $c_s = (\vec{F}, T, \mathcal{C}) \in \mathcal{F}$ such that $c_s \Vdash^\alpha_\nu \varphi(G)$, where $\nu = P(s)$.

- If $\varphi \in \Sigma^0_1$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 1 of Definition 4.8, there exists a $w \in \omega$ such that $\psi(F^\alpha_\nu, w)$ holds. As $\nu = P(s)$, $F^\alpha_\nu = F^\alpha_{P(s)} \subseteq G$ and $G \setminus F^\alpha_\nu \subseteq (max(F^\alpha_\nu), +\infty)$, so $\psi(G, w)$ holds by continuity, hence $\varphi(G)$ holds.
- If $\varphi \in \Pi^0_1$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_0$. By clause 2 of Definition 4.8, for every $\sigma \in T$, every $w < |\sigma|$ and every set $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$, $\psi(F^\alpha_\nu \cup F', w)$ holds. For every $F' \subseteq G \setminus F^\alpha_\nu$, and $w \in \omega$ there exists a $\sigma \in T$ such that $w < |\sigma|$ and $F' \subseteq \text{dom}(T) \cap \text{set}_\nu(\sigma)$. Hence $\psi(F^\alpha_\nu \cup F', w)$ holds. Therefore, for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.

□

Lemma 4.19 Suppose that $\mathcal{F}$ is sufficiently generic and let α and G be the generic side and the generic real, respectively. For every Σ^0_{n+2} (Π^0_{n+2}) formula $\varphi(G)$, $\varphi(G)$ holds iff $c_s \Vdash^\alpha \varphi(G)$ for some $c_s \in \mathcal{F}$.

Proof. This lemma uses validity implicitely by calling Lemma 4.18 which itself uses it explicitely. Still following the explanation in Lemma 2.13, it suffices to prove that if $c_s \Vdash^\alpha \varphi(G)$ for some $c_s \in \mathcal{F}$ then $\varphi(G)$ holds. Let P be the generic path induced by the generic filter $\mathcal{F}$. Fix a condition $c_s = (\vec{F}, T, \mathcal{C}) \in \mathcal{F}$ such that $c_s \Vdash^\alpha \varphi(G)$. We proceed by case analysis on φ.

- If $\varphi \in \Sigma^0_2$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_1$. By clause 1 of Definition 4.11, for every part ν of T such that $(\nu, T) \in \mathcal{C}$, there exists a $w < \text{dom}(T)$ such that $c_s \Vdash^\alpha_\nu \psi(G, w)$. Since $(P(s), T) \in \mathcal{C}$, $c_s \Vdash^\alpha_{P(s)} \psi(G, w)$. By Lemma 4.18, $\psi(G, w)$ holds, hence $\varphi(G)$ holds.
- If $\varphi \in \Pi^0_2$ then $\varphi(G)$ can be expressed as $(\forall x)\psi(G, x)$ where $\psi \in \Sigma^0_1$. By clause 2 of Definition 4.11, for every infinite k'-partition tree S, every function $f : \text{parts}(S) \to \text{parts}(T)$, every w and $\vec{E}$ smaller than the code of S such that the following hold
  i) $(E_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F_{f(\nu)}, \text{dom}(T) \cap R_\alpha)$ for each $\nu < \text{parts}(S)$
  ii) S f-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E_\nu]}$

  for every $(\mu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \nVdash^\alpha_\mu \neg\psi(G, w)$. Suppose for the sake of absurd that $\psi(G, w)$ does not hold for some $w \in \omega$. Then by Lemma 4.18, there exists a $c_t \in \mathcal{F}$ such that $c_t \Vdash^\alpha_{P(t)} \neg\psi(G, w)$. Since $\mathcal{F}$ is a filter, there is a condition $c_e = (\vec{E}, S, \mathcal{D}) \in \mathcal{F}$ extending c_s and c_t. By choice of P, $(P(e), S) \in \mathcal{C}$, so by clause ii), $(\vec{E}, S, \mathcal{C}[S]) \nVdash^\alpha_{P(e)} \psi(G, w)$, hence by Lemma 4.10, $c_e \nVdash^\alpha_{P(e)} \psi(G, w)$. However, since part $P(e)$ of c_e refines part $P(t)$ of c_t, then by Lemma 4.9, $c_e \Vdash^\alpha_{P(e)} \psi(G, w)$. Contradiction. Hence, for every $w \in \omega$, $\psi(G, w)$ holds, so $\varphi(G)$ holds.
- If $\varphi \in \Sigma^0_{n+3}$ then $\varphi(G)$ can be expressed as $(\exists x)\psi(G, x)$ where $\psi \in \Pi^0_{n+2}$. By clause 3 of Definition 4.11, there exists a $w \in \omega$ such that $c_s \Vdash^\alpha \psi(G, w)$. By induction hypothesis, $\psi(G, w)$ holds, hence $\varphi(G)$ holds.

