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Abstract. Ramsey's theorem states that for any coloring of the n-element subsets of $\mathbb{N}$ with finitely many colors, there is an infinite set H such that all n-element subsets of H have the same color. The strength of consequences of Ramsey's theorem has been extensively studied in reverse mathematics and under various reducibilities, namely, computable reducibility and uniform reducibility. Our understanding of the combinatorics of Ramsey's theorem and its consequences has been greatly improved over the past decades. In this paper, we state some questions which naturally arose during this study. The inability to answer those questions reveals some gaps in our understanding of the combinatorics of Ramsey's theorem.

1. Introduction

Ramsey's theory is a branch of mathematics studying the conditions under which some structure appears among a sufficiently large collection of objects. Perhaps the most well-known example is Ramsey's theorem, which states that for any coloring of the n-element subsets of $\mathbb{N}$ with finitely many colors, there is an infinite set H such that all n-element subsets of H have the same color. Consequences of Ramsey's theorem have been extensively studied in reverse mathematics and under various reducibilities, among which, computable reducibility.

Reverse mathematics is a vast mathematical program whose goal is to classify ordinary theorems in terms of their provability strength. It uses the framework of subsystems of second-order arithmetic, which is sufficiently rich to express many theorems in a natural way. The base system, RCA_0 (standing for Recursive Comprehension Axiom), contains basic first-order Peano arithmetic together with the Δ^0_1 comprehension scheme and the Σ^0_1 induction scheme. The early study of reverse mathematics revealed that most "ordinary", i.e., non set-theoretic, theorems are equivalent to five main subsystems, known as the "Big Five" [60]. However, Ramsey's theory provides a large class of theorems escaping this observation, making it an interesting research subject in reverse mathematics [2, 8, 28, 29, 50, 58]. The book of Hirschfeldt [26] is an excellent introduction to reverse mathematics and in particular the reverse mathematics of Ramsey's theorem.

Many theorems are Π^1_2 statements P of the form $(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X, Y)]$. A set X such that $\Phi(X)$ holds is called a P-*instance* and a set Y such that $\Psi(X, Y)$ holds is a *solution to* X. A theorem P is computably reducible to Q (written $\mathsf{P} \leq_c \mathsf{Q}$) if for every P-instance X, there is an X-computable Q-instance Y such that every solution to Y computes relative to X a solution to X. Computable reducibility provides a more fine-grained analysis of theorems than reverse mathematics, in the sense that it is sensitive to the number of applications of the theorem P in a proof that P implies Q over RCA_0 [13, 16, 27, 56]. For example, Ramsey's theorem for $(k+1)$-colorings of the n-element subsets of $\mathbb{N}$ is not computably equivalent to Ramsey's theorem for k-colorings, whereas those statements are equivalent over RCA_0 [56].

1.1. Ramsey's theorem

The strength of Ramsey-type statements is notoriously hard to tackle in the setting of reverse mathematics. The separation of Ramsey's theorem for pairs (RT^2_2) from the arithmetic comprehension axiom (ACA_0) was a long-standing open problem, until Seetapun solved it [58] using the notion of cone avoidance.

Definition 1.1 (Ramsey's theorem) A subset H of $\mathbb{N}$ is *homogeneous* for a coloring $f : [\mathbb{N}]^n \to k$ (or *f-homogeneous*) if all n-tuples over H are given the same color by f. RT^n_k is the statement "Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite f-homogeneous set".

Jockusch [32] studied the computational strength of Ramsey's theorem, and Simpson [60, Theorem III.7.6] built upon his work to prove that whenever $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{ACA}_0$. Ramsey's theorem for pairs is probably the most famous example of a statement escaping the Big Five. Seetapun [58] proved that RT^2_2 is strictly weaker than ACA_0 over RCA_0. Because of the complexity of the related separations, RT^2_2 received particular attention from the reverse mathematics community [8, 58, 32]. Cholak, Jockusch and Slaman [8] and Liu [43] proved that RT^2_2 is incomparable with weak König's lemma. Dorais, Dzhafarov, Hirst, Mileti and Shafer [13], Dzhafarov [17], Hirschfeldt and Jockusch [27], Rakotoniaina [57] and the author [56] studied the computational strength of Ramsey's theorem according to the number of colors, when fixing the number of applications of the principle.

In this paper, we state some remaining open questions which naturally arose during the study of Ramsey's theorem. Many of them are already stated in various papers. These questions cover a much thinner branch of reverse mathematics than the paper of Montalban [47] and are of course influenced by the own interest of the author. We put our focus on a few central open questions, motivate them and try to give some insights about the reason for their complexity. We also detail some promising approaches and ask more technical questions, driven by the resolution of the former ones. The questions are computability-theoretic oriented, and therefore mainly involve the relations between statements over ω-models, that is, models where the first-order part is composed of the standard integers.

2. Cohesiveness and partitions

In order to better understand the combinatorics of Ramsey's theorem for pairs, Cholak et al. [8] decomposed it into a cohesive and a stable version. This approach has been fruitfully reused in the analysis of various consequences of Ramsey's theorem [29].

Definition 2.1 (Stable Ramsey's theorem) A function $f : [\mathbb{N}]^2 \to k$ is *stable* if for every $x \in \mathbb{N}$, $\lim_s f(x, s)$ exists. SRT^2_k is the restriction of RT^2_k to stable colorings. D^2_k is the statement "Every Δ^0_2 k-partition has an infinite subset in one of its parts".

By Shoenfield's limit lemma [59], a stable coloring $f : [\mathbb{N}]^2 \to 2$ can be seen as the Δ^0_2 approximation of the Δ^0_2 set $A = \lim_s f(\cdot, s)$. Cholak et al. [8] proved that D^2_k and SRT^2_k are computably equivalent and that the proof is formalizable over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, where $\mathsf{B}\Sigma^0_2$ is the Σ^0_2-bounding statement. Later, Chong et al. [9] proved that D^2_2 implies $\mathsf{B}\Sigma^0_2$ over RCA_0, showing therefore that $\mathsf{RCA}_0 \vdash \mathsf{D}^2_k \leftrightarrow \mathsf{SRT}^2_k$ for every $k \geq 2$. The statement D^2_k can be seen as a variant of RT^1_k, where the instances are Δ^0_2. It happens that many statements of the form "Every computable instance of SRT^2_2 has a solution satisfying some properties" are proven by showing the following stronger statement "Every instance of RT^1_2 (without any effectiveness restriction) has a solution satisfying some properties". This is for example the case for closed sets avoidances [19, 44, 43], and various preservation notions [53, 56, 64]. This observation shows that the weakness of RT^1_2 has a combinatorial nature, whereas SRT^2_2 is only effectively weak.

