NON-ASYMPTOTIC CONCENTRATION INEQUALITY FOR AN APPROXIMATION OF THE INVARIANT DISTRIBUTION OF A DIFFUSION DRIVEN BY COMPOUND POISSON PROCESS

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A. GLOTER, I. HONORÉ, AND D. LOUKIANOVA

Abstract. In this article we approximate the invariant distribution ν of an ergodic Jump Diffusion driven by the sum of a Brownian motion and a Compound Poisson process with sub-Gaussian jumps. We first construct an Euler discretization scheme with decreasing time steps, particularly suitable in cases where the driving Lévy process is a Compound Poisson. This scheme is similar to those introduced by Lamberton and Pagès in [LP02] for a Brownian diffusion and extended by Panloup in [Pan08b] to the Jump Diffusion with Lévy jumps. We obtain a non-asymptotic Gaussian concentration bound for the difference between the invariant distribution and the empirical distribution computed with the scheme of decreasing time step along a appropriate test functions f such that $f - \nu(f)$ is is a coboundary of the infinitesimal generator.

1. Introduction

1.1. **Setting.** Let $(\mathbf{X}_t)_{t\geq 0}$ be a d-dimensional càdlàg process solution of the stochastic differential equation:

$$d\mathbf{X}_t = b(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dW_t + \kappa(\mathbf{X}_{t^-})d\mathbf{Z}_t. \tag{E}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$ and $\kappa : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$ are Lipschitz continuous, $(W_t)_{t\geq 0}$ is a Wiener process of dimension r, and $(\mathbf{Z}_t)_{t\geq 0}$ is a $\mathbb{R}^r$ -valued compound Poisson process (CPP), $\mathbf{Z}_t = \sum_{k=1}^{N_t} \mathbf{Y}_k$, where $(\mathbf{Y}_k)_{k\in\mathbb{N}}$ are i.i.d. r -dimensional random vectors with common distribution π on $\mathcal{B}(\mathbb{R}^r)$ and $(N_t)_{t\geq 0}$ is a Poisson process, independent of $(\mathbf{Y}_k)_{k\in\mathbb{N}}$. The processes $(W_t)_{t\geq 0}$ and $(\mathbf{Z}_t)_{t\geq 0}$ are assumed to have the same dimension for the sake of simplicity. Moreover, $(N_t)_{t\geq 0}$, $(\mathbf{Y}_k)_{k\in\mathbb{N}}$ and $(W_t)_{t\geq 0}$ are independent and defined on a given filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$. We assume that b, σ, and κ satisfy a suitable Lyapunov condition (assumption $(\mathcal{L}_{\mathbf{V}})$ in Section 1.3) which ensures the existence of an invariant distribution ν of $(\mathbf{X}_t)_{t\geq 0}$ (see [Pan08b]). For the sake of simplicity we also assume the uniqueness of the invariant distribution. We refer to [Mas07] under irreductibility and Lyapunov conditions for the existence and uniqueness of the invariant distribution for a diffusion driven by Lévy process.

The aim of this paper is to establish a non-asymptotic bound on the probability of the deviation $\nu_n(f) - \nu(f)$, where ν_n is an appropriate empirical measure such that $\lim_{n\to\infty} \nu_n(f) = \nu(f)$ $a.s.$ for all suitable test functions f.

The algorithm that we define in this article is based on an Euler-like discretization scheme with decreasing time step $(\gamma_n)_{n\geq 1}$ s.t. $\lim_n \gamma_n = 0$. Lamberton and Pagès first introduced such a scheme in [LP02] for a Brownian diffusion. They showed that the empirical measure of their scheme converges to the invariant measure of the diffusion and that it satisfies the Central Limit Theorem. The decreasing steps allows to the empirical measure to directly converge towards the invariant one. If we choose a constant time step $\gamma_k = h > 0$ in the scheme, the expected ergodic theorem is $\nu_n(f) \xrightarrow[n]{a.s.} \nu^h(f) = \int_{\mathbb{R}^d} f(x)\nu^h(dx)$, where ν^h is the invariant distribution of the scheme which is

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supposed to converge toward the invariant measure of the diffusion (E) when $h \to 0$ (see e.g. For more details about this approach we refer to [TT90], [Tal02] and [MT06]).

Next, Panloup in [Pan08a] and [Pan08b] adapted the algorithm of [LP02] to the Jump Diffusion with Lévy jumps [Pan08b] and also showed the convergence and the Central Limit Theorem for the empirical measure in this case. In the same way as the questions of the convergence of the empirical measure ν_n or of its limiting distribution, the natural question is that of the nature of the deviations $\nu_n(f) - \nu(f)$ along appropriate test functions f. In the case of the Brownian diffusion this question was considered in [HMP18] and [Hon17]. Note that in the Brownian diffusion case the innovations of the Euler scheme are designed in order to "mimic" Brownian increments, hence it is natural to assume that they satisfy some Gaussian Concentration property (assumption **(GC)** in Section 1.3). In particular this Gaussian Concentration property is satisfied by Gaussian or symmetric Bernoulli law. Taken as an assumption on the Brownian innovations of the scheme, it allows to show a non-asymptotic Gaussian Concentration bound for the probability of the deviations of $\nu_n(f)$ from $\nu(f)$, see [HMP18] and [Hon17] with sharp constants. The deviation $\nu_n(f) - \nu(f)$ is evaluated along the functions f such that $f - \nu(f)$ is a coboundary of the infinitesimal generator of the diffusion.

When the diffusion contains Lévy jumps, it is not generally expected that these deviations will show a Gaussian behaviour. But such a behaviour seems natural if we suppose that the driving Levy process is a Compound Poisson process and the jump size vectors $(\mathbf{Y}_k)_{k\in\mathbb{N}}$ satisfy a Gaussian Concentration property **(GC)**. In this paper, we focus on this situation. Before giving its precise formulation we need to introduce some notations. First of all, we introduce our discretization scheme. In general, for a Euler scheme corresponding to a Jump Diffusion with Lévy jumps, one has to define a numerically computable jump vectors designed to "mimic" the increments of the driving Lévy process. In most cases, the increments of a Lévy process are not numerically computable, that is why it is important to propose different ways to approximate these increments according to the nature of the driving Lévy process. In this paper we introduce a scheme (S) particularly suitable in the case where a driven Lévy process is a Compound Poisson. Note that our scheme is close to the scheme (C) of [Pan08b]. Like in the previously mentioned articles, we denote time steps $(\gamma_k)_{k\geq 1}$, and for all $n \geq 0$, we define:

$$X_{n+1} = X_n + \gamma_{n+1} b(X_n) + \sqrt{\gamma_{n+1}}\sigma(X_n)U_{n+1} + \kappa(X_n)Z_{n+1}, \tag{S}$$

where X_0 is an $\mathbb{R}^d$ valued random variables such that $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, $(U_n)_{n\geq 1}$ is an i.i.d. sequence of centered random variables matching the moments of the Gaussian law on $\mathbb{R}^r$ up to order three, independent of X_0. Furthermore, for all $n \geq 1$ we put

$$Z_n := B_n Y_n, \tag{1.1}$$

where $(B_n)_{n\geq 1}$ are one-dimensional independent Bernoulli random variables, independent of X_0, $(U_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$, s.t. $B_n \overset{(\text{law})}{=} \mathcal{B}ern(\mu\gamma_n)$, where μ is an intensity of the Poisson process driving the CPP $(\mathbf{Z}_t)_{t\geq 0}$. Without loss of generality we can suppose from now on that $\mu = 1$. The choice (1.1) of the innovations $Z_n, n \in \mathbb{N}$ is motivated by the following heuristic reasoning: Z_n has to "mimic" the increment of the CPP $\mathbf{Z}_t = \sum_{k=1}^{N_t} Y_k$ on the small time interval of the length γ_n. The probability that the CPP does not jump on this interval is equal to $\exp(-\gamma_n) = 1 - \gamma_n + o(\gamma_n)$, and if the CPP jumps on this interval, it will most probably have only one jump. Hence we approximate the increment $\mathbf{N}_{\gamma_n}$ of the CPP by a $\{0,1\}$ random variable with the probability of 1 equal to γ_n.

We also introduce the empirical (random) measure of the scheme: for all $A \in \mathcal{B}(\mathbb{R}^d)$ (where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$):

$$\nu_n(A) := \nu_n(\omega, A) := \frac{\sum_{k=1}^n \gamma_k \delta_{X_{k-1}(\omega)}(A)}{\sum_{k=1}^n \gamma_k}. \tag{1.2}$$

Obviously, to study long time behaviour, we have to consider steps $(\gamma_k)_{k\geq 1}$ such that the current time of the scheme $\Gamma_n := \sum_{k=1}^n \gamma_k \underset{n}{\to} +\infty$. We recall as well that $\gamma_k \underset{k}{\downarrow} 0$. We suppose that both jump amplitudes $(Y_n)_{n\geq 1}$ and Brownian innovations $(U_n)_{n\geq 1}$ satisfy a Gaussian concentration (see further the assumption **(GC)**). As we already mentioned, the aim of the paper is to show that this assumption implies a non-asymptotic Gaussian Concentration inequality for the probability of the deviations of $\nu_n(f)$ from $\nu(f)$ (see Theorem 2, Section 2). The main argument in the proof of Theorem 2 is the fact that the **(GC)** property of jumps sizes $Y_k, k \in \mathbb{N}$, permits to show the similar Gaussian Concentration property for the jump innovations $Z_k, k \in \mathbb{N}$. This result is given in Proposition 1. However the Gaussian Concentration property of jump innovations depends on the dimension of the jump heights. This dependence survives in the main Theorem 2 giving Gaussian Concentration of the deviation of ν_n from ν.

The paper is organized as follows. In Section 1.2, we introduce some useful notations. The assumptions required for our main results are outlined in Section 1.3. In this part, we formulate a Gaussian concentration property of the jump innovation Z_n, the proof is given in Section 2.4. We state in Section 1.4 some already known results connected with the approximation scheme. Our main results are in Section 2, and the demonstration is located in Section 2.3. Section 3 is dedicated to the analysis of the exponential integrability of Lyapunov function. Technical lemmas are stated in Section 2.2, but their proofs are postponed to Section 4. Eventually, we propose a numerical illustration of our main result in Section 5.

1.2. **General notations.** We set for any step sequence $(\gamma_n)_{n\geq 1}$:

$$\forall \ell \in (0,+\infty),\ \Gamma_n^{(\ell)} := \sum_{k=1}^{n} \gamma_k^{\ell},\ \Gamma_n := \sum_{k=1}^{n} \gamma_k = \Gamma_n^{(1)},$$

where Γ_n corresponds to the current time, hence $\Gamma_n \underset{n\to+\infty}{\to} +\infty$. For the sake of simplicity, from now on, the time step sequence will have the form: $\gamma_n \asymp \frac{1}{n^\theta}$ with $\theta \in (0,1]$, where for two sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ the notation $u_n \asymp v_n$ means that $\exists n_0 \in \mathbb{N},\ \exists C \geq 1$ s.t. $\forall n \geq n_0,\ C^{-1}v_n \leq u_n \leq Cv_n$.

Henceforth, C will be a non negative constant, and $(e_n)_{n\geq 1}, (\mathscr{R}_n)_{n\geq 1}$ will be deterministic sequences s.t. $e_n \to_n 0$ and $\mathscr{R}_n \to_n 1$, that may change from line to line. The constant C as well as the sequences $(e_n)_{n\geq 1}, (\mathscr{R}_n)_{n\geq 1}$ depend on known parameters appearing in the hypotheses set in Section 1.3 (which will be called **(A)** further). Other possible dependencies will be explicitly specified.

We denote by I_m, $m \in \{d, r\}$ the identity matrix of dimension m.

Through the article, for any smooth enough function f, for $k \in \mathbb{N}$ we will denote $D^k f$ the tensor of the k^{th} derivatives of f. Namely $D^k f = (\partial_{i_1} \ldots \partial_{i_k} f)_{1\leq i_1,\ldots,i_k\leq d}$. Yet, for a multi-index $\alpha \in \mathbb{N}_0^d := (\mathbb{N}\cup\{0\})^d$, we set $D^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d} f : \mathbb{R}^d \to \mathbb{R}$.

For a β-Hölder continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we denote by

$$[f]_\beta := \sup_{x\neq x'} \frac{|f(x) - f(x')|}{|x - x'|^\beta} < +\infty,$$

its Hölder modulus of continuity. Here, $|x - x'|$ stands for the Euclidean norm of $x - x' \in \mathbb{R}^d$.

We define for $(p, d, m) \in \mathbb{N}^3$, $\mathcal{C}^p(\mathbb{R}^d, \mathbb{R}^m)$ the space of p-times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^m$. Furthermore, for $f \in \mathcal{C}^p(\mathbb{R}^d, \mathbb{R}^m)$, $p \in \mathbb{N}$, we set for $\beta \in (0,1]$ the Hölder modulus:

$$[f^{(p)}]_\beta := \sup_{x\neq x', |\alpha|=p} \frac{|D^\alpha f(x) - D^\alpha f(x')|}{|x - x'|^\beta} \leq +\infty,$$

where $\alpha \in \mathbb{N}^d$ is a multi-index of length p, namely $|\alpha| := \sum_{i=1}^d \alpha_i = p$. In other words, in the above definition, the $|\cdot|$ in the numerator is the usual absolute value. We will also use the notation $[\![n,p]\!]$, $(n,p) \in (\mathbb{N}_0)^2, n \le p$, for the set of integers being between n and p.

Let us introduce for $k \in \mathbb{N}_0, \beta \in (0,1]$ and $m \in \{1, d, d \times r\}$ the Hölder spaces

$$\mathcal{C}^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m) := \{f \in \mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^m) : \forall \alpha \in \mathbb{N}^d, |\alpha| \in [\![1,k]\!], \sup_{x\in\mathbb{R}^d} |D^\alpha f(x)| < +\infty, [f^{(k)}]_\beta < +\infty\},$$
$$\mathcal{C}_b^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m) := \{f \in \mathcal{C}^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m) : \|f\|_\infty < +\infty\}. \tag{1.3}$$

In the above definition, we denote for all bounded mapping $\zeta : \mathbb{R}^d \to \mathbb{R}^m$, $m \in \{1, d, d\times r\}$, the uniform norm $\|\zeta\|_\infty := \sup_{x\in\mathbb{R}^d} \|\zeta\zeta^*(x)\|$ with $\|\zeta(x)\| = \mathrm{Tr}\,(\zeta\zeta^*(x))^{1/2}$, where for $M \in \mathbb{R}^m \otimes \mathbb{R}^m$, $\mathrm{Tr}(M)$ is the trace of M. In particular, $\|\cdot\|$ is the Fröbenius norm [1].

Practically, with these notations, $\mathcal{C}^{k,\beta}(\mathbb{R}^d, \mathbb{R}^m)$ stands for the subset of $\mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^m)$ whose elements have bounded derivatives up to order k and β-Hölder continuous k^{th} derivatives. In particular, for $k = 0$, the space of Lipschitz continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^m$ is denoted by $\mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R}^m)$.

For a given Borel function $f : \mathbb{R}^d \to E$, where E can be $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}^d \otimes \mathbb{R}^r, \mathbb{R}^d \otimes \mathbb{R}^d$, we set for $k \in \mathbb{N}_0$:

$$f_k := f(X_k).$$

Moreover, for $k \in \mathbb{N}_0$, we denote

$$\mathcal{F}_k := \boldsymbol{\sigma}\big(X_0, (U_j, Z_j)_{j\in[\![1,k]\!]}\big) \quad \text{and} \quad \widetilde{\mathcal{F}}_k := \boldsymbol{\sigma}\big(X_0, (U_j, Z_j)_{j\in[\![1,k]\!]}, U_{k+1}\big). \tag{1.4}$$

Eventually, we define the infinitesimal generator associated with the diffusion (E) which writes for all $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$:

$$\begin{aligned} \mathcal{A}\varphi(x) &= b(x)\nabla\varphi(x) + \frac{1}{2} Tr\left(\sigma\sigma^*(x) D^2\varphi(x)\right) + \int_{\mathbb{R}^d} \left(\varphi(x + \kappa(x)y) - \varphi(x)\right)\pi(dy) \\ &=: \widetilde{\mathcal{A}}\varphi(x) + \int_{\mathbb{R}^d} \left(\varphi(x + \kappa(x)y) - \varphi(x)\right)\pi(dy), \end{aligned} \tag{1.5}$$

where π stands for the distribution of Y_1, and $\widetilde{\mathcal{A}}$ is the infinitesimal generator of the continuous part of the diffusion.

