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Abstract

In this article we consider a linear finite dimensional system. Our aim is to design a control such that the output of the system reach a given target at a final time $T > 0$. This notion is known as output controllability.

We extend this notion to the one of *long-time output controllability*. More precisely, we consider the question: is it possible to steer the output of the system to some prescribed value in time $T > 0$ and then keep the output of the system at this prescribed value for all times $t > T$? We provide a necessary and sufficient condition for this property to hold. Once the condition is satisfied, one can apply a feedback control that keeps the average fixed during a given time period. We also address the L^2-norm optimality of such controls.

We apply our results to *(long-time) averaged control problems*.

Key words: Output controllability, Controlled invariant subspace, Linear optimal control, Open-loop control systems, Averaged control.

1 Introduction

In this paper, we consider the output controllability of finite-dimensional control systems. In many applications, state variables are not relevant from the point of view of the practical control application. To give a basic idea, if one aim to model a car, the wheel angle, will be one the state variables, but this state variable is not relevant for the main purpose of a car, i.e., moving from one point to another. In this context, the system output could be the orientation and the position of the mass center of the car. Our main goal is to only control the output of the system to some desired target in a given time $T > 0$, and then keep this output fixed for the remaining times $t > T$.

In order to clearly present the problem, let us first give some notations. $\mathbb{R}$ (respectively $\mathbb{R}_+$) is the set of real numbers (respectively nonnegative real numbers). $\mathbb{N}$ (respectively $\mathbb{N}^*$) is the set of nonnegative (respectively positive) integers. $M_{n,m}(\mathbb{R})$ is the set of real matrix of n lines and m columns and $M_n(\mathbb{R})$ stands for $M_{n,n}(\mathbb{R})$. $\mathrm{I}_n \in M_n(\mathbb{R})$ is the $n \times n$ identity matrix, and $GL_n(\mathbb{R})$ denotes the set of regular matrices of $M_n(\mathbb{R})$. Elements of $M_{n,m}(\mathbb{R})$ are identified to linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$. For C a linear operator from E to F, E and F being two vectorial spaces, $\operatorname{Ran} C = \{Cx \mid x \in E\} \subset F$ is the range of C, $\operatorname{rank} C$ is the dimension of $\operatorname{Ran} C$, and $\operatorname{Ker} C = \{x \in E \mid Cx = 0\}$ is the kernel of C. For $y \in F$, we also denote by $C^{-1}\{y\} = \{x \in E \mid Cx = y\}$, the preimage of y by C. For $T > 0$, We also denote $L^2(0, T)$ the set of real measurable functions defined on $(0, T)$ which are square integrable. This space is a Hilbert space endowed with the norm $\|f\|_{L^2(0,T)} = \sqrt{\int_0^T |f(t)|^2 \, \mathrm{d}t}$. $L^2_{loc}(\mathbb{R}_+)$ is the set of real and measurable functions defined on $\mathbb{R}_+$ such that, for every $T > 0$, their restriction to the interval $(0, T)$ belongs to $L^2(0, T)$.

In this paper, we consider the system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx. \tag{1b}$$

Here, $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the control variable and $y(t) \in \mathbb{R}^q$ is the output variable, and we have $A \in M_n(R)$, $B \in M_{n,m}(\mathbb{R})$ and $C \in M_{q,n}(\mathbb{R})$, with $m, n, q \in \mathbb{N}^*$. We recall the notion of

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output-controllability.

Definition 1 *We say that the system* (1) *is output controllable, if for every $x^0 \in \mathbb{R}^n$ and every $\bar{y} \in \operatorname{Ran} C$, there exist a time $T > 0$ and a control $u \in L^2(0,T)^m$ such that the solution of* (1)*, with initial condition $x(0) = x^0$ satisfies $y(T) = \bar{y}$.*

Let us recall that the (state) controllability of the system (1) is defined in the way that for every x^0 and every $x^1 \in \mathbb{R}^n$, there exist a time $T > 0$ and a control $u \in L^2(0,T)^m$ such that the solution of (1a), with initial condition $x(0) = x^0$ satisfies $x(T) = x^1$.

In this paper, we will study condition for having *long-time output controllability*. This controllability notion express the ability to steer the system output to a target and then stay on the target for later times and is defined as follows.

Definition 2 *Given $\bar{y} \in \operatorname{Ran} C$, we say that the system* (1) *is long-time output controllable (on $\bar{y}$), if for every $x^0 \in \mathbb{R}^n$, there exist a time $T > 0$ and a control $u \in L^2_{loc}(\mathbb{R}_+)^m$ such that the solution of* (1)*, with initial condition $x(0) = x^0$ satisfies $y(t) = \bar{y}$ for every $t \in [T, +\infty)$.*

Obviously, state and long-time output controllability are stronger notion than output controllability, and in this paper, we will only assume that the system (1) is output controllable.

Let us recall that state-controllability of (1) is equivalent to the well-known Kalman rank condition [2],

$$\operatorname{rank}\left(B \; AB \; \dots \; A^{n-1}B\right) = n. \tag{2}$$

By adapting [4, Theorem III], it is also easy to prove that output-controllability of (1) is equivalent to the rank condition

$$\operatorname{rank} C\left(B \; AB \; \dots \; A^{n-1}B\right) = \operatorname{rank} C. \tag{3}$$

Note that when C is a full rank matrix, then the above condition is exactly the one given in [4, Theorem III].
Note also that the time T in the definition of state or output controllability can be chosen arbitrarily.

Let us note that the output controllability notion, given in Definition 1, deals with a property in a single time point. In this paper our main concern is whether it is possible or not to find a control such that the system output, once brought to its desired value, remains constant for the subsequent time period. We must emphasise an essential difference compared to classical control problems. Namely, application of a control steering the output of a system to zero at time T does not imply the system output will remain zero for the subsequent time period. Contrarily to classical control problems, the null control applied for $t > T$ does not lead to a constant null output. This is because a null output does not necessarily coincide with a null state. More precisely, with null control, even if the initial state x^0 is such that $Cx^0 = 0$, the system state $x(t) = e^{tA}x^0$ does not necessarily satisfy $Cx(t) = 0$ for every $t > 0$.
For discretized time systems, the notion of long-time output controllability is referred as dead-beat control, see for instance [1,3,11].

In this paper, our first goal is to present a rank condition, similar to (2) and (3), for the long-time output controllability notion. We will also describe the structure of L^2-norm optimal controls for long-time output controllability.