Definition 2.2 (Cohesiveness) An infinite set C is $\vec{R}$-cohesive for a sequence of sets $R_0, R_1, \dots$ if for each $i \in \mathbb{N}$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$. A set C is *p-cohesive* if it is $\vec{R}$-cohesive where $\vec{R}$ is an enumeration of all primitive recursive sets. COH is the statement "Every uniform sequence of sets $\vec{R}$ has an $\vec{R}$-cohesive set."

Jockusch & Stephan [37] studied the degrees of unsolvability of cohesiveness and proved that COH admits a universal instance, i.e., an instance whose solutions compute solutions to every other instance. This instance consists of the primitive recursive sets. They characterized the degrees of p-cohesive sets as those whose jump is PA relative to $\emptyset'$. Later, the author [56] refined this correspondence by showing that for every computable sequence of sets $R_0, R_1, \dots,$ there is a $\Pi^{0,\emptyset'}_1$ class $\mathcal{C}$ of reals such that the degrees bounding an $\vec{R}$-cohesive set are exactly those whose jump bounds a member of $\mathcal{C}$. Moreover, for every $\Pi^{0,\emptyset'}_1$ class $\mathcal{C}$, we can construct a computable sequence of sets $\vec{R}$ satisfying the previous property. In particular, if some instance of COH has no computable solution, then it has no low solution. This shows that COH is a statement about the Turing jump.

Cholak, Jockusch and Slaman [8] claimed that RT^2_2 is equivalent to $\mathsf{SRT}^2_2 + \mathsf{COH}$ over RCA_0 with an erroneous proof. Mileti [45] and Jockusch & Lempp [unpublished] independently fixed the proof. Cholak et al. [8] proved that COH does not imply SRT^2_2 over RCA_0. The question of the other direction has been a long-standing open problem. Recently, Chong et al. [11] proved that SRT^2_2 is strictly weaker than RT^2_2 over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. However they used non-standard models to separate the statements and the question whether SRT^2_2 and RT^2_2 coincide over ω-models remains open. See the survey of Chong, Li and Yang [10] for the approach of non-standard analysis applied to reverse mathematics.

Question 2.3 Does SRT^2_2 imply RT^2_2 (or equivalently COH) over ω-models?

Jockusch [32] constructed a computable instance of RT^2_2 with no Δ^0_2 solution. Cholak et al. [8] suggested building an ω-model of SRT^2_2 composed only of low sets. However, Downey et al. [15] constructed a Δ^0_2 set with no low subset of either it or its complement. As often, the failure of an approach should not be seen as a dead-end, but as a starting point. The construction of Downey et al. revealed that SRT^2_2 carries some additional computational power, whose nature is currently unknown. Indeed, all the natural computability-theoretic consequences of SRT^2_2 known hitherto admit low solutions. Answering the following question would be a significant step towards understanding the strength of SRT^2_2.

Question 2.4 Is there a natural computable SRT^2_2-instance with no low solution?

Here, by "natural", we mean an instance which carries more informational content than having no low solution. Interestingly, Chong et al. [11] constructed a non-standard model of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{SRT}^2_2$ containing only low sets. This shows that the argument of Downey et al. [15] requires more than Σ^0_2-bounding, and suggests that such an instance has to use an elaborate construction.

Hirschfeldt et al. [28] proposed a very promising approach using an extension of Arslanov's completeness criterion [1, 34]. A set X is *1-CEA over Y* if c.e. in and above Y. A set X is *$(n+1)$-CEA over Y* if it is 1-CEA over a set Z which is n-CEA over Y. In particular the sets which are 1-CEA over $\emptyset$ are the c.e. sets. By [34], every set n-CEA over $\emptyset$ computing a set of PA degree is complete. Hirschfeldt et al. asked the following question.

Question 2.5 Does every Δ^0_2 set admit an infinite subset of either it or its complement which is both low$_2$ and Δ^0_2?

A positive answer to Question 2.5 would enable one to build an ω-model $\mathcal{M}$ of SRT^2_2 such that for every set $X \in \mathcal{M}$, X is low$_2$ and X' is n-CEA over $\emptyset'$. By Jockusch & Stephan [37], if some $X \in \mathcal{M}$ computes some p-cohesive set, then X' is of PA degree relative to $\emptyset'$. By a relativization of Jockusch et al. [34], X' would compute $\emptyset''$, so would be high, which is impossible since X is low$_2$. Therefore, $\mathcal{M}$ would be an ω-model of SRT^2_2 which is not a model of COH, answering Question 2.3. Note that the argument also works if we replace low$_2$ by low$_n$, where n may even depend on the instance. Hirschfeldt et al. [28] proved that every Δ^0_2 set has an infinite incomplete Δ^0_2 subset of either it or its complement. This is the best upper bound currently known. They asked the following question which is the strong negation of Question 2.5.

Question 2.6 Is there a Δ^0_2 set such that every infinite subset of either it or its complement has a jump of PA degree relative to $\emptyset'$?

By Arslanov's completeness criterion, a positive answer to Question 2.6 would also provide one to the following question.

Question 2.7 Is there a Δ^0_2 set such that every Δ^0_2 infinite subset of either it or its complement is high?

Cohesiveness can be seen as a sequential version of RT^1_2 with finite errors ($\mathsf{Seq}^*(\mathsf{RT}^1_2)$). More formally, given some theorem P, $\mathsf{Seq}^*(\mathsf{P})$ is the statement "For every uniform sequence of P-instances $X_0, X_1, \dots$, there is a set Y which is, up to finite changes, a solution to each of the X's." One intuition about the guess that SRT^2_2 does not imply COH could be that by the equivalence between SRT^2_2 and D^2_2, SRT^2_2 is nothing but a non-effective instance of RT^1_2, and that one cannot encode in a single instance of RT^1_2 countably many RT^1_2-instances. Note that this argument is of combinatorial nature, as it does not make any effectiveness assumption on the instance of RT^1_2. We express reservations concerning the validity of this argument, as witnessed by what follows.

Definition 2.8 (Thin set theorem) Given a coloring $f : [\mathbb{N}]^n \to k$ (resp. $f : [\mathbb{N}]^n \to \mathbb{N}$), an infinite set H is *thin* for f if $|f([H]^n)| \leq k-1$ (resp. $f([H]^n) \neq \mathbb{N}$), that is, f avoids at least one color over H. For every $n \geq 1$ and $k \geq 2$, TS^n_k is the statement "Every coloring $f : [\mathbb{N}]^n \to k$ has a thin set" and TS^n is the statement "Every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has a thin set".