1.3. **Hypotheses.** We assume the following set of hypothesis about the coefficients of the SDE (E) and the parameters of the scheme (S):

(C0) The functions $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$ and $\kappa : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$ are globally Lipschitz continuous.

(C1) The first value of the scheme X_0 is sub-Gaussian: there exists $\lambda_0 \in \mathbb{R}_+^*$ such that

$$\forall \lambda < \lambda_0, \quad \mathbb{E}\left[\exp(\lambda |X_0|^2)\right] < +\infty.$$

(C2) Defining for all $x \in \mathbb{R}^d$, $\Sigma(x) := \sigma\sigma^*(x)$, $K(x) = \kappa\kappa^*(x)$, we suppose that

$$\sup_{x\in\mathbb{R}^d} \mathrm{Tr}(\Sigma(x)) = \sup_{x\in\mathbb{R}^d} \|\sigma(x)\|^2 =: \|\sigma\|_\infty^2 < +\infty, \quad \sup_{x\in\mathbb{R}^d} \mathrm{Tr}(K(x)) = \sup_{x\in\mathbb{R}^d} \|\kappa(x)\|^2 =: \|\kappa\|_\infty^2 < +\infty.$$

1. Remark that this notation is also available for vector norms. Indeed, any $\mathbb{R}^d$ vectors can be regarded as line vectors, and we define similarly for a vector or for a matrix the uniform norm $\|\cdot\|_\infty$.

(GM) The sequences of random variables $(U_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are respectively i.i.d., such that

$$\mathbb{E}\,[U_1] = \mathbb{E}\,[Y_1] = 0; \quad \mathbb{E}[(U_1^i U_1^j)_{1\leq i,j\leq r}] =: \mathbb{E}[U_1^{\otimes 2}] = I_r,$$
$$\mathbb{E}[Y_1^{\otimes 2}] = I_r; \quad \mathbb{E}[(U_1^i U_1^j U_1^k)_{1\leq i,j;k\leq r}] =: \mathbb{E}[U_1^{\otimes 3}] = 0^{\,2}.$$

Also, $(U_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ and X_0 are independent.

(GC) *We say that r.v.* $G \in \mathbb{L}^1$ *satisfies Gaussian concentration property, if for every Lipschitz continuous function* $g : \mathbb{R}^r \to \mathbb{R}$ *and every* $\lambda > 0$*:*

$$\mathbb{E}\big[\exp(\lambda g(G))\big] \leq \exp\left(\lambda \mathbb{E}\,[g(G)] + \frac{\lambda^2 [g]_1^2}{2}\right), \tag{1.6}$$

We assume that U_1 and Y_1 satisfy the Gaussian concentration property.

$(\mathcal{L}_{\mathbf{V}})$ We assume the following Lyapunov like stability condition:

There exists a non-negative function $V : \mathbb{R}^d \longrightarrow [v^*, +\infty)$ with $v^* > 0$ such that

i) V is a $\mathcal{C}^2$ continuous function s.t. $\|D^2V\|_\infty < +\infty$, and $\lim_{|x|\to\infty} V(x) = +\infty$.

ii) There is $C_V \in (0, +\infty)$ such that for all $x \in \mathbb{R}^d$:

$$|\nabla V(x)|^2 + |b(x)|^2 \leq C_V V(x).$$

iii) There exist $\alpha_V > 0$, $\beta_V \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}^d$,

$$\mathcal{A}V(x) \leq -\alpha_V V(x) + \beta_V.$$

(U) There is a unique invariant distribution ν to equation (E).

(S) We assume that the sequence $(\gamma_k)_{k\geq 1}$ is small enough, namely for all $k \geq 1$,

$$\gamma_k \leq \frac{1}{2}\min(2, \frac{\alpha_V}{4\big(\frac{[b]_1 C_V}{8} + (\frac{\sqrt{C_V}[b]_1}{2} + \frac{\sqrt{C_V}\|D^2V\|_\infty}{2} + \frac{C_V}{4})\sqrt{C_V}\big)}),$$

where C_V is given by the assumption $(\mathcal{L}_{\mathbf{V}})$. For $\beta \in (0, 1]$, we introduce:

($\mathbf{T}_\beta$) We choose a test function φ such that

i) $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$,

ii) $x \mapsto \langle \nabla\varphi(x), b(x)\rangle$ is Lipschitz continuous,

we further assume that there exist $C_{V,\varphi} > 0$ s.t. for all $x \in \mathbb{R}^d$:

iii) $|\varphi(x)| \leq C_{V,\varphi}(1 + \sqrt{V(x)})$.

Remark 1. *Under the assumption* **(C0)** *the equation* (E) *admits a unique non-explosive solution, cf* [App09] *(Theorem 6.2.9.).*

Remark 2. *The assumption* **(GC)** *is central for this paper. Note that the laws* $\mathcal{N}(0, I_r)$ *and* $(\frac{1}{2}(\delta_1 + \delta_{-1}))^{\otimes r}$*, i.e. for Gaussian or symmetrized Bernoulli increments which are the most commonly used sequences for the sub-Gaussian innovations, satisfy* **(GC)***. Moreover, inequality* (1.6) *yields that for all* $r \geq 0$*,* $\mathbb{P}[|U_1| \geq r] \leq 2\exp(-\frac{r^2}{2})$ *(sub-Gaussian concentration of the innovation, see e.g.* [BGL14]*).*

Remark 3. *The assumption* $(\mathcal{L}_{\mathbf{V}})$ *together with* **(C2)** *ensure, following* [Pan08a] *(Proposition 1) the existence of at least one invariant distribution of the SDE* (E)*. Note that this Lyapunov assumption* $(\mathcal{L}_{\mathbf{V}})$ *is equivalent to the similar Lyapunov assumption for the continuous part of the* *equation* (E). *Indeed, using second order Taylor expansion, the fact that* $\pi(\cdot) = 0$ *and* $\pi(|\cdot|^2) = r < \infty$, *we get that*

$$|\int_{\mathbb{R}^r} (V(x+\kappa(x)y) - V(x))\pi(dy)| \leq \frac{\|\kappa\|_\infty^2 r \|D^2 V\|_\infty}{2}.$$

Hence the condition iii) of $(\mathcal{L}_{\mathbf{V}})$ *is equivalent that the generator of the diffusion without jumps satisfies*

$$\mathcal{A}V(x) \leq -\widetilde{\alpha}_V V(x) + \widetilde{\beta}_V, \tag{1.7}$$

with $\tilde{\alpha}_V = \alpha_V$, $\tilde{\beta}_V = \beta_V + \frac{\|\kappa\|_\infty^2 r \|D^2 V\|_\infty}{2}$. *For more information about Lyapunov function existence, see e.g.* [GP14].

Moreover, it is classic to see that this assumption constraints the drift coefficient b *to be under a linear map. Indeed, this is the consequence of the fact that the Lyapunov function* V *has to be lower than the square norm, i.e. there exist constants* $K, \bar{c} > 0$ *such that for all* $|x| \geq K$, $|V(x)| \leq \bar{c}|x|^2$ *and hence using ii) of (*$\mathcal{L}_V$*)* $|b(x)| \leq \sqrt{C_V \bar{c}}|x|$.

Remark 4. *The assumption* $(\mathbf{T}_\beta)$ *allows to substantially simplify the proof of our results. The condition iii) is natural for* φ *Lipschitz continuous, which is obviously lower than the square root of a quadratic function (potentially* V*). Whilst the condition ii) is a direct consequence if* φ *is the solution of the Poisson equation:*

$$\mathcal{A}\varphi = f, \tag{1.8}$$

where $f \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R})$ *s.t.* $\nu(f) = 0$. *If* $\sigma, \kappa \in \mathcal{C}_b^{1,\beta}(\mathbb{R}^d, \mathbb{R}^{d\times r}), b \in \mathcal{C}^{1,\beta}(\mathbb{R}^d, \mathbb{R}^d)$ *and* $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$, *then both sides of the following identity:*

$$\langle \nabla\varphi, b\rangle \;=\; f - \frac{1}{2}\mathrm{Tr}\big(\Sigma D^2\varphi\big) - \int_{\mathbb{R}^d} \big(\varphi(\cdot + \kappa(\cdot)y) - \varphi(\cdot)\big)\pi(dy),$$

are Lipschitz continuous.

From now on, we identify assumptions **(C0)**, **(C1)**, **(C2)**, **(GM)**, **(GC)**, $(\mathcal{L}_{\mathbf{V}})$, **(U)**, **(S)** and $(\mathbf{T}_\beta)$ for some $\beta \in (0,1]$ to **(A)**. Except when explicitly indicated, we assume throughout the paper that assumption **(A)** is in force.

We suppose that the step sequence $(\gamma_k)_{k\geq 1}$ is taken such that $\gamma_k \asymp k^{-\theta}$, $\theta \in (0,1]$. This pick yields for any $\ell \geq 0$, $\Gamma_n^{(\ell)} \asymp n^{1-\ell\theta}$ if $\ell\theta < 1$, $\Gamma_n^{(\ell)} \asymp \ln(n)$ if $\ell\theta = 1$ and $\Gamma_n^{(\ell)} \asymp 1$ if $\ell\theta > 1$.

The corner stone of our analysis is the fact that the jumps innovations $(Z_n)_{n\in\mathbb{N}}$ inherit the Gaussian concentration property of $(Y_n)_{n\in\mathbb{N}}$:

Proposition 1 (Gaussian concentration of the jumps innovation)**.** *Let* $g : \mathbb{R}^r \to \mathbb{R}$ *uniform Lipschitz continuous function,* $\varepsilon \in (0,1)$ *and*

$$\rho(r) = \sqrt{r}(3+r) + \frac{1}{8} + 4\exp(\sqrt{r} + 1 + r/2). \tag{1.9}$$

Then for all $0 < \lambda < \frac{\varepsilon}{6[g]_1\rho(r)}$ *the following inequality holds for all* $n \in \mathbb{N}$

$$\mathbb{E}\exp(\lambda g(Z_n)) \leq \exp(\lambda\mathbb{E}g(Z_n) + \frac{\lambda^2\gamma_n(1+r+\varepsilon)[g]_1^2}{2}). \tag{1.10}$$

Remark 5. *Let us point out that the concentration inequality is only valid for* λ *on a compact set. This constraint is due to the difficulty to approximate a compound Poisson process which has actually a sub-exponential tail (and not a sub-Gaussian one).*

The proof of this proposition is given in Subsection 2.4.

1.4. **Existing results.** The natural next question concerns the rate of that convergence. In a Brownian diffusion framework, a Central Limit Theorem (CLT) was established by Lamberton and Pagès [LP02] for functions f of the form $f-\nu(f)=\widetilde{\mathcal{A}}\varphi$, namely $f-\nu(f)$ is a *coboundary* for $\widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ denotes the continuous part of the infinitesimal generator (see (1.5) further). This choice of functions class comes from the characterization of the invariant distribution ν by a solution in the distribution sense of the stationary Fokker-Planck equation: $\widetilde{\mathcal{A}}^*\nu=0$ (where $\widetilde{\mathcal{A}}^*$ stands for the adjoint of $\widetilde{\mathcal{A}}$). In other words, for all functions $\varphi\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R})$, we have $\nu(\widetilde{\mathcal{A}}\varphi)=\int_{\mathbb{R}^d}\widetilde{\mathcal{A}}\varphi(x)\nu(dx)=0$.

In [Pan08a], the author also provided the rate of convergence through a Central Limit Theorem (CLT) for the already mentioned general scheme:

Theorem 1 (CLT)**.** *Under* **(C2)**, **(U)** *and* $(\mathcal{L}_{\mathbf{V}})$, *if* $\mathbb{E}|Z_t|^{2p}<+\infty$ *for* $p>2$, *if* $\mathbb{E}[U_1^{\otimes 3}]=0$, $\mathbb{E}[|U_1|^{2p}]<+\infty$ *and* $\lim_n \frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}=0$ *then for all function* $\varphi\in\mathcal{C}^{3,1}(\mathbb{R}^d,\mathbb{R})$ *we have the following results (with* $(\mathcal{L})$ *denoting the weak convergence):*

$$\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\xrightarrow{(\mathcal{L})}\mathcal{N}\left(0,\sigma_\varphi^2\right), \tag{1.11}$$

with

$$\sigma_\varphi^2:=\int_{\mathbb{R}^d}\left(|\sigma^*\nabla\varphi|^2(x)+\int_{\mathbb{R}^d}|\varphi(x+\kappa(x)y)-\varphi(x)|^2\pi(dy)\right)\nu(dx). \tag{1.12}$$

In the Brownian diffusion context ($\kappa=0$), under some confluence and non-degeneracy or regularity assumptions Honoré, Menozzi and Pagès [HMP18] established suitable derivatives controls for the Poisson problem (e.g. Schauder estimates). With a compound Poisson process, we think that a similar analysis may work. It will be a future research. Let us mention [Pri10] for some Schauder estimates for Poisson equation, with a potential, associated with a SDE purely driven by stable processes but with a constant drift.

In [HMP18], the authors have established a non-asymptotic Gaussian concentration with $\kappa=0$ there are explicit sequences $c_n\le 1\le C_n$ converging to 1 such that for all $n\in\mathbb{N}$, for all $a>0$ and $\gamma_k\asymp k^{-\theta},\ \theta\in(\frac{1}{3},1]$,

$$\mathbb{P}[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\ge a]\le C_n\exp\left(-c_n\frac{a^2}{2\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty}\right), \tag{1.13}$$

which is our goal for a diffusion with jump contributions. Remark that, in [Hon17], a non-asymptotic Gaussian concentration was established with the asymptotically best constants for a particular large deviation called "Gaussian deviations" therein. In other words, for $a=o(\sqrt{\Gamma_n})$:

$$\mathbb{P}[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\ge a]\le C_n\exp\left(-c_n\frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}\right). \tag{1.14}$$

In this present work, we aim to obtain a Gaussian deviations bound like (1.13) for the scheme (S). To do so, we will perform the so-called martingales increments method which was exploited successfully by Frikha and Menozzi [FM12]. It was also the backbone of the analysis in [HMP18] and [Hon17]. Here, we adapt their techniques for the stochastic differential equation (E) driven by the compound Poisson with Jump heigh sizes satisfying Gaussian concentration.

2. Main results

2.1. Result of non-asymptotic Gaussian concentration.

Our main result is stated below.

Theorem 2. *For* $\theta\in(\frac{1}{2+\beta},1]$, $\beta\in(0,1]$, *assume that* **(A)** *is in force. For all positive sequence* $(\chi_n)_{n\ge1}$ *with* $\lim_{n\to\infty}\chi_n=0$, *there are two non-negative sequences* $(c_n)_{n\ge1}$ *and* $(C_n)_{n\ge1}$ *satisfying*

$c_n \nearrow 1$ and $C_n \searrow 1$ as $n \to \infty$, such that for all $n \in \mathbb{N}$, $a > 0$, satisfying $a \le \chi_n \frac{\sqrt{\Gamma_n}}{\Gamma_n^{(2)}}$, the following bound holds:

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \ge a\big] \le 2\, C_n \exp\big(- c_n \frac{a^2}{2\sigma_\infty^2}\big),$$

where $\sigma_\infty^2 := (1+r)\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2 + \|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2$ and $C_n = \exp\big(\frac{[D^3\varphi]_\beta\|\sigma\|_\infty^{3+\beta}\mathbb{E}[|U_1|^{3+\beta}]}{(1+\beta)(2+\beta)(3+\beta)} \frac{\Gamma_n^{(\frac{3+\beta}{2})}}{\sqrt{\Gamma_n}} + p_n \frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}\big)$ for $p_n \ge 1$ such that $p_n \to_n +\infty$ and $p_n \frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}} \to_n 0$.