These notions will be applied to averaged controllability property of finite-dimensional, parameter dependent systems. The notion of averaged controllability has been introduced by E. Zuazua in [14], and afterwards generalised to PDE setting in [5,7,8]. Its goal is to control the average (or more generally a suitable linear combination) of system components by a single control. The problem is relevant in practice when the system depends on a number of variable parameters, and the control has to be chosen independently of their values.
More precisely, let us consider d realisation of control systems,

$$\dot{x}_i = A_i x_i + B_i u, \quad x_i(0) = x_i^0 \quad (i \in \{1, \cdots, d\}), \tag{4}$$

with $A_i \in M_n(\mathbb{R})$, $x_i(t) \in \mathbb{R}^n$, $B_i \in M_{n,m}(\mathbb{R})$ and $u(t) \in \mathbb{R}^m$, and d parameters $\sigma_1, \cdots, \sigma_d \in [0,1]$ such that $\sum \sigma_i = 1$. Note that, here, no controllability assumptions on the pairs (A_i, B_i) are made. The only assumption we make is the *averaged controllability* of this system.

Definition 3 *We say that the system* (4) *is controllable in average in some time $T > 0$ for the weights $\sigma_1, \cdots, \sigma_d$ if for every $x_1^0, \cdots, x_d^0 \in \mathbb{R}^n$ and every $\bar{y} \in \mathbb{R}^n$, there exist a control $u \in L^2(0,T)^m$ such that the solution of* (4) *satisfies: $\sum_{i=1}^d \sigma_i x_i(T) = \bar{y}$.*

Note that the average controllability notion is exactly the output controllability of (1) with matrices $A \in M_{nd}(\mathbb{R})$, $B \in M_{nd,m}(\mathbb{R})$ and $C \in M_{n,nd}(\mathbb{R})$ given by:

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_d \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_d \end{pmatrix}$$

$$\text{and} \quad C = \left(\sigma_1 \mathrm{I}_n \; \dots \; \sigma_d \mathrm{I}_n\right), \tag{5}$$

and state $x = \begin{pmatrix} x_1^\top & \dots & x_d^\top \end{pmatrix}^\top \in \mathbb{R}^{nd}$.
In the context of averaged controllability, the state (respectively long-time output) controllability of (1) is referred as the simultaneous (respectively long-time averaged) controllability of (4).

It is obvious that if the system (4) is simultaneously controllable, then it is controllable in average. More detailed relations between these notions have been studied in [6], in which the authors analyse deviations of each system component from the averaged value. To this effect they identify the optimal averaged control as the one minimising a quadratic functional which, together with the control norm, contains a penalisation term measuring deviation of each system component from the average. Under suitable assumptions they show that by increasing the penalisation constant, the averaged control converges to the simultaneous one.

This paper is organised as follows. In Section 2, we prove the main result of this paper (Theorem 3). This result gives a necessarily and sufficient condition for long-time output controllability in terms of a rank condition. In Section 3, we give the formal expression of L^2-norm optimal control. More precisely, for a system having the long-time output controllability property, we fix T_0 and T_1, two positive times, and we aim to find the minimal L^2-norm optimal control such that the system output is constant and equal to 0 for every $t \in [T_0, T_0 + T_1]$. The Section 4 is an application of the results of Section 2 to long-time average controllability. In particular, we focus to the case where $d = 2$ and $\sigma_1 = \sigma_2 = 1/2$. In this section, we also provide an illustrative example. Finally, the Section 5 concludes this article, with some remarks and open questions.

2 Long-time output controllability condition

Recall that we assume in this paper that the matrices A, B and C satisfy the rank condition (3).

The aim of this section is to determine weather it is possible or not to find a control u such that (1) has the long-time output controllability property.

Remark 1 *Of course, if* $\operatorname{rank} C = n \leqslant q$*, the constraint* $y(t) = \bar{y}$ *for every* $t \geqslant T$ *immediately gives* $x(t) = \bar{x}$ *for every* $t \geqslant T$*, where* $\bar{x} \in \mathbb{R}^n$ *is such that* $C\bar{x} = \bar{y}$*. Thus, in particular, we have* $\dot{x}(t) = 0$*, implying* $A\bar{x} = -Bu(t)$ *for every* $t > T$*. Thus, there exists a control* u *(which can be chosen constant) ensuring that* $y(t) = \bar{y}$ *for every* $t \geqslant T$ *if and only if* $A\bar{x} \in \operatorname{Ran} B$*, i.e.,* $AC^{-1}\{\bar{y}\} \cap \operatorname{Ran} B \neq \emptyset$*.*

In order to consider the long-time output controllability notion, we are going to assume that the initial condition of (1) is some $\bar{x} \in C^{-1}\{\bar{y}\}$. In fact, this is enough since the system (1) is autonomous, and it is assumed that the condition (3) is fulfilled, this ensures that the system output $y(T)$ can reach the output target $\bar{y}$ in any time $T > 0$. Our aim is then to find a control u such that

$$Cx(t) = \bar{y} \qquad (t \geqslant 0), \tag{6}$$

with x solution of (1a), with initial condition $x(0) = \bar{x} \in C^{-1}\{y\}$. Note that (6) can be written as

$$Cx(t) = C\bar{x} \qquad (t \geqslant 0), \tag{7}$$

In order to treat a more general case, than the one described in Remark 1, we develop the following procedure. By time derivation of (7), we have:

$$CAx(t) + CBu(t) = 0 \qquad (t > 0). \tag{8}$$

To satisfy the last relation, one need to have $CAx(t) \in \operatorname{Ran}(CB)$ for every $t \geqslant 0$.
More precisely, let us define $P \in M_q(\mathbb{R})$ the orthogonal projector of $\mathbb{R}^q$ on $\operatorname{Ker}(CB)^\top \subset \mathbb{R}^q$, so that we have $\operatorname{rank} P = q - \operatorname{rank}(CB)$.

Remark 2 *Note that the projector* P *has the following form,*

$$P = \mathrm{I}_q - Q(CB)(CB)^\top Q^\top,$$

where Q *is the Gram-Schmidt matrix ensuring that* $\operatorname{Ran} Q(CB) = \operatorname{Ran}(CB)$ *and the columns of* $Q(CB)$ *are orthonormal.*

Consequently, (8) is

$$-PCAx(t) = PCBu(t) = 0 \qquad (t > 0)\,.$$

Since x is continuous, by taking the limit $t \to 0$, we observe that the initial state $\bar{x}$ shall satisfy $PCA\bar{x} = 0$. Thus, in order to satisfy (7), one also has to satisfy:

$$\begin{pmatrix} C \\ PCA \end{pmatrix} x(t) = \begin{pmatrix} C\bar{x} \\ 0 \end{pmatrix} = \begin{pmatrix} C \\ PCA \end{pmatrix} \bar{x} \qquad (t \geqslant 0). \tag{9}$$

This leads to the condition (7) with C replaced by $C_1 = \begin{pmatrix} C \\ PCA \end{pmatrix}$.
Iterating this process, we define the sequence $(C_k)_{k\in\mathbb{N}}$ by:

$$C_0 = C \in M_{q,n}(\mathbb{R}), \tag{10a}$$

and for every $k \in \mathbb{N}$,

$$C_{k+1} = \begin{pmatrix} C_0 \\ \Xi_k \end{pmatrix} \in M_{(k+2)q,n}(\mathbb{R}), \tag{10b}$$

where

$$\Xi_k = P_k C_k A, \tag{11}$$

with P_k, the orthogonal projector of $\mathbb{R}^{(k+1)q}$ on $\operatorname{Ker}(C_k B)^\top$.