The thin set theorem is a natural weakening of Ramsey's theorem. Its reverse mathematical analysis started with Friedman [23, 24]. It has been studied by Cholak et al. [7], Wang [65] and the author [50, 52, 56] among others. According to the definition, $\mathsf{Seq}^*(\mathsf{TS}^1_\mathsf{k})$ is the statement "For every uniform sequence of functions $f_0, f_1, \dots : \mathbb{N} \to k$, there is an infinite set H which is, up to finite changes, thin for all the f's."

Definition 2.9 (DNC functions) A function $f : \mathbb{N} \to \mathbb{N}$ is *diagonally non-computable relative to* X if for every e, $f(e) \neq \Phi^X_e(e)$. For every $n \geq 1$, n-DNC is the statement "For every set X, there is a function DNC relative to $X^{(n-1)}$. We write DNC for 1-DNC.

DNC functions are central notions in algorithmic randomness, as their degrees coincide with the degrees of infinite subsets of Martin-Löf randoms (see Kjos-Hanssen [39] and Greenberg & Miller [25]). Moreover, Jockusch & Soare [35] and Solovay [unpublished] proved that the degrees of $\{0,1\}$-valued DNC functions are exactly the PA degrees. DNC functions naturally appear in reverse mathematics, the most suprising example being the equivalence between the rainbow Ramsey theorem for pairs and 2-DNC proven by Miller [46]. The rainbow Ramsey theorem asserts that every coloring of $[\mathbb{N}]^n$ in which each color appears at most k times admits an infinite set in which each color appears at most once. It has been studied by Csima & Mileti [12], Wang [62, 63] and the author [55], among others.

Theorem 3.5 in Jockusch & Stephan [37] can easily be adapted to prove that the degrees of solutions to primitive recursive instances of $\mathsf{Seq}^*(\mathsf{TS}^1)$ (resp. $\mathsf{Seq}^*(\mathsf{TS}^1_\mathsf{k})$) are exactly those whose jump is of DNC (resp. k-valued DNC) degree relative to $\emptyset'$. By a relativization of Friedberg [33], the degrees whose jump is PA and those whose jump bounds a k-valued DNC function coincide. Therefore COH and $\mathsf{Seq}^*(\mathsf{TS}^1_\mathsf{k})$ are computably equivalent. However, $\mathsf{Seq}^*(\mathsf{TS}^1)$ is a strictly weaker statement, as for any computable instance of $\mathsf{Seq}^*(\mathsf{TS}^1)$, the measure of oracles computing a solution to it is positive.

Recall our intuition that a single instance of RT^1_2 cannot encode the information of countably many instances of RT^1_2. This intuition is false when considering TS^1. Indeed, there is a (non-Δ^0_2) instance of TS^1 (and *a fortiori* one of RT^1_2) whose solutions all bound a function DNC relative to $\emptyset'$, and therefore computes a solution to any computable instance of $\mathsf{Seq}^*(\mathsf{TS}^1)$. Therefore, before asking whether COH is a consequence of $\emptyset'$-effective instances of RT^1_2, it seems natural to ask whether COH is a consequence of any coloring over singletons in a combinatorial sense, that is, with no effectiveness restriction at all.

Question 2.10 Is there any RT^1_2-instance whose solutions have a jump of PA degree relative to $\emptyset'$?

Note that a negative answer to Question 2.10 would have practical reverse mathematical consequences. There is an ongoing search for natural statements strictly between RT^2_2 and SRT^2_2 over RCA_0 [18]. Dzhafarov & Hirst [18] introduced the increasing polarized Ramsey's theorem for pairs (IPT^2_2), and proved it to be between RT^2_2 and SRT^2_2 over RCA_0. The author [55] proved that IPT^2_2 implies the existence of a function DNC relative to $\emptyset'$, therefore showing that SRT^2_2 does not imply IPT^2_2 over RCA_0. The statement IPT^2_2 is equivalent to 2-RWKL, a relativized variant of the Ramsey-type weak König's lemma, over RCA_0, and therefore is a combinatorial consequence of RT^1_2. An iterable negative answer to Question 2.10 would prove that IPT^2_2 does not imply RT^2_2, hence is strictly between RT^2_2 and SRT^2_2 over RCA_0.

We have seen that $\mathsf{Seq}^*(\mathsf{TS}^1)$ is a combinatorial consequence of TS^1. This information helps us for tackling the following question. Indeed, if proven false, it must be answered by effective means and not by combinatorial ones.

Question 2.11 Does SRT^2_2 imply $\mathsf{Seq}^*(\mathsf{TS}^1)$ over ω-models?

Question 2.12 Is there a Δ^0_2 set such that every infinite subset of either it or its complement has a jump of DNC degree relative to $\emptyset'$?

Although those questions are interesting in their own right, a positive answer to Question 2.5 would provide a negative answer to Question 2.11, and a positive answer to Question 2.12 would provide a positive answer to Question 2.7. Indeed, the extended version of Arslanov's completeness criterion states in fact that every set n-CEA over $\emptyset$ computing a set of DNC degree is complete. In particular, if Question 2.5 has a positive answer, then there is an ω-model of SRT^2_2 which is not a model of the rainbow Ramsey theorem for pairs.

Let us finish this section by a discussion about why those problems are so hard to tackle. There are different levels of answers, starting from the technical one which is more objective, but probably also less informative, to the meta discussion which tries to give more insights, but can be more controversial.

From a purely technical point of view, all the forcing notions used so far to produce solutions to Ramsey-type statements are variants of Mathias forcing. In particular, they restrict the future elements to a "reservoir". Any sufficiently generic filters for those notions of forcing yield cohesive sets. Therefore, one should not expect to obtain a diagonalization against instances of COH by exhibiting a particular dense set of conditions. Indeed, one would derive a contradiction by taking a set sufficiently generic to meet both those diagonalizing sets, and the dense sets producing a cohesive solution. More generally, as long as we use a forcing notion where we restrict the future elements to a reservoir, any diagonalization against COH has to strongly rely on some effectiveness of the overall construction. The first and second jump control of Cholak et al. [8] form a case in point of how to restrict the amount of genericity to obtain some stronger properties, which are provably wrong when taking any sufficiently generic filter.

In a higher level, we have mentioned that COH is a statement about the jump of Turing degrees. In other words, by Shoenfield's limit lemma [59], COH is a statement about some limit behavior, and is therefore non-sensitive to any local modification. However, the computability-theoretic properties used so far to separate statements below ACA_0 are mainly acting "below the first jump", in the sense that the diagonalization occurs after a finite amount of time. With COH, there will be some need for a "continuous diagonalization", that is, a diagonalization which has to be maintained all along the construction.