The proof of Theorem 2 is given in Section 2.3.

Remark 6. *Note that for all $\theta \in (\frac{1}{2+\beta}, 1]$, $\frac{\sqrt{\Gamma_n}}{\Gamma_n^{(2)}} = +\infty$, then we can choose χ_n s.t. $\chi_n \frac{\sqrt{\Gamma_n}}{\Gamma_n^{(2)}} \underset{n}{\to} +\infty$. In other words, we can pick $a = a(n) \to_n +\infty$. We have unsurprisingly that $\sigma_\varphi^2 \le \sigma_\infty^2$, where σ_φ^2 is the asymptotic variance of $\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)$ defined in (1.12). Moreover, the difficulty to adapt a Gaussian Concentration result to compound Poisson process yields that the upper-bound variance σ_∞^2 depends on the dimension.*

2.2. **Strategy.** For the analysis of $\nu_n(\mathcal{A}\varphi)$, we will first perform an appropriate Taylor expansion (equation (2.3) below). An expansion of this kind is standard in this context, and analogous decompositions were already used in [HMP18], [Hon17] and [Pan08b], [Pan08a] with a jump component. It can be viewed as a kind of Itô formula for Euler scheme, because it permits to write the difference $\varphi(X_n) - \varphi(X_0)$ as a sum of a martingale, a term involving the generator and a remainder term. Recall that $\mathcal{F}_k = \boldsymbol{\sigma}\big(X_0, (U_j, Z_j)_{j\in[\![1,k]\!]}\big)$, $k \in \mathbb{N}^*$. Let us define the contributions of the decomposition of $\nu_n(\mathcal{A}\varphi)$ in the following lemma.

$$\begin{aligned}
\psi_k^\varphi(X_{k-1}, U_k) &:= \sqrt{\gamma_k}\sigma_{k-1}U_k \cdot \nabla\varphi(X_{k-1} + \gamma_k b_{k-1}) \\
&\quad + \gamma_k \int_0^1 (1-t)\mathrm{Tr}\Big(D^2\varphi(X_{k-1} + \gamma_k b_{k-1} + t\sqrt{\gamma_k}\sigma_{k-1}U_k)\sigma_{k-1}U_k \otimes U_k\sigma_{k-1}^* \\
&\quad - D^2\varphi(X_{k-1} + \gamma_k b_{k-1})\Sigma_{k-1}\Big)dt, \\
\Delta_k^\varphi(X_{k-1}, U_k) &:= \psi_k^\varphi(X_{k-1}, U_k) - \mathbb{E}\big[\psi_k^\varphi(X_{k-1}, U_k)|\mathcal{F}_{k-1}\big], \\
\widetilde{\Delta}_k^\varphi(X_{k-1}, Z_k) &:= \varphi(X_{k-1} + \kappa_{k-1}Z_k) - \varphi(X_{k-1}) - \gamma_k \int_{\mathbb{R}^r} \big[\varphi(X_{k-1} + \kappa_{k-1}y) - \varphi(X_{k-1})\big]\pi(dy).
\end{aligned} \tag{2.1}$$

Moreover, we define the remainder contributions in the decomposition of $\nu_n(\mathcal{A}\varphi)$.

$$\begin{aligned}
D_{2,b}^{k,\varphi}(X_{k-1}) &:= \gamma_k \int_0^1 \langle \nabla\varphi(X_{k-1} + t\gamma_k b_{k-1}) - \nabla\varphi(X_{k-1}), b_{k-1}\rangle dt, \\
D_{2,\Sigma}^{k,\varphi}(X_{k-1}) &:= \frac{\gamma_k}{2}\mathrm{Tr}\Big(\big(D^2\varphi(X_{k-1} + \gamma_k b_{k-1}) - D^2\varphi(X_{k-1})\big)\Sigma_{k-1}\Big), \\
D_j^{k,\varphi}(X_{k-1}, U_k, Z_k) &:= \varphi(X_k) - \varphi(X_{k-1} + \gamma_k b_{k-1} + \sqrt{\gamma_k}\sigma_{k-1}U_k) - \left(\varphi(X_{k-1} + \kappa_{k-1}Z_k) - \varphi(X_{k-1})\right).
\end{aligned} \tag{2.2}$$

Lemma 1 (Local decomposition of the empirical measure)**.** *For all $\varphi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, $k \in \mathbb{N}^*$ the following decomposition holds:*

$$\varphi(X_k) - \varphi(X_{k-1}) = \gamma_k\mathcal{A}\varphi(X_{k-1}) + \Delta_k^\varphi(X_{k-1}, U_k) + \widetilde{\Delta}_k^\varphi(X_{k-1}, Z_k) + \mathbf{R}_k^\varphi(X_{k-1}, U_k, Z_k), \tag{2.3}$$

where

$$\mathbf{R}_k^\varphi(X_{k-1}, U_k, Z_k) := D_{2,b}^{k,\varphi}(X_{k-1}) + D_{2,\Sigma}^{k,\varphi}(X_{k-1}) + D_j^{k,\varphi}(X_{k-1}, U_k, Z_k) + \mathbb{E}\big[\psi_k^\varphi(X_{k-1}, U_k)|\mathcal{F}_{k-1}\big]. \tag{2.4}$$

Furthermore, we have the following properties:

i): *For all $k \in \mathbb{N}^*$, the functions $u \mapsto \Delta_k^\varphi(X_{k-1}, u)$ and $z \mapsto \widetilde{\Delta}_k^\varphi(X_{k-1}, z)$ are Lipschitz, satisfying*

$$[\Delta_k^\varphi(X_{k-1}, \cdot)]_1 \le \sqrt{\gamma_k}\|\sigma_{k-1}\|\|\nabla\varphi\|_\infty \le \sqrt{\gamma_k}\|\sigma\|_\infty\|\nabla\varphi\|_\infty,$$

$$[\widetilde{\Delta}_k^\varphi(X_{k-1}, \cdot)]_1 \le \|\kappa_{k-1}\|\|\nabla\varphi\|_\infty \le \|\kappa\|_\infty\|\nabla\varphi\|_\infty.$$

ii): *For all $k \in \mathbb{N}^*$, $\Delta_k^\varphi(X_{k-1}, U_k)$ and $\widetilde{\Delta}_k^\varphi(X_{k-1}, Z_k)$ are martingale increments with respect to $\mathcal{F}_k$, namely:*

$$\mathbb{E}\big[\Delta_k^\varphi(X_{k-1}, U_k)\big|\mathcal{F}_{k-1}\big] = 0, \quad \mathbb{E}\big[\widetilde{\Delta}_k^\varphi(X_{k-1}, Z_k)\big|\mathcal{F}_{k-1}\big] = 0.$$

The proof of Lemma 1 is given in Section 4. Now we introduce the martingales associated to these martingale increments:

$$M_n^\varphi := \sum_{k=1}^n \Delta_k^\varphi(X_{k-1}, U_k), \quad \widetilde{M}_n^\varphi := \sum_{k=1} \widetilde{\Delta}_k^\varphi(X_{k-1} Z_k). \tag{2.5}$$

Summing (2.3) over k we obtain the following global decomposition of the empirical measure:

$$\nu_n(\mathcal{A}\varphi) = -\frac{1}{\Gamma_n}(M_n^\varphi + \widetilde{M}_n^\varphi + \mathfrak{R}_n^\varphi), \tag{2.6}$$

where we denoted

$$\mathfrak{R}_n^\varphi := \sum_{k=1}^n \mathbf{R}_k^\varphi(X_{k-1}, U_k, Z_k) - (\varphi(X_n) - \varphi(X_0)). \tag{2.7}$$

Using the definition (2.2) we can write $\mathfrak{R}_n^\varphi = -L_n^\varphi + D_{2,b,n}^\varphi + D_{2,\Sigma,n}^\varphi + D_{j,n}^\varphi + \bar{G}_n^\varphi$, with

$$\begin{aligned} L_n^\varphi &:= \varphi(X_n) - \varphi(X_0), \quad D_{2,b,n}^\varphi := \sum_{k=1}^n D_{2,b}^{k,\varphi}(X_{k-1}), \quad D_{2,\Sigma,n}^\varphi := \sum_{k=1}^n D_{2,\Sigma}^{k,\varphi}(X_{k-1}), \\ D_{j,n}^\varphi &:= \sum_{k=1}^n D_j^{k,\varphi}(X_{k-1}, U_k, Z_k), \quad \bar{G}_n^\varphi := \sum_{k=1}^n \mathbb{E}\left[\psi_k^\varphi(X_{k-1}, U_k)|\mathcal{F}_{k-1}\right]. \end{aligned} \tag{2.8}$$

In the proof of Theorem 2, we need some key results stated below. The proofs of all these statements are postponed to Section 4.

The main contribution in the decomposition (2.6) is given by the martingales M_n^φ and $\widetilde{M}_n^\varphi$. Their analysis is given with the help of the Gaussian Concentration inequality (1.6) and (1.10), trough the following lemma:

Lemma 2 (Concentration of the martingale increments)**.** *Let Δ_n^φ and $\widetilde{\Delta}_n^\varphi$ given by* (2.1).

i): *For all $\Lambda > 0$ we have*

$$\mathbb{E}\left[\exp\left(-\frac{\Lambda}{\Gamma_n}\Delta_n^\varphi(X_{n-1}, U_n)\right)\Big|\mathcal{F}_{n-1}\right] \le \exp\left(\gamma_n\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\frac{\Lambda^2}{2\Gamma_n^2}\right).$$

ii): *For all $0 < \varepsilon < 1, n \in \mathbb{N}^*$, for all $\Lambda > 0$ s.t. $\frac{\Lambda}{\Gamma_n} < \frac{\varepsilon}{6\|\kappa\|_\infty\|\nabla\varphi\|_\infty\rho(r)}$, where $\rho(r)$ is defined in* (1.9) *, we have*

$$\mathbb{E}\left[\exp\left(-\frac{\Lambda}{\Gamma_n}\widetilde{\Delta}_n^\varphi(X_{n-1}, Z_n)\right)\Big|\mathcal{F}_{n-1}\right] \le \exp\left(\gamma_n\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1+r+\varepsilon)\frac{\Lambda^2}{2\Gamma_n^2}\right).$$

Now we formulate several propositions and lemmas that are used to control the components of the remainder term $\mathfrak{R}_n^\varphi$. The following proposition is the counterpart to the jumps diffusion of the useful Proposition 1 in [HMP18].

Proposition 2. *Under* **(A)**, *there is a constant* $c_V := c_V((\mathbf{A})) > 0$ *such that for all* $\lambda \in [0, c_V]$:

$$I_V^{\frac{1}{2}} := \sup_{n\geq 0} \mathbb{E}\big[\exp\big(\lambda\sqrt{V}(X_n)\big)\big] < +\infty. \tag{2.9}$$

Remark 7. *In particular, we easily see that for all* $\lambda \in [0, c_V]$ *and* $\xi \in [0, \frac{1}{2}]$:

$$I_V^{\xi} := \sup_{n\geq 0} \mathbb{E}\big[\exp\big(\lambda V(X_n)^{\xi}\big)\big] < +\infty. \tag{2.10}$$

Note that for $\kappa = 0$ *(purely continuous case), the integrability of* $\exp\big(\lambda V(X_n)^{\xi}\big)$ *is available until* $\xi = 1$ *(see Proposition 1 in* [HMP18]*). The lost of integrability is the consequence of the bound condition over* λ *in the Gaussian Concentration result of Proposition 1.*

We have the following results for the initial term appearing in (2.3) which is handled thanks to the below result.

Lemma 3 (Initial term)**.** *For all* $\Lambda > 0$ *s.t.* $\frac{\Lambda}{\Gamma_n} < \frac{c_V}{2C_{V,\varphi}}$:

$$\mathbb{E}\exp\Big(\Lambda\frac{|L_n^{\varphi}|}{\Gamma_n}\Big) \leq \exp\Big(2C_{V,\varphi}\frac{\Lambda}{\Gamma_n}\Big)(I_V^{\frac{1}{2}})^{\frac{2C_{V,\varphi}\Lambda}{c_V\Gamma_n}}$$

with c_V, $I_V^{\frac{1}{2}}$ *given in Proposition 2.*

Next the last remainders are controlled as following:

Lemma 4 (Remainders)**.** *For all* $\Lambda > 0$ *s.t.* $\frac{\Lambda}{\Gamma_n} < \frac{2c_V}{\big(\|\nabla\varphi\|_\infty[b]_1 + [\langle\nabla\varphi, b\rangle]_1\big)\sqrt{C_V}}\frac{1}{\Gamma_n^{(2)}}$:

$$\mathbb{E}\exp\Big(\frac{\Lambda}{\Gamma_n}\big|D_{2,b,n}^{\varphi}\big|\Big) \leq (I_V^{1/2})^{\frac{\Lambda\big(\|\nabla\varphi\|_\infty[b]_1+[\langle\nabla\varphi,b\rangle]_1\big)\sqrt{C_V}\Gamma_n^{(2)}}{2c_V\Gamma_n}}. \tag{2.11}$$

We also have, for all $\Lambda > 0$ *s.t.* $\frac{\Lambda}{\Gamma_n} \leq \frac{c_V}{\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}}\frac{1}{\Gamma_n^{(2)}}$:

$$\mathbb{E}\exp\Big(\frac{\Lambda}{\Gamma_n}\big|D_{2,\Sigma,n}^{\varphi}\big|\Big) \leq (I_V^{1/2})^{\frac{\|\sigma\|_\infty^2\|D^3\varphi\|_\infty C_V^{\frac{1}{2}}\Lambda\Gamma_n^{(2)}}{2c_V\Gamma_n}}. \tag{2.12}$$

Lemma 5 (Bounds for the Conditional expectations)**.** *With the notations* (2.8)*, and for* $\theta \in (\frac{1}{2+\beta}, 1]$*, we have that*

$$\frac{|\bar{G}_n^{\varphi}|}{\sqrt{\Gamma_n}} \overset{a.s.}{\leq} \alpha_n := \frac{[\varphi^{(3)}]_\beta\|\sigma\|_\infty^{(3+\beta)}\mathbb{E}\big[|U_1|^{3+\beta}\big]}{(1+\beta)(2+\beta)(3+\beta)}\frac{\Gamma_n^{(\frac{3+\beta}{2})}}{\sqrt{\Gamma_n}} \underset{n}{\to} 0.$$

Remark 8. *The strongest condition over* θ *comes from this remainder term. Indeed, for* $\theta < \frac{2}{3+\beta}$, $\frac{\Gamma_n^{(\frac{3+\beta}{2})}}{\sqrt{\Gamma_n}} \asymp n^{\frac{1-(2+\beta)\theta}{2}}$ *which goes to* 0 *if and only if* $\theta > \frac{1}{2+\beta}$*. Whilst for the other remainders, for* $\theta < \frac{1}{2}$*, we need to have* $\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}} \asymp n^{\frac{1-3\theta}{2}} \to_n 0$ *which is implied by* $\theta > \frac{1}{3}$.

Now, let us deal with the remainder term $D_{j,n}$ due to the jump vector $(Z_k)_{k\geq 1}$.

Lemma 6 (Remainder term due to the jumps)**.** *If* $0 < \frac{\Lambda}{\Gamma_n} < \frac{1}{12\|\kappa\|_\infty\|\nabla\varphi\|_\infty\rho(r)}$*, then we have:*

$$\mathbb{E}\Big[\exp\Big(\frac{\Lambda}{\Gamma_n}D_{j,n}^{\varphi}\Big)\Big] \leq \exp\Big(\big(\frac{\Lambda}{\sqrt{\Gamma_n}} + \frac{\Lambda^2}{\Gamma_n}\big)e_n\Big), \tag{2.13}$$

where we recall that $e_n = e_n((\mathbf{A}))$*,* $n \geq 1$*, is a sequence such that* $e_n \to_n 0$.

The proof of this lemma is one of the most intricate of this article, we the decided to postponed it to the end of Section 4.

2.3. **Proof of our main result.**

Proof of Theorem 2. Through the following analysis, we deal with $\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\geq a\big]$. The term $\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\leq -a\big]$ can be handled readily by symmetry.