  Conversely, if $\varphi(G)$ holds, then there exists a $w \in \omega$ such that $\psi(G, w)$ holds, so by induction hypothesis $c_s \Vdash^\alpha \psi(G, w)$ for some $c_s \in \mathcal{F}$, so by clause 3 of Definition 4.11, $c_s \Vdash^\alpha \varphi(G)$.
- If $\varphi \in \Pi^0_{n+3}$ then $\varphi(G)$ can be expressed as $\neg\psi(G)$ where $\psi \in \Sigma^0_{n+3}$. By clause 4 of Definition 4.11, for every $d \in \text{Ext}(c_s)$, $d \nVdash^\alpha \psi(G)$. By Lemma 4.12, $d \nVdash^\alpha \psi(G)$ for every $d \in \mathcal{F}$, and by a previous case, $\psi(G)$ does hold, so $\varphi(G)$ holds.

□

4.5. Preserving definitions

The following (and last) lemma shows that every sufficiently generic real preserves higher definitions. This preservation property cannot be proved in the case of non-Σ^0_1 sets since the weak forcing relation does not have the good density property in general.

Lemma 4.20 If $A \not\in \Sigma^0_{n+2}$ and $\varphi(G,x)$ is Σ^0_{n+2}, then the set of $c \in \mathbb{P}$ satisfying the following property is dense:

$$(\forall \alpha < 2)[(\exists w \in A) c \Vdash^\alpha \neg\varphi(G,w)] \vee [(\exists w \not\in A) c \Vdash^\alpha \varphi(G,w)]$$

Proof. It is sufficient to find, given a condition c and a side $\alpha < 2$, an extension d of c such that the following holds:

$$[(\exists w \in A) c \Vdash^\alpha \neg\varphi(G,w)] \vee [(\exists w \not\in A) c \Vdash^\alpha \varphi(G,w)]$$

Fix a condition $c = (\vec{F}, T, \mathcal{C})$ and a side $\alpha < 2$.

- In case $n = 0$, $\varphi(G,w)$ can be expressed as $(\exists x)\psi(G,w,x)$ where $\psi \in \Pi^0_1$. Let U be the set of integers w such that there exists an infinite p.r. k'-partition tree S for some $k' \in \omega$, a function $f : \text{parts}(S) \to \text{parts}(T)$ and a $2k'$-tuple of finite sets $\vec{E}$ such that
  i) $E^\beta_\nu = F^\beta_{f(\nu)}$ for each $\nu < \text{parts}(S)$ and $\beta \neq \alpha$
  ii) $(E^\alpha_\nu, \text{dom}(S) \cap R_\alpha)$ Mathias extends $(F^\alpha_{f(\nu)}, \text{dom}(T))$ for each $\nu < \text{parts}(S)$.
  iii) S f-refines $\bigcap_{\nu < \text{parts}(S)} T^{[f(\nu), E^\alpha_\nu]}$
  iv) for each non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$, $(\vec{E}, S, \mathcal{C}[S]) \Vdash^\alpha_\nu \psi(G,w,u)$ for some $u < \#S$

  By Lemma 4.15 and Lemma 3.10, $U \in \Sigma^0_2$, thus $U \neq A$. Let $w \in U \Delta A$.