Lemma 1 *We have* $\operatorname{Ker} C_{k+1} \subset \operatorname{Ker} C_k \subset \mathbb{R}^n$ *for every* $k \in \mathbb{N}$*, and there exist* $K \in \{0, \dots, n\}$ *such that* $\operatorname{Ker} C_{K+1} = \operatorname{Ker} C_K$*. Furthermore, for every* $i \in \mathbb{N}$*, we have* $\operatorname{Ker} C_{K+i} = \operatorname{Ker} C_K$*.*

PROOF. Indeed, from the definition of the operator C_1 it follows $\operatorname{Ker} C_1 = \operatorname{Ker} C_0 \cap \operatorname{Ker} \Xi_0$, trivially implying $\operatorname{Ker} C_1 \subseteq \operatorname{Ker} C_0$. Now suppose that $\operatorname{Ker} C_k \subseteq \operatorname{Ker} C_{k-1}$ for some $k \in \mathbb{N}^*$. We want to show that it implies $\operatorname{Ker} C_{k+1} \subseteq \operatorname{Ker} C_k$. From the definition of C_k, the last inclusion follows if we show that $\operatorname{Ker} \Xi_k \subseteq \operatorname{Ker} \Xi_{k-1}$. To this effect, notice that by the induction hypothesis it follows

$$\begin{aligned} \Xi_k x = 0 &\Leftrightarrow C_k(Ax - Bu) = 0, \quad \text{for some } u \in \mathbb{R}^m \\ &\Rightarrow C_{k-1}(Ax - Bu) = 0 \Leftrightarrow \Xi_{k-1} x = 0 \end{aligned}$$

thus obtaining $\operatorname{Ker} C_{k+1} \subseteq \operatorname{Ker} C_k \subseteq \mathbb{R}^n$ for every $k \in \mathbb{N}$. It is then easy to show the existence of $K \in \{0, \dots, n\}$ such that $\operatorname{Ker} C_{K+1} = \operatorname{Ker} C_K$. In particular, from the structure of operators C_k, we have $\operatorname{Ker} \Xi_{K+1} = \operatorname{Ker} \Xi_K$, and hence $\operatorname{Ker} C_{K+2} = \operatorname{Ker} C_{K+1}$ and inductively, we obtain the final claim. □

Note also that, due to the structure of the projectors P_k (see Remark 2), we have $\operatorname{Ran} \Xi_k \subset \operatorname{Ran} C_k$, for every $k \in \mathbb{N}$.

We can now state the conditions by which we are able to stay in $\operatorname{Ker} C$ if we start from a point in $\operatorname{Ker} C$.

Proposition 2 *Let* $(C_k)_{k \in \mathbb{N}}$ *be the sequence defined by* (10)*, and let* $K \in \{0, \dots, n\}$ *such that* $\operatorname{Ker} C_{K+1} = \operatorname{Ker} C_K$*. Then there exists a control* $u \in L^2_{loc}(\mathbb{R}_+)^m$ *such that* $Cx(t) = C\bar{x}$ *for every* $t \geqslant 0$*, where* x *is solution of* (1a) *with initial condition* $x(0) = \bar{x}$*, if and only if* $\Xi_K \bar{x} = 0$*, with* Ξ_K *defined by* (11) *with* $k = K$*.*

PROOF. The existence of K is ensured by Lemma 1. Note that using the relation (9), and repeating the arguments behind it for different values of k, without loosing generality we may replace the condition $Cx(t) = C\bar{x}$ for every $t \geqslant 0$ by $C_K x(t) = \begin{pmatrix} C\bar{x} \\ 0 \end{pmatrix}$, for every $t \geqslant 0$. Of course, the rows of this equality imply $\Xi_K \bar{x} = 0$.

Assume now that $\Xi_K \bar{x} = 0$. Since x is solution of (1a), for every control u we have

$$C_K \dot{x} = C_K A x + C_K B u. \tag{12}$$

The claim follows if we show that $C_K \dot{x} = 0$.

As for the iteration procedure, we set P_K the projection on $\operatorname{Ker}(C_K B)^\top$. Then we have

$$C_K \dot{x} = P_K C_K A x + (\mathrm{I}_{(K+1)q} - P_K) C_K A x + C_K B u.$$

Since, by construction, $(\mathrm{I}_{(K+1)q} - P_K) C_K A x$ belongs to $\operatorname{Ran}(C_K B)$, whatever x is, one can find $u = u(x)$ such that $(\mathrm{I}_{(K+1)q} - P_K) C_K A x + C_K B u = 0$. For such a control u, the relation (12) reduces to

$$C_K \dot{x} = P_K C_K A x = \Xi_K x.$$

Using the decomposition

$$x(t) - \bar{x} = x_0(t) + x_1(t) \qquad (t \geqslant 0), \tag{13}$$

with $x_0(t) \in \operatorname{Ker} C_K$ and $x_1(t) \in \operatorname{Ran} C_K^\top$, we have

$$C_K \dot{x} = C_K \dot{x}_1 = \Xi_K \left(\bar{x} + x_0 + x_1\right). \tag{14}$$

Recall that, by assumption, $\bar{x} \in \operatorname{Ker} \Xi_K$. Recall also that $\operatorname{Ker} C_K = \operatorname{Ker} C_{K+1} = \operatorname{Ker} C_0 \cap \operatorname{Ker} \Xi_K$, implying that $\operatorname{Ker} C_K \subset \operatorname{Ker} \Xi_K$. Thus, we deduce that

$$C_K \dot{x}_1 = \Xi_K x_1. \tag{15}$$

Set $z = C_K x_1$, and noticing that C_K, seen as a linear operator from $\operatorname{Ran} C_K^\top \to \operatorname{Ran} C_K$ is regular, hence, there exist an invertible and linear operator $\Theta : \operatorname{Ran} C_K \to \operatorname{Ran} C_K^\top$ such that $\Theta z = x_1$. Thus, (15) can be expressed as

$$\dot{z} = \Xi_K \Theta z.$$

But, (13) gives $x_1(0) = 0$ and hence $z(0) = 0$ implying $z(t) = 0$ for every $t \geqslant 0$. The conclusion follows. □