From notations introduced in (2.6), $\nu_n(\mathcal{A}\varphi)=-\frac{1}{\Gamma_n}(\mathfrak{R}_n^\varphi+M_n^\varphi+\widetilde{M}_n^\varphi)$. The idea is now to write for $a,\lambda>0$:

$$\begin{aligned}\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\geq a\big] &\leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\mathbb{E}\Big[\exp\big(-\frac{\lambda}{\Gamma_n}(\mathfrak{R}_n^\varphi+M_n^\varphi+\widetilde{M}_n^\varphi)\big)\Big]\\ &\leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}(M_n^\varphi+\widetilde{M}_n^\varphi)\big)\Big]^{1/q}\mathbb{E}\Big[\exp\big(\frac{p\lambda}{\Gamma_n}|\mathfrak{R}_n^\varphi|\big)\Big]^{1/p},\end{aligned}\tag{2.14}$$

for $\frac{1}{p}+\frac{1}{q}=1,\ p,q>1$. We will choose later $p=p(n)\to_n+\infty$ slowly enough, which implies that $q=q(n)\to 1$. Let $\lambda>0$. Recall that

$$\mathfrak{R}_n^\varphi=-L_n^\varphi+D_{2,b,n}^\varphi+D_{2,\Sigma,n}^\varphi+D_{j,n}^\varphi+\bar{G}_n^\varphi$$

By Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\mathbb{E}\Big[\exp\big(\frac{p\lambda}{\Gamma_n}|\mathfrak{R}_n^\varphi|\big)\Big]^{1/p}\leq\Big(\mathbb{E}\exp\Big(\frac{2p\lambda}{\Gamma_n}\big|L_n^\varphi\big|\Big)\Big)^{\frac{1}{2p}}\Big(\mathbb{E}\exp\Big(\frac{4p\lambda}{\Gamma_n}|\bar{G}_n^\varphi|\Big)\Big)^{\frac{1}{4p}}\\ \Big(\mathbb{E}\exp\Big(\frac{8p\lambda}{\Gamma_n}\big|D_{2,b,n}^\varphi\big|\Big)\Big)^{\frac{1}{8p}}\Big(\mathbb{E}\exp\Big(\frac{16p\lambda}{\Gamma_n}\big|D_{2,\Sigma,n}^\varphi\big|\Big)\Big)^{\frac{1}{16p}}\Big(\mathbb{E}\exp\Big(\frac{16p\lambda}{\Gamma_n}\big|D_{j,n}^\varphi\big|\Big)\Big)^{\frac{1}{16p}}.\end{aligned}\tag{2.15}$$

We recall that all the long of our analysis, $C>0$ denotes a generic constant, $(\mathscr{R}_n)_{n\geq1}$ and $(e_n)_{n\geq1}$ are generic non-negative sequences, depending on coefficients of assumption **(A)**, which may change frome line to line, such that $\lim_{n\to\infty}\mathscr{R}_n=1$, $\lim_{n\to\infty}e_n=0$. For the term associated with L_n in (2.15), if $\frac{2p\lambda}{\Gamma_n}<\frac{c_V}{2C_{V,\varphi}}$, by Lemma 3 we can write:

$$\Big(\mathbb{E}\exp\big(2p\lambda\frac{|L_n^\varphi|}{\Gamma_n}\big)\Big)^{\frac{1}{2p}}\leq\exp\Big(2C_{V,\varphi}\frac{\lambda}{\Gamma_n}\Big)(I_V^{\frac12})^{\frac{2C_{V,\varphi}\lambda}{c_V\Gamma_n}}=\exp(C\frac{\lambda}{\Gamma_n}).\tag{2.16}$$

From Lemma 5, with α_n defined there and by Young inequality we obtain:

$$\Big(\mathbb{E}\exp\big(\frac{4p\lambda}{\Gamma_n}|\bar{G}_n^\varphi|\big)\Big)^{\frac{1}{4p}}\leq\exp\big(\frac{\lambda}{\sqrt{\Gamma_n}}\alpha_n\big)\leq\exp\big(\frac{\lambda^2}{\Gamma_n 2p}+\frac{\alpha_n^2p}{2}\big)=\mathscr{R}_n\exp\big(\frac{\lambda^2}{\Gamma_n}e_n\big).\tag{2.17}$$

In the last equality, $\mathscr{R}_n=\exp(\alpha_n^2p_n/2)$ and $e_n=1/2p_n$. Recall that $\alpha_n\to_n 0$. We need to choose $p=p_n\to_n+\infty$. We choose p_n such that $p_n\alpha_n^2\to_n 0$.

For the term involving $D_{2,\Sigma,n}^\varphi$ from (2.15), if $\frac{16p\lambda}{\Gamma_n}\leq\frac{2c_V}{\Gamma_n^{(2)}\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}}$, using Lemma 4, we can write

$$\Big(\mathbb{E}\exp\big(\frac{16p\lambda}{\Gamma_n}\big|D_{2,\Sigma,n}^\varphi\big|\big)\Big)^{\frac{1}{16p}}\leq(I_V^{1/2})^{\frac{\|\sigma\|_\infty^2\|D^3\varphi\|_\infty C_V^{\frac12}\lambda\Gamma_n^{(2)}}{c_V\Gamma_n}}=\exp(C\frac{\lambda\Gamma_n^{(2)}}{\Gamma_n})=\exp(C\frac{\lambda}{\sqrt{\Gamma_n}}e_n),\tag{2.18}$$

where in the last equality we take $e_n=\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}$ and recall that for all $\theta\in(\frac13,1]$, $\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}\underset{n}{\to}0$.

For the remainder depending on $D_{2,b,n}^\varphi$ from (2.15), if $\frac{4p\lambda}{\Gamma_n}<\frac{2c_V}{\Gamma_n^{(2)}\big(\|\nabla\varphi\|_\infty[b]_1+[\langle\nabla\varphi,b\rangle]_1\big)\sqrt{C_V}}$, thanks to Lemma 4 we have again with $e_n=\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}$:

(2.19)
$$\Big(\mathbb{E}\exp\big(\frac{4p\lambda}{\Gamma_n}\big|D^{\varphi}_{2,b,n}\big|\big)\Big)^{\frac{1}{4p}} \leq (I_V^{1/2})^{\frac{\lambda\big(\|\nabla\varphi\|_\infty[b]_1+[\langle\nabla\varphi,b\rangle]_1\big)\sqrt{c_V}\Gamma_n^{(2)}}{2c_V\Gamma_n}} = \exp(C\frac{\lambda\Gamma_n^{(2)}}{\Gamma_n}) = \exp(C\frac{\lambda}{\sqrt{\Gamma_n}}e_n).$$

Finally, Lemma 6 yields that if $0<\frac{16p\lambda}{\Gamma_n}<\frac{1}{12\|\kappa\|_\infty\|\nabla\varphi\|_\infty\rho(r)}$, then:

$$\mathbb{E}\left[\exp\left(\frac{16p\lambda}{\Gamma_n}D^{\varphi}_{j,n}\right)\right] \leq \exp\left(\frac{\lambda}{\sqrt{\Gamma_n}}e_n+\frac{\lambda^2}{\Gamma_n}e_n\right). \tag{2.20}$$

We gather (2.16), (2.17), (2.18), (2.19) and (2.20) into (2.15) and finally from (2.14) we obtain:

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\geq a\big] \leq \exp\big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\big)\mathbb{E}\big[\exp\big(-\frac{q\lambda}{\Gamma_n}(M_n^{\varphi}+\widetilde{M}_n^{\varphi})\big)\big]^{\frac{1}{q}}\exp\big((\frac{\lambda}{\sqrt{\Gamma_n}}+\frac{\lambda^2}{\Gamma_n})e_n\big)\mathscr{R}_n. \tag{2.21}$$

Now, let us control the martingale terms thanks to Lemma 2. Let $0<\varepsilon<1$, and $\lambda>0$ s.t. $q\lambda/\Gamma_n<\frac{C\varepsilon}{6\|\kappa\|_\infty\|\nabla\varphi\|_\infty\rho(r)}$, we recall that $\rho(r)$ is defined in (1.9).

Thanks to Lemma 2 and the independence of Z_n and U_n conditionally to $\mathcal{F}_{n-1}$ we can write

$$\begin{aligned}
&\mathbb{E}\exp\Big(-\frac{q\lambda}{\Gamma_n}(M_n^{\varphi}+\widetilde{M}_n^{\varphi})\Big)= \\
&\mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}(M_{n-1}^{\varphi}+\widetilde{M}_{n-1}^{\varphi})\big)\mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}(\Delta_n^{\varphi}(X_{n-1},U_n)+\widetilde{\Delta}_n^{\varphi}(X_{n-1},Z_n))\Big|\mathcal{F}_{n-1}\Big]\Big] \\
&\leq \mathbb{E}\Big[\exp\big(-\frac{q\lambda}{\Gamma_n}(M_{n-1}^{\varphi}+\widetilde{M}_{n-1}^{\varphi})\big)\exp\Big(\frac{q^2\lambda^2\gamma_n}{2\Gamma_n^2}(\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2+\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1+r+\varepsilon))\Big)\Big].
\end{aligned}$$

By induction we obtain

$$\begin{aligned}
\mathbb{E}\exp\Big(-\frac{q\lambda}{\Gamma_n}(M_n^{\varphi}+\widetilde{M}_n^{\varphi})\Big) &\leq \exp\left(\Big(\frac{q^2\lambda^2}{2\Gamma_n^2}\big(\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1+r+\varepsilon)+\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)\Big)\sum_{k=1}^n\gamma_k\right) \\
&= \exp\left(\frac{q^2\lambda^2}{2\Gamma_n}\big(\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1+r+\varepsilon)+\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)\right).
\end{aligned}$$

Plugging this inequality into (2.21) yields:

$$\begin{aligned}
&\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)\geq a\big]\leq\exp\Big(-\frac{a\lambda}{\sqrt{\Gamma_n}}\Big) \\
&\times\exp\left(\frac{q\lambda^2}{2\Gamma_n}\big(\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1+r+\varepsilon)+\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)\right)\exp((\frac{\lambda}{\sqrt{\Gamma_n}}+\frac{\lambda^2}{\Gamma_n})e_n)\mathscr{R}_n.
\end{aligned}\tag{2.22}$$

Next, we optimize the polynomial $-\frac{a\lambda}{\sqrt{\Gamma_n}}+\frac{q\lambda^2}{2\Gamma_n}\big(\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1+r+\varepsilon)+\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)$ over λ which leads to consider:

$$\lambda:=\lambda_n:=\frac{a\sqrt{\Gamma_n}}{q(\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1+r+\varepsilon)+\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2)}. \tag{2.23}$$

We check first the assumptions over λ in all lemmas that we used in this proof. In (2.16) and (2.20) we need $p\lambda_n/\Gamma_n<C$. And for (2.18) and (2.19) we need $p\lambda_n/\Gamma_n<\frac{C}{\Gamma_n^{(2)}}$. Finally $q\lambda_n/\Gamma_n<C\varepsilon_n$ is required to apply Lemma 2. We recall that we will choose $p\to\infty$ and $q\to1$ and finally $\varepsilon\to0$.

We recall also that from the statement of the theorem $a=a(n)$ can depends of n in such a way that $\frac{a}{\sqrt{\Gamma_n}}\leq\frac{\chi_n}{\Gamma_n^{(2)}}\to_n 0$. But if $q\to1$, for n big enough $\frac{\lambda_n}{\Gamma_n}\asymp\frac{a}{\sqrt{\Gamma_n}}<\frac{\chi_n}{\Gamma_n^{(2)}}\to0$. Hence, the condition

$$\frac{p\lambda_n}{\Gamma_n}\asymp\frac{pa}{\sqrt{\Gamma_n}}<\frac{p\chi_n}{\Gamma_n^{(2)}}<C/\Gamma_n^{(2)} \tag{2.24}$$

has to be satisfied. Let us calibrate $p = p(n) \to \infty$ depending on $\chi_n \to 0$ s.t. $\limsup_n p\chi_n < C$. This pick of p yields (2.24). We can also choose, for $C > 0$ large enough $\varepsilon_n = C\frac{\chi_n}{\Gamma_n^{(2)}}$ such that all conditions over $\lambda_n, p, \varepsilon$ are satisfied with these choices.

The inequality (2.22) yields then for $\lambda = \lambda_n$:

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a\big] \leq \mathscr{R}_n \exp\Big(-\frac{\tilde{c}_n a^2}{2\big((1+r)\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2 + \|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)} + \frac{\lambda_n}{\sqrt{\Gamma_n}}e_n\Big),$$

with $\tilde{c}_n = \frac{(1+r)\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2+\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}{q\big((1+r+\varepsilon)\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2+\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)} \underset{n\to+\infty}{\to} 1$, and $\frac{\lambda_n}{\sqrt{\Gamma_n}}e_n \leq Cae_n$.

If $a \leq 1$, we take $C_n = \mathscr{R}_n\exp(Cae_n) \underset{n\to+\infty}{\to} 1$, otherwise if $a > 1$ then we set $c_n = 1 - \frac{Ce_n}{a^2} \underset{n\to+\infty}{\to} 1$. In any case, we write the result:

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geq a\big] \leq C_n \exp\Big(-\frac{c_n a^2}{2\big((1+r)\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2 + \|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2\big)}\Big).$$

□

2.4. Proof of the Gaussian property of the jump innovation.

Proof of Proposition 1. Suppose first that $g : \mathbb{R}^r \to \mathbb{R}$ is Lipschitz continuous with $[g]_1 \leq 1$. The case of $[g]_1 > 1$ follows by considering $\tilde{\lambda} = \lambda[g]_1$ and $\tilde{g} = \frac{g}{[g]_1}$. We suppose w.l.o.g. that $g(0) = 0$. We recall that thank to the definition (1.1) the law of Z_n is the same that the law of B_nY, where B_n is a Bernoulli variable with parameter γ_n, independent of the random vector Y with distribution π on $\mathcal{B}(\mathbb{R}^r)$. We will establish first that for all $\varepsilon \in (0,1)$ and $0 < \lambda < \frac{\varepsilon}{\rho(r)}$ (see (1.9)) we have

$$\mathbb{E}\exp(\lambda g(Z_n)) \leq \exp(\lambda\mathbb{E}g(Z_n) + \frac{\lambda^2\gamma_n(1 + (\mathbb{E}g(Y))^2 + \varepsilon)}{2}). \tag{2.25}$$

Denote for this proof $m_g := \mathbb{E}g(Y)$. Using **(GC)** property of Y we can write

$$\mathbb{E}\exp(\lambda g(Z_n)) = \gamma_n\mathbb{E}\exp(\lambda g(Y)) + (1-\gamma_n) \leq \gamma_n\exp(\lambda m_g + \lambda^2/2) + (1-\gamma_n).$$

Denote

$$\Delta_n^{\exp} := \gamma_n\exp(\lambda m_g + \frac{\lambda^2}{2}) + (1-\gamma_n) - \exp(\lambda\gamma_n m_g + \lambda^2\frac{\gamma_n(1+m_g^2+\varepsilon)}{2}). \tag{2.26}$$

Here, the second exponential corresponds to the right hand side in (2.25). We will show that $\Delta_n^{\exp} < 0$. Indeed, let us develop the difference $\Delta_n^{\exp}$ by power series expansion:

$$\begin{aligned}
\Delta_n^{\exp} &= \gamma_n(\lambda m_g + \frac{\lambda^2}{2}) - (\gamma_n\lambda m_g + \lambda^2\frac{\gamma_n(1+m_g^2+\varepsilon)}{2}) \\
&+ \frac{1}{2}\gamma_n(\lambda m_g + \frac{\lambda^2}{2})^2 - \frac{1}{2}(\gamma_n\lambda m_g + \frac{\gamma_n\lambda^2(1+m_g^2+\varepsilon)}{2})^2 + Q(\lambda) \\
&= -\frac{\gamma_n\lambda^2}{2}(\varepsilon + \gamma_n m_g^2) + \frac{\gamma_n\lambda^3}{2}\big(m_g(1-\gamma_n(1+m_g^2+\varepsilon))\big) \\
&+ \frac{1}{8}\gamma_n\lambda^4\big(1-\gamma_n(1+m_g^2+\varepsilon)^2\big) + Q(\lambda),
\end{aligned}$$

where

$$Q(\lambda) := \gamma_n\sum_{k\geq 3}\frac{1}{k!}(\lambda m_g + \frac{\lambda^2}{2})^k - \sum_{k\geq 3}\frac{1}{k!}(\gamma_n\lambda m_g + \frac{\gamma_n\lambda^2(1+m_g^2+\varepsilon)}{2})^k.$$

