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Abstract. No natural principle is currently known to be strictly between the arithmetic comprehension axiom (ACA_0) and Ramsey's theorem for pairs (RT^2_2) in reverse mathematics. The tree theorem for pairs (TT^2_2) is however a good candidate. The tree theorem states that for every finite coloring over tuples of comparable nodes in the full binary tree, there is a monochromatic subtree isomorphic to the full tree. The principle TT^2_2 is known to lie between ACA_0 and RT^2_2 over RCA_0, but its exact strength remains open. In this paper, we prove that RT^2_2 together with weak König's lemma (WKL_0) does not imply TT^2_2, thereby answering a question of Montálban. This separation is a case in point of the method of Lerman, Solomon and Towsner for designing a computability-theoretic property which discriminates between two statements in reverse mathematics. We therefore put the emphasis on the different steps leading to this separation in order to serve as a tutorial for separating principles in reverse mathematics.

1. Introduction

"Every sufficiently large collection of objects has an arbitrarily large sub-collection whose objects satisfy some structural properties". This general statement reflects the main idea of Ramsey's theory. This theory has connections with many areas of mathematics, namely, combinatorics, model theory or set theory. One of the most well-known statements is Ramsey's theorem, stating that for every coloring of the k-tuples of integers in finitely many colors, there is an infinite monochromatic subset. In this paper, we are interested in the tree theorem for pairs, a strengthening of Ramsey's theorem for pairs stating that for every finite coloring over pairs of comparable nodes in the full binary tree, there is a monochromatic subtree isomorphic to the full tree. Our main theorem states that the tree theorem for pairs is strictly stronger than Ramsey's theorem for pairs in the sense of *reverse mathematics*.

Reverse mathematics is a mathematical program whose goal is to classify theorems in terms of their provability strength. It uses the framework of subsystems of second-order arithmetic, with the base theory RCA_0 (recursive comprehension axiom). RCA_0 is composed of the basic first-order Peano axioms, together with Δ^0_1-comprehension and Σ^0_1-induction schemes. RCA_0 is usually thought of as capturing *computational mathematics*. This program led to two important observations: First, most "ordinary" (i.e. non set-theoreric) theorems require only very weak set existence axioms. Second, many of those theorems are actually *equivalent* to one of five main subsystems over RCA_0, known as the "Big Five".

Ramsey's theory, among others, provides a large class of theorems escaping this phenomenon. Indeed, consequences of Ramsey's theorem for pairs usually belong to their own subsystem and their study is still an active research subject within reverse mathematics. This article focuses on Ramseyan principles below ACA_0, the arithmetic comprehension axiom. See [14] for a good introduction to reverse mathematics.

1.1. Ramsey's theorem

The strength of Ramsey-type statements is notoriously hard to tackle in the setting of reverse mathematics. The separation of Ramsey's theorem for pairs (RT^2_2) from the arithmetic comprehension axiom (ACA_0) was a long-standing open problem, until Seetapun solved it [27] using the notion of cone avoidance.

Definition 1.1 (Ramsey's theorem) A subset H of ω is *homogeneous* for a coloring $f : [\omega]^k \to n$ (or *f-homogeneous*) if each k-tuples over H are given the same color by f. RT^k_n is the statement "Every coloring $f : [\omega]^k \to n$ has an infinite f-homogeneous set".

Simpson [29, Theorem III.7.6] proved that whenever $k \geq 3$ and $n \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^k_n \leftrightarrow \mathsf{ACA}_0$. Ramsey's theorem for pairs is probably the most famous example of statement escaping the Big Five. Seetapun [27] proved that RT^2_2 is strictly weaker than ACA_0 over RCA_0. Because of the complexity of the related separations, RT^2_2 received a particular attention from the reverse mathematics community [2, 27, 16]. Cholak, Jockusch and Slaman [2] and Liu [18] proved that RT^2_2 is incomparable with weak König's lemma. Dorais, Dzhafarov, Hirst, Mileti and Shafer [6], Dzhafarov [7], Hirschfeldt and Jockusch [15] and the author [25] studied the computational strength of Ramsey's theorem according to the number of colors, when fixing the number of applications of the principle.

1.2. The tree theorem

There is no natural principle currently known to be strictly between ACA_0 and RT^2_2. The tree theorem for pairs is however a good candidate. The tree theorem is a strengthening of Ramsey's theorem in which we do not consider colorings over tuples of integers, but colorings over tuples of nodes over a binary tree. Ramsey's theorem can be recovered from the tree theorem by identifying all nodes at every given level of the tree.

Given a set of binary strings $S \subseteq 2^{<\omega}$, we denote by $[S]^n$ the collection of *linearly ordered* subsets of S of size n, that is, n-sets of strings $\{\sigma_0, \dots, \sigma_{n-1}\} \subseteq S$ such that $\sigma_i \prec \sigma_{i+1}$ for each $i < n - 1$.

Definition 1.2 (Tree theorem) A subtree $S \subseteq 2^{<\omega}$ is *order isomorphic* to $2^{<\omega}$ (written $S \cong 2^{<\omega}$) if there is a bijection $g : 2^{<\omega} \to S$ such that for all $\sigma, \tau \in 2^{<\omega}$, $\sigma \preceq \tau$ if and only if $g(\sigma) \preceq g(\tau)$. Given a coloring $f : [2^{<\omega}]^n \to k$, a tree S is f-homogeneous if $S \cong 2^{<\omega}$ and $f \restriction [S]^n$ is monochromatic. TT^n_k is the statement "Every coloring $f : [2^{<\omega}]^n \to k$ has an f-homogeneous tree."

Note that if $S \cong 2^{<\omega}$, witnessed by the bijection $g : 2^{<\omega} \to S$, then S is g-computable. Therefore we can consider that TT^n states the existence of the bijection g instead of the pair $\langle S, g \rangle$. The tree theorem was first analyzed by McNicholl [19] and by Chubb, Hirst, and McNicholl [4]. They proved that TT^2_2 lies between ACA_0 and RT^2_2 over RCA_0, and left open whether any of the implications is strict. Further work was done by Corduan, Groszek, and Mileti [5]. Dzhafarov, Hirst and Lakins [8] studied stability notions for the tree theorem and introduced a polarized variant. Montálban [21] asked whether RT^2_2 implies TT^2_2 over RCA_0. We give a negative answer by proving the following stronger theorem, where WKL_0 stands for weak König's lemma.

Theorem 1.3 (Main result) $\mathsf{RT}^2_2 \wedge \mathsf{WKL}_0$ does not imply TT^2_2 over RCA_0.

The separation builds on the forcing method introduced by Lerman, Solomon and Towsner [17], and enhanced by the author [23], for designing a computability-theoretic property which discriminates between two statements in reverse mathematics. The construction being quite complex, we present the proof step by step, hoping that our exposition can serve as a tutorial for separating principles in reverse mathematics.

1.3. Separating principles in reverse mathematics

An *ω-structure* is a structure $\mathcal{M} = (\omega, \mathcal{S}, +, \cdot, <)$ where ω is the set of standard integers, $+$, $\cdot$ and $<$ are the standard operations over integers and $\mathcal{S}$ is a set of reals such that $\mathcal{M}$ satisfies the axioms of RCA_0. Friedman [12] characterized the second-order parts $\mathcal{S}$ of ω-structures as those forming a *Turing ideal*, that is, a set of reals closed under Turing join and downward-closed under the Turing reduction. Given two principles P and Q, proving that P does not imply Q over RCA_0 usually consists in constructing a Turing ideal $\mathcal{I}$ in which P holds, but not Q.

Many theorems in reverse mathematics are Π^1_2 statements, i.e., of the form $(\forall X)(\exists Y)\Phi(X,Y)$ where Φ is an arithmetic formula. They can be considered as *problems* which usually come with a natural class of *instances*. Given an instance X, a set Y such that $\Phi(X,Y)$ holds is called a *solution* to X. For example, an instance of RT^2_2 is a coloring $f : [\omega]^2 \to 2$, and a solution to f is an infinite f-homogeneous set. In this setting, the construction of an ω-model of P which is not a model of Q consists in creating a Turing ideal $\mathcal{I}$ together with a fixed Q-instance $I \in \mathcal{I}$, such that every P-instance $J \in \mathcal{I}$ has a solution in $\mathcal{I}$, whereas I contains no solution in $\mathcal{I}$. Building a Turing ideal is usually achieved via the following technique.

1. Choose a particular Q-instance I admitting no I-computable solution.
2. Start with the Turing ideal $\mathcal{I}_0 = \{Z : Z \leq_T I\}$.
3. Given a Turing ideal $\mathcal{I}_n$ containing no solution to I, take any P-instance $J \in \mathcal{I}_n$ having no solution in $\mathcal{I}_n$ and add a solution Y to J. The closure by Turing reducibility and join gives $\mathcal{I}_{n+1}$.
4. Repeat step 3 to obtain a Turing ideal $\mathcal{I} = \bigcup_n \mathcal{I}_n$ such that every P-instance in $\mathcal{I}$ admits a solution in $\mathcal{I}$.

The difficulty of such a construction is to avoid adding a solution to the instance I in $\mathcal{I}_{n+1}$ during step 3. One needs to ensure that every P-instance in $\mathcal{I}_n$ admits a solution Y such that $Y \oplus C$ *avoids* computing a solution to I for each $C \in \mathcal{I}_n$.