3. A Ramsey-type weak König's lemma

König's lemma asserts that every infinite, finitely branching tree has an infinite path. Weak König's lemma (WKL_0) is the restriction of König's lemma to infinite binary trees. WKL_0 plays a central role in reverse mathematics. It is one of the Big Five and informally captures compactness arguments [60]. Weak König's lemma is involved in many constructions of solutions to Ramsey-type statements, e.g., cone avoidance [58, 65] or control of the jump [8, 51]. The question of whether RT^2_2 implies WKL_0 over RCA_0 was open for decades, until Liu [43] solved it by proving that PA degrees are not a combinatorial consequence of RT^1_2.

Recently, Flood [21] clarified the relation between Ramsey-type theorems and WKL_0, by introducing a Ramsey-type variant of weak König's lemma (RWKL). Informally, seeing a set as a 2-coloring of the integers, for every Π^0_1 class of 2-colorings, RWKL states the existence of an infinite set homogeneous for one of them. The exact statement of RWKL has to be done with some care, as we do not want to state the existence of a member of the Π^0_1 class.

Definition 3.1 (Ramsey-type weak König's lemma) A set $H \subseteq \mathbb{N}$ is *homogeneous* for a $\sigma \in 2^{<\mathbb{N}}$ if $(\exists c < 2)(\forall i \in H)(i < |\sigma| \to \sigma(i) = c)$, and a set $H \subseteq \mathbb{N}$ is *homogeneous* for an infinite tree $T \subseteq 2^{<\mathbb{N}}$ if the tree $\{\sigma \in T : H \text{ is homogeneous for } \sigma\}$ is infinite. RWKL is the statement "for every infinite subtree of $2^{<\mathbb{N}}$, there is an infinite homogeneous set."

The Ramsey-type weak König's lemma was introduced by Flood in [21] under the name RKL, and later renamed RWKL by Bienvenu, Patey and Shafer. Flood [21] proved that RWKL is a strict consequence of both SRT^2_2 and WKL_0, and that RWKL implies DNC over RCA_0. Bienvenu et al. [2] and Flood & Towsner [22] independently separated DNC from RWKL. They furthermore proved that DNC coincides over RCA_0 to the restriction of RWKL to trees of positive measure. Very little is currently known about RWKL. Despite its complicated formulation, RWKL is a natural statement which is worth being studied due to the special status of Ramsey-type theorems in reverse mathematics.

The statements analysed in reverse mathematics are collections of problems (instances) coming with a class of solutions. Sometimes, it happens that one problem is maximally difficult. In this case, the strength of the whole statement can be understood by studying this particular instance.

Definition 3.2 (Universal instance) A computable P-instance X is *universal* if for every computable P-instance Y, every solution to X computes a solution to Y. A degree $\mathbf{d}$ is P-*bounding relative to a degree* $\mathbf{e}$ if every $\mathbf{e}$-computable instance has a $\mathbf{d}$-computable solution. The degree $\mathbf{d}$ is P-*bounding* if it is P-bounding relative to $\mathbf{0}$.

WKL_0 is known to admit universal instances, e.g., the tree whose paths are completions of Peano arithmetic, whereas Mileti [45] proved that RT^2_2 and SRT^2_2 do not admit one. The author [52] studied extensively which theorems in reverse mathematics admit a universal instance, and which do not. It happens that most consequences of RT^2_2 in reverse mathematics do not admit a universal instance. The most notable exceptions are the rainbow Ramsey theorem for pairs [12, 55, 62, 63], the finite intersection property [5, 14, 20] and DNC. It is natural to wonder, given the fact that RWKL is a consequence of both SRT^2_2 and of WKL_0, whether RWKL admits a universal instance.

Question 3.3 Does RWKL admit a universal instance?

There is a close link between the P-bounding degrees and the existence of a universal P-instance. Indeed, the degrees of the solutions to a universal P-instance are P-bounding. Using the contraposition, one usually proves that a statement P admits no universal instance by showing that every computable P-instance has a solution of degree belonging to a class $\mathcal{C}$, and that for every degree $\mathbf{d} \in \mathcal{C}$, there is a computable P-instance to which $\mathbf{d}$ bounds no solution [45, 52].

Interestingly, Question 3.3 has some connections with the SRT^2_2 vs COH question. The only construction of solutions to instances of RWKL which do not produce sets of PA degree are variants of Mathias forcing which produce solutions to $\mathsf{Seq}^*(\mathsf{RWKL})$. In both cases, the solutions are RWKL-bounding, and in the latter case, they have a jump of PA degree relative to $\emptyset'$. By the previous discussion, one should not expect to prove that RWKL admits no universal instance in the usual way, unless by finding a new forcing notion.

Some statements do not admit a universal instance because their class of instances is too restrictive, but they have a natural strengthening which does admit one. This is for example the case of the rainbow Ramsey theorem for pairs, which by Miller [46] admit a universal instance whose solutions are of DNC degree relative to $\emptyset'$, but stable variants of the rainbow Ramsey theorem do not admit one [52]. It is therefore natural to wonder whether there is some strengthening of RWKL still below RT^2_2 which admits a universal instance. The Erdős-Moser theorem, defined in the next section, is a good candidate. The following question is a weakening of this interrogation.

Question 3.4 Is there some instance of RT^1_2 or some computable instance of RT^2_2 whose solutions are RWKL-bounding?

Weak König's lemma is equivalent to several theorems over RCA_0 [60]. However, the Ramsey-type versions of those theorems do not always give statements equivalent to RWKL over RCA_0. For instance, the existence of a separation of two computably inseparable c.e. sets is equivalent to WKL_0 over RCA_0, whereas it is easy to compute an infinite subset of a separating set. Among those statements, the Ramsey-type version of the graph coloring problem [31] is of particular interest in our study of RWKL. Indeed, the RWKL-instances built in [2, 22, 54] are only of two kinds: trees of positive measure, and trees whose paths code the k-colorings of a computable k-colorable graph. The restriction of RWKL to the former class of instances is equivalent to DNC over RCA_0. We shall discuss further the latter one.

Definition 3.5 (Ramsey-type graph coloring) Let $G = (V, E)$ be a graph. A set $H \subseteq V$ is *k-homogeneous for G* if every finite $V_0 \subseteq V$ induces a subgraph that is k-colorable by a coloring that colors every vertex in $V_0 \cap H$ color 0. RCOLOR_k is the statement "for every infinite, locally k-colorable graph $G = (V, E)$, there is an infinite $H \subseteq V$ that is k-homogeneous for G."

The Ramsey-type graph coloring statements have been introduced by Bienvenu et al. [2]. They proved by an elaborate construction that RCOLOR_n is equivalent to RWKL for $n \geq 3$, and that DNC (or even weak weak König's lemma) does not imply RCOLOR_2 over ω-models.