Remark 3 *The recursive construction of the operators* C_k *introduced above resembles the one presented in [13, Theorem 4.3] for construction of a supremal controlled invariant subspace of an arbitrary space* W*.*
To be recalled, a subspace $V \subset W$ *is controlled invariant if it has the property that if* $x(0) \in V$ *then there exists a control* u *such that the solution to system* (1a) *satisfies* $x(t) \in V$ *for all* $t \geqslant 0$*.*
When $W = \operatorname{Ker} C$ *the supremal invariant subspace is exactly the kernel of the operator* C_K *constructed above. This confirms the optimality of the result obtained in Proposition 2 in the sense that* $\operatorname{Ker} C_K$ *is the largest space we can start our system from, still being sure that we shall be able to remain within* $\operatorname{Ker} C_K$*, and consequently within* $\operatorname{Ker} C$*, for all subsequent time periods.*

Let us now consider the complete problem, i.e., we consider a time $T > 0$ and any starting point $x_0 \in \mathbb{R}^n$, and we aim to steer the solution of (1a) to a point $x(T) \in C^{-1}\{\bar{y}\}$ and then for $t \geqslant T$ keep the trajectory fixed with respect to C. Merging the condition (3) and the result of Proposition 2 we can formulate the following.

Theorem 3 *Given $\bar{y} \in \operatorname{Ran} C$ and $T > 0$. For every $x^0 \in \mathbb{R}^n$ there exists a control $u \in L^2_{loc}(\mathbb{R}_+)^m$ such that the solution to the system (1) satisfies $Cx(t) = \bar{y}$ for every $t \geqslant T$ if and only if*

$$\begin{pmatrix} \bar{y} \\ 0 \end{pmatrix} \in \operatorname{Ran} C_{K+1} \quad \textit{and}$$
$$\operatorname{rank} C_K \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = \operatorname{rank} C_K, \quad (16)$$

where $(C_k)_{k\in\mathbb{N}}$ is the sequence defined by (10) and $K \in \{0, \dots, n\}$ is defined by Lemma 1.

PROOF. According to Proposition 2 the required control exists if and only if the system is steered to a state $x(T)$ such that $C_{K+1}x(T) = \begin{pmatrix} \bar{y} \\ 0 \end{pmatrix}$. Using [4, Theorem III] (see (3)), this is possible if and only if

$$\operatorname{rank} C_{K+1} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = \operatorname{rank} C_{K+1}. \quad (17)$$

However, by construction $\operatorname{Ker} C_K = \operatorname{Ker} C_{K+1}$, the operator C_{K+1} in (17) can be replaced by C_K, thus yielding the result. □

Remark 4 *In practice, constructing the sequence of matrices C_k can be a challenging computation. Note however, that all the statements are still valid, if at step $k+1$ we consider any matrix $C_{k+1} = \begin{pmatrix} RC \\ \tilde{\Xi}_k \end{pmatrix}$, with $R \in GL_q(\mathbb{R})$ and $\tilde{\Xi}_k$ a matrix of n columns such that $\operatorname{Ker} C \cap \operatorname{Ker} \Xi_k = \operatorname{Ker} C \cap \operatorname{Ker} \tilde{\Xi}_k$, where Ξ_k is defined by (11). With this modification, $\bar{y}$ has to be moditfied in $R\bar{y}$.*

3 Norm optimal controls

In this section, we analyse L^2-norm optimal controls. More precisely, given two positive times T_0 and T_1, and given some initial condition $x^0 \in \mathbb{R}^n$, an operator $C \in M_{q,n}(\mathbb{R})$ and $\bar{y} \in \operatorname{Ran} C$, we aim to find the control of minimal L^2-norm such that the output y of the system (1) satisfies

$$y(t) = \bar{y} \qquad (t \in [T_0, T_0 + T_1]). \quad (18)$$

Recall that Theorem 3 ensures that this problem admits a solution for every starting point x^0 if and only if (16) is satisfied.

According to Bellman principle, we have

$$\min_{u \in \mathcal{U}(x^0, T_0, T_1)} \|u\|^2_{L^2(0, T_0+T_1)} = \min_{\substack{\bar{x} \in C^{-1}\{\bar{y}\} \\ \bar{x} \in \operatorname{Ker} \Xi_K \cap R_{T_0}(x^0)}} \left(J_0(\bar{x}; T_0) + J_1(\bar{x}; T_1) \right), \quad (19)$$

where Ξ_K is defined by (11) (with K given in Lemma 1), $R_{T_0}(x^0)$ is the set of reachable points, starting from x^0 in time T_0,

$$R_{T_0}(x^0) = \Big\{ e^{T_0 A} x^0 + \int_0^{T_0} e^{(T_0 - t)A} Bu(t) \, \mathrm{d}t \;\; | \;\; u \in L^2(0, T_0)^m \Big\},$$

$\mathcal{U}(x^0, T_0, T_1)$ is the set of *admissible* controls,

$$\mathcal{U}(x^0, T_0, T_1) = \{ u \in L^2(0, T_0 + T_1)^m \;\; | \;\; Cx(t) = \bar{y}, \quad \forall t \in [T_0, T_0 + T_1] \},$$

while x is the solution of (1a) with initial condition x^0 and control u, and where J_0 and J_1 are defined by

$$J_0(\bar{x}; T_0) = \min \; \|u\|^2_{L^2(0,T_0)} \;\Bigg|\; \begin{array}{l} u \in L^2(0, T_0)^m, \\ x(T_0) = \bar{x}, \\ \quad \text{with } x \text{ is the solution of (1a) with} \\ \quad \text{initial condition } x^0 \text{ and control } u, \end{array} \quad (20)$$

and

$$J_1(\bar{x}; T_1) = \min \; \|u\|^2_{L^2(0,T_1)} \;\Bigg|\; \begin{array}{l} u \in L^2(0, T_1)^m, \\ Cx(t) = C\bar{x} \qquad (\forall t \in [0, T_1]), \\ \quad \text{with } x \text{ is the solution of (1a) with} \\ \quad \text{initial condition } \bar{x} \text{ and control } u. \end{array} \quad (21)$$

Let us first give some properties on J_1.