Assuming that $\mathcal{I}_n$ does not contain a solution to I is sometimes not sufficient to ensure the existence of a solution Y to the next P-instance such that the ideal closure of $\mathcal{I}_n \cup \{Y\}$ does not contain such a solution as well. A core step of the separation of P from Q consists in designing the computability-theoretic property that we will propagate from $\mathcal{I}_n$ to $\mathcal{I}_{n+1}$ and which will ensure in particular that I has no solution in $\mathcal{I}_{n+1}$. This property strongly depends on the nature of the principles P and Q.

Lerman, Solomon and Towsner [17] introduced a general technique for designing such a property. Their framework has been successfuly reused to separate various principles in reverse mathematics [11, 22, 24]. Recently, the author [23] refined their technique to make it more lightweight and modular. Once simplified, a separation between two statements P and Q using the framework of Lerman, Solomon and Towsner yields a computability-theoretic property called *fairness property*. This property is closed downward under the Turing reduction and is *preserved* by the statement P, that is, for every *fair* set X and every X-computable P-instance I, there is a solution Y to I such that $X \oplus Y$ is fair. This property is designed so that it is not preserved by Q, which enables one to build an ω-model of P in which Q does not hold. Note that "fairness property" is a generic appelation for the computability-theoretic property yielded by the construction of Lerman, Solomon and Towsner. Different statements give different fairness properties, such as hyperimmunity [23], CAC-fairness [26] or again TT-fairness [13].

In this paper, we shall take the case of the separation of Ramsey's theorem for pairs from the tree theorem for pairs to make explicit the different steps leading to the separation of two principles. In particular, we shall focus on the design of the discriminating property.

1.4. **Definitions and notation**

String, sequence. Fix an integer $k \in \omega$. A *string* (over k) of length n is an ordered tuple of integers $a_0, \dots, a_{n-1}$ (such that $a_i < k$ for every $i < n$). The empty string is written ε. A *sequence* (over k) is an infinite listing of integers $a_0, a_1, \dots$ (such that $a_i < k$ for every $i \in \omega$). Given $s \in \omega$, k^s is the set of strings of length s over k and $k^{<s}$ is the set of strings of length $< s$ over k. Similarly, $k^{<\omega}$ is the set of finite strings over k and k^ω is the set of sequences (i.e. infinite strings) over k. Given a string $\sigma \in k^{<\omega}$, we denote by $|\sigma|$ its length. Given two strings $\sigma, \tau \in k^{<\omega}$, σ is a *prefix* of τ (written $\sigma \preceq \tau$) if there exists a string $\rho \in k^{<\omega}$ such that $\sigma\rho = \tau$. Given a sequence X, we write $\sigma \prec X$ if $\sigma = X{\restriction}n$ for some $n \in \omega$, where $X{\restriction}n$ denotes the restriction of X to its first n elements. A *binary string* (resp. binary sequence) is a *string* (resp. sequence) over 2. We may identify a binary sequence with a set of integers by considering that the sequence is its characteristic function.

Tree, path. A tree $T \subseteq k^{<\omega}$ is a set downward-closed under the prefix relation. A *binary* tree is a tree $T \subseteq 2^{<\omega}$. A sequence $P \in \omega^\omega$ is a *path* though T if for every $\sigma \prec P$, $\sigma \in T$. A string $\sigma \in k^{<\omega}$ is a *stem* of a tree T if every $\tau \in T$ is comparable with σ. Given a tree T and a string $\sigma \in T$, we denote by $T^{[\sigma]}$ the subtree $\{\tau \in T : \tau \preceq \sigma \vee \tau \succeq \sigma\}$.

Sets, partitions. Given two sets A and B, we denote by $A < B$ the formula $(\forall x \in A)(\forall y \in B)[x < y]$ and by $A \subseteq^* B$ the formula $(\forall^\infty x \in A)[x \in B]$, meaning that A is included in B *up to finitely many elements*. Given a set X and some integer k, a *k-cover of X* is a k-uple $A_0, \dots, A_{k-1}$ such that $A_0 \cup \dots \cup A_{k-1} = X$. We may simply say *$k$-cover* when the set X is unambiguous. A *k-partition* is a k-cover whose sets are pairwise disjoint. A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition (F_1, X_1) *extends* (F, X) (written $(F_1, X_1) \leq (F, X)$) if $F \subseteq F_1$, $X_1 \subseteq X$ and $F_1 \setminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \setminus F \subseteq X$.

2. Partitioning trees and strong reducibility

In order to get progressively into the framework used to separate Ramsey's theorem for pairs from the tree theorem for pairs, we shall first study the singleton version of the considered principles. The fairness property that we shall design during the next section is directly obtained by abstracting and generalizing the diagonalization argument of this section. Ramsey's theorem for singletons is simply the infinite pigeonhole principle, stating that for every finite partition of an infinite set, one of its parts has an infinite subset. Both RT^1_k and TT^1_k are computably true and provable over RCA_0. We shall therefore study non-computable instances of RT^1_k and TT^1_k to see how their combinatorics differ.

As explained in subsection 1.3, a proof of implication from P to Q over RCA_0 may involve multiple applications of P. Therefore, if we want to prove that P does not imply Q over RCA_0, we need to create an instance of Q diagonalizing against successive applications of P. In order to simplify our argument, we shall first describe a one-step diagonalization between a Δ^0_2 instance of TT^1_2 and arbitrary instances of RT^1_k, that is, with no effectiveness restriction. This is the notion of *strong computable non-reducibility*.

Definition 2.1 (Computable reducibility) Fix two Π^1_2 statements P and Q. P is *strongly computably reducible* to a Q (written $\mathsf{P} \leq_{sc} \mathsf{Q}$) if every P-instance I computes a Q-instance J such that every solution to J computes a solution to I.

The remainder of this section will be dedicated to proving that $\mathsf{TT}^1_2 \not\leq_{sc} \mathsf{RT}^1_2$. More precisely, we shall prove the following stronger theorem.

Theorem 2.2 There exists a Δ^0_2 TT^1_2-instance $A_0 \cup A_1 = 2^{<\omega}$ such that for every (non-necessarily computable) RT^1_2-instance $B_0 \cup B_1 = \omega$, there is an infinite set homogeneous for the B's which does not compute a TT^1_2-solution to the A's.

In section 4, we will prove a theorem which implies Theorem 2.2. Therefore we shall focus on the key ideas of the construction rather than on the technical details.

Requirements. Let us first assume that we have constructed our TT^1_2 instance $A_0 \cup A_1 = 2^{<\omega}$. Fix some 2-partition $B_0 \cup B_1 = \omega$. We will construct by forcing an infinite set G such that both $G \cap B_0$ and $G \cap B_1$ are infinite. Either $G \cap B_0$, or $G \cap B_1$ will be taken as our solution to the RT^1_2-instance $B_0 \cup B_1 = \omega$. We only need one solution to the RT^1_2-instance. However, we will be only able to ensure that either $G \cap B_0$, or $G \cap B_1$ will not compute a solution to the TT^1_2-instance $A_0 \cup A_1 = 2^{<\omega}$. Therefore, among $G \cap B_0$ and $G \cap B_1$, only the one which does not compute a solution to the TT^1_2-instance will be the desired solution to our RT^1_2-instance. Here, by "solution to the TT^1_2-instance", we mean an infinite subtree isomorphic to $2^{<\omega}$ which is included in A_0 or A_1.

Let $\Phi_0, \Phi_1, \dots$ be an enumeration of all partial tree functionals isomorphic to $2^{<\omega}$, that is, if $\Phi^X(n)$ halts, then $\Phi^X(n)$ outputs 2^n pairwise incomparable strings representing the nth level

of the tree. We ensure that the following requirements hold for every pair of indices e_0, e_1.

$$\mathcal{Q}_{e_0,e_1} : \qquad \mathcal{R}_{e_0}^{G\cap B_0} \qquad \vee \qquad \mathcal{R}_{e_1}^{G\cap B_1}$$

where $\mathcal{R}_e^H$ is the statement

> Either Φ_e^H is partial, or $\Phi_e^H(n)$ is a set D of 2^n incomparable strings intersecting both A_0 and A_1 for some n.

We call any $\mathcal{Q}_{e_0,e_1}$ a *$\mathcal{Q}$-requirement* and for a given H (being either $G\cap B_0$, or $G\cap B_1$), we call any $\mathcal{R}_e^H$ an *$\mathcal{R}$-requirement for H*. If every $\mathcal{Q}$-requirement is satisfied, then by the usual pairing argument, either every $\mathcal{R}$-requirement is satisfied for $G\cap B_0$, or every $\mathcal{R}$-requirement is satisfied for $G\cap B_1$. Call H a set among $G\cap B_0$ and $G\cap B_1$ for which every $\mathcal{R}$-requirement is satisfied. We claim that H does not compute a solution to the TT^1_2-instance $A_0\cup A_1 = 2^{<\omega}$. Suppose for the sake of contradiction that H computes a tree $S\cong 2^{<\omega}$ using some procedure Φ_e. By the requirement $\mathcal{R}_e^H$, S intersects both A_0 and A_1, and therefore S is not a TT^1_2-solution to the A's.

Forcing. The forcing conditions are Mathias conditions, that is, an ordered pair (F, X), where F is a finite set of integers, X is an infinite set belonging to some fixed Scott set $\mathcal{S}$, and such that $max(F) < min(X)$. A *Scott set* Turing ideal satisfying weak König's lemma. By Simpson [29, Theorem VIII.2.17], we can choose $\mathcal{S}$ so that $\mathcal{S} = \{X_i : C = \bigoplus_i X_i\}$ for some low set C. This precision will be useful during the construction of the TT^1_2-instance. We furthermore assume that C does not compute a TT^1_2-solution to the A's, and therefore that there is no C-computable infinite set homogeneous for the B's, otherwise we are done.