Question 3.6 Does RCOLOR_2 imply RWKL or even DNC over RCA_0?

Separating RCOLOR_2 from RWKL may require constructing an RWKL-instance of a new kind, i.e., but not belonging to any of the two classes of instances mentioned before. The question about the existence of a universal instance also holds for RCOLOR_2.

4. The Erdős-Moser theorem

The Erdős-Moser theorem is a statement from graph theory. It provides together with the ascending descending principle (ADS) an alternative decomposition of Ramsey's theorem for pairs. Indeed, every coloring $f : [\mathbb{N}]^2 \to 2$ can be seen as a tournament R such that $R(x, y)$ holds if $x < y$ and $f(x, y) = 1$, or $x > y$ and $f(y, x) = 0$. Every infinite transitive subtournament induces a linear order whose infinite ascending or descending sequences are f-homogeneous.

Definition 4.1 (Erdős-Moser theorem) A tournament T on a domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation on D such that for all $x, y \in D$ with $x \neq y$, exactly one of $T(x, y)$ or $T(y, x)$ holds. A tournament T is *transitive* if the corresponding relation T is transitive in the usual sense. A tournament T is *stable* if $(\forall x \in D)[(\forall^\infty s)T(x, s) \vee (\forall^\infty s)T(s, x)]$. EM is the statement

"Every infinite tournament T has an infinite transitive subtournament." SEM is the restriction of EM to stable tournaments.

Bovykin and Weiermann [3] introduced the Erdős-Moser theorem in reverse mathematics and proved that $\mathsf{EM} + \mathsf{ADS}$ is equivalent to RT^2_2 over RCA_0. Lerman et al. [42] proved that EM is strictly weaker than RT^2_2 over ω-models. This separation has been followed by various refinements of the weakness of EM by Wang [64] and the author [49, 48]. Bienvenu et al. [2] and Flood & Towsner [22] independently proved that SEM strictly implies RWKL over RCA_0.

Cholak et al. [8] proved that every computable RT^2_2-instance has a low$_2$ solution, while Mileti [45] and the author [52] showed that various consequences P of RT^2_2 do not have P-bounding low$_2$ degrees, showing therefore that P does not have a universal instance. This approach does not apply to EM since there is a low$_2$ EM-bounding degree [52]. The Erdős-Moser theorem is, together with RWKL, one of the last Ramsey-type theorems for which the existence of a universal instance is unknown. A positive answer to the following question would refine our understanding of RWKL-bounding degrees. Note that, like RWKL, the only known forcing notion for building solutions to EM-instances produces EM-bounding degrees.

Question 4.2 Does EM admit a universal instance?

Due to the nature of the decomposition of RT^2_2 into EM and ADS, the Erdős-Moser theorem shares many features with RT^2_2. In particular, there is a computable SEM-instance with no low solution [40]. The forcing notion used to construct solutions to EM-instances is very similar to the one used to construct solutions to RT^2_2-instances. The main difference is that in the EM case, only one object (a transitive subtournament) is constructed, whereas in the RT^2_2 case, both a set homogeneous with color 0 and a set homogeneous with color 1 are constructed. As a consequence, the constructions in the EM case remove the disjunction appearing in almost every construction of solutions to RT^2_2, and therefore simplifies many arguments, while preserving some computational power. In particular, the author proved that $\mathsf{COH} \leq_c \mathsf{SRT}^2_2$ if and only if $\mathsf{COH} \leq_c \mathsf{SEM}$.

Considering that SEM behaves like SRT^2_2 with respect to COH, one would wonder whether EM, like RT^2_2, implies COH over RCA_0. The closest result towards an answer is the proof that EM implies $[\mathsf{STS}^2 \vee \mathsf{COH}]$ over RCA_0, where STS^2 is the restriction of TS^2 to stable functions [55]. The Erdős-Moser theorem is known not to imply STS^2 over ω-models [49, 64], but the following question remains open.

Question 4.3 Does EM imply COH over RCA_0?

A natural first step in the study of the computational strength of a principle consists in looking at how it behaves with respect to "typical" sets. Here, by typical, we mean randomness and genericity. By Liu [44], EM does not imply the existence of a Martin-Löf random. However, it implies the existence of a function DNC relative to $\emptyset'$ [38, 55], which corresponds to the computational power of an infinite subset of a 2-random. On the genericity hand, EM implies the existence of a hyperimmune set [42] and does not imply Π^0_1-genericity [30, 48]. The relation between EM and 1-genericity is currently unclear.

Definition 4.4 (Genericity) Fix a set of strings $S \subseteq 2^{<\mathbb{N}}$. A real G *meets* S if it has some initial segment in S. A real G *avoids* S if it has an initial segment with no extension in S. Given an integer $n \in \mathbb{N}$, a real is *n-generic* if it meets or avoids each Σ^0_n set of strings. n-GEN is the statement "For every set X, there is a real n-generic relative to X".

1-genericity received particular attention from the reverse mathematical community recently as it happened to have close connections with the finite intersection principle (FIP). Dzhafarov & Mummert [20] introduced FIP in reverse mathematics. Day, Dzhafarov, and Miller [unpublished] and Hirschfeldt and Greenberg [unpublished] independently proved that it is a consequence of the atomic model theorem (and therefore of SRT^2_2) over RCA_0. Downey et al. [14] first established a link between 1-genericity and FIP by proving that $\mathsf{1\text{-}GEN}$ implies FIP over RCA_0. Later, Cholak et al. [5] proved the other direction.

Question 4.5 Does EM imply $\mathsf{1\text{-}GEN}$ over RCA_0?

Every 1-generic bounds an ω-model of $\mathsf{1\text{-}GEN}$. This property is due to a Van Lambalgen-like theorem for genericity [66], and implies in particular that there is no 1-generic of minimal degree. Cai et al. [4] constructed an ω-model of DNC such that the degrees of the second-order part belong to a sequence $\mathbf{0}, \mathbf{d}_1, \mathbf{d}_2, \dots$ where $\mathbf{d}_1$ is a minimal degree and $\mathbf{d}_{n+1}$ is a strong minimal cover of $\mathbf{d}_n$. By the previous remark, such a model cannot contain a 1-generic real. Construcing a similar model of EM would answer negatively Question 4.5. The following question is a first step towards an answer. In particular, answering positively would provide a proof that $\mathsf{1\text{-}GEN} \not\leq_{\mathsf{c}} \mathsf{EM}$.

Question 4.6 Does every computable EM-instance (or even RWKL-instance) admit a solution of minimal degree?