Proposition 4 *Let $T_1 > 0$, $\bar{y} \in \mathbb{R}^q$ and $\bar{x} \in C^{-1}\{\bar{y}\}$ such that (16) is fulfilled. Then the minimization problem (21) admits a minimizer u given by*

$$u(t) = \left(F(BF)^\top E(t) - (CB)^\top MCA \right) x(t),$$

where $M \in M_q(\mathbb{R})$ is a symmetric operator such that for every $w \in \operatorname{Ran}(CB)$, $M(CB)(CB)^\top w = w$, and $F \in M_{m,\mathrm{d}_{CB}}(\mathbb{R})$, where $\mathrm{d}_{CB} = \dim \operatorname{Ker}(CB)$, is such that $\operatorname{Ran} F = \operatorname{Ker}(CB)$ and $|Fw| = |w|$ for every $w \in \mathbb{R}^{d_{CB}}$, *and finally, $E(t) \in M_n(\mathbb{R})$ is solution of the backward Riccati equation:*

$$\begin{aligned}\dot{E} &= (CA)^\top MCA - EBF(BF)^\top E \\ &\qquad - \left(A - B(CB)^\top MCA\right)^\top E \\ &\qquad - E\left(A - B(CB)^\top MCA\right), \qquad (22a)\\ E(T_1) &= 0. \qquad (22b)\end{aligned}$$

In addition, we have,

$$J_1(\bar{x}; T_1) = -\bar{x}^\top E(0)\bar{x},$$

PROOF. Let us decompose $\mathbb{R}^m$ in $\operatorname{Ker}(CB) \oplus (\operatorname{Ker}(CB))^\perp = \operatorname{Ran} F \oplus \operatorname{Ran}(CB)^\top$ and set $u(t) = Fw(t) + (CB)^\top v(t)$, with $w(t) \in \mathbb{R}^{\mathrm{d}_{CB}}$ and $v(t) \in \mathbb{R}^q$. Consequently, we aim to minimise

$$\|u\|^2_{L^2(0,T_1)} = \int_0^{T_1} \left(|w(t)|^2 + |(CB)^\top v(t)|^2\right) \, \mathrm{d}t$$

under the constraint $Cx(t) = C\bar{x}$ for every $t \in [0, T_1]$. Note that, without modifications, we can assume that $v(t) \in \left(\operatorname{Ker}(CB)^\top\right)^\perp = \operatorname{Ran}(CB)$. Furthermore, the constraint $Cx(t) = \mathit{const}$ is satisfied if and only if $-CAx = (CB)(CB)^\top v$. For every $x \in \mathbb{R}^n$ the last equation admits one and only one solution $v = v(x) = -MCAx \in \operatorname{Ran}(CB)$. Consequently, the problem is to find $w \in L^2(0,T)^{\mathrm{d}_{CB}}$ such that it minimises

$$\int_0^{T_1} \left(|w(t)|^2 + \langle MCAx(t), CAx(t)\rangle\right) \, \mathrm{d}t,$$

where x is the solution of

$$\dot{x} = \left(A - B(CB)^\top MCA\right)x + BFw, \qquad x(0) = \bar{x}.$$

This is an unconstrained linear quadratic problem and according to [2], see also [12, Theorem 4.11], the corresponding minimizer is given by $w(t) = (BF)^\top E(t)x(t)$, with E defined by (22). Furthermore, we have,

$$\min_{w \in L^2(0,T)^{\mathrm{d}(CB)}} \int_0^T \left(|w(t)|^2 + \langle MCAx(t), CAx(t)\rangle\right) \mathrm{d}t = -\bar{x}^\top E(0)\bar{x} \qquad (23)$$

and $E(0)$ is a non-positive matrix. □

Remark 5 *In order to solve the Riccati equation (22), a classical approach (see for instance [9,10]) is to perform the change of variables $E(t) = P(t)Q(t)^{-1}$, with $P(t) \in M_n(\mathbb{R})$ and $Q(t) \in GL_n(\mathbb{R})$, and where P and Q are solutions of*

$$\begin{aligned}\dot{P} &= (CA)^\top MCAQ - \left(A - B(CB)^\top MCA\right)^\top P,\\ \dot{Q} &= BB^\top P + \left(A - B(CB)^\top MCA\right)Q,\end{aligned}$$

with the terminal conditions $P(T_1) = 0$ and $Q(T_1) = \mathrm{I}_n$.

The problem (19) can then reset as the minimization problem:

$$\begin{aligned}&\min \int_0^{T_0} |u(t)|^2 \, \mathrm{d}t - x(T_0)^\top E(0)x(T_0)\\ &\quad \Big| \; u \in L^2(0,T_0)^m, \qquad (24)\\ &\quad \Big| \; Cx(T_0) = \bar{y} \quad \text{and} \quad \Xi_K x(T_0) = 0,\end{aligned}$$

where x is the solution of (1) with initial condition x^0 and control u and $E(0)$ is given by (22).

Remark 6 *Let us also point out that $J_0(\cdot, T_0)$ can also be expressed as a quadratic function. More precisely, set $\bar{x} \in R_{T_0}(x^0)$, we aim to find the optimal control u of minimal L^2-norm such that the solution x of (1a) satisfies $x(T_0) = \bar{x}$.*
Following classical optimal control results, a norm optimal control u is given by

$$u(t) = B^\top p(t),$$

where $p \in \mathbb{R}^n$ is the solution of the adjoint problem

$$-\dot{p} = A^\top p, \qquad p(T_0) = \bar{p}$$

whose terminal datum $\bar{p} \in \mathbb{R}^n$ is given as solution to

$$\Gamma_{T_0}\bar{p} = \bar{x} - e^{T_0 A}x^0,$$

with $\Gamma_{T_0} = \int_0^{T_0} e^{(T_0-t)A}BB^\top e^{(T_0-t)A^\top} \, \mathrm{d}t$, the control Gramian. Consequently, the norm of the optimal control is given by $\|u\|^2_{L^2(0,T_0)^m} = \bar{p}^\top \Gamma_{T_0}\bar{p}$.
Note that here, since the controllability of the pair (A, B) is not assumed, the matrix Γ_{T_0} is not necessarily regular. However, the fact that $\bar{x}$ belongs to $R_{T_0}(x^0)$, ensures that $\bar{x} - e^{T_0 A}x^0 \in \operatorname{Ran}\Gamma_{T_0}$, and consequently, the existence of a particular $\bar{p}_p \in \mathbb{R}^n$ such that $\Gamma_{T_0}\bar{p}_p = x^1 - e^{T_0 A}x^0$. In addition, we have

$$\left\{\bar{p} \in \mathbb{R}^n \;\middle|\; \Gamma_{T_0}\bar{p} = \bar{x} - e^{T_0 A}x^0\right\} = \{\bar{p}_p\} + \operatorname{Ker}\Gamma_{T_0}$$

and it is easy to check that the L^2-norm of the control u given by $u(t) = B^\top e^{(T_0-t)A^\top}\bar{p}$, for $\bar{p} \in \{\bar{p}_p\} + \operatorname{Ker}\Gamma_{T_0}$ is independent of $\bar{p}$. Furthermore, there exist a non-negative matrix $Q_{T_0} \in M_n(\mathbb{R})$ such that $Q_{T_0}\Gamma_{T_0}\bar{p} = \bar{p}$ for $\bar{p} \in \operatorname{Ran}\Gamma_{T_0}$, and

$$\|u\|^2_{L^2(0,T_0)^m} = \left(\bar{x} - e^{T_0 A}x^0\right)^\top Q_{T_0}\left(\bar{x} - e^{T_0 A}x^0\right).$$

In particular, the optimal control u of minimal L^2-norm such that the solution x of (1a) *satisfies $x(T_0) = \bar{x}$ is given by*

$$u(t) = B^\top e^{(T_0-t)A^\top} Q_{T_0} \left(\bar{x} - e^{T_0 A} x^0\right). \qquad (25)$$

Several remarks are in order.