The following lemma ensures that we can force both $G\cap B_0$ and $G\cap B_1$ to be infinite, assuming that the B's have no infinite C-computable homogeneous set.

Lemma 2.3 Given a condition $c = (F, X)$ and some side $i < 2$, there is an extension $d = (E, Y)$ such that $|E\cap B_i| > |F\cap B_i|$.

Proof. If $X\cap B_i = \emptyset$ then X is an infinite C-computable subset of B_{1-i}, contradicting our assumption. So there is some $x\in X\cap B_i$. Take $d = (F\cup\{x\}, X\setminus[0,x])$ as the desired extension. □

The next step consists in forcing the $\mathcal{Q}$-requirements to be satisfied. A condition c *forces* a requirement $\mathcal{Q}_{e_0,e_1}$ if $\mathcal{Q}_{e_0,e_1}$ holds for every set G satisfying c. Of course, we cannot force the $\mathcal{Q}$-requirements for any TT^1_2-instance $A_0\cup A_1 = 2^{<\omega}$ since some of them admit computable solutions. We must therefore choose our TT^1_2-instance carefully. For now, simply assume that we managed to build a TT^1_2-instance $A_0\cup A_1 = 2^{<\omega}$ satisfying the following property (P). We will detail its construction later.

> (P) Given a condition $c = (F, X)$ and some indices e_0, e_1, there is an extension d of c forcing $\mathcal{Q}_{e_0,e_1}$.

Assuming that the property (P) holds, we now show how to build our infinite set G from it. After that, we will construct a TT^1_2-instance $A_0\cup A_1 = 2^{<\omega}$ so that the property (P) is satisfied.

Construction. Thanks to Lemma 2.3 and the property (P), we can define an infinite, decreasing sequence of conditions $(\emptyset,\omega)\geq c_0\geq c_1\ldots$ such that for each $s\in\omega$

(i) $|F_s\cap B_0|\geq s$ and $|F_s\cap B_1|\geq s$
(ii) c_s forces $\mathcal{Q}_{e_0,e_1}$ if $s = \langle e_0, e_1\rangle$

where $c_s = (F_s, X_s)$. The set $G = \bigcup_s F_s$ is such that both $G\cap B_0$ and $G\cap B_1$ are infinite by (i), and either $G\cap B_0$ or $G\cap B_1$ does not compute a TT^1_2-solution to the A's by (ii). The set among $G\cap B_0$ and $G\cap B_1$ which does not compute a TT^1_2-solution to the A's is our desired RT^1_2-solution to the B's. We now need to satisfy the property (P).

Satisfying the property (P). Given a condition, the extension stated in the property (P) cannot be ensured for an arbitrary TT^1_2-instance $A_0\cup A_1 = 2^{<\omega}$. We must design the A's so that the property (P) holds. To do so, we will apply the ideas developped by Lerman, Solomon and Towsner [17]. We can see the construction of the set G as a game. The opponent is the TT^1_2-instance which will try everything, not to be diagonalized against. However, the opponent is

fair, in the sense that if we have infinitely many occasions to diagonalize against him, then he will let us do it. More precisely, if given a condition $c = (F, X)$ and some indices e_0, e_1, we can find extensions which makes both $\Phi_{e_0}^{G\cap B_0}$ and $\Phi_{e_1}^{G\cap B_1}$ produce arbitrarily large initial segments of the output, then one of those outputs will intersect both A_0 and A_1. In this case, we will have succeeded to satisfy (P) for the condition c by producing an extension d forcing $\Phi_{e_i}^{G\cap B_i}$ to intersect both A_0 and A_1 for some $i < 2$, and therefore forcing $\mathcal{Q}_{e_0,e_1}$. In the other case, we will also have vacuously succeeded since we will not be able to find extensions making both $\Phi_{e_0}^{G\cap B_0}$ and $\Phi_{e_1}^{G\cap B_1}$ produce arbitrarily large initial segments of the output tree, and therefore c is already a condition forcing one $\Phi_{e_0}^{G\cap B_0}$ and $\Phi_{e_1}^{G\cap B_1}$ to be partial, so forcing $\mathcal{Q}_{e_0,e_1}$.

We now describe how to construct a fair TT^1_2-instance. The construction of the A's will be $\Delta^{0,C}_2$, hence Δ^0_2 since C is low. The access to the oracle C enables us to *code* the conditions $c = (F, X)$ into finite objects, namely, pairs (F, i) so that $C = \bigoplus_i X_i$ and $X = X_i$, and to enumerate them C-computably. More precisely, we will enumerate all 6-tuples $\langle F, i, E_0, E_1, e_0, e_1\rangle$, where (F, X_i) is a *preconditions*, that is, a condition where we drop the constraint that X_i is infinite since it requires too much computational power to know it, $E_0 \sqcup E_1 = F$ represents a *guess* of the sets $F \cap B_0$ and $F \cap B_1$, and e_0, e_1 denote the $\mathcal{Q}_{e_0,e_1}$ we want to force. In particular, among those 6-tuples enumerated, we will enumerate $\langle F, i, F \cap B_0, F \cap B_1, e_0, e_1\rangle$ for all the true conditions (F, X_i).

The construction of the A's is done by stages. At stage s, we have constructed two sets $A_{0,s} \cup A_{1,s} = 2^{<q}$ for some $q \in \omega$. We want to satisfy the property (P) given a 6-tuple $\langle F, i, E_0, E_1, e_0, e_1\rangle$, that is, given a precondition $c = (F, X)$, a guess of $F\cap B_0$ and $F\cap B_1$, and a pair of indices e_0, e_1. If any of $\Phi_{e_0}^{E_0}(2)$ and $\Phi_{e_1}^{E_1}(2)$ is not defined, do nothing and go to the next stage. We can restrict ourselves without loss of generality to preconditions such that both $\Phi_{e_0}^{E_0}(2)$ and $\Phi_{e_1}^{E_1}(2)$ are defined. Indeed, if in the property (P), the condition c has no such extension, then c already forces either $\Phi_{e_0}^{G\cap B_0}$ or $\Phi_{e_1}^{G\cap B_1}$ to be partial and therefore vacuously forces $\mathcal{Q}_{e_0,e_1}$. The choice of "2" as input seems arbitrary. It has not been picked randomly and this choice will be justified in the next paragraph.

Let D_0 and D_1 be the 4-sets of pairwise incomparable strings outputted by $\Phi_{e_0}^{E_0}(2)$ and $\Phi_{e_1}^{E_1}(2)$, respectively. Altough the strings are pairwise incomparable *within* D_0 or D_1, there may be two comparable strings in $D_0 \cup D_1$. However, by a simple combinatorial argument, we may always find two strings $\sigma_0, \tau_0 \in D_0$ and $\sigma_1, \tau_1 \in D_1$ such that $\sigma_0, \tau_0, \sigma_1$ and τ_1 are pairwise incomparable (see Lemma 3.2). Here, we use the fact that on input 2, the sets have cardinality 4, which is enough to apply Lemma 3.2. We are now ready to ask the main question.

"Is it true that for every 2-partition $Z_0 \cup Z_1 = X$, there is some side $i < 2$ and some set $G \subseteq Z_i$ such that $\Phi_{e_i}^{E_i\cup G}(q)$ halts?"

Note that the question looks $\Pi^{1,X}_2$, but is in fact $\Sigma^{0,X}_1$ by a compactness argument. It is therefore C'-decidable since $X \in \mathcal{S}$ and so can be uniformly decided during the construction. We have two cases.

Case 1: The answer is negative. In this case, the $\Pi^{0,X}_1$ class $\mathcal{C}$ of all sets $Z_0 \oplus Z_1$ such that $Z_0 \cup Z_1 = X$ and for every $i < 2$ and every set $G \subseteq Z_i$, $\Phi_{e_i}^{E_i\cup G}(q) \uparrow$ is non-empty. In this case, we do nothing and claim that the property (P) holds for c. Indeed, since $\mathcal{S}$ is a Scott set containing X, there is some $Z_0 \oplus Z_1 \in \mathcal{C} \cap \mathcal{S}$ such that $Z_0 \cup Z_1 = X$. As X is infinite, there is some $i < 2$ such that Z_i is infinite. In this case, if $E_0 = F \cap B_0$ and $E_1 = F \cap B_1$, $d = (F, Z_i)$ is an extension forcing $\Phi_{e_i}^{(G\cap B_i)}(q) \uparrow$ and therefore forcing $\mathcal{Q}_{e_0,e_1}$. Note that this extension cannot be found C'-computably since it requires to decide which of Z_0 and Z_1 is infinite. However, we do not need to uniformly provide this extension. The property (P) simply states the *existence* of such an extension.