5. Ramsey-type hierarchies

Jockusch [32] proved that the hierarchy of Ramsey's theorem collapses at level 3 over ω-models, that is, for every $n, m \geq 3$, RT^n_2 and RT^m_2 have the same ω-models. Simpson [60, Theorem III.7.6] formalized Jockusch's results within reverse mathematics and proved that RT^n_2 is equivalent to the arithmetic comprehension axiom (ACA_0) over RCA_0 for every $n \geq 3$. Since RT^1_2 is provable over RCA_0 and RT^2_2 is strictly between RT^3_2 and RCA_0, the status of the whole hierarchy of Ramsey's theorem is known.

However, some consequences of Ramsey's theorem form hierarchies for which the question of the strictness is currently unanswered. This is for example the case of the thin set theorems, the free set theorem and the rainbow Ramsey theorem. The thin set theorems, already introduced, are natural weakenings of Ramsey's theorem, where the solutions are allowed to use more than one color. The free set theorem is a strengthening of the thin set theorem over ω colors, in which every member of a free set is a witness of thinness of the same set. Indeed, if H is an infinite f-free set for some function f, for every $a \in H$, $H \setminus \{a\}$ is f-thin with witness color a. See Theorem 3.2 in [7] for a formal version of this claim.

Definition 5.1 (Free set theorem) Given a coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, an infinite set H is *free* for f if for every $\sigma \in [H]^n$, $f(\sigma) \in H \rightarrow f(\sigma) \in \sigma$. For every $n \geq 1$, FS^n is the statement "Every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ has a free set". FS is the statement $(\forall n)\, \mathsf{FS}^n$.

The free set theorem has been introduced by Friedman [23] together with the thin set theorem. Cholak et al. [7] proved that TS^n is a consequence of FS^n for every $n \geq 2$. Wang [65] proved that the full free set hierarchy (hence the thin set hierarchy) lies strictly below ACA_0 over RCA_0. This result was improved by the author [50] who showed that FS does not even imply WKL_0 (and in fact weak weak König's lemma) over RCA_0.

Question 5.2 Does the free set theorem (resp. the thin set theorem with ω colors) form a strict hierarchy?

Jockusch [32] proved for every $n \geq 2$ that every computable RT^n_2-instance has a Π^0_n solution and constructed a computable RT^n_2-instance with no Σ^0_n solution. Cholak et al. [7] proved that FS^n and TS^n satisfy the same bounds. In particular, there is no ω-model of FS^n or TS^n containing only Δ^0_n sets. By Cholak et al. [8], every computable instance of FS^2 and TS^2 admit a low$_2$ solution. Therefore FS^2 (hence TS^2) are strictly weaker than TS^3 (hence FS^3) over RCA_0. Using this approach, a way to prove the strictness of the hierarchies would be to answer positively the following question.

Question 5.3 For every $n \geq 3$, does every computable instance of FS^n (resp. TS^n) admit a low$_n$ solution?

The complexity of controlling the nth iterate of the Turing jump grows very quickly with n. While the proof of the existence of low$_2$ solutions to computable RT^2_2-instances using first jump control is relatively simple, the proof using the second jump control requires already a much more elaborate framework [8]. Before answering Question 5.3, one may want to get rid of the technicalities due to the specific combinatorics of the free set and thin set theorems, and focus on the control of iterated Turing jumps by constructing simpler objects.

Non-effective instances of Ramsey's theorem for singletons is a good starting point, since the only combinatorics involved are the pigeonhole principle. Moreover, RT^1_2 can be seen as a bootstrap principle, above which the other Ramsey-type statements are built. For instance, cohesiveness is proven by making ω applications of the RT^1_2 principles, and RT^2_2 is obtained by making one more application of RT^1_2 over a non-effective instance. The proofs of the free set and thin set theorems also make an important use of non-effective instances of RT^1_2 [50, 65].

Question 5.4 For every $n \geq 3$, does every Δ^0_n set admit an infinite low$_n$ subset of either it or its complement?

The solutions to Ramsey-type instances are usually built by forcing. In order to obtain low$_n$ solutions, one has in particular to $\emptyset^{(n)}$-effectively decide the Σ^0_n theory of the generic set. The forcing relation over a partial order is defined inductively, and intuitively expresses whether, whatever the choices of conditions extensions we make in the future, we will still be able to make some progress in satisfying the considered property.

This raises the problem of talking about the "future" of a condition c. To do that, one needs to be able to describe effectively the conditions extending c. The problem of the forcing notions used to build solutions to Ramsey-type instances is that they use variants of Mathias forcing, whose conditions cannot be described effectively. For instance, let us take the simplest notion of Mathias forcing: pairs (F, X) where F is a finite set of integers representing the finite approximation of the solution, X is a computable infinite "reservoir" of integers and $max(F) < min(X)$. Given a condition $c = (F, X)$, the extensions of c are the pairs (H, Y) such that $F \subseteq H$, $Y \subseteq X$ and $H \setminus F \subseteq X$. Deciding whether a Turing index is the code of an infinite computable subset of a fixed computable set requires a lot of computational power. Cholak et al. [6] studied computable Mathias forcing and proved that the forcing relation for deciding Σ^0_n properties is not Σ^0_n in general. This is why we need to be more careful in the design of the forcing notions.

In some cases, the reservoir has a particular shape. Through their second jump control, Cholak et al. [8] first used this idea by noticing that the only operations over the reservoir were partitioning and finite truncation. This idea has been reused by Wang [63] to prove that every computable $\mathsf{Seq}^*(\mathsf{D}^2_2)$-instance has a solution of low$_3$ degree. Recently, the author [51] designed new forcing notions for various Ramsey-type statements, e.g., COH, EM and D^2_2, in which the forcing relation for deciding Σ^0_n properties is Σ^0_n.

6. Proofs

In this last section, we provide the proofs supporting some claims which have been made throughout the discussion. We first clarify the links between sequential variants of the thin set theorem and diagonally non-computable functions.

Lemma 6.1 For every instance $\vec{f}$ of $\mathsf{Seq}^*(\mathsf{TS}^1)$ and every set C whose jump is of DNC degree relative to the jump of $\vec{f}$, $C \oplus \vec{f}$ computes a solution to $\vec{f}$.

Proof. Fix a $\mathsf{Seq}^*(\mathsf{TS}^1)$-instance $f_0, f_1, \dots : \mathbb{N} \to \mathbb{N}$. Let $A_0, A_1, \dots$ be a uniformly $\vec{f}$-computable sequence of sets containing $\mathbb{N}$, the sets $\{x : f_s(x) \neq i\}$ for each $s, i \in \mathbb{N}$, and that is closed under the intersection and complementation operations. Let g be the partial $\vec{f}'$-computable function defined for each $s, i \in \mathbb{N}$ by $g(i,s) = \lim_{n \in A_i} f_s(n)$ if it exists. The jump of C computes a function $h(\cdot,\cdot)$ such that $h(s,i) \neq g(s,i)$ for each $s, i \in \mathbb{N}$.