In particular, using $\gamma_n \leq 1$ from **(S)**, and $\varepsilon < 1$, we can roughly estimate:

$$\Delta_n^{\exp} \leq -\frac{\gamma_n\lambda^2}{2}\varepsilon + \frac{\gamma_n\lambda^3}{2}\big(|m_g|(3+m_g^2)\big) + \frac{1}{8}\gamma_n\lambda^4 + Q(\lambda), \tag{2.27}$$

Because g is 1-Lipschitz continuous and from the assumption **(GM)** we obtain:

$$m_g^2 = |\mathbb{E}g(Y_n)|^2 \leq \mathbb{E}|g(Y_n)|^2 = \mathbb{E}|g(Y_n) - g(0)|^2 \leq [g]_1 \mathbb{E}\|Y_n\|^2 = [g]_1 \sum_{k=1}^{r} |Y^k|^2 \leq r.$$

Using again $\gamma_n \leq 1$, $\lambda \leq 1$, $\varepsilon \leq 1$ we get

$$Q(\lambda) \leq \gamma_n \sum_{k\geq 3} \frac{\lambda^k}{k!}(\sqrt{r} + \frac{1}{2})^k + \gamma_n \sum_{k\geq 3} \frac{\lambda^k \gamma_n^{k-1}}{k!}(\sqrt{r} + \frac{(2+r)}{2})^k \leq 2\gamma_n \lambda^3 \exp(\sqrt{r} + 1 + r/2)$$

Thus combined with (2.27) gives :

$$\begin{aligned}\Delta_n^{\exp} &\leq -\frac{\gamma_n \lambda^2}{2}\varepsilon + \frac{\gamma_n \lambda^3}{2}\big(|m_g|(3+m_g^2)\big) + \frac{1}{8}\gamma_n \lambda^4 + 2\gamma_n \lambda^3 \exp(\sqrt{r} + 1 + r/2) \\ &\leq \gamma_n \lambda^2/2\big\{ -\varepsilon + \lambda\sqrt{r}(3+r) + \frac{1}{8}\lambda^2 + 4\lambda \exp(\sqrt{r}+1+r/2)\big\} \leq \gamma_n\lambda^2/2\big\{ -\varepsilon + \lambda\rho(r)\big\}.\end{aligned}$$

which is negative if $\lambda\rho(r) < \varepsilon$, with $\rho(r)$ defined in (1.9). This proves the (2.25). Together with the inequality $m_g^2 \leq r$ this proves the concentration inequality in the case $[g]_1 \leq 1$.

□

Remark 9. *Note that $r \to \rho(r)$ is increasing, hence the condition we need to put on λ in order to propagate the Gaussian concentration from $Y \in \mathbb{R}^r$ to Z became stronger if the dimension r increase.*

3. Exponential integrability of the square root of Lyapunov function.

In [HMP18], the exponential moments of the Lyapunov function was used to control the remainder terms of the decomposition of the empirical measure. In this article, we also use the Lyapunov function for this purpose. But our framework yields more constraints over the analysis. Namely, we cannot directly use $\exp(CV_n)$ which is not *a priori* integrable. Indeed, let us consider the Compound Poisson process $\tilde{Z}_t := \sum_{k=1}^{N_t} \tilde{\mathbf{Y}}_k$ where $(\tilde{\mathbf{Y}}_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of a standard normal variables independent of N_t wich follows a Poisson law, which is the typical jump random variables that we aim to approximate. Conditionally to N_t, $\tilde{Z}_t \sim \mathcal{N}(0, N_t)$. So if we choose the Lyapunov function to be the standard quadratic map, i.e. for all $x \in \mathbb{R}^d$, $V(x) = |x|^2 + 1$. We obtain *in fine*:

$$\mathbb{E}[\exp(\lambda V(\tilde{Z}_t)] = e^{\lambda}\mathbb{E}\big[\mathbb{E}[\exp(\lambda \tilde{Z}_t^2)|N_t]\big] \geq e^{\lambda}\mathbb{E}[\int_{\mathbb{R}^d} \exp(\lambda|y_t|^2)\exp(-\frac{|y_t|^2}{2N_t})\frac{dy_t}{(2\pi N_t)^{1/2}}\mathbb{1}_{N_t\geq 1}],$$

this is integrable if almost surely $N_t < \frac{1}{2\lambda}$ which is not true for $\lambda > 0$.

Proof of Proposition 2. Preliminarily to the proof of this proposition, we write some useful controls thanks to assumption $(\mathcal{L}_{\mathbf{V}})$, for all $x \in \mathbb{R}^d$,

$$|\nabla\sqrt{V}(x)| = |\frac{\nabla V(x)}{2\sqrt{V}(x)}| \leq \frac{\sqrt{C_V}}{2}, \tag{3.1}$$

$$\|D^2\sqrt{V}(x)\| = \|\frac{D^2V(x)}{2\sqrt{V}(x)} - \frac{\nabla V \nabla V^*(x)}{4V^{3/2}(x)}\| \leq \frac{\|D^2V\|_\infty}{2\sqrt{v^*}} + \frac{C_V}{4\sqrt{v^*}}. \tag{3.2}$$

To begin we check that $\sqrt{V}$ satisfies assumption $(\mathcal{L}_V)$ *iii)*. We have readily that:

$$\widetilde{\mathcal{A}}\sqrt{V} = \frac{1}{2\sqrt{V}}\widetilde{\mathcal{A}}V - \frac{1}{8V^{\frac{3}{2}}}|\sigma^*\nabla V|^2 \leq \frac{1}{2\sqrt{V}}(-\tilde{\alpha}_V V + \tilde{\beta}_V) \leq -\frac{\alpha_V}{2}\sqrt{V} + \bar{\beta}_V, \tag{3.3}$$

with $\bar{\beta}_V := \frac{\tilde{\beta}_V}{2\sqrt{v^*}}$. The first inequality is a consequence of Remark 3. Furthermore, for the purely jump part of the infinitesimal generator we write:

$$\mathcal{A}\sqrt{V}(x) - \widetilde{\mathcal{A}}\sqrt{V}(x) = \int_{\mathbb{R}^r} (\sqrt{V(x+\kappa(x)y)} - \sqrt{V(x)})\pi(dy) \leq \frac{\sqrt{C_V}\|\kappa\|_\infty \pi(|\cdot|)}{2},$$

using (3.1). The previous inequality and (3.3) implies that:

$$\mathcal{A}\sqrt{V} \leq -\frac{\alpha_V}{2}\sqrt{V} + \bar{\beta}'_V, \tag{3.4}$$

where $\bar{\beta}'_V = \bar{\beta}_V + \frac{\sqrt{C_V}\|\kappa\|_\infty \pi(|\cdot|)}{2}$.

Next, let us decompose the Lyapunov function $\sqrt{V}$ with a Taylor expansion similarly to Lemma 1. We again use a splitting between the deterministic contributions and those involving the innovation. We write for all $n \in \mathbb{N}$:

$$\begin{aligned}
&\sqrt{V}(X_n) - \sqrt{V}(X_{n-1}) = \sqrt{V}(X_{n-1} + \gamma_n b_{n-1} + \sqrt{\gamma_n}\sigma_{n-1}U_n) - \sqrt{V}(X_{n-1})\\
&+\sqrt{V}(X_n) - \sqrt{V}(X_{n-1} + \gamma_n b_{n-1} + \sqrt{\gamma_n}\sigma_{n-1}U_n)\\
&= \gamma_n \mathcal{A}\sqrt{V}(X_{n-1}) + \Big[\gamma_n \int_0^1 \langle b_{n-1}, \big[\nabla\sqrt{V}(X_{n-1} + t\gamma_n b_{n-1}) - \nabla\sqrt{V}(X_{n-1})\big]\rangle dt\Big]\\
&-\Big[\frac{\gamma_n}{2}\mathrm{Tr}\big(D^2\sqrt{V}(X_{n-1}))\Sigma_{n-1}\big)\Big] + \Big[\sqrt{\gamma_n}\sigma_{n-1}U_n \cdot \nabla\sqrt{V}(X_{n-1} + \gamma_n b_{n-1})\\
&+\gamma_n \int_0^1 (1-t)\mathrm{Tr}\Big(D^2\sqrt{V}(X_{n-1} + \gamma_n b_{n-1} + t\sqrt{\gamma_n}\sigma_{n-1}U_n)\sigma_{n-1}U_n \otimes U_n\sigma^*_{n-1}\Big)dt\Big]\\
&+\Big[\sqrt{V}(X_n) - \sqrt{V}(X_{n-1} + \gamma_n b_{n-1} + \sqrt{\gamma_n}\sigma_{n-1}U_n) - \gamma_n\pi\big(\sqrt{V(X_{n-1} + \kappa(X_{n-1})\cdot)} - \sqrt{V(X_{n-1})}\big)\Big]\\
&=: \gamma_n\mathcal{A}\sqrt{V}(X_{n-1}) + \mathcal{V}_1(X_{n-1}) + \mathcal{V}_2(X_{n-1}) + \mathcal{V}_3(X_{n-1}, U_n) + \mathcal{V}_4(X_{n-1}, U_n, Z_n),
\end{aligned} \tag{3.5}$$

where for all $x \in \mathbb{R}^d$, the first term is such that:

$$\begin{aligned}
\mathcal{V}_1(x) &= \gamma_n \int_0^1 \langle b(x), \frac{\nabla V}{2\sqrt{V}}(x + t\gamma_n b(x)) - \frac{\nabla V}{2\sqrt{V}}(x)\rangle dt\\
&= \gamma_n \int_0^1 \langle b(x) - b(x + t\gamma_n b(x)), \frac{\nabla V}{2\sqrt{V}}(x + t\gamma_n b(x))\rangle dt\\
&+\gamma_n \int_0^1 \langle b, \frac{\nabla V}{2\sqrt{V}}\rangle(x + t\gamma_n b(x)) - \langle b, \frac{\nabla V}{2\sqrt{V}}\rangle(x) dt =: \mathcal{V}_1^1(x) + \mathcal{V}_1^2(x).
\end{aligned} \tag{3.6}$$

Because b is supposed to be Lipschitz continuous and thanks to $(\mathcal{L}_V)$ ii), we readily writes:

$$\mathcal{V}_1^1(x) \leq \frac{\gamma_n^2[b]_1}{4}|b(x)| \int_0^1 \frac{|\nabla V|}{2\sqrt{V}}(x + t\gamma_n b(x))dt \leq \frac{\gamma_n^2[b]_1 C_V}{8}\sqrt{V}(x). \tag{3.7}$$

Whilst the next term is more subtle. Indeed, observe that thanks to $(\mathcal{L}_V)$ ii) the following term is bounded:

$$|\nabla\langle b, \frac{\nabla V}{2\sqrt{V}}\rangle| \leq \big|Db\frac{\nabla V}{2\sqrt{V}}\big| + \big|b\frac{D^2 V}{2\sqrt{V}}\big| + \big|\frac{(\nabla V)(\nabla V)^* b}{4V^{\frac{3}{2}}}\big| \leq \frac{\sqrt{C_V}[b]_1}{2} + \frac{\sqrt{C_V}\|D^2 V\|_\infty}{2} + \frac{C_V^{3/2}}{4} =: C_{(3.8)}, \tag{3.8}$$

which directly yields again thanks to $(\mathcal{L}_\mathbf{V})$ ii) that

$$\mathcal{V}_1^2(x) \leq \gamma_n^2 C_{(3.8)} \int_0^1 t|b(x)|dt \leq \frac{\gamma_n^2 C_{(3.8)}\sqrt{C_V}\sqrt{V}(x)}{2}. \tag{3.9}$$

Hence plugging (3.7) and (3.9) into (3.6) implies that:

$$\mathcal{V}_1(x) \le \gamma_n^2\Big(\frac{[b]_1 C_V}{8} + C_{(3.8)}\sqrt{C_V}\Big)\sqrt{V}(x). \tag{3.10}$$

The second term is handled by (3.2):

$$\mathcal{V}_2(x) \le \frac{\gamma_n}{2}\|\sigma\|_\infty^2\Big(\frac{\|D^2V\|_\infty}{2\sqrt{v^*}} + \frac{C_V}{4\sqrt{v^*}}\Big). \tag{3.11}$$

The third term satisfies the following identity:

$$\begin{aligned}
&\mathcal{V}_3(x,U_n) = \sqrt{\gamma_n}\sigma(x)U_n\cdot\nabla\sqrt{V}(x+\gamma_n b(x))\\
&+\gamma_n\int_0^1(1-t)\mathrm{Tr}\Big(D^2\sqrt{V}(x+\gamma_n b(x)+t\sqrt{\gamma_n}\sigma(x)U_n)\sigma(x)U_n\otimes U_n\sigma(x)^*\Big)dt\\
&\overset{(3.2)}{\le} \sqrt{\gamma_n}\sigma(x)U_n\cdot\nabla\sqrt{V}(x+\gamma_n b(x)) + \frac{\gamma_n}{2}\Big(\frac{\|D^2V\|_\infty}{2\sqrt{v^*}} + \frac{C_V}{4\sqrt{v^*}}\Big)\|\sigma\|_\infty^2|U_n|^2\\
&=: \mathcal{V}_3^1(x,U_n) + \mathcal{V}_3^2(U_n), \qquad (3.12)
\end{aligned}$$

and the last term is:

$$\begin{aligned}
&\mathcal{V}_4(x,U_n,Z_n)\\
=\ &\sqrt{V}\big(x+\gamma_n b(x)+\sqrt{\gamma_n}\sigma(x)U_n+\kappa(x)Z_n\big) - \sqrt{V}\big(x+\gamma_n b(x)+\sqrt{\gamma_n}\sigma(x)U_n\big)\\
&-\gamma_n\pi\big(\sqrt{V(x+\kappa(x)\cdot)}-\sqrt{V(x)}\big)\\
\overset{(3.1)}{\le}\ &\sqrt{V}(x+\gamma_n b(x)+\sqrt{\gamma_n}\sigma(x)U_n+\kappa(x)Z_n) - \sqrt{V}(x+\gamma_n b(x)+\sqrt{\gamma_n}\sigma(x)U_n)\\
&+\frac{\gamma_n\|\kappa\|_\infty\sqrt{C_V}\pi(|\cdot|)}{2}\\
=:\ &\mathcal{V}_4^1(x,U_n,Z_n) + \frac{\gamma_n\|\kappa\|_\infty\sqrt{C_V}\pi(|\cdot|)}{2}. \qquad (3.13)
\end{aligned}$$