- The problem (19) is a quadratic minimization problem determined by positive semi-definite matrices, thus allowing for a solution which in general does not have to be unique. However, the norm of the corresponding optimal control is independent of a choice of a solution to (19).
- The minimizer $\bar{x}$ of the problem (19) does not have to produce the control $u \in L^2(0, T_0)^m$ minimising the energy required to steer the system to the pre-image of $\bar{y}$ under C. This is because here we consider the global minimization problem, in which we also take into account the cost of keeping the solution fixed with respect to C for $t > T_0$ (this is the terminal cost in (24)). Putting a larger effort in the initial time period might be compensated with a lower cost in the subsequent period.
- If $\bar{y} = 0$ then the set $\operatorname{Ker} \Xi_K \cap C^{-1}\{\bar{y}\}$ coincides with the kernel of the operator C_{K+1}.

4 Application to long-time averaged control problem

In this section, we apply the Theorem 3 to a long-time averaged control problem. Recall that averaged controllability is a particular situation of output controllability with matrices given by (5).

For the sake of example, we restrict the analysis to the case of the null control, i.e. the goal is to steer and keep the average equal to zero. We also consider the matrices (5) with $d = 2$ components, and we chose $\sigma_1 = \sigma_2 = 1/2$, and $B_1 = B_2 = \tilde{B}$. In the sequel, P denotes the orthogonal projector of $\mathbb{R}^n$ on $\operatorname{Ker}(CB)^\top = \operatorname{Ker} \tilde{B}^\top$, and we set $D = (A_1 - A_2)/2 \in M_n(\mathbb{R})$, $S = (A_1 + A_2)/2 \in M_n(\mathbb{R})$, and $K \in M_{2n,2nm}(\mathbb{R})$, the Kalman controllability matrix,

$$\mathcal{K} = \begin{pmatrix} B & AB & \dots & A^{2n-1}B \end{pmatrix} = \begin{pmatrix} \tilde{B} & A_1\tilde{B} & \dots & A_1^{2n-1}\tilde{B} \\ \tilde{B} & A_2\tilde{B} & \dots & A_2^{2n-1}\tilde{B} \end{pmatrix} \qquad (26)$$

Instead of considering the sequence $(C_k)_k$ introduced in (10), we are going to use Remark 4, and to consider the sequence $(L_k)_{k\in\mathbb{N}}$ defined by

$$L_k = \begin{pmatrix} \mathrm{I}_n & \mathrm{I}_n \\ PD & -PD \\ PDS & -PDS \\ \vdots & \vdots \\ PDS^{k-1} & -PDS^{k-1} \end{pmatrix} \in M_{(k+1)n,2n}(\mathbb{R}). \qquad (27)$$

Let us check that the sequence $(L_k)_k$ can be chosen in place of the sequence $(C_k)_k$ introduced in (10). In fact,

- **for** $k = 0$**,** we choose $L_0 = 2C$;
- **for** $k = 1$**,** let P_0 be the orthogonal projector of $\mathbb{R}^n$ on $\operatorname{Ker}(L_0 B)^\top = \operatorname{Ker} \tilde{B}^\top$, P_0 actually coincides with P. Then we set $\tilde{L}_1 = \begin{pmatrix} L_0 \\ PL_0A \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & \mathrm{I}_n \\ PA_1 & PA_2 \end{pmatrix}$. We conclude that the matrix L_1, given by (27), is suitable by noticing that $\operatorname{Ker} \tilde{L}_1 = \operatorname{Ker} L_1$;
- **recurrence relation,** we assume that at step k, the matrix L_k, given by (27), is suitable. We define P_k, the orthogonal projector of $\mathbb{R}^{(k+1)n}$ on $\operatorname{Ker}(L_k B)^\top = \operatorname{Ker} B^\top \times \mathbb{R}^{kn}$. Thus, $P_k \in M_{(k+1)n}(\mathbb{R})$ is the block diagonal matrix formed by the matrices $P, \mathrm{I}_n, \dots, \mathrm{I}_n$. Then, we set

$$\tilde{L}_{k+1} = \begin{pmatrix} L_0 \\ P_k L_k A \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & \mathrm{I}_n \\ PA_1 & PA_2 \\ PDA_1 & -PDA_2 \\ PDSA_1 & -PDSA_2 \\ \vdots & \vdots \\ PDS^{k-1}A_1 & -PDS^{k-1}A_2 \end{pmatrix}.$$

  We conclude that the matrix L_{k+1}, given by (27), is suitable by noticing that $\operatorname{Ker} \tilde{L}_{k+1} = \operatorname{Ker} L_{k+1}$.

Remark 7 *As for Lemma 1, we have* $\operatorname{Ker} L_{k+1} \subset \operatorname{Ker} L_k \subset \operatorname{Ker} L_0 \subset \mathbb{R}^{2n}$*. Since* $\dim \operatorname{Ker} L_0 = n$*, there exist* $K \in \{0, \dots, n\}$ *such that* $\operatorname{Ker} L_{K+1} = \operatorname{Ker} L_K$*, and we have* $\operatorname{Ker} L_K = \operatorname{Ker} L_n$ *(see Lemma 1 again).*

As direct consequence of Theorem 3 and the above comments, we obtain the following result.