Let $e_0, e_1, \dots$ be a C'-computable sequence defined inductively by $A_{e_0} = \mathbb{N}$ and $A_{e_{s+1}} = A_{e_s} \cap \{x : f_s(x) \neq g(e_s, s)\}$. By definition of g, if A_{e_s} is infinite, then so is $A_{e_{s+1}}$. By Shoenfield's limit lemma [59], the e's have a uniformly C-computable approximation $e_{s,0}, e_{s,1}, \dots$ such that $\lim_t e_{s,t} = e_s$.

We now use the argument of Jockusch & Stephan in [37, Theorem 2.1] to build an infinite set X such that $X \subseteq^* A_{e_s}$ for all $s \in \mathbb{N}$. Let

$$x_0 = 0 \quad \text{and} \quad x_{s+1} = min\{x > x_s : (\forall m \leq s)[x \in A_{e_{m,x}}]\}$$

Assuming that x_s is defined, we prove that x_{s+1} is found. Since $A_{e_{s+1}} \subseteq A_{e_m}$ for all $m \leq s$, any sufficiently large element satisfies $x > x_s : (\forall m \leq s)[x \in A_{e_{m,x}}]$. Therefore x_{s+1} is defined. The set $X = \{x_0, x_1, \dots\}$ is a solution to $\vec{f}$ by definition of the sequence $A_{e_0}, A_{e_1}, \dots$ □

Lemma 6.2 Fix a set X and let $\vec{f}$ be the instance of $\mathsf{Seq}^*(\mathsf{TS}^1)$ consisting of all X-primitive recursive functions. For every solution C to $\vec{f}$, $(X \oplus C)'$ is of DNC degree relative to X'.

Proof. Fix a set X, and consider the uniformly X-primitive recursive sequence of colorings $f_0, f_1, \dots : \mathbb{N} \to \mathbb{N}$ defined for each $e \in \mathbb{N}$ by $f_e(s) = \Phi^{X'_s}_{e,s}(e)$, where X'_s is the approximation of the jump of X at stage s. Let C be a solution to the $\mathsf{Seq}^*(\mathsf{TS}^1)$-instance $\vec{f}$. By definition, for every $e \in \mathbb{N}$, there is some i such that $(\forall^\infty s) f_e(s) \neq i$. Let $g : \mathbb{N} \to \mathbb{N}$ be the function which on input e makes a $(X \oplus C)'$-effective search for such i and outputs it. The function g is DNC relative to $\emptyset'$. □

Theorem 6.3 For every standard k, $\mathsf{Seq}^*(\mathsf{TS}^1_k) =_{\mathsf{c}} \mathsf{COH}$.

Proof. $\mathsf{COH} \leq_c \mathsf{Seq}^*(\mathsf{TS}^1_k)$: Let $R_0, R_1, \dots$ be a COH-instance. Let $f_0, f_1, \dots : \mathbb{N} \to k$ be the uniformly $\vec{R}$-computable sequence of functions defined for each $e \in \mathbb{N}$ by $f_e(s) = \Phi^{\emptyset'_s}_{e,s}(e) \mod k$. By slightly modifying the proof of Lemma 6.2, for every solution C to the $\mathsf{Seq}^*(\mathsf{TS}^1_k)$-instance $\vec{f}$, $(\vec{R} \oplus C)'$ computes a k-valued function DNC relative to $\vec{R}'$. By Friedberg [33] in a relativized form, $(\vec{R} \oplus C)'$ is of PA degree relative to $\emptyset'$. Jockusch & Stephan [37] proved that every set whose jump is of PA degree relative to $\emptyset'$ computes a solution to $\vec{R}$. Using this fact, $\vec{R} \oplus C$ computes an infinite $\vec{R}$-cohesive set.

$\mathsf{Seq}^*(\mathsf{TS}^1_k) \leq_{\mathsf{c}} \mathsf{COH}$: Let $f_0, f_1, \dots$ be a $\mathsf{Seq}^*(\mathsf{TS}^1_k)$-instance. Let $R_0, R_1, \dots$ be the $\vec{f}$-computable sequence of sets defined for each $e, x \in \mathbb{N}$ by $x \in R_e(x)$ if and only if $f_e(x) = 0$. Every infinite $\vec{R}$-cohesive set is a solution to $\vec{f}$. □

Theorem 6.4 There is an ω-model of $\mathsf{Seq}^*(\mathsf{TS}^1)$ that is not a model of COH.

Proof. By Kučera [41] or Jockusch & Soare [36] in a relativized form, the measure of oracles whose jump is of PA degree relative to $\emptyset'$ is null. Let Z be a 2-random (a Martin-Löf random relative to $\emptyset'$) whose jump is not of PA degree relative to $\emptyset'$. By Van Lambalgen [61], Z bounds the second-order part of an ω-model of the statement "For every set X, there is a Martin-Löf random relative to X'.". In particular, by Kjos-Hanssen [39] and Greenberg & Miller [25], $\mathcal{M} \models \mathsf{2\text{-}DNC}$. By Lemma 6.1, $\mathcal{M} \models \mathsf{Seq}^*(\mathsf{TS}^1)$, and by Jockusch & Stephan [37], $\mathcal{M} \not\models \mathsf{COH}$. □

Theorem 6.5 For every set X, there is an X'-computable TS^1-instance $f : \mathbb{N} \to \mathbb{N}$ such that every infinite f-thin set computes a function DNC relative to X.

Proof. Fix a set X and let $g : \mathbb{N} \to \mathbb{N}$ be the X'-computable function such that $g(e) = \Phi^X_e(e)$ if $\Phi^X_e(e) \downarrow$, and $g(e) = 0$ otherwise. Fix an enumeration of all sets $(D_{e,i} : e, i \in \mathbb{N})$ such that $D_{e,i}$ is of size $2\langle e, i\rangle + 1$, where $\langle e, i\rangle$ is the standard pairing function. We define our TS^1-instance $f : \mathbb{N} \to \mathbb{N}$ by an X'-computable sequence of finite approximations $f_0 \subseteq f_1 \subseteq \dots$, such that at each stage s, f_s is defined on a domain of size at most $2s$. Furthermore, to ensure that $f = \bigcup_s f_s$ is X'-computable, we require that $s \in dom(f_{s+1})$.