Hence plugging (3.4), (3.10), (3.11), (3.12) and (3.13) into (3.5):

$$\begin{aligned}
&\sqrt{V}(X_n)-\sqrt{V}(X_{n-1}) \le -\gamma_n\Big[\frac{\alpha_V}{2}\sqrt{V}(X_{n-1})-\bar\beta_V'\Big] + \gamma_n^2\Big(\frac{[b]_1C_V}{8}+C_{(3.8)}\sqrt{C_V}\Big)\sqrt{V}(X_{n-1})\\
&+\frac{\gamma_n}{2}\|\sigma\|_\infty^2\Big(\frac{\|D^2V\|_\infty}{2\sqrt{v^*}}+\frac{C_V}{4\sqrt{v^*}}\Big) + \mathcal{V}_3(X_{n-1},U_n) + \frac{\gamma_n\|\kappa\|_\infty\sqrt{C_V}\pi(|\cdot|)}{2} + \mathcal{V}_4^1(X_{n-1},U_n)\\
&\le -\gamma_n\frac{\alpha_V}{4}\sqrt{V}(X_{n-1}) + \gamma_n\hat\beta_V + \mathcal{V}_3^1(X_{n-1},U_n) + \mathcal{V}_3^2(U_n) + \mathcal{V}_4^1(X_{n-1},U_n,Z_n), \qquad (3.14)
\end{aligned}$$

for

$$\gamma_n \le \frac{\alpha_V}{4\big(\frac{[b]_1C_V}{8}+C_{(3.8)}\sqrt{C_V}\big)} = \frac{\alpha_V}{4\Big(\frac{[b]_1C_V}{8}+\big(\frac{\sqrt{C_V}[b]_1}{2}+\frac{\sqrt{C_V}\|D^2V\|_\infty}{2}+\frac{C_V}{4}\big)\sqrt{C_V}\Big)}$$

which corresponds to assumption **(S)** and

$$\hat\beta_V := \bar\beta_V' + \frac{1}{2}\|\sigma\|_\infty^2\Big(\frac{\|D^2V\|_\infty}{2\sqrt{v^*}}+\frac{C_V}{4\sqrt{v^*}}\Big) + \frac{\|\kappa\|_\infty\sqrt{C_V}\pi(|\cdot|)}{2}.$$

We control the contribution of $\mathcal{V}_3^1(X_{n-1},U_n)$ and $\mathcal{V}_3^2(U_n)$ (defined in (3.12)) in the exponential moment of $\sqrt{V(X_n)}$ by the Gaussian concentration hypothesis **(GC)** and $\mathcal{V}_4^1(X_{n-1},U_n,Z_n)$ (see (3.13)) thanks to Proposition 1. We define for all $x\in\mathbb{R}^d$ and $\lambda>0$:

$$\begin{aligned}
I_1(\lambda,x) &:= \mathbb{E}\Big[\exp\big(\lambda\mathcal{V}_3^1(x,U_n)\big)\Big],\ I_2(\lambda) := \mathbb{E}\Big[\exp\big(\lambda\mathcal{V}_3^2(U_n)\big)\Big],\\
I_3(\lambda,x) &:= \mathbb{E}\left[\exp\big(\lambda\mathcal{V}_4^1(x,U_n,Z_n)\big)\right].
\end{aligned}$$

Indeed, by **(GC)**, we first write:

$$I_1(\lambda,x) \le \exp\Big(\frac{\lambda^2\gamma_n|\sigma^*(x)\nabla\sqrt{V}(x+\gamma_n b(x))|^2}{2}\Big) \overset{(3.1)}{\le} \exp\Big(\frac{\lambda^2\gamma_n C_V\|\sigma\|_\infty^2}{4}\Big). \tag{3.15}$$

Next, it is well known that under **(GC)**, for all $c<\frac{1}{2}$, $I_c := \mathbb{E}\left[\exp(c|U_n|^2)\right] < +\infty$. So we have for all $\lambda < \frac{2c\sqrt{v^*}}{(C_V/2+\|D^2V\|_\infty)\|\sigma\|_\infty^2\gamma_1}$, by Jensen's inequality:

$$I_2(\lambda) \;\le\; \Big[\mathbb{E}\exp\big(c|U_n|^2\big)\Big]^{\frac{\lambda\gamma_n}{2c}(\frac{\|D^2V\|_\infty}{2\sqrt{v^*}}+\frac{C_V}{4\sqrt{v^*}})\|\sigma\|_\infty^2} = \exp\Big(\gamma_n\ln(I_c)\frac{\lambda}{2c}(\frac{\|D^2V\|_\infty}{2\sqrt{v^*}}+\frac{C_V}{4\sqrt{v^*}})\|\sigma\|_\infty^2\Big). \tag{3.16}$$

Now, let us deal with the third term $I_3(\lambda,x)$. First of all, note that from definition in (3.13) and (3.1) the function $z\mapsto \mathcal{V}_4^1(X_{n-1},U_n,z)$ is $\|\kappa\|_\infty\sqrt{C_V}$-Lipschitz continuous.

Furthermore, we have that $|\mathbb{E}[\mathcal{V}_4^1(x,U_n,Z_n)|U_n]| = \gamma_n|\mathbb{E}[\mathcal{V}_4^1(x,U_n,Y_n)|U_n]| \le \gamma_n\sqrt{C_V}\|\kappa\|_\infty\pi(|\cdot|)$. Hence, by the Proposition 1, and for all $0<\lambda<\frac{1}{6\|\kappa\|_\infty\sqrt{C_V}\rho(r)}$ (see (1.9)), for the corresponding notation of Proposition 1 we take $\varepsilon=1$, and we get:

$$\begin{aligned} I_3(\lambda,x) &\le \mathbb{E}\left[\mathbb{E}\left[\exp\big(\lambda\mathcal{V}_4^1(x,U_n,Z_n)\big)\,|U_n\right]\right] \\ &\le \mathbb{E}\Big[\exp\Big(\lambda\mathbb{E}[\mathcal{V}_4^1(x,U_n,Z_n)|U_n] + \frac{(2+r)\gamma_n\lambda^2[\mathcal{V}_4^1(x,U_n,\cdot)]_1^2}{2}\Big)\Big] \\ &\le \exp\big(\lambda\gamma_n\|\kappa\|_\infty\sqrt{C_V}\pi(|\cdot|) + (2+r)\|\kappa\|_\infty^2 C_V\frac{\lambda^2\gamma_n}{2}\big). \end{aligned} \tag{3.17}$$

From now on, we assume that for all

$$\lambda < \lambda_V := \min\Big(1, \frac{\lambda_0}{2\bar{c}}, \frac{2c\sqrt{v^*}}{(C_V/2+\|D^2V\|_\infty)\|\sigma\|_\infty^2\gamma_1}, \frac{1}{6\|\kappa\|_\infty\sqrt{C_V}\rho(r)}\Big).$$

Gathering identities (3.14), (3.15) and (3.16), and by the Cauchy-Schwarz inequality, we obtain that for all $\lambda<\lambda_V$,

$$\begin{aligned} &\mathbb{E}\exp\big(\lambda\sqrt{V}(X_n)\big) \\ =\;& \mathbb{E}\Big[\exp\big(\lambda\sqrt{V}(X_{n-1})\big)\mathbb{E}\big[\exp\big(\lambda(\sqrt{V}(X_n)-\sqrt{V}(X_{n-1}))\big)\big|\mathcal{F}_{n-1}\big]\Big] \\ \le\;& \mathbb{E}\Big[\exp\big(\lambda[\sqrt{V}(X_{n-1})(1-\frac{\alpha_V}{4}\gamma_n)+\hat\beta_V\gamma_n]\big)I_1(2\lambda,X_{n-1})^{1/2}I_2(4\lambda)^{1/4}I_3(4\lambda,X_{n-1})^{1/4}\Big] \\ \le\;& \exp\big(\lambda\gamma_n\hat\beta_V'\big)\mathbb{E}\Big[\exp\big(\lambda(1-\gamma_n\hat\alpha_V)\sqrt{V}(X_{n-1})\big)\Big], \end{aligned}$$

where we have defined:

$$\hat\beta_V' := \hat\beta_V + \frac{C_V\|\sigma\|_\infty^2}{2} + \ln(I_c)\frac{(\frac{\|D^2V\|_\infty}{2\sqrt{v^*}}+\frac{C_V}{4\sqrt{v^*}})\|\sigma\|_\infty^2}{2c} + \|\kappa\|_\infty\sqrt{C_V}2\pi(|\cdot|) + 2(2+r)\|\kappa\|_\infty^2 C_V,$$

and

$$\hat\alpha_V := \min\big(\frac{1}{\gamma_1},\frac{\alpha_V}{4}\big) \in (0,\frac{1}{\gamma_1}].$$

So $(1-\gamma_n\hat\alpha_V)\in[0,1)$ and we deduce by Jensen inequality:

$$\mathbb{E}\exp\big(\lambda\sqrt{V}(X_n)\big) \le \exp\big(\lambda\gamma_n\hat\beta_V'\big)\mathbb{E}\big[\exp\big(\lambda\sqrt{V}(X_{n-1})\big)\big]^{(1-\gamma_n\hat\alpha_V)}. \tag{3.18}$$

For all $\lambda>0$, we introduce

$$C_{V,\lambda} := \max\big(\mathbb{E}[e^{\lambda\sqrt{V}(X_0)}], e^{\lambda\hat\beta_V'/\hat\alpha_V}\big).$$

In particular, we have $\mathbb{E}[e^{\lambda\sqrt{V}(X_0)}]\le C_{V,\lambda}$.

Let us check by induction that for all $n \in \mathbb{N}$:

$$\mathbb{E}[e^{\lambda\sqrt{V}(X_n)}] \leq C_{V,\lambda}.$$

We deduce from (3.18) and by induction assumption that:

$$\mathbb{E}\exp\left(\lambda\sqrt{V}(X_n)\right) \leq \exp\left(\lambda\gamma_n\hat{\beta}'_V\right)C_{V,\lambda}^{(1-\gamma_n\hat{\alpha}_V)} \leq C_{V,\lambda}.$$

We pick $c_V < \lambda_V$ and the proof is completed. □

Remark 10. *Observe also that* $v^* := \inf_{x\in\mathbb{R}^d} V(x) > 0$, *we have that for all* $(n,\xi) \in \mathbb{N}\times[0,\frac{1}{2}]$, $\lambda < \lambda_V(v^*)^{1-\xi}$*:*

$$\mathbb{E}\exp(\lambda V_n^\xi) = \mathbb{E}\exp\Big(\lambda(v^*)^\xi \underbrace{\left(\frac{V_n}{v^*}\right)^\xi}_{\geq 1}\Big) \leq \mathbb{E}\exp\left(\lambda(v^*)^{\xi-1}V_n\right) \leq C_{V,\lambda(v^*)^{\xi-1}} < +\infty.$$

Hence, $\xi\in[0,1], \lambda < \lambda_V(v^*)^{1-\xi}$, $\sup_{n\in\mathbb{N}}\mathbb{E}\exp(\lambda\sqrt{V}_n^\xi) < +\infty$.

4. Proof of the Technical Lemmas

Proof of Lemma 1. For $k \in [\![1,n]\!]$, we first write:

$$\begin{aligned}
&\varphi(X_k)-\varphi(X_{k-1}) = (\varphi(X_k)-\varphi(X_{k-1}+\gamma_k b_{k-1}+\sqrt{\gamma_k}\sigma_{k-1}U_k))\\
&+(\varphi(X_{k-1}+\gamma_k b_{k-1}+\sqrt{\gamma_k}\sigma_{k-1}U_k)-\varphi(X_{k-1}+\gamma_k b_{k-1}))+(\varphi(X_{k-1}+\gamma_k b_{k-1})-\varphi(X_{k-1}))\\
&=: T_{k-1,j}(\varphi)+T_{k-1,r}(\varphi)+T_{k-1,d}(\varphi),
\end{aligned} \tag{4.1}$$

with

$$\begin{aligned}
T_{k-1,j}(\varphi) &= (\varphi(X_k)-\varphi(X_{k-1}+\gamma_k b_{k-1}+\sqrt{\gamma_k}\sigma_{k-1}U_k))\\
T_{k-1,d}(\varphi) &= \gamma_k\langle\nabla\varphi(X_{k-1}),b_{k-1}\rangle+\gamma_k\int_0^1\langle\nabla\varphi(X_{k-1}+t\gamma_k b_{k-1})-\nabla\varphi(X_{k-1}),b_{k-1}\rangle dt,\\
T_{k-1,r}(\varphi) &= \sqrt{\gamma_k}\sigma_{k-1}U_k\cdot\nabla\varphi(X_{k-1}+\gamma_k b_{k-1})\\
&\quad+\gamma_k\int_0^1(1-t)\mathrm{Tr}\Big(D^2\varphi(X_{k-1}+\gamma_k b_{k-1}+t\sqrt{\gamma_k}\sigma_{k-1}U_k)\sigma_{k-1}U_k\otimes U_k\sigma_{k-1}^*\Big)dt.
\end{aligned}$$

Thanks to this splitting, we are able to isolate the deterministic, the sub-Gaussian random variable approximating Brownian increments and the jump contributions. Then we proceed by Taylor expansion up to the order 2 for the function φ in the two last terms of the r.h.s. of (4.1),

$$\begin{aligned}
&\varphi(X_k)-\varphi(X_{k-1})=\gamma_k\mathcal{A}\varphi(X_{k-1})\\
&+\varphi(X_k)-\varphi(X_{k-1}+\gamma_k b_{k-1}+\sqrt{\gamma_k}\sigma_{k-1}U_k)-\gamma_k\int_{\mathbb{R}^r}\left(\varphi(X_{k-1}+\kappa_{k-1}y)-\varphi(X_{k-1})\right)\pi(dy)\\
&+[\gamma_k\int_0^1\langle\nabla\varphi(X_{k-1}+t\gamma_k b_{k-1})-\nabla\varphi(X_{k-1}),b_{k-1}\rangle dt]+[\frac{\gamma_k}{2}(D^2\varphi(X_{k-1}+\gamma_k b_{k-1})-D^2\varphi(X_{k-1}))]\\
&+[\sqrt{\gamma_k}\sigma_{k-1}U_k\cdot\nabla\varphi(X_{k-1}+\gamma_k b_{k-1})\\
&+\gamma_k\int_0^1(1-t)\mathrm{Tr}\Big(D^2\varphi(X_{k-1}+\gamma_k b_{k-1}+t\sqrt{\gamma_k}\sigma_{k-1}U_k)\sigma_{k-1}U_k\otimes U_k\sigma_{k-1}^*\\
&-D^2\varphi(X_{k-1}+\gamma_k b_{k-1})\Sigma_{k-1}\Big)dt]\\
&=\gamma_k\mathcal{A}\varphi(X_{k-1})+D_{2,b}^{k,\varphi}(X_{k-1})+D_{2,\Sigma}^{k,\varphi}+\psi_k^{\varphi}(X_{k-1},U_k)\\
&+\varphi(X_k)-\varphi(X_{k-1}+\gamma_k b_{k-1}+\sqrt{\gamma_k}\sigma_{k-1}U_k)-\gamma_k\int_{\mathbb{R}^r}\left(\varphi(X_{k-1}+\kappa_{k-1}y)-\varphi(X_{k-1})\right)\pi(dy)\\
&=\gamma_k\mathcal{A}\varphi(X_{k-1})+D_{2,b}^{k,\varphi}(X_{k-1})+D_{2,\Sigma}^{k,\varphi}+\psi_k^{\varphi}(X_{k-1},U_k)\\
&+[\varphi(X_{k-1}+\kappa_{k-1}Z_k)-\varphi(X_{k-1})-\gamma_k\int_{\mathbb{R}^r}\left(\varphi(X_{k-1}+\kappa_{k-1}y)-\varphi(X_{k-1})\right)\pi(dy)]\\
&+[\varphi(X_k)-\varphi(X_{k-1}+\gamma_k b_{k-1}+\sqrt{\gamma_k}\sigma_{k-1}U_k)-(\varphi(X_{k-1}+\kappa_{k-1}Z_k)-\varphi(X_{k-1}))]\\
&=\gamma_k\mathcal{A}\varphi(X_{k-1})+D_{2,b}^{k,\varphi}(X_{k-1})+D_{2,\Sigma}^{k,\varphi}(X_{k-1})+\psi_k^{\varphi}(X_{k-1},U_k)+\widetilde{\Delta}_k^{\varphi}(X_{k-1},Z_k)+D_j^{k,\varphi}(X_{k-1},U_k,Z_k).
\end{aligned}\tag{4.2}$$

Note finally, that by definition of $D_j^{k,\varphi}(X_{k-1},U_k,Z_k)$ in the previous expansion (4.2):

$$\begin{aligned}
&\psi_k^{\varphi}(X_{k-1},U_k)=\varphi(X_k)-\varphi(X_{k-1})-\gamma_k\mathcal{A}\varphi(X_{k-1})-\widetilde{\Delta}_k^{\varphi}(X_{k-1},Z_k)\\
&-\big(D_{2,b}^{k,\varphi}(X_{k-1})+D_{2,\Sigma}^{k,\varphi}(X_{k-1})+D_j^{k,\varphi}(X_{k-1},U_k,Z_k)\big)\\
&=\varphi(X_{k-1}+\gamma_k b_{k-1}+\sqrt{\gamma_k}\sigma_{k-1}U_k)+\varphi(X_{k-1}+\kappa_{k-1}Z_k)-\gamma_k\mathcal{A}\varphi(X_{k-1})-2\varphi(X_{k-1})\\
&-\widetilde{\Delta}_k^{\varphi}(X_{k-1},Z_k)-\big(D_{2,b}^{k,\varphi}(X_{k-1})+D_{2,\Sigma}^{k,\varphi}(X_{k-1})\big),
\end{aligned}$$

hence after differentiating, we see that $u\mapsto\psi_k^{\varphi}(X_{k-1},u)$ and hence $u\mapsto\Delta_k^{\varphi}(X_{k-1},u)$ are Lipschitz continuous with a modulus bounded by $\sqrt{\gamma_{k-1}}\|\sigma_{k-1}\|\|\nabla\varphi\|_\infty\le\sqrt{\gamma_{k-1}}\|\sigma\|_\infty\|\nabla\varphi\|_\infty$.