Corollary 5 *Let $d = 2$, $n \in \mathbb{N}^*$ and let $A_1, A_2 \in M_n(\mathbb{R})$ and $B_1 = B_2 = \tilde{B} \in M_{n,m}(\mathbb{R})$. Then for every $x_1^0, x_2^0 \in \mathbb{R}^n$, the system* (4) *is long-time average controllable to* 0 *for the parameters $\sigma_1 = \sigma_2 = 1/2$ if and only if*

$$\operatorname{rank} L_n\mathcal{K} = \operatorname{rank} L_n,$$

with $\mathcal{K}$ defined by (26)*, and the operator L_n given by* (27)*, for $k = n$.*

Remark 8 *The Corollary 5 ensures that the solutions x_1 and x_2 of* (4) *(with $d = 2$ and $B_1 = B_2 = \tilde{B}$) can be steered to some $\begin{pmatrix} x_1(T)^\top & x_2(T)^\top \end{pmatrix}^\top \in \operatorname{Ker} L_n$. This condition can be equivalently rewritten as*

$$x_1(T) + x_2(T) = 0$$
$$x_1(T) - x_2(T) \in \Big\{\delta \in \mathbb{R}^n \ \mid\ \forall k \in \{0, \cdots, n-1\}, \; DS^k \delta \in \operatorname{Ran} \tilde{B}\Big\}.$$

The last inclusion implies the existence of u such that

$$\tilde{B}u(t) = -De^{tS}(x_1 - x_2)(T) \qquad t \geqslant T \qquad (28)$$

and it is easy to check that such control ensures long-time averaged controllability. Indeed, denoting $s = (x_1 + x_2)/2$ and $\delta = (x_1 - x_2)/2$, the system (4) *(with $d = 2$ and $B_1 = B_2 = \tilde{B}$) can be rewritten as*

$$\dot{s} = Ss + D\delta + \tilde{B}u$$
$$\dot{\delta} = Ds + S\delta.$$

Now it becomes obvious that $s(t) = 0$ for $t \geq T$ if and only if there exists a control of the form (28).

Let us conclude this section with a complete example.

Example 1 *Set $n = 2$ and $m = 1$, and consider the matrices*

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \tilde{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that the matrix P, the orthogonal projector of $\mathbb{R}^2$ on $\operatorname{Ker} \tilde{B}^\top$, is given by $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Recall that we set $D = (A_1 - A_2)/2$ and $S = (A_1 + A_2)/2$. Let us also define $L = L_0 = 2C$. We compute

$$\mathcal{K} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad L\mathcal{K} = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

and we observe that $\operatorname{rank} \mathcal{K} = 3$ *and* $\operatorname{rank} L\mathcal{K} = 2 = \operatorname{rank} L$*. This ensures that this system is controllable in average, but not simultaneously controllable.*
In order to check the long-time averaged controllability, we compute the matrix L_n ($n = 2$) given by (27)*. We obtain*

$$L_n = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{rank} L_n = 3$ *and* $\operatorname{Ker} L_n = \big(1,\, 1,\, -1,\, -1\big)^\top \mathbb{R}$.
Similarly, we check that $\operatorname{rank} L_n\mathcal{K} = 3 = \operatorname{rank} L_n$*, which ensures this system has the long-time averaged controllability property.*

Let us now consider the norm optimal control problem. To this end, we set $T_0, T_1 > 0$, and we recall that we aim to minimise the L^2-norm of the controls u such that the solution of the system (1a) *satisfies $Lx(t) = 0$ for every $t \in [T_0, T_0 + T_1]$. By Proposition 2 the last condition is equivalent to $L_n x(T) = 0$, i.e., we have to steer the state of the system into* $\operatorname{Ker} L_n$ *. Denoting $\bar{x} = x(T_0) \in \mathbb{R}^4$, the reached point at time T_0, $\bar{x} \in \operatorname{Ker} L_n$ means that there exist $\xi \in \mathbb{R}$ such that $\bar{x} = (\xi,\, \xi,\, -\xi,\, -\xi)^\top$.*

Minimal L^2-norm control steering x^0 to $\bar{x}$ in time T_0.
In order to compute explicitly this L^2-norm, we follow the lines of Remark 6. To this end, we compute the Gramian matrix, $\Gamma_{T_0} = \int_0^{T_0} e^{tA}BB^\top e^{tA^\top}\,\mathrm{d}t$. This matrix (not displayed here) takes the form $\begin{pmatrix} G_{T_0} & 0 \\ 0 & 0 \end{pmatrix} \in M_4(\mathbb{R})$, with G_{T_0} a regular 3×3 matrix. In particular, we have $\operatorname{rank} \Gamma_{T_0} = \operatorname{rank} G_{T_0} = 3$ *and* $\operatorname{Ran} \Gamma_{T_0} = \mathbb{R}^3 \times \{0\}$*. Since $\bar{x}$ should be a reachable point, it must hold that $\bar{x} - e^{T_0 A}x^0 \in \operatorname{Ran} \Gamma_{T_0}$. Denoting by x_4^0 the 4^{th}-component of x^0, we conclude that we shall have,*

$$\xi = -e^{T_0} x_4^0 \qquad (29)$$

and hence, $\bar{x} = -e^{T_0} x_4^0 (1,\, 1,\, -1,\, -1)^\top$.
Let us now compute a solution $\bar{p} \in \mathbb{R}^4$ such that $\Gamma_{T_0}\bar{p} = \bar{x} - e^{T_0 A}x^0$. It is easy to see that all the solution $\bar{p}$ of $\Gamma_{T_0}\bar{p} = \bar{x} - e^{T_0 A}x^0$ are of the form $\bar{p} = \begin{pmatrix} \bar{q} \\ \bar{p}_4 \end{pmatrix}$, with $\bar{p}_4 \in \mathbb{R}$ and $\bar{q} \in \mathbb{R}^3$ is solution of

$$G_{T_0}\bar{q} = \hat{x},$$

with $\hat{x}$ the vector of $\mathbb{R}^3$ formed by the first three components of $\bar{x} - e^{T_0 A}x^0$. We then deduce that the norm-optimal control u is given by

$$u(t) = B^\top e^{(T_0 - t)A^\top} \begin{pmatrix} G_{T_0}^{-1}\hat{x} \\ 0 \end{pmatrix}$$

and the corresponding L^2-norm is,

$$\|u\|^2_{L^2(0,T_0)} = \hat{x}^\top G_{T_0}^{-1} \hat{x}.$$

Minimal L^2-norm control in order to stay in $\operatorname{Ker} L$.
In order to compute this L^2-norm, we follow the lines of Section 3. First of all, we assume we have reached a point $\bar{x} \in \operatorname{Ker} L_n$, i.e., $\bar{x} = (\xi, \xi, -\xi, -\xi)^\top$ for ξ given by (29). We aim to find the minimal L^2-norm control such that the solution of (4), starting from $\bar{x}$ at time T_0 satisfies $Lx(t) = 0$ for every $t \in [T_0, T_0 + T_1]$.
One can easily compute that $\operatorname{Ker} LB = \{0\}$. Consequently, the control keeping the solution in the kernel of L is uniquely determined and takes the form

$$u(t) = (LB)^\top v(t),$$

with $v(t) \in \operatorname{Ran} LB$ solution of

$$-LAx(t) = (LB)(LB)^\top v(t). \tag{30}$$

We have, $LB = (2 \quad 0)^\top$ and $\operatorname{Ran} LB = \mathbb{R} \times \{0\}$. Thus, for $M = \begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix}$ it holds $M(LB)(LB)^\top w = w$ for every $w \in \operatorname{Ran} LB$. Acting by M on (30) we deduce that the required optimal control is a feedback control of the form

$$u = (LB)^\top MLAx \tag{31}$$

and x is solution of

$$\dot{x} = \left(A - B(LB)^\top MLA\right) x, \qquad x(0) = (\xi, \xi, -\xi, -\xi)^\top \in \mathbb{R}^4.$$

After some computation, one can check that $(1, 1, -1, -1)^\top$ is an eigenvector of $A - B(LB)^\top MLA$ for the eigenvalue 1. Consequently, we have

$$x(t) = \xi e^{(t-T_0)} (1, 1, -1, -1)^\top \qquad (t \geqslant T_0)$$

and finally, the L^2-norm of the control is

$$\|u\|^2_{L^2(T_0,T_0+T_1)} = \int_{T_0}^{T_0+T_1} \langle MLAx(t), LAx(t) \rangle \, \mathrm{d}t = 2\xi^2 \left(e^{2T_1} - 1\right),$$

with ξ given by (29).