At stage 0, f_0 is the empty function. At stage $s+1$, if $s = \langle e, i \rangle$, take an element $x \in D_{g(e),i} \setminus dom(f_s)$, and set $f_{s+1} = f_s \sqcup \{x \mapsto i\}$ if $s \in dom(f_s) \cup \{x\}$ and $f_{s+1} = f_s \sqcup \{x, s \mapsto i\}$ otherwise. Such an x must exist as $|D_{g(e),i}| = 2\langle e, i \rangle + 1 = 2s + 1 > dom(f_s)$. Then go to the next stage.

Let H be an infinite f-thin set, with witness color i. For each $e \in \mathbb{N}$, let $h(e)$ be such that $D_{h(e),i} \subseteq H$. We claim that $h(e) \neq \Phi_e^X(e)$. Suppose that it is not the case. In particular, $h(e) = g(e)$. Let $s = \langle g(e), i \rangle$. At stage $s+1$, $f_{s+1}(x) = i$ for some $x \in D_{g(e),i} = D_{h(e),i} \subseteq H$, contradicting the fact that H is f-thin. Therefore h is a function DNC relative to X. $\square$

Corollary 6.6 For every instance $\vec{f}$ of $\mathsf{Seq}^*(\mathsf{TS}^1)$, there is an instance g of TS^1 such that for every solution H to g, $H \oplus \vec{f}$ computes a solution to $\vec{f}$.

Proof. Fix a $\mathsf{Seq}^*(\mathsf{TS}^1)$-instance $\vec{f}$. By Theorem 6.5, there is a TS^1-instance g such that every infinite g-thin set H computes a function DNC relative to $\vec{f}$. By Lemma 6.1, $H \oplus \vec{f}$ computes a solution to $\vec{f}$. $\square$

The increasing polarized Ramsey's theorem has been introduced by Dzhafarov and Hirst [18] to find new principles between stable Ramsey's theorem for pairs and Ramsey's theorem for pairs. We prove that the relativized Ramsey-type weak König's lemma and the increasing polarized Ramsey's theorem are equivalent over RCA_0.

Definition 6.7 (Relativized Ramsey-type weak König's lemma) Given an infinite set of strings $S \subseteq 2^{<\mathbb{N}}$, let T_S denote the downward closure of S, that is, $T_S = \{\tau \in 2^{<\mathbb{N}} : (\exists \sigma \in S)[\tau \preceq \sigma]\}$. 2-$\mathsf{RWKL}$ is the statement "For every set of strings S, there is an infinite set which is homogeneous for T_S".

Note that the statement 2-RWKL slightly differs from a relativized variant of RWKL where the tree would have a Δ^0_2 presentation. However, those two formulations are equivalent over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ (see Flood [21]), and therefore over RCA_0 since $\mathsf{B}\Sigma^0_2$ is a consequence of both statements.

Definition 6.8 (Increasing polarized Ramsey's theorem) A set *increasing p-homogeneous* for $f : [\mathbb{N}]^n \to k$ is a sequence $\langle H_1, \dots, H_n \rangle$ of infinite sets such that for some color $c < k$, $f(x_1, \dots, x_n) = c$ for every increasing tuple $\langle x_1, \dots, x_n \rangle \in H_1 \times \dots \times H_n$. IPT^n_k is the statement "Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite increasing p-homogeneous set".

Theorem 6.9 $\mathsf{RCA}_0 \vdash \mathsf{IPT}^2_2 \leftrightarrow$ 2-RWKL

Proof. $\mathsf{IPT}^2_2 \to$ 2-RWKL: Let $S = \{\sigma_0, \sigma_1, \dots\}$ be an infinite set of strings such that $|\sigma_i| = i$ for each i. Define the coloring $f : [\mathbb{N}]^2 \to 2$ for each $x < y$ by $f(x, y) = \sigma_y(x)$. By IPT^2_2, let $\langle H_1, H_2 \rangle$ be an infinite set increasing p-homogeneous for f with some color c. We claim that H_1 is homogeneous for T_S with color c. We will prove that the set $I = \{\sigma \in T_S : H_1 \text{ is homogeneous for } \sigma\}$ is infinite. For each $y \in \mathbb{N}$, let τ_y be the string of length y defined by $\tau_y(x) = f(x, y)$ for each $x < y$. By definition of f, $\tau_y \in S$ for each $y \in \mathbb{N}$. By definition of $\langle H_1, H_2 \rangle$, $\tau_y(x) = c$ for each $x \in H_1$ and $y \in H_2$. Therefore, H_1 is homogeneous for τ_y with color c for each $y \in H_2$. As $\{\tau_y : y \in H_2\} \subseteq I$, the set I is infinite and therefore H_1 is homogeneous for T_S with color c.

2-$\mathsf{RWKL} \to \mathsf{IPT}^2_2$: Let $f : [\mathbb{N}]^2 \to 2$ be a coloring. For each y, let σ_y be the string of length y such that $\sigma_y(x) = f(x, y)$ for each $x < y$, and let $S = \{\sigma_i : i \in \mathbb{N}\}$. By 2-$\mathsf{RWKL}$, let H be an infinite set homogeneous for T_S with some color c. Define $\langle H_1, H_2 \rangle$ by stages as follows. At stage 0, $H_{1,0} = H_{2,0} = \emptyset$. Suppose that at stage s, $|H_{1,s}| = |H_{2,s}| = s$, $H_{1,s} \subseteq H$ and $\langle H_{1,s}, H_{2,s} \rangle$ is a finite set increasing p-homogeneous for f with color c. Take some $x \in H$ such that $x > max(H_{1,s}, H_{2,s})$ and set $H_{1,s+1} = H_{1,s} \cup \{x\}$. By definition of H, there exists a string $\tau \prec \sigma_y$ for some $y > x$, such that $|\tau| > x$ and H is homogeneous for τ with color c. Set $H_{2,s+1} = H_{2,s} \cup \{y\}$. We now check that the finite set $\langle H_{1,s+1}, H_{2,s+1}\rangle$ is an increasing p-homogeneous for f with color c. By induction hypothesis, we need only to check that $f(z, y) = c$ for every $z \in H_{1,s+1}$. By definition of homogeneity and as $H_{1,s+1} \subset H$, $\sigma_y(z) = c$ for every $y \in H_{1,s+1}$. By definition of σ_y, $f(z, y) = c$ for every $z \in H_{1,s+1}$. This finishes the proof. $\square$

Acknowledgements. The author is thankful to his PhD advisor Laurent Bienvenu for interesting comments and discussions. The author is funded by the John Templeton Foundation ('Structure and Randomness in the Theory of Computation' project). The opinions expressed in this publication are those of the author(s) and do not necessarily reflect the views of the John Templeton Foundation.

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