Moreover, from the definition $\mathbb{E}[\Delta_k^{\varphi}(X_{k-1},U_k)|\mathcal{F}_{k-1}]=0$ and using the definition of Z_n we get

$$\begin{aligned}
&\mathbb{E}[\widetilde{\Delta}_k^{\varphi}(X_{k-1},Z_k)|\mathcal{F}_{k-1}]=\\
&\mathbb{E}\big[\varphi(X_{k-1}+\kappa_{k-1}Z_k)-\varphi(X_{k-1})\big|\mathcal{F}_{k-1}\big]-\gamma_k\int_{\mathbb{R}^r}\big[\varphi(X_{k-1}+\kappa_{k-1}y)-\varphi(X_{k-1})\big]\pi(dy)=0.
\end{aligned}\tag{4.3}$$

□

Proof of Lemma 2. We first prove the point **ii)**.

For all $\varepsilon\in(0,1)$ and $0<\frac{\Lambda}{\Gamma_n}<\frac{\varepsilon}{6[\widetilde{\Delta}_n(X_{n-1},\cdot)]_1\rho(r)}$ ($\rho(r)$ set in (1.9)), thanks to Proposition 1, we have for all $n\in\mathbb{N}$:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{\Lambda}{\Gamma_n}\widetilde{\Delta}_n^{\varphi}(X_{n-1},Z_n)\right)\Big|\mathcal{F}_{n-1}\right]\\
&\le\exp\left(-\frac{\Lambda}{\Gamma_n}\mathbb{E}[\widetilde{\Delta}_n^{\varphi}(X_{n-1},Z_n)|\mathcal{F}_{n-1}]+\gamma_n\frac{\Lambda^2}{2\Gamma_n^2}[\widetilde{\Delta}_n^{\varphi}(X_{n-1},\cdot)]_1^2(1+r+\varepsilon)\right).
\end{aligned}\tag{4.4}$$

By definition of $\widetilde{\Delta}_n^{\varphi}(X_{n-1}, Z_n)$ in (2.1), and from (4.3) we have:

$$\mathbb{E}[\frac{\Lambda}{\Gamma_n}\widetilde{\Delta}_n^{\varphi}(X_{n-1}, Z_n)\big|\mathcal{F}_{n-1}] = 0, \text{ and } [\widetilde{\Delta}_n^{\varphi}(X_{n-1}, \cdot)]_1^2 \leq \|\kappa\|_\infty^2 \|\nabla\varphi\|_\infty^2.$$

The previous control with (4.4) directly yield:

$$\mathbb{E}\left[\exp\left(\frac{\Lambda}{\Gamma_n}\widetilde{\Delta}_n^{\varphi}(X_{n-1}, Z_n)\right)\Big|\mathcal{F}_{n-1}\right] \leq \exp\left(\frac{\gamma_n\Lambda^2}{2\Gamma_n^2}\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2(1 + r + \varepsilon)\right), \tag{4.5}$$

with the constraint $0 < \frac{\Lambda}{\Gamma_n} < \frac{\varepsilon}{6\|\kappa\|_\infty\|\nabla\varphi\|_\infty\rho(r)}$.

The demonstration of the point **i)**, is a direct consequence of the previous analysis without using Proposition 1, which yields no restriction on λ. □

Proof of Lemma 3. By assumption $(\mathbf{T}_\beta)$, we know that there exists $C_{V,\varphi} > 0$ such that for all $x \in \mathbb{R}^d$, $|\varphi(x)| \leq C_{V,\varphi}\big(1 + \sqrt{V(x)}\big)$, so we obtain:

$$\begin{aligned}
&\mathbb{E}\exp\left(\Lambda\frac{|\varphi(X_0) - \varphi(X_n)|}{\Gamma_n}\right) \leq \mathbb{E}\exp\left(\Lambda\frac{C_{V,\varphi}(2 + \sqrt{V(X_0)} + \sqrt{V(X_n)})}{\Gamma_n}\right)\\
&\leq \exp\left(2C_{V,\varphi}\frac{\Lambda}{\Gamma_n}\right)\left[\mathbb{E}\exp\left(2C_{V,\varphi}\Lambda\frac{\sqrt{V(X_0)}}{\Gamma_n}\right)\right]^{\frac{1}{2}}\left[\mathbb{E}\exp\left(2C_{V,\varphi}\Lambda\frac{\sqrt{V(X_n)}}{\Gamma_n}\right)\right]^{\frac{1}{2}}\\
&\leq \exp\left(2C_{V,\varphi}\frac{\Lambda}{\Gamma_n}\right)(I_V^{\frac{1}{2}})^{\frac{2C_{V,\varphi}\Lambda}{c_V\Gamma_n}}.
\end{aligned}$$

The last inequality is obtained by Jensen's inequality for $\frac{\Lambda}{\Gamma_n} < \frac{c_V}{2C_{V,\varphi}}$ and by Proposition 2. □

Proof of Lemma 4. From the definition (2.2) we can write:

$$\begin{aligned}
D_{2,b}^{k,\varphi} &= \gamma_k\int_0^1 \langle\nabla\varphi(X_{k-1} + t\gamma_k b_{k-1}) - \nabla\varphi(X_{k-1}), b_{k-1}\rangle dt\\
&= \gamma_k\Big[\int_0^1 \langle\nabla\varphi(X_{k-1} + t\gamma_k b_{k-1}), b_{k-1} - b(X_{k-1} + t\gamma_k b_{k-1})\rangle dt\\
&\quad + \int_0^1 \big(\langle\nabla\varphi, b\rangle(X_{k-1} + t\gamma_k b_{k-1}) - \langle\nabla\varphi, b\rangle(X_{k-1})\big)dt\Big].
\end{aligned}$$

From the boundedness of $\nabla\varphi$, Lipschitz property of the mapping $x \mapsto b(x)$ (assumption (C_0)) and Lipschitz property of the mapping $x \mapsto \langle\nabla\varphi(x), b(x)\rangle$ (assumption $(\mathbf{T}_\beta)$), using the assumption $(\mathcal{L}_\mathbf{V})$, ii) one derives that:

$$|D_{2,b}^{k,\varphi}| \leq \gamma_k^2\Big(\|\nabla\varphi\|_\infty[b]_1 + [\langle\nabla\varphi, b\rangle]_1\Big)\frac{|b_{k-1}|}{2} \leq C_{(4.6)}\gamma_k^2\sqrt{V_{k-1}}, \tag{4.6}$$

for $C_{(4.6)} := (\|\nabla\varphi\|_\infty[b]_1 + [\langle\nabla\varphi, b\rangle]_1)\frac{\sqrt{C_V}}{2}$. Hence

$$|D_{2,b,n}^{\varphi}| \leq \sum_{k=1}^n C_{(4.6)}\gamma_k^2\sqrt{V_{k-1}}.$$

Next, by the Jensen inequality (for the exponential function with $\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}\gamma_k^2\delta_k$ as a measure), we deduce that:

$$\begin{aligned}\mathbb{E}\exp\Big(\frac{\Lambda}{\Gamma_n}|D^{\varphi}_{2,b,n}|\Big) &\le \frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^{n-1}\gamma_k^2\mathbb{E}\Big[\exp\Big(\Gamma_n^{(2)}\frac{\Lambda}{\Gamma_n}C_{(4.6)}\sqrt{V_{k-1}}\Big)\Big]\\ &\le \frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^{n-1}\gamma_k^2\mathbb{E}\Big[\exp\Big(c_V\sqrt{V_{k-1}}\Big)\Big]^{\frac{C_{(4.6)}\Lambda\Gamma_n^{(2)}}{c_V\Gamma_n}},\end{aligned}$$

for $\frac{\Lambda}{\Gamma}\le\frac{c_V}{C_{(4.6)}\Gamma_n^{(2)}}=\frac{2c_V}{(\|\nabla\varphi\|_\infty[b]_1+[\langle\nabla\varphi,b\rangle]_1)\sqrt{C_V}\Gamma_n^{(2)}}<1$, and c_V is introduced in Proposition 2 which readily yields:

$$\mathbb{E}\exp\Big(\frac{\Lambda}{\Gamma_n}|D^{\varphi}_{2,b,n}|\Big)\le\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^{n-1}\gamma_k^2(I_V^{1/2})^{\frac{C_{(4.6)}\Lambda\Gamma_n^{(2)}}{c_V\Gamma_n}}=(I_V^{1/2})^{\frac{C_{(4.6)}\Lambda\Gamma_n^{(2)}}{c_V\Gamma_n}}.$$

For the second inequality, we first use a Taylor expansion:

$$\begin{aligned}|D^{k,\varphi}_{2,\Sigma}| &= \frac{\gamma_k}{2}\Big|\mathrm{Tr}\Big(\big(D^2\varphi(X_{k-1}+\gamma_k b_{k-1})-D^2\varphi(X_{k-1})\big)\Sigma_{k-1}\Big)\Big|\\ &\le\frac{1}{2}\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\gamma_k^2|b_{k-1}|\le\frac{1}{2}\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}\gamma_k^2|V_{k-1}|^{\frac12}\end{aligned}\tag{4.7}$$

So

$$|D^{\varphi}_{2,\Sigma,n}|\le\frac{1}{2}\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}\sum_{k=1}^{n}\gamma_k^2|V_{k-1}|^{\frac12}.$$

Hence, like previously, by Jensen inequality and Proposition 2 for $\frac{\Lambda}{\Gamma_n}\le\frac{2c_V}{\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}\Gamma_n^{(2)}}<1$ we obtain

$$\begin{aligned}\mathbb{E}\exp\Big(\frac{\Lambda}{\Gamma_n}|D^{\varphi}_{2,\Sigma,n}|\Big) &\le \mathbb{E}\exp\Big(\frac{\Lambda}{2\Gamma_n}\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}\sum_{k=1}^{n}\gamma_k^2|V_{k-1}|^{\frac12}\Big)\\ &\le \frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^{n}\gamma_k^2\mathbb{E}\exp\Big(\frac{\Lambda\Gamma_n^{(2)}}{2\Gamma_n}\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}|V_{k-1}|^{\frac12}\Big)\\ &\le (I_V^{1/2})^{\frac{\Lambda\Gamma_n^{(2)}\|\sigma\|_\infty^2\|D^3\varphi\|_\infty\sqrt{C_V}}{2c_V\Gamma_n}}.\end{aligned}$$

□

Proof of Lemma 5. The proof is similar to the analysis of Lemma 3 in [HMP18]. By the definition (2.1), and because U_k, $k\in[\![1,n]\!]$, has the same moments as the standard Gaussian random variable up to order three (see **(GM)**) we have for all $k\in[\![1,n]\!]$:

$$\begin{aligned}\mathbb{E}\left[\psi_k^{\varphi}(X_{k-1},U_k)|\mathcal{F}_{k-1}\right] = \gamma_k\int_0^1(1-t)\mathrm{Tr}\Big(&\mathbb{E}\big[D^2\varphi(X_{k-1}+\gamma_k b_{k-1}+t\sqrt{\gamma_k}\sigma_{k-1}U_k)\sigma_{k-1}U_k\otimes U_k\sigma_{k-1}^*\\ &-D^2\varphi(X_{k-1}+\gamma_k b_{k-1})\Sigma_{k-1}|\mathcal{F}_{k-1}\big]\Big)dt,\end{aligned}$$

where

$$\mathbb{E}\Big[\mathrm{Tr}\Big(D^2\varphi(X_{k-1}+\gamma_k b_{k-1}+t\sqrt{\gamma_k}\sigma_{k-1}U_k)\sigma_{k-1}U_k\otimes U_k\sigma_{k-1}^* - D^2\varphi(X_{k-1}+\gamma_k b_{k-1})\Sigma_{k-1}\Big)\Big|\mathcal{F}_{k-1}\Big]$$
$$= t\sqrt{\gamma_k}\int_0^1 \mathbb{E}\Big[\mathrm{Tr}\Big(\big([D^3\varphi(X_{k-1}+\gamma_k b_{k-1}+ut\sqrt{\gamma_k}\sigma_{k-1}U_k) - D^3\varphi(X_{k-1}+\gamma_k b_{k-1})]\sigma_{k-1}U_k\big)$$
$$\big(\sigma_{k-1}U_k\otimes U_k\sigma_{k-1}^*\big)\Big)\Big|\mathcal{F}_{k-1}\Big]du.$$

Then,

$$\begin{aligned} |\mathbb{E}\,[\psi_k^\varphi(X_{k-1},U_k)|\mathcal{F}_{k-1}]| &\le \gamma_k\int_0^1(1-t)t^{1+\beta}[\varphi^{(3)}]_\beta\mathbb{E}\Big[\gamma_k^{\frac{1+\beta}{2}}\|\sigma_{k-1}\|^{3+\beta}|U_k|^{3+\beta}\int_0^1 u^\beta du\Big|\mathcal{F}_{k-1}\Big]dt \\ &= \frac{[\varphi^{(3)}]_\beta\gamma_k^{\frac{3+\beta}{2}}\|\sigma_{k-1}\|^{3+\beta}\mathbb{E}\,[|U_k|^{3+\beta}]}{(1+\beta)(2+\beta)(3+\beta)}. \end{aligned} \tag{4.8}$$

We sum over k to get the result. □

Proof of Lemma 6. Recall that we have denoted for $n\in\mathbb{N}_0$, $\mathcal{F}_n := \boldsymbol{\sigma}\big(X_0,(U_j,Z_j)_{j\in[\![1,n]\!]}\big)$ and $\widetilde{\mathcal{F}}_n = \mathcal{F}_n\vee\boldsymbol{\sigma}(U_{n+1})$.

$$\begin{aligned} &\mathbb{E}\Big[\exp\big(\frac{\Lambda}{\Gamma_n}\sum_{k=1}^n D_j^{k,\varphi}(X_{k-1},U_k,Z_k)\big)\Big] \\ &= \mathbb{E}\Big[\exp\big(\frac{\Lambda}{\Gamma_n}\sum_{k=1}^{n-1} D_j^{k,\varphi}(X_{k-1},U_k,Z_k)\big)\mathbb{E}\Big[\exp\big(\frac{\Lambda}{\Gamma_n}D_{j,n}^{\varphi}(X_{n-1},U_n,Z_n)\big)\Big|\widetilde{\mathcal{F}}_{n-1}\Big]\Big]. \end{aligned} \tag{4.9}$$

The idea is to control the last conditional expectation using Proposition 2. Recall that

$$\begin{aligned} D_{j,n}^{\varphi}(X_{n-1},U_n,Z_n) = {}&\varphi(X_{n-1}+\gamma_n b_{n-1}+\sqrt{\gamma_n}\sigma_{n-1}U_n+\kappa_{n-1}Z_n) - \varphi(X_{n-1}+\gamma_n b_{n-1}+\sqrt{\gamma_n}\sigma_{n-1}U_n) \\ &-\big[\varphi(X_{n-1}+\kappa_{n-1}Z_n)-\varphi(X_{n-1})\big]. \end{aligned}$$