Numerical illustration.
Let us fix $T_0 = 2$ and $T_1 = 1$ and consider $x^0 = (1, 2, -1, 3)^\top$ as initial condition. Then we compute the control of minimal L^2-norm, and obtain $\|u\|_{L^2(0,T_0+T_1)} \simeq 86.39$. The corresponding state trajectories and control are displayed on Figure 1.

The system is first governed by the control steering it to the unique accessible point of the $\operatorname{Ker} L_n$. Reaching it at time T_0, the feedback control (31) is turned on, resulting in a zero average for $t > T_0$ (Figure 1c). Recall that in order to keep the average constant, we have to remain within $\operatorname{Ker} L_n$ during the entire second time period. This is clearly visible from (Figure 1b).

Remark 9 *With the matrices $\tilde{B}$ and A_1 used in Example 1, one can also define a matrix A_2 such that the system (4) is:*

- *not controllable in average ($A_2 = -A_1$);*
- *controllable in average but does not have the long-time averaged controllability property ($A_2 = \mathrm{I}_2$);*
- *simultaneously controllable ($A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$).*

5 Concluding remarks and open questions

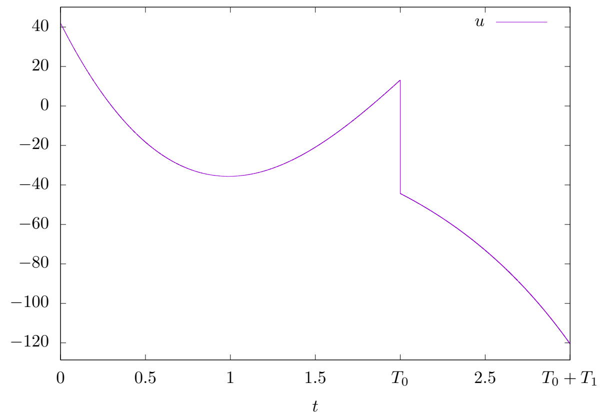
In this paper, we have extended the notion output controllability to the notion of long-time output controllability. We provide necessary and sufficient condition for this property to hold, as well as the explicit formula for the optimal control keeping the average fixed.

Although we present a rather complete theory for the proposed problem, there are some interesting open questions arising from the obtained results.

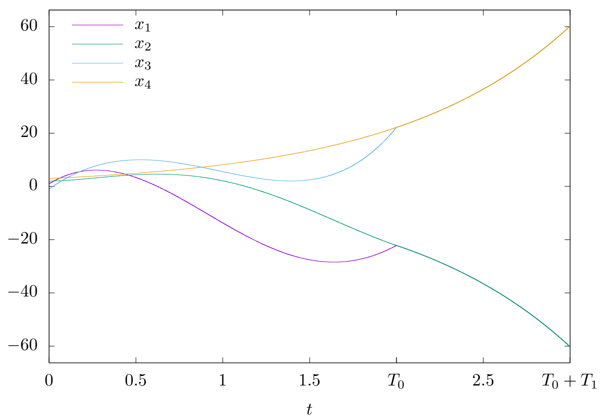
- If the system is output controllable, then for every initial condition x^0 and every positive time T, one can find a minimal L^2-norm control u such that $y(T) = 0$. However, the behaviour of this L^2-norm with respect to T is not clear at all.
- Does output controllability imply output feed-back stabilisation? More precisely, if the system is output controllable, does it exist a feed-back control $u(t) = Ky(t)$ such that $y(t)$ goes to 0 as t goes to $+\infty$? Note that when the system is controllable, it is automatically stabilisable, with the help of the pole placement Theorem (see for instance [13, Theorem 2.1]).

Let us also point out some open questions related to averaged controllability problems.

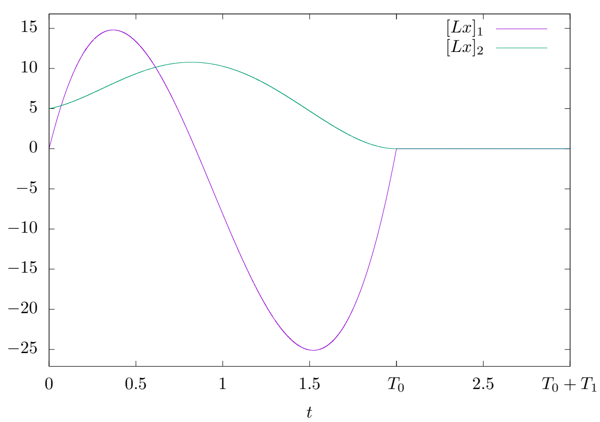
- The results provided here deal with a finite number of parameters. The same question has to be addressed to the situation where an infinite number of parameters (either discrete or continuous) are involved. In this case, we consider a system $\dot{x}_\zeta(t) = A_\zeta x_\zeta(t) + B_\zeta u(t)$ with initial condition $x_\zeta(0) = x^0_\zeta$, where $\zeta \in \Omega$ is an unknown parameter and with $(\Omega, \mathcal{F}, \mu)$ a probability space. If we assume that $\int_\Omega x^0_\zeta \, \mathrm{d}\mu_\zeta = 0$ under which conditions does it exist a control u independent of ζ such that $\int_\Omega x_\zeta(t) \, \mathrm{d}\mu_\zeta = 0$ for every $t > 0$?



(a) Control.



(b) State.



(c) Average of the state.

Fig. 1. State trajectories for minimal L^2-norm control such that $Lx(t) = 0$ for $t \in [T_0, T_0 + T_1]$. We considered $T_0 = 1$, $T_1 = 2$ and the system (1) with matrices A and B given by (5) (precise values are defined in Example 1), and initial conditions $x^0 = (1,\ 2,\ -1,\ 3)^\top$.

- A similar question could have also been addressed for partial differential equations. In this case, the algebraic relation we used surely fails.

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