Case 2: The answer is positive. Given a string $\sigma \in 2^{<\omega}$, let $S_\sigma = \{\tau \succeq \sigma\}$. Since the Φ's are tree functionals and $\Phi_{e_i}^{E_i\cup G}(2)$ outputs (among others) the strings σ_i and τ_i, whenever $\Phi_{e_i}^{E_i\cup G}(q)$ halts, it outputs a finite set D of size 2^q intersecting both S_{σ_i} and S_{τ_i}. Therefore, by compactness, there are finite sets $U_0 \subseteq S_{\sigma_0}$, $V_0 \subseteq S_{\tau_0}$, $U_1 \subseteq S_{\sigma_1}$ and $V_1 \subseteq S_{\tau_1}$ such that for every 2-partition $Z_0 \cup Z_1 = X$, there is some side $i < 2$ and some set $G \subseteq Z_i$ such that $\Phi_{e_i}^{E_i\cup G}(q)$

intersects both U_i and V_i. In particular, whenever $E_0 = F \cap G_0$, $E_1 = F \cap B_1$, $Z_0 = X \cap B_0$ and $Z_1 = X \cap B_1$, there is some $i < 2$ and some finite set $G \subseteq X \cap B_i$ such that $\Phi_{e_i}^{(F\cap B_i)\cup G}(q)$ intersects both U_i and V_i. Notice that all the strings in U_i and V_i have length at least q and therefore are not yet colored by the A's. Put the U's in $A_{0,s+1}$ and the V's in $A_{1,s+1}$ and complete the coloring so that $A_{0,s+1} \cup A_{1,s+1} = 2^{<r}$ for some $r \geq q$. Then go to the next step. We claim that the property (P) holds for c. Indeed, let $G \subseteq X \cap B_i$ be the finite set witnessing $\Phi_{e_i}^{(F\cap B_i)\cup G}(q) \cap U_i \neq \emptyset$ and $\Phi_{e_i}^{(F\cap B_i)\cup G}(q) \cap V_i \neq \emptyset$. The condition $d = (F \cup G, X \setminus [0, max(E)])$ is an extension forcing $\mathcal{Q}_{e_0,e_1}$ by its ith side. This finishes the construction of the TT^1_2-instance and the proof of Theorem 2.2.

3. The fairness property

In this section, we analyse the one-step separation proof of section 2 in order to extract the core of the argument. Then, we use the framework of Lerman, Solomon and Towsner to design the computability-theoretic property which will enable us to discriminate RT^2_2 from TT^2_2.

The multiple-step case. We have seen how to diagonalize against one application of RT^1_2. The strength of TT^1_2 comes from the fact that when we build a solution S to some TT^1_2-instance $A_0 \cup A_1 = 2^{<\omega}$, we must provide finite subtrees $S_n \cong 2^{<n}$ for arbitrarily large n. However, as soon as we have outputted S_n, we *commit* to provide arbitrarily large extensions to each leaf of S_n. Since the leaves in S_n are pairwise incomparable, the sets of their extensions are mutually disjoint. During the construction of the TT^1_2-instance, we can pick any pair σ, τ of incomparable leaves in S_n, and put the extensions of σ in A_0 and the extensions of τ in A_1 since they are disjoint.

In the proof of $\mathsf{TT}^1_2 \not\leq_{sc} \mathsf{RT}^1_2$, when we create a solution to some RT^1_2-instance $B_0 \cup B_1 = \omega$, we build two candidate solutions $G \cap B_0$ and $G \cap B_1$ at the same time. For each pair of tree functionals Φ_{e_0} and Φ_{e_1}, we must prevent one of $\Phi_{e_0}^{G\cap B_0}$ and $\Phi_{e_1}^{G\cap B_1}$ from being a TT^1_2-solution to the A's. However, the finite subtrees S_0 and S_1 outputted respectively by the left side and the right side may have comparable leaves. We cannot take any 2 leaves of S_0 and 2 leaves of S_1 to obtain 4 pairwise incomparable strings. Thankfully, if S_0 and S_1 contain enough leaves (4 is enough), we can find such strings.

If we try to diagonalize against two applications of RT^1_2, below each side $G \cap B_0$ and $G \cap B_1$ of the first RT^1_2-instance, we will have again two sides corresponding to the second RT^1_2-instance. We will have then to diagonalize against four candidate subtrees S_0, S_1, S_2 and S_3. We need therefore to wait until the subtrees have enough leaves, so that we can find 8 pairwise incomparable leaves $\sigma_0, \tau_0, \ldots, \sigma_3, \tau_3$ such that $\sigma_i, \tau_i \in S_i$ for each $i < 4$.

In the general case, we will then have to diagonalize against an arbitrarily large number of subtrees, and want to ensure that if they contain enough leaves, we can find two leaves in each, such that they form a set of pairwise incomparable strings. This leads to the notion of disjoint matrix.

Definition 3.1 (Disjoint matrix) An m-by-n *matrix* M is a rectangular array of strings $\sigma_{i,j} \in 2^{<\omega}$ such that $i < m$ and $j < n$. The ith *row* $M(i)$ of the matrix M is the n-tuple of strings $\sigma_{i,0}, \ldots, \sigma_{i,n-1}$. An m-by-n matrix M is *disjoint* if for each row $i < m$, the strings $\sigma_{i,0}, \ldots, \sigma_{i,n-1}$ are pairwise incomparable.

The following combinatorial lemma gives an explicit bound on the number of leaves we require on each subtree to obtain our desired sequence of pairwise incomparable strings.

Lemma 3.2 For every m-by-$2m$ disjoint matrix M, there are pairwise incomparable strings $\sigma_0, \tau_0, \ldots, \sigma_{m-1}, \tau_{m-1}$ such that $\sigma_i, \tau_i \in M(i)$ for every $i < m$.

Proof. Consider the following greedy algorithm. At each stage, we maintain a set P of *pending rows* which is initially the whole matrix M. Among those rows, some strings are flagged as *invalid*. Initially, all the strings are valid. Pick a string ρ of maximal length among all the valid strings of all the pending rows. Let $M(i)$ be a pending row to which ρ belongs. If we have

already chosen the value of σ_i, set $\tau_i = \rho$ and remove $M(i)$ from the pending rows. Otherwise, set $\sigma_i = \rho$. In any case, flag every prefix of ρ from any row of M as invalid, and go to the next step.

Notice that at any step, we flag as invalid at most one string from each row of M since the strings in each row are pairwise incomparable. Moreover, since we want to construct a sequence of $2m$ pairwise incomparable strings, and at each step we add one string to this sequence, there are at most $2m$ steps. The algorithm gets stuck at some points only if all pending rows contain only invalid strings, which cannot happen since each row contains at least $2m$ strings.

We claim that the chosen strings are pairwise incomparable. Indeed, when at some stage, we pick a string ρ, it is of shorter length than any string we have picked so far, and cannot be a prefix of any of them since each time we pick a string, we flag all its prefixes as invalid in the matrix. □

Abstracting the requirements. The first feature of the framework of Lerman, Solomon and Towsner that we already exploited is the "fairness" of the TT^1_2-instance which allows each RT^1_2-instance to diagonalize it as soon as the RT^1_2-instance gives him enough occasions to do it. We will now use the second aspect of this framework which consists in getting rid of the complexity of the requirements by replacing them with arbitrary computable predicates (or blackboxes).

Indeed, consider the case of two successive applications of RT^1_2. Say that the first instance is $B_0 \cup B_1 = \omega$, and the second $C_0 \cup C_1 = \omega$. We need to design the TT^1_2-instance $A_0 \cup A_1 = 2^{<\omega}$ so that there is an infinite set $G \cap B_i$ for some $i < 2$ and an infinite set $H \cap B_j$ for some $j < 2$ such that $(G \cap B_i) \oplus (H \cap B_j)$ does not compute a solution to the A's. While constructing the A's, we enumerate two levels of conditions. We first enumerate the conditions $c = (F, X)$ used for constructing the set G, but we also enumerate the conditions $c_0 = (F_0, X_0)$ and $c_1 = (F_1, X_1)$ such that c_i is used to construct a solution H to the second RT^1_2-instance $C_0 \cup C_1 = \omega$ below $G \cap B_i$. The question that the TT^1_2-instance asks during its construction becomes

"For every 2-partition $Z_0 \cup Z_1 = X$, is there some side $i < 2$ and some set $G \subseteq Z_i$ such that for every 2-partition $W_0 \cup W_1 = X_i$, there is some side $j < 2$ and some set $H \subseteq W_j$ such that $\Phi_{e_{i,j}}^{((F \cap B_i) \cup G) \oplus ((F_i \cap C_j) \cup H)}(q)$ halts?"

While staying Σ^0_1 (with parameters), the question becomes arbitrarily complicated to formulate. Moreover, looking at the shape of the question, we see that the first iteration can box any Σ^0_1 question asked about the second iteration. We can therefore abstract the question and make the fairness property independent of the specificities of the forcing notion used to solve the RT^1_2-instances. See [23] for detailed explanations about this abstraction process.

Definition 3.3 (Formula, valuation) An *m-by-n formula* is a formula φ with distinguished set variables $U_{i,j}$ for each $i < m$ and $j < n$. Given an m-by-n matrix $M = \{\sigma_{i,j} : i < m, j < n\}$, an *$M$-valuation* V is a tuple of finite sets $A_{i,j} \subseteq \{\tau \in 2^{<\omega} : \tau \succeq \sigma_{i,j}\}$ for each $i < m$ and $j < n$. The valuation V *satisfies* φ if $\varphi(A_{i,j} : i < m, j < n)$ holds. We write $\varphi(V)$ for $\varphi(A_{i,j} : i < m, j < n)$.

Given some valuation $V = (A_{i,j} : i < m, j < n)$ and some integer s, we write $V > s$ to say that for every $\tau \in A_{i,j}$, $|\tau| > s$. Moreover, we denote by $V(i)$ the n-tuple $A_{i,0}, \dots, A_{i,n-1}$. Following the terminology of [17], we define the notion of essentiality for a formula (an abstract requirement), which corresponds to the idea that there is room for diagonalization since the formula is satisfied for arbitrarily far valuations.