  Suppose that $w \in U \setminus A$. Let $\mathcal{D} = \mathcal{C}[S] \setminus \{(\nu, S') \in \mathcal{C} : \text{ part } \nu \text{ of } S' \text{ is empty}\}$. As $\mathcal{C}$ is an $\emptyset'$-p.r. promise for T, $\mathcal{C}[S]$ is an $\emptyset'$-p.r. promise for S. As $\mathcal{D}$ is obtained from $\mathcal{C}[S]$ by removing only empty parts, $\mathcal{D}$ is also an $\emptyset'$-p.r. promise for S. By Lemma 4.10, for every part ν of S such that $(\nu, S) \in \mathcal{D} \subseteq \mathcal{C}$, $(\vec{E}, S, \mathcal{D}) \Vdash^\alpha_\nu \psi(G,w,u)$ for some $u < \text{dom}(S)$, hence by clause 1 of Definition 4.11, $d = (\vec{E}, S, \mathcal{D}) \Vdash^\alpha (\exists x)\psi(G,w,x)$. In other words, $d \Vdash^\alpha \varphi(G)$ for some $w \not\in A$.

  We may choose a coding of the p.r. trees such that such that the code of S is sufficiently large to witness w and $\vec{E}$. So suppose now that $w \in A \setminus U$. Then for every infinite p.r. k'-partition tree S, every function $f : \text{parts}(S) \to \text{parts}(T)$ and every $\vec{E}$ smaller than the code of S such that properties i-iii) hold, there exists a non-empty part ν of S such that $(\nu, S) \in \mathcal{C}$ and $(\vec{E}, S, \mathcal{C}[S]) \not\Vdash^\alpha_\nu \psi(G,w,u)$ for every $u < \#S$. Let $\mathcal{D}$ be the collection of all such (ν, S). $\mathcal{D}$ is $\emptyset'$-p.r. By Lemma 4.9 and since $\#S \geq \#T$ whenever $S \leq T$, $\mathcal{D}$ is upward-closed under the refinement relation, hence is a promise for T. By clause 2 of Definition 4.11, $d = (\vec{F}, T, \mathcal{D}) \Vdash^\alpha (\forall x)\neg\psi(G,w,x)$, hence $d \Vdash^\alpha \neg\varphi(G,w)$ for some $w \in A$.
- In case $n > 0$, let $U = \{w \in \omega : (\exists d \in \text{Ext}(c)) d \Vdash^\alpha \varphi(G,w)\}$. By Lemma 3.24 and Lemma 3.25, $U \in \Sigma^0_{n+2}$, thus $U \neq A$. Fix $w \in U \Delta A$. If $w \in U \setminus A$ then by definition of U, there exists a condition d extending c such that $d \Vdash^\alpha \varphi(G,w)$. If $w \in A \setminus U$, then for every $d \in \text{Ext}(c)$, $d \not\Vdash^\alpha \varphi(G,w)$ so by clause 4 of Definition 4.11, $c \Vdash^\alpha \neg\varphi(G,w)$.

□

We are now ready to reprove Corollary 3.29 from Wang [21].

Theorem 4.21 (Wang [21]) RT^2_2 admits preservation of Ξ definitions simultaneously for all Ξ in $\{\Sigma^0_{n+2}, \Pi^0_{n+2}, \Delta^0_{n+2} : n \in \omega\}$.

Proof. Since $\mathsf{RCA}_0 \vdash \mathsf{COH} \wedge \mathsf{D}^2_2 \to \mathsf{RT}^2_2$, and COH admits preservation of the arithmetic hierarchy, it suffices to prove that D^2_2 admits preservation of Ξ definitions simultaneously for all Ξ in $\{\Sigma^0_{n+2}, \Pi^0_{n+2}, \Delta^0_{n+2} : n \in \omega\}$. Fix some set C and a $\Delta^{0,C}_2$ 2-partition $R_0 \cup R_1 = \omega$. Let $\mathcal{C}_0$ be the C'-p.r. set of all $(\nu, T) \in \mathbb{U}$ such that $(\nu, T) \leq (0, 1^{<\omega})$. Let $\mathcal{F}$ be a sufficiently generic filter containing $c_0 = (\{\emptyset, \emptyset\}, 1^{<\omega}, \mathcal{C}_0)$. Let G be the corresponding generic real. By definition of a condition, the set G is $\vec{R}$-cohesive. By Lemma 4.20 and Lemma 4.15, G preserves non-Σ^0_{n+2} definitions relative to C for every $n \in \omega$. Therefore, by Proposition 2.2 of [21], G preserves Ξ definitions relative to C simultaneously for all Ξ in $\{\Sigma^0_{n+2}, \Pi^0_{n+2}, \Delta^0_{n+2} : n \in \omega\}$. □

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