Definition 3.4 (Essential formula) An m-by-n formula φ is *essential* in an m-by-n matrix M if for every $s \in \omega$, there is an M-valuation $V > s$ such that $\varphi(V)$ holds.

The notion of fairness is defined accordingly. If some formula is essential, that is, gives enough room for diagonalization, then there is an actual valuation which will diagonalize against the TT^1_2-instance.

Definition 3.5 (Fairness) Fix two sets $A_0, A_1 \subseteq 2^{<\omega}$. Given an m-by-n disjoint matrix M, an M-valuation V *diagonalizes* against $A_0, A_1 \subseteq 2^{<\omega}$ if for every $i < m$, there is some $L, R \in V(i)$

such that $L \subseteq A_0$ and $R \subseteq A_1$. A set X is *n-fair* for A_0, A_1 if for every m and every $\Sigma^{0,X}_1$ m-by-$2^n m$ formula φ essential in some disjoint matrix M, there is an M-valuation V diagonalizing against A_0, A_1 such that $\varphi(V)$ holds. A set X is *fair* for A_0, A_1 if it is n-fair for A_0, A_1 for some $n \geq 1$.

Of course, if $Y \leq_T X$, then every $\Sigma^{0,Y}_1$ formula is $\Sigma^{0,X}_1$. As an immediate consequence, if X is n-fair for some A_0, A_1 and $Y \leq_T X$, then Y is n-fair for A_0, A_1. Moreover, if X is n-fair for A_0, A_1 and $p > n$, X is also p-fair for A_0, A_1 as witnessed by cropping the rows.

Definition 3.6 (Fairness preservation) Fix a Π^1_2 statement P.

1. P admits *fairness (resp. n-fairness) preservation* if for all sets $A_0, A_1 \subseteq 2^{<\omega}$, every set C which is fair (resp. n-fair) for A_0, A_1 and every C-computable P-instance X, there is a solution Y to X such that $Y \oplus C$ is fair (resp. n-fair) for A_0, A_1.
2. P admits strong *fairness (resp. n-fairness) preservation* if for all sets $A_0, A_1 \subseteq 2^{<\omega}$, every set C which is fair (resp. n-fair) for A_0, A_1 and every P-instance X, there is a solution Y to X such that $Y \oplus C$ is fair (resp. n-fair) for A_0, A_1.

Note that a principle P may admit fairness preservation without preserving n-fairness for any fixed n, as this is the case with RT^2_2 (see Theorem 4.11 and Theorem 4.13). On the other hand, if P admits n-fairness preservation for every n, then it admits fairness preservation. The notion of fairness preservation has been designed so that it is closed under the implication over RCA_0.

Lemma 3.7 If P admits fairness preservation but not Q, then P does not imply Q over RCA_0.

Proof. Since Q does not admit fairness preservation, there is a set C which is n-fair for some $A_0, A_1 \subseteq 2^{<\omega}$ and a C-computable Q-instance J such that for every solution Y to J, $C \oplus Y$ is not fair for A_0, A_1. We build an infinite sequence of sets $X_0, X_1, \dots$ starting with $X_0 = C$ and such that for every $s \in \omega$,

(i) X_{s+1} is a solution to the P-instance $I_s^{X_0 \oplus \cdots \oplus X_s}$
(ii) $X_0 \oplus \cdots \oplus X_{s+1}$ is fair for A_0, A_1

where $I_0, I_1, \dots$ is a (non-computable) enumeration of all P-instance functionals. Let $\mathcal{M}$ be the ω-model whose second-order part is the Turing ideal

$$\mathcal{I} = \{Z : (\exists s)[Z \leq_T X_0 \oplus \cdots \oplus X_s]\}$$

By Friedman [12], $\mathcal{M} \models \mathsf{RCA}_0$ since $\mathcal{I}$ is a Turing ideal. By (i), $\mathcal{M} \models \mathsf{P}$. As $J \leq_T C = X_0$, $J \in \mathcal{I}$. However, for every $Z \in \mathcal{I}$, $Z \oplus C$ is fair for A_0, A_1 by downward closure of fairness under the Turing reducibility, so Z is not a solution to the Q-instance J. Therefore $\mathcal{M} \not\models \mathsf{Q}$. □

Now we have introduced the necessary terminology, we create a Δ^0_2 instance of TT^1_2 which will serve as a bootstrap for fairness preservation.

Lemma 3.8 There exists a Δ^0_2 partition $A_0 \cup A_1 = 2^{<\omega}$ such that $\emptyset$ is 1-fair for A_0, A_1.

Proof. The proof is done by a no-injury priority construction. Let $\varphi_0, \varphi_1, \dots$ be a computable enumeration of all m-by-$2m$ Σ^0_1 formulas and $M_0, M_1, \dots$ be an enumeration of all m-by-$2m$ disjoint matrices for every m. We want to satisfy the following requirements for each pair of integers e, k.

$\mathcal{R}_{e,k}$: If φ_e is essential in M_k, then $\varphi_e(V)$ holds for some M_k-valuation V diagonalizing against A_0, A_1.

The requirements are ordered via the standard pairing function $\langle \cdot, \cdot \rangle$. The sets A_0 and A_1 are constructed by a $\emptyset'$-computable list of finite approximations $A_{i,0} \subseteq A_{i,1} \subseteq \dots$ such that all elements added to $A_{i,s+1}$ from $A_{i,s}$ are strictly greater than the maximum of $A_{i,s}$ for each $i < 2$. We then let $A_i = \bigcup_s A_{i,s}$ which will be a Δ^0_2 set. At stage 0, set $A_{0,0} = A_{1,0} = \emptyset$. Suppose that at stage s, we have defined two disjoint finite sets $A_{0,s}$ and $A_{1,s}$ such that

(i) $A_{0,s} \cup A_{1,s} = 2^{<b}$ for some integer $b \geq s$

(ii) $\mathcal{R}_{e',k'}$ is satisfied for every $\langle e', k' \rangle < s$

Let $\mathcal{R}_{e,k}$ be the requirement such that $\langle e, k \rangle = s$. Decide $\emptyset'$-computably whether there is some M_k-valuation $V > b$ such that $\varphi_e(V)$ holds. If so, computably fetch such a V and let d be an upper bound on the length of the strings in V. By Lemma 3.2, there are pairwise incomparable strings $\sigma_0, \tau_0, \dots, \sigma_{m-1}, \tau_{m-1}$ such that $\sigma_i, \tau_i \in M(i)$ for every $i < m$. For each $i < m$, let $A_{i,l}$ and $A_{i,r}$ be the sets in V corresponding to σ_i and τ_i, respectively. Set $A_{0,s+1} = A_{0,s} \bigcup_{i<m} A_{i,l}$ and $A_{1,s+1} = 2^{<d} \setminus A_{0,s+1}$. This way, $A_{0,s+1} \cup A_{1,s+1} = 2^{<d}$. Since the σ's and τ's are pairwise incomparable, the sets $A_{i,l}$ and $A_{i,r}$ are disjoint, so $\bigcup_{i<m} A_{i,r} \subseteq 2^{<d} \setminus A_{0,s+1}$ and the requirement $\mathcal{R}_{e,i}$ is satisfied. If no such M_k-valuation is found, the requirement $\mathcal{R}_{e,k}$ is vacuously satisfied. Set $A_{0,s+1} = A_{0,s} \cup 2^b$ and $A_{1,s+1} = A_{1,s}$. This way, $A_{0,s+1} \cup A_{1,s+1} = 2^{<(b+1)}$. In any case, go to the next stage. This finishes the construction. □

Theorem 3.9 TT^2_2 does not admit fairness preservation.

Proof. Let $A_0 \cup A_1 = 2^{<\omega}$ be the Δ^0_2 partition constructed in Lemma 3.8. By Schoenfield's limit lemma [28], there is a computable function $h : 2^{<\omega} \times \omega \to 2$ such that for each $\sigma \in 2^{<\omega}$, $\lim_s h(\sigma, s)$ exists and $\sigma \in A_{\lim_s h(\sigma,s)}$. Let $f : [2^{<\omega}]^2 \to 2$ be the computable coloring defined by $f(\sigma, \tau) = h(\sigma, |\tau|)$ for each $\sigma \prec \tau \in 2^{<\omega}$. Let $S \cong 2^{<\omega}$ be a TT^2_2-solution to f with witness isomorphism $g : 2^{<\omega} \to S$ and witness color $c < 2$. Note that $S \subseteq A_c$.

Fix any $n \geq 1$. We claim that S is not n-fair for A_0, A_1. For this, we construct a 1-by-2^n $\Sigma^{0,S}_1$ formula and a 1-by-2^n disjoint matrix M such that φ is essential in M, but such that every M-valuation V satisfying φ is included in A_c.

Let $\varphi(U_j : j < 2^n)$ be the 1-by-2^n $\Sigma^{0,S}_1$ formula which holds if for each $j < 2^n$, U_j is a non-empty subset of S. Let $M = (\sigma_j : j < 2^n)$ be the 1-by-2^n disjoint matrix defined for each $j < 2^n$ by $\sigma_j = g(\tau_j)$ where τ_j is the jth string of length n. In other words, σ_j is the jth node at level n in S. For every s, let V_s be the M-valuation defined by $B_j = \{g(\rho)\}$ such that ρ is the least string of length $max(n, s)$ extending τ_j. Notice that $V_s > s$ and $\varphi(V_s)$ holds. Therefore, the formula φ is essential in M. For every M-valuation $V = (B_j : j < 2^n)$ such that $\varphi(V)$ holds, there is no $j < 2^n$ such that $B_j \subseteq A_{1-c}$. Indeed, since $\varphi(V)$ holds, B_j is a non-empty subset of S, which is itself a subset of A_c. Therefore S is not n-fair for A_0, A_1. □

Notice that we actually proved a stronger statement. Dzhafarov, Hirst and Lakins defined in [8] various notions of stability for the tree theorem for pairs. A coloring $f : [2^{<\omega}]^2 \to r$ is *1-stable* if for every $\sigma \in 2^{<\omega}$, there is some threshold t and some color $c < r$ such that $f(\sigma, \tau) = c$ for every $\tau \succ \sigma$ such that $|\tau| \geq t$. In the proof of Theorem 3.9, we showed in fact that TT^2_2 restricted to 1-stable colorings does not admit fairness preservation. In the same paper, Dzhafarov et al. studied an increasing polarized version of the tree theorem for pairs, and proved that its 1-stable restriction coincides with the 1-stable tree theorem for pairs over RCA_0. Therefore the increasing polarized tree theorem for pairs does not admit fairness preservation.

4. Separating principles in reverse mathematics

In this section, we prove fairness preservation for various principles in reverse mathematics, namely, weak König's lemma, cohesiveness and RT^2_2. We prove independently that they admit fairness preservation, and then use the compositional nature of the notion of preservation to deduce that the conjunction of these principles do not imply TT^2_2 over RCA_0.

Definition 4.1 (Weak König's lemma) WKL_0 is the statement "Every infinite binary tree has an infinite path".

Weak König's lemma is one of the "Big Five". It can be thought of as capturing compactness arguments. The question of its relation with RT^2_2 has been a long standing open problem, until Cholak, Jockusch and Slaman [2] and Liu [18] proved that RT^2_2 is incomparable with weak König's lemma. Although the above mentioned results show that compactness is not really necessary in the proof of RT^2_2, WKL_0 preserves many computability-theoretic notions

and is therefore involved in many effective constructions related to RT^2_2. Flood [10] introduced recently a Ramsey-type version of König's lemma (RWKL). This strict weakening of WKL_0 is sufficient in most applications of WKL_0 involved in proofs of RT^2_2. The statement RWKL has been later studied by Bienvenu, Patey and Shafer [1] and by Flood and Towsner [11].

Theorem 4.2 For every $n \geq 1$, WKL_0 admits n-fairness preservation.

Proof. Let C be a set n-fair for some sets $A_0, A_1 \subseteq 2^{<\omega}$, and let $T \subseteq 2^{<\omega}$ be a C-computable infinite binary tree. We construct an infinite decreasing sequence of computable subtrees $T = T_0 \supseteq T_1 \supseteq \dots$ such that for every path P through $\bigcap_s T_s$, $P \oplus C$ is n-fair for A_0, A_1. Note that the intersection $\bigcap_s T_s$ is non-empty since the T's are infinite trees. More precisely, if we interpret s as a tuple $\langle m, \varphi, M\rangle$ where $\varphi(G, U)$ is an m-by-$2^n m$ $\Sigma^{0,C}_1$ formula $\varphi(G, U)$ and M is an m-by-$2^n m$ disjoint matrix M, we want to satisfy the following requirement.

> $\mathcal{R}_s$: For every path P through T_{s+1}, either $\varphi(P, U)$ is not essential in M, or $\varphi(P, V)$ holds for some M-valuation V diagonalizing against A_0, A_1.

Given two M-valuations $V_0 = (B_{i,j} : i < m, j < 2^n m)$ and $V_1 = (D_{i,j} : i < m, j < 2^n m)$, we write $V_0 \subseteq V_1$ to denote the pointwise subset relation, that is, for every $i < m$ and every $j < 2^n m$, $B_{i,j} \subseteq D_{i,j}$. At stage $s = \langle m, \varphi, M\rangle$, given some infinite, computable binary tree T_s, define the m-by-$2^n m$ $\Sigma^{0,C}_1$ formula

$$\psi(U) = (\exists n)(\forall \tau \in T_s \cap 2^n)(\exists \tilde{V} \subseteq U)\varphi(\tau, \tilde{V})$$

We have two cases. In the first case, $\psi(U)$ is not essential in M with some witness t. By compactness, the following set is an infinite C-computable subtree of T_s:

$$T_{s+1} = \{\tau \in T_s : (\text{for every } M\text{-valuation } V > t)\neg\varphi(\tau, V)\}$$

The tree T_{s+1} has been defined so that $\varphi(P, U)$ is not essential in M for every $P \in [T_{s+1}]$.

In the second case, $\psi(U)$ is essential in M. By n-fairness of C for A_0, A_1, there is an M-valuation V diagonalizing against A_0, A_1 such that $\psi(V)$ holds. We claim that for every path $P \in [T_s]$, $\varphi(P, \tilde{V})$ holds for some M-valuation $\tilde{V}$ diagonalizing against A_0, A_1. Fix some path $P \in [T_s]$. Unfolding the definition of $\psi(V)$, there is some n such that $\varphi(P\upharpoonright n, \tilde{V})$ holds for some M-valuation $\tilde{V} \subseteq V$. Since V is diagonalizing against A_0, A_1, for every $i < m$, there is some $L, R \in V(i)$ such that $L \subseteq A_0$ and $R \subseteq A_1$. Let $\tilde{L}, \tilde{R} \in \tilde{V}(i)$ be such that $\tilde{L} \subseteq L$ and $\tilde{R} \subseteq R$. In particular, $\tilde{L} \subseteq A_0$ and $\tilde{R} \subseteq A_1$ so $\tilde{V}$ diagonalizes against A_0, A_1. Take $T_{s+1} = T_s$ and go to the next stage. This finishes the proof of Theorem 4.2. □

As previously noted, preserving n-fairness for every n implies preserving fairness. However, we really need the fact that WKL_0 admits n-fairness preservation and not only fairness preservation in the proof of Theorem 4.8.

Corollary 4.3 WKL_0 admits fairness preservation.

Cholak, Jockusch and Slaman [2] studied extensively Ramsey's theorem for pairs in reverse mathematics, and introduced their cohesive and stable variants.

Definition 4.4 (Cohesiveness) An infinite set C is $\vec{R}$-cohesive for a sequence of sets $R_0, R_1, \dots$ if for each $i \in \omega$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$. COH is the statement "Every uniform sequence of sets $\vec{R}$ has an $\vec{R}$-cohesive set."

A coloring $f : [\omega]^{k+1} \to n$ is *stable* if for every k-tuple $\sigma \in [\omega]^k$, $\lim_s f(\sigma, s)$ exists. SRT^k_n is the restriction of RT^k_n to stable colorings. Mileti [20] and Jockusch & Lempp [unpublished] proved that RT^2_2 is equivalent to $\mathsf{SRT}^2_2 + \mathsf{COH}$ over RCA_0. Recently, Chong et al. [3] proved that SRT^2_2 is strictly weaker than RT^2_2 over RCA_0. However they used non-standard models to separate the statements and the question whether SRT^2_2 and RT^2_2 coincide over ω-models remains open.

Cohesiveness can be seen as a sequential version of RT^1_2 with finite errors. There is a natural decomposition of RT^2_2 between COH and Δ^0_2 instances of RT^1_2. Indeed, given a computable instance $f : [\omega]^2 \to 2$ of RT^2_2, COH states the existence of an infinite set H such that $f : [H]^2 \to 2$ is stable. By Schoenfield's limit lemma [28], the stable coloring $f : [H]^2 \to 2$ can be seen as the Δ^0_2 approximation of a $\emptyset'$-computable instance $\tilde{f} : H \to 2$ of RT^1_2. Moreover, we can H-compute an infinite f-homogeneous set from any $\tilde{f}$-homogeneous set. We shall therefore prove independently fairness preservation of COH and strong fairness preservation of RT^1_2 to deduce that RT^2_2 admits fairness preservation.

Theorem 4.5 For every $n \geq 1$, COH admits n-fairness preservation.

Proof. Let C be a set n-fair for some sets $A_0, A_1 \subseteq 2^{<\omega}$, and let $R_0, R_1, \dots$ be a C-computable sequence of sets. We will construct an $\vec{R}$-cohesive set G such that $G \oplus C$ is n-fair for A_0, A_1. The construction is done by a Mathias forcing, whose conditions are pairs (F, X) where X is a C-computable set. The result is a direct consequence of the following lemma.

Lemma 4.6 For every condition (F, X), every m-by-$2^n m$ $\Sigma^{0,C}_1$ formula $\varphi(G, U)$ and every m-by-$2^n m$ disjoint matrix M, there exists an extension $d = (E, Y)$ such that either $\varphi(G, U)$ is not essential for every set G satisfying d, or $\varphi(E, V)$ holds for some M-valuation V diagonalizing against A_0, A_1.

Proof. Define the m-by-$2^n m$ $\Sigma^{0,C}_1$ formula $\psi(U) = (\exists G \supseteq F)[(G \subseteq F \cup X) \wedge \varphi(G, U)]$. By n-fairness of C for A_0, A_1, either $\psi(U)$ is not essential in M, or $\psi(V)$ holds for some M-valuation V diagonalizing against A_0, A_1. In the former case, the condition (F, X) already satisfies the desired property. In the latter case, by the finite use property, there exists a finite set E satisfying (F, X) such that $\varphi(E, V)$ holds. Let $Y = X \setminus [0, max(E)]$. The condition (E, Y) is a valid extension. □

Using Lemma 4.6, define an infinite descending sequence of conditions $c_0 = (\emptyset, \omega) \geq c_1 \geq \dots$ such that for each $s \in \omega$

(i) $|F_s| \geq s$
(ii) $X_{s+1} \subseteq R_s$ or $X_{s+1} \subseteq \overline{R}_s$
(iii) $\varphi(G, U)$ is not essential in M for every set G satisfying c_{s+1}, or $\varphi(F_{s+1}, V)$ holds for some M-valuation V diagonalizing against A_0, A_1 if $s = \langle \varphi, M \rangle$

where $c_s = (F_s, X_s)$. The set $G = \bigcup_s F_s$ is infinite by (i), $\vec{R}$-cohesive by (ii) and $G \oplus C$ is n-fair for A_0, A_1 by (iii). This finishes the proof of Theorem 4.5. □

Corollary 4.7 COH admits fairness preservation.

The next theorem is the reason why we use the notion of fairness instead of n-fairness in our separation of RT^2_2 from TT^2_2. Indeed, given an instance of RT^1_2 and a set C which is n-fair for some sets A_0, A_1, the proof constructs a solution H such that $H \oplus C$ is $(n+1)$-fair for A_0, A_1. We shall see in Corollary 4.14 that the proof is optimal, in the sense that RT^1_2 does not admit strong n-fairness preservation.

Theorem 4.8 RT^1_2 admits strong fairness preservation.

Proof. Let C be a set n-fair for some sets $A_0, A_1 \subseteq 2^{<\omega}$, and let $B_0 \cup B_1 = \omega$ be a (non-necessarily computable) 2-partition of ω. Suppose that there is no infinite set $H \subseteq B_0$ or $H \subseteq B_1$ such that $H \oplus C$ is n-fair for A_0, A_1, since otherwise we are done. We construct a set G such that both $G \cap B_0$ and $G \cap B_1$ are infinite. We need therefore to satisfy the following requirements for each $p \in \omega$.

$$\mathcal{N}_p : \qquad (\exists q_0 > p)[q_0 \in G \cap B_0] \qquad \wedge \qquad (\exists q_1 > p)[q_1 \in G \cap B_1]$$

Furthermore, we want to ensure that one of $(G \cap B_0) \oplus C$ and $(G \cap B_1) \oplus C$ is fair for A_0, A_1. To do this, we will satisfy the following requirements for every integer m, every m-by-$2^{n+1}m$ $\Sigma^{0,C}_1$ formulas $\varphi_0(H, U)$ and $\varphi_1(H, U)$ and every m-by-$2^{n+1}m$ disjoint matrices M_0 and M_1.

$$\mathcal{Q}_{\varphi_0,M_0,\varphi_1,M_1} : \qquad \mathcal{R}^{G\cap B_0}_{\varphi_0,M_0} \qquad \vee \qquad \mathcal{R}^{G\cap B_1}_{\varphi_1,M_1}$$

where $\mathcal{R}^{H}_{\varphi,M}$ holds if $\varphi(H, U)$ is not essential in M or $\varphi(H, V)$ holds for some M-valuation V diagonalizing against A_0, A_1. We first justify that if every $\mathcal{Q}$-requirement is satisfied, then either $(G \cap B_0) \oplus C$ or $(G \cap B_1) \oplus C$ is $(n + 1)$-fair for A_0, A_1. By the usual pairing argument, for every m, there is some side $i < 2$ such that the following property holds:

> (P) For every m-by-$2^{n+1}m$ $\Sigma^{0,C}_1$ formula $\varphi(G \cap B_i, U)$ and every m-by-$2^{n+1}m$ disjoint matrix M, either $\varphi(G \cap B_i, U)$ is not essential in M, or $\varphi(G \cap B_i, V)$ holds for some M-valuation V diagonalizing against A_0, A_1.

By the infinite pigeonhole principle, there is some side $i < 2$ such that (P) holds for infinitely many m. By a cropping argument, if (P) holds for m and $q < m$, then (P) holds for q. Therefore (P) holds for every m on side i. In other words, $(G \cap B_i) \oplus C$ is $(n + 1)$-fair for A_0, A_1.

We construct our set G by forcing. Our conditions are Mathias conditions (F, X), such that $X \oplus C$ is n-fair for A_0, A_1. We now prove the progress lemma, stating that we can force both $G \cap B_0$ and $G \cap B_1$ to be infinite.

Lemma 4.9 For every condition $c = (F, X)$, every $i < 2$ and every $p \in \omega$ there is some extension $d = (E, Y)$ such that $E \cap B_i \cap (p, +\infty) \neq \emptyset$.

Proof. Fix c, i and p. If $X \cap B_i \cap (p, +\infty) = \emptyset$, then $X \cap (p, +\infty)$ is an infinite subset of B_{1-i}. Moreover, $X \cap (p, +\infty)$ is n-fair for A_0, A_1, contradicting our hypothesis. Thus, there is some $q > p$ such that $q \in X \cap B_i \cap (p, +\infty)$. Take $d = (F \cup \{q\}, X \setminus [0, q])$ as the desired extension. □

Next, we prove the core lemma stating that we can satisfy each $\mathcal{Q}$-requirement. A condition c *forces* a requirement $\mathcal{Q}$ if $\mathcal{Q}$ is holds for every set G satisfying c. This is the place where we really need the fact that $\mathsf{WKL_0}$ admits n-fairness preservation and not only fairness preservation.

Lemma 4.10 For every condition $c = (F, X)$, every integer m, every m-by-$2^{n+1}m$ $\Sigma^{0,C}_1$ formulas $\varphi_0(H, U)$ and $\varphi_1(H, U)$ and every m-by-$2^{n+1}m$ disjoint matrices M_0 and M_1, there is an extension $d = (E, Y)$ forcing $\mathcal{Q}_{\varphi_0,M_0,\varphi_1,M_1}$.

Proof. Let $\psi(U_0, U_1)$ be the $2m$-by-$2^{n+1}m$ $\Sigma^{0,X\oplus C}_1$ formula which holds if for every 2-partition $Z_0 \cup Z_1 = X$, there is some $i < 2$, some finite set $E \subseteq Z_i$ and an m-by-$2^{n+1}m$ M_i-valuation $V \subseteq U_i$ such that $\varphi_i((F \cap B_i) \cup E, V)$ holds. By n-fairness of $X \oplus C$, we have two cases.

In the first case, $\psi(U_0, U_1)$ is not essential in M_0, M_1, with some witness t. By compactness, the $\Pi^{0,X\oplus C}_1$ class $\mathcal{C}$ of sets $Z_0 \oplus Z_1$ such that $Z_0 \cup Z_1 = \omega$ and for every $i < 2$ and every finite set $E \subseteq Z_i$, there is no M_i-valuation $V > t$ such that $\varphi_i((F \cap B_i) \cup E, V)$ holds is non-empty. By n-fairness preservation of $\mathsf{WKL_0}$ (Theorem 4.2), there is a 2-partition $Z_0 \oplus Z_1 \in \mathcal{C}$ such that $Z_0 \oplus Z_1 \oplus C$ is n-fair for A_0, A_1. Since $Z_0 \cup Z_1 = X$, there is some $i < 2$ such that Z_i is infinite. Take such an i. The condition $d = (F, Z_i)$ is an extension forcing $\mathcal{Q}_{\varphi_0,M_0,\varphi_1,M_1}$ by the ith side.

In the second case, $\psi(V_0, V_1)$ holds for some (M_0, M_1)-valuation (V_0, V_1) diagonalizing against A_0, A_1. Let $Z_0 = X \cap B_0$ and $Z_1 = X \cap B_1$. By hypothesis, there is some $i < 2$, some finite set $E \subseteq Z_i = X \cap B_i$ and some M_i-valuation $V \subseteq V_i$ such that $\varphi_i((F \cap B_i) \cup E, V)$ holds. Since $V \subseteq V_i$, the M_i-valuation V diagonalizes against A_0, A_1. The condition $d = (F \cup E, X \setminus [0, max(E)])$ is an extension forcing $\mathcal{Q}_{\varphi_0,M_0,\varphi_1,M_1}$ by the ith side. □

Using Lemma 4.9 and Lemma 4.10, define an infinite descending sequence of conditions $c_0 = (\emptyset, \omega) \geq c_1 \geq \dots$ such that for each $s \in \omega$

(i) $|F_s \cap B_0| \geq s$ and $|F_s \cap B_1| \geq s$
(ii) c_{s+1} forces $\mathcal{Q}_{\varphi_0,M_0,\varphi_1,M_1}$ if $s = \langle \varphi_0, M_0, \varphi_1, M_1 \rangle$

where $c_s = (F_s, X_s)$. Let $G = \bigcup_s F_s$. The sets $G \cap B_0$ and $G \cap B_1$ are both infinite by (i) and one of $G \cap B_0$ and $G \cap B_1$ is fair for A_0, A_1 by (ii). This finishes the proof of Theorem 4.8. □

Theorem 4.11 RT^2_2 admits fairness preservation.

Proof. Fix any set C fair for some sets $A_0, A_1 \subseteq 2^{<\omega}$ and any C-computable coloring $f : [\omega]^2 \to 2$. Consider the uniformly C-computable sequence of sets $\vec{R}$ defined for each $x \in \omega$ by

$$R_x = \{s \in \omega : f(x, s) = 1\}$$

As COH admits fairness preservation, there is some $\vec{R}$-cohesive set G such that $G \oplus C$ is fair for A_0, A_1. The set G induces a $(G \oplus C)'$-computable coloring $\tilde{f} : \omega \to 2$ defined by:

$$(\forall x \in \omega)\tilde{f}(x) = \lim_{s \in G} f(x, s)$$

As RT^1_2 admits strong fairness preservation, there is an infinite $\tilde{f}$-homogeneous set H such that $H \oplus G \oplus C$ is fair for A_0, A_1. The set $H \oplus G \oplus C$ computes an infinite f-homogeneous set. □

Corollary 4.12 $\mathsf{RT}^2_2 \wedge \mathsf{WKL}_0$ does not imply TT^2_2 over RCA_0.

Proof. By Theorem 4.11 and Corollary 4.3, RT^2_2 and WKL_0 admit fairness preservation. By Theorem 3.9, TT^2_2 does not admit fairness preservation. We conclude by Lemma 3.7. □

We now prove the optimality of Theorem 4.8 and Theorem 4.11 by showing that n-fairness cannot be preserved.

Theorem 4.13 SRT^2_2 does not admit n-fairness preservation for any $n \geq 1$.

Proof. Let $A_0 \cup A_1 = 2^{<\omega}$ be the Δ^0_2 partition constructed in Lemma 3.8. By Schoenfield's limit lemma [28], there is a stable computable function $f : [\omega]^2 \to 2$ such that $x \in A_{\lim_s f(x,s)}$ for every x. Fix some $n \geq 1$. For each $\sigma \in 2^{n+1}$, apply SRT^2_2 to the coloring f restricted to the set $S_\sigma = \{\tau \succeq \sigma\}$ to obtain an infinite f-homogeneous set H_σ for some color $c_\sigma < 2$. By definition of f, $H_\sigma \subseteq A_{c_\sigma}$. By the finite pigeonhole principle, there is a color $c < 2$ and a set $M \subseteq 2^{n+1}$ of size 2^n such that $c_\sigma = c$ for every $\sigma \in M$. We can see M as a 1-by-2^n disjoint matrix. Let $H = \bigoplus_{\sigma \in M} H_\sigma$ and let $\varphi(U_\sigma : \sigma \in M)$ be the 1-by-2^n $\Sigma^{0,H}_1$ formula which holds if for every $\sigma \in M$, U_σ is a non-empty subset of H_σ. Note that $H_\sigma \subseteq A_c$ for every $\sigma \in M$. The formula $\varphi(U)$ is essential in M but there is no M-valuation $V = (V_\sigma : \sigma \in M)$ such that $\varphi(V)$ holds and $V_\sigma \subseteq A_{1-c}$ for some $\sigma \in M$. Therefore H is not n-fair for A_0, A_1. □

Corollary 4.14 RT^1_2 does not admit n-fairness preservation for every $n \geq 1$.

Proof. Fix some $n \geq 2$. By Theorem 4.13, there is some set C n-fair some A_0, A_1 and a stable C-computable function $f : [\omega]^2 \to 2$ such that for every infinite f-homogeneous set H, $H \oplus C$ is not n-fair for A_0, A_1. Let $\tilde{f} : \omega \to 2$ be defined by $\tilde{f}(x) = \lim_s f(x, s)$. Every infinite $\tilde{f}$-homogeneous set H C-computes an infinite f-homogeneous set H_1 such that $H_1 \oplus C$ is not n-fair for A_0, A_1. Therefore $H \oplus C$ is not n-fair for A_0, A_1. □

5. Questions

In this last section, we state some remaining open questions. The tree theorem for pairs is known to lie between ACA_0 and RT^2_2 over RCA_0. By Corollary 4.12, TT^2_2 is strictly stronger than RT^2_2 over RCA_0. However, it is unknown whether TT^2_2 is strictly weaker than ACA_0 over RCA_0.

Question 5.1 Does TT^2_2 imply ACA_0 over RCA_0?

From a computability-theoretic point of view, Seetapun [27] proved that for every non-computable set C, every computable instance of RT^2_2 has a solution which does not compute C. This is the notion of *cone avoidance*.

Question 5.2 Does TT^2_2 admit cone avoidance?

Dzhafarov and Jockusch [9] simplified Seetapun's argument and proved that for every non-computable set C, every arbitrary, that is, non-necessarily computable, instance of RT^1_2 has a solution which does not compute C. This strengthening is called *strong cone avoidance* and is usually joined with the cone avoidance of the cohesive version of the principle to obtain cone avoidance for the principle over pairs.

Question 5.3 Does TT^1_2 admit strong cone avoidance?

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References

[1] Laurent Bienvenu, Ludovic Patey, and Paul Shafer. On the logical strengths of partial solutions to mathematical problems. Submitted. Available at `http://arxiv.org/abs/1411.5874`, 2015.

[2] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 66(01):1–55, 2001.

[3] Chitat Chong, Theodore Slaman, and Yue Yang. The metamathematics of stable Ramsey's theorem for pairs. *Journal of the American Mathematical Society*, 27(3):863–892, 2014.

[4] Jennifer Chubb, Jeffry L. Hirst, and Timothy H. McNicholl. Reverse mathematics, computability, and partitions of trees. *Journal of Symbolic Logic*, 74(01):201–215, 2009.

[5] Jared Corduan, Marcia J. Groszek, and Joseph R. Mileti. Reverse mathematics and Ramsey's property for trees. *Journal of Symbolic Logic*, 75(03):945–954, 2010.

[6] François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer. On uniform relationships between combinatorial problems. *arXiv preprint arXiv:1212.0157*, 2012.

[7] Damir D. Dzhafarov. Cohesive avoidance and strong reductions. *Proceedings of the American Mathematical Society*, 2014.

[8] Damir D. Dzhafarov, Jeffry L. Hirst, and Tamara J. Lakins. Ramsey's theorem for trees: the polarized tree theorem and notions of stability. *Archive for Mathematical Logic*, 49(3):399–415, 2010.

[9] Damir D. Dzhafarov and Carl G. Jockusch. Ramsey's theorem and cone avoidance. *Journal of Symbolic Logic*, 74(2):557–578, 2009.

[10] Stephen Flood. Reverse mathematics and a Ramsey-type König's lemma. *Journal of Symbolic Logic*, 77(4):1272–1280, 2012.

[11] Stephen Flood and Henry Towsner. Separating principles below WKL_0, 2014. Submitted. Available at `http://arxiv.org/abs/1410.4068`.

[12] Harvey M. Friedman. Some systems of second order arithmetic and their use. In *Proceedings of the International Congress of Mathematicians, Vancouver*, volume 1, pages 235–242, 1974.

[13] Emanuele Frittaion and Ludovic Patey. Coloring the rationals in reverse mathematics. Submitted. Available at `http://arxiv.org/abs/1508.00752`., 2015.

[14] Denis R. Hirschfeldt. Slicing the truth. *Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore*, 28, 2014.

[15] Denis R. Hirschfeldt and Carl G. Jockusch. On notions of computability theoretic reduction between Π^1_2 principles. *To appear*.

[16] Carl G. Jockusch. Ramsey's theorem and recursion theory. *Journal of Symbolic Logic*, 37(2):268–280, 1972.

[17] Manuel Lerman, Reed Solomon, and Henry Towsner. Separating principles below Ramsey's theorem for pairs. *Journal of Mathematical Logic*, 13(02):1350007, 2013.

[18] Lu Liu. RT^2_2 does not imply WKL_0. *Journal of Symbolic Logic*, 77(2):609–620, 2012.

[19] T.H. McNicholl. *The Inclusion Problem for Generalized Frequency Classes*. PhD thesis, George Washington University, 1995.

[20] Joseph R. Mileti. *Partition theorems and computability theory*. PhD thesis, Carnegie Mellon University, 2004.

[21] Antonio Montalbán. Open questions in reverse mathematics. *Bulletin of Symbolic Logic*, 17(03):431–454, 2011.

[22] Ludovic Patey. A note on "Separating principles below Ramsey's theorem for pairs". Unpublished. Available at `http://ludovicpatey.com/media/research/note-em-sts.pdf`, 2013.

[23] Ludovic Patey. Iterative forcing and hyperimmunity in reverse mathematics. In Arnold Beckmann, Victor Mitrana, and Mariya Soskova, editors, *Evolving Computability*, volume 9136 of *Lecture Notes in Computer Science*, pages 291–301. Springer International Publishing, 2015.
[24] Ludovic Patey. Ramsey-type graph coloring and diagonal non-computability. *Archive for Mathematical Logic*, 54(7-8):899–914, 2015.
[25] Ludovic Patey. The weakness of being cohesive, thin or free in reverse mathematics. Submitted. Available at `http://arxiv.org/abs/1502.03709`, 2015.
[26] Ludovic Patey. *The reverse mathematics of Ramsey-type theorems*. PhD thesis, Université Paris Diderot, 2016.
[27] David Seetapun and Theodore A. Slaman. On the strength of Ramsey's theorem. *Notre Dame Journal of Formal Logic*, 36(4):570–582, 1995.
[28] Joseph R. Shoenfield. On degrees of unsolvability. *Annals of Mathematics*, 69(03):644–653, May 1959.
[29] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Cambridge University Press, 2009.