Moreover, we have for all $z\in\mathbb{R}^r$:

$$\begin{aligned} &|\nabla_z D_{j,n}^{\varphi}(X_{n-1},U_n,z)| \\ = {}&|\kappa_{n-1}^*\big(\nabla\varphi(X_{n-1}+\gamma_n b_{n-1}+\sqrt{\gamma_n}\sigma_{n-1}U_n+\kappa_{n-1}z)-\nabla\varphi(X_{n-1}+\kappa_{n-1}z)\big)| \le 2\|\kappa\|_\infty\|\nabla\varphi\|_\infty. \end{aligned} \tag{4.10}$$

Hence for all X_{n-1}, U_n fixed the function $z\to D_{j,\varphi}^{n,\varphi}(X_{n-1},U_n,z)$ is Lipschitz continuous satisfying

$$[D_{j,n}^{\varphi}(X_{n-1},U_n,z)]_1 \le 2\|\kappa\|_\infty\|\nabla\varphi\|_\infty.$$

This estimation is used to bound Λ for which we can apply the Proposition 1. However, we need a more subtle control of the last Lipschitz modulus. Namely, using Taylor expansion we can write

$$\begin{aligned} |\nabla_z D_{j,n}^{\varphi}(X_{n-1},U_n,z)| &\le |\kappa_{n-1}^*\big(\nabla\varphi(X_{n-1}+\gamma_n b_{n-1}+\kappa_{n-1}z)-\nabla\varphi(X_{n-1}+\kappa_{n-1}z)\big)| \\ &+|\kappa_{n-1}^*\big(\nabla\varphi(X_{n-1}+\gamma_n b_{n-1}+\sqrt{\gamma_n}\sigma_{n-1}U_n+\kappa_{n-1}z)-\nabla\varphi(X_{n-1}+\gamma_n b_{n-1}+\kappa_{n-1}z)\big)| \\ &\le \|\kappa\|_\infty\big(\sqrt{2}\|\nabla\varphi\|_\infty^{\frac12}|\nabla\varphi(X_{n-1}+\gamma_n b_{n-1}+\kappa_{n-1}z)-\nabla\varphi(X_{n-1}+\kappa_{n-1}z)|^{\frac12}+\sqrt{\gamma_n}\|D^2\varphi\|_\infty\|\sigma\|_\infty|U_n|\big) \\ &\le \sqrt{\gamma_n}\|\kappa\|_\infty\big(\sqrt{2}C_V^{\frac14}\|\nabla\varphi\|_\infty^{\frac12}\|D^2\varphi\|_\infty^{\frac12}V^{\frac14}(X_{n-1})+\|D^2\varphi\|_\infty\|\sigma\|_\infty|U_n|\big). \end{aligned} \tag{4.11}$$

Now for all Λ satisying $0 < \frac{\Lambda}{\Gamma_n} < \frac{1}{12\|\kappa\|_\infty \|\nabla\varphi\|_\infty \rho(r)}$, we get

$$\begin{aligned}
(4.12)\qquad & \mathbb{E}\left[\exp\left(\frac{\Lambda}{\Gamma_n} D^{\varphi}_{j,n}(X_{n-1},U_n,Z_n)\right)\Big|\widetilde{\mathcal{F}}_{n-1}\right] \\
&\le \exp\Big(\frac{\Lambda}{\Gamma_n}\mathbb{E}[D^{\varphi}_{j,n}(X_{n-1},U_n,Z_n)|\widetilde{\mathcal{F}}_{n-1}] + \frac{\gamma_n\Lambda^2}{2\Gamma_n^2}(1+r+\varepsilon)[D^{\varphi}_{j,n}(X_{n-1},U_n,\cdot)]_1^2\Big) \\
&\le \exp\left(\frac{\Lambda}{\Gamma_n}\gamma_n\mathbb{E}[D^{\varphi}_{j,n}(X_{n-1},U_n,Y_n)|\widetilde{\mathcal{F}}_{n-1}] + \frac{\gamma_n^2\Lambda^2}{2\Gamma_n^2}\Big(C_1\sqrt{V}(X_{n-1}) + C_2|U_n|^2\Big)\right),
\end{aligned}$$

where we have denoted

$$C_1 := (1+r+\varepsilon)\|\kappa\|_\infty^2 4\sqrt{C_V}\|\nabla\varphi\|_\infty\|D^2\varphi\|_\infty, \quad C_2 := 2(1+r+\varepsilon)\|\kappa\|_\infty^2\|D^2\varphi\|_\infty^2\|\sigma\|_\infty^2$$

and used the following identities:

$$D^{\varphi}_{j,n}(X_{n-1},U_n,0) = 0,$$

and

$$\mathbb{E}[D^{\varphi}_{j,n}(X_{n-1},U_n,Z_n)|\widetilde{\mathcal{F}}_{n-1}] = \gamma_n\mathbb{E}[D^{\varphi}_{j,n}(X_{n-1},U_n,Y_n)|\widetilde{\mathcal{F}}_{n-1}],$$

which is a consequence of the definition of Z_n in (1.1). To control $\mathbb{E}[D^{\varphi}_{j,n}(X_{n-1},U_n,Y_n)|\widetilde{\mathcal{F}}_{n-1}]$ we introduce for all $(x,y)\in(\mathbb{R}^d)^2$ the function:

$$\bar\varphi(x,y) := \mathbb{E}[\varphi(x+\kappa(y)Y_n)] - \varphi(x),$$

which readily implies that:

$$\mathbb{E}[D^{\varphi}_{j,n}(X_{n-1},U_n,Y_n)|\widetilde{\mathcal{F}}_{n-1}] = \bar\varphi(X_{n-1}+\gamma_n b_{n-1}+\sqrt{\gamma_n}\sigma_{n-1}U_n, X_{n-1}) - \bar\varphi(X_{n-1},X_{n-1}).$$

The idea in the following is to apply the expansion of Lemma 1 with $\kappa = 0$ to the function $x \to \bar\varphi(x,y)$, which also corresponds to the expansion of Lemma 1 in [HMP18] for diffusion without jumps. If $\kappa = 0$, then $X_n = X_{n-1}+\gamma_n b_{n-1}+\sqrt{\gamma_n}\sigma_{n-1}U_n$, we can write using (2.3) and the definition (2.1) of $\Delta_k^{\bar\varphi}$ with $\widetilde{M}_n^{\bar\varphi} = D^{\bar\varphi}_{j,n} = 0$:

$$\bar\varphi(X_n,X_{n-1}) - \bar\varphi(X_{n-1},X_{n-1}) = \gamma_n\widetilde{\mathcal{A}}\bar\varphi(X_{n-1}) + D^{\bar\varphi}_{2,b,n}(X_{n-1}) + D^{\bar\varphi}_{2,\Sigma,n}(X_{n-1}) + \psi_n^{\bar\varphi}(X_{n-1},U_n). \tag{4.13}$$

All the terms in the right have obviously the same properties as the corresponding terms in the similar expansion of φ given by Lemma 1 with $\kappa = 0$. In particular, for all $y\in\mathbb{R}^d$, the map $\bar\varphi(\cdot,y)$ is Lipschitz continuous with

$$\|\nabla\bar\varphi(\cdot,y)\|_\infty \le 2\|\nabla\varphi\|_\infty, \quad \|D^2\bar\varphi(\cdot,y)\|_\infty \le 2\|D^2\varphi\|_\infty, \quad \|D^3\bar\varphi(\cdot,y)\|_\infty \le 2\|D^3\varphi\|_\infty. \tag{4.14}$$

Furthermore, $D^{\bar\varphi}_{2,b,n}$ and $D^{\bar\varphi}_{2,\Sigma,n}$ satisfy similar inequalities as (4.6) and (4.7) where φ is replaced by $\bar\varphi$. We directly have thanks to the definitions (1.5), (2.2), identities (4.6), (4.7) and (4.14):

$$\begin{aligned}
\gamma_n|\widetilde{\mathcal{A}}\bar\varphi(X_{n-1})| &= \gamma_n|\langle b_{n-1},\nabla\bar\varphi(X_{n-1})\rangle + \frac{1}{2}Tr\left(\sigma\sigma^* D^2\bar\varphi(X_{n-1})\right)| \\
&\le \gamma_n(2\sqrt{C_V}\sqrt{V}(X_{n-1})\|\nabla\varphi\|_\infty + \|\sigma\|_\infty^2\|D^2\varphi\|_\infty) \le \gamma_n C_3\sqrt{V}(X_{n-1}), \\
|D^{\bar\varphi}_{2,b,n}(X_{n-1})| &= \gamma_n|\int_0^1 \langle\nabla\bar\varphi(X_{n-1}+t\gamma_n b_{n-1}) - \nabla\bar\varphi(X_{n-1}), b_{n-1}\rangle dt| \\
&\le \gamma_n^2\Big(\|\nabla\bar\varphi\|_\infty[b]_1 + [\langle\nabla\bar\varphi,b\rangle]_1\Big)\frac{|b_{n-1}|}{2} \le \gamma_n^2 2C_{(4.6)}\sqrt{V}(X_{n-1}), \\
|D^{\bar\varphi}_{2,\Sigma,n}(X_{n-1})| &= \Big|\frac{\gamma_n}{2}\mathrm{Tr}\Big(\big(D^2\bar\varphi(X_{n-1}+\gamma_n b_{n-1}) - D^2\bar\varphi(X_{n-1})\big)\Sigma_{n-1}\Big)\Big| \\
&\le \gamma_n^2\|\sigma\|_\infty^2\|D^3\varphi\|_\infty C_V^{\frac{1}{2}}\sqrt{V}(X_{n-1}),
\end{aligned}$$

where $C_3 := 2\sqrt{C_V}\|\nabla\varphi\|_\infty + \|\sigma\|_\infty^2\|D^2\varphi\|_\infty(v*)^{-\frac{1}{2}}$. Therefore, from the previous controls and using (4.12) and (4.13), we get that there is a constant $C = C((\mathbf{A})) > 0$ such that:

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{\Lambda}{\Gamma_n}D^{\varphi}_{j,n}(X_{n-1},U_n,Z_n)\right)\Big|\widetilde{\mathcal{F}}_{n-1}\right]\\ &\leq \exp\Big(\frac{C\Lambda}{\Gamma_n}\gamma_n\big(\gamma_n\sqrt{V}(X_{n-1})+\psi_n^{\bar{\varphi}}(X_{n-1},U_n)\big)+\frac{C\Lambda^2}{\Gamma_n^2}\gamma_n^2\big(\sqrt{V}(X_{n-1})+|U_n|^2\big)\Big).\end{aligned}\tag{4.15}$$

Next, the idea is to separate the unbounded contribution from the terms involving $(\sqrt{V}(X_k))_{k\in[\![1,n]\!]}$, $(|U_k|^2)_{k\in[\![1,n]\!]}$ by a global Cauchy-Schwarz inequality:

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{\Lambda}{\Gamma_n}\sum_{k=1}^n D_j^{k,\varphi}(X_{k-1},U_k,Z_k)\right)\right]\\ &\leq \mathbb{E}\left[\exp\left(\frac{2\Lambda}{\Gamma_n}\sum_{k=1}^n\left[D_j^{k,\varphi}(X_{k-1},U_k,Z_k)-C\gamma_k^2(1+\frac{2\Lambda}{\Gamma_n})\sqrt{V}(X_{k-1})-\frac{2C\Lambda}{\Gamma_n}\gamma_k^2|U_k|^2\right]\right)\right]^{\frac{1}{2}}\\ &\times\mathbb{E}\Big[\exp\left(2\sum_{k=1}^n\left[C\gamma_k^2\left(\frac{\Lambda}{\Gamma_n}+\frac{2\Lambda^2}{\Gamma_n^2}\right)\sqrt{V}(X_{k-1})+\frac{2C\Lambda^2}{\Gamma_n^2}\gamma_k^2|U_k|^2\right]\right)\Big]^{\frac{1}{2}} =: \Upsilon_1^{\frac{1}{2}}\times\Upsilon_2^{\frac{1}{2}}.\end{aligned}\tag{4.16}$$

Again by the Cauchy-Schwarz inequality, we get :

$$\Upsilon_2^{1/2} \leq \mathbb{E}\Big[\exp\Big(4C(\frac{\Lambda}{\Gamma_n}+\frac{2\Lambda^2}{\Gamma_n^2})\sum_{k=1}^n\gamma_k^2\sqrt{V}(X_{k-1})\Big)\Big]^{\frac{1}{4}}\mathbb{E}\Big[\exp\Big(4\frac{C\Lambda^2}{\Gamma_n^2}\sum_{k=1}^n\gamma_k^2|U_k|^2\Big)\Big]^{1/4}.\tag{4.17}$$

We control the second expected value under condition $\frac{4C\Lambda^2\gamma_1^2}{\Gamma_n^2} < 1$ using Jensen inequality:

$$\mathbb{E}\Big[\exp\Big(4\frac{C\Lambda^2}{\Gamma_n^2}\sum_{k=1}^n\gamma_k^2|U_k|^2\Big)\Big]^{1/4} \leq \sum_{k=1}^n\frac{\gamma_k^2}{\Gamma_n^{(2)}}\mathbb{E}\Big[\exp\Big(\frac{4C\Lambda^2\Gamma_n^{(2)}}{\Gamma_n^2}|U_k|^2\Big)\Big]^{1/4} \leq \mathbb{E}\Big[\exp\Big(\frac{|U_1|^2}{4}\Big)\Big]^{\frac{16C\Lambda^2\Gamma_n^{(2)}}{\Gamma_n^2}}.\tag{4.18}$$

Because U_1 satisfies **(GC)**, $\mathbb{E}[\exp(\frac{|U_1|^2}{4})] < +\infty$. We handle the first expectation in (4.17) by the same method, using Jensen inequality under condition $\frac{\Lambda}{\Gamma_n} < \frac{c_V}{2C\Gamma_n^{(2)}}$, $\frac{\Lambda^2}{\Gamma_n^2} < \frac{c_V}{16C\Gamma_n^{(2)}}$ and Proposition 2 and we obtain:

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(4C\left(\frac{\Lambda}{\Gamma_n}+\frac{2\Lambda^2}{\Gamma_n^2}\right)\sum_{k=1}^n\gamma_k^2\sqrt{V}(X_{k-1})\Big)\Big]^{\frac{1}{4}}\\ \leq\ &\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n\gamma_k^2\mathbb{E}\Big[\exp\Big(4C\Gamma_n^{(2)}(\frac{\Lambda}{\Gamma_n}+\frac{2\Lambda^2}{\Gamma_n^2})\sqrt{V}(X_{k-1})\Big)\Big]^{\frac{1}{4}}\\ \leq\ &(I_V^{1/2})^{C\Gamma_n^{(2)}\left(\frac{\Lambda}{c_V\Gamma_n}+\frac{2\Lambda^2}{c_V\Gamma_n^2}\right)}.\end{aligned}\tag{4.19}$$

Gathering (4.17), (4.18) and (4.19), recalling that $\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}} \to 0$, we deduce that:

$$\Upsilon_2^{1/2} \leq \exp\Big((\frac{\Lambda}{\sqrt{\Gamma_n}}+\frac{\Lambda^2}{\Gamma_n})e_n\Big).\tag{4.20}$$

The first term in (4.16), Υ_1, is handled by identity (4.15).

$$\begin{aligned}
\Upsilon_1 \leq & \mathbb{E}\Big[\exp\Big(\frac{2\Lambda}{\Gamma_n}\gamma_n\psi_n^{\bar\varphi}(X_{n-1},U_n) \\
&+\frac{2\Lambda}{\Gamma_n}\sum_{k=1}^{n-1}\big[D_j^{k,\varphi}(X_{k-1},U_k,Z_k)-C\gamma_k^2\big(1+\frac{2\Lambda}{\Gamma_n}\big)\sqrt{V}(X_{k-1})-\frac{2C\Lambda^2}{\Gamma_n^2}\gamma_k^2|U_k|^2\big]\Big)\Big] \\
&=\mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(\frac{2\Lambda}{\Gamma_n}\gamma_n\psi_n^{\bar\varphi}(X_{n-1},U_n)|\mathcal{F}_{n-1}\Big] \\
&\times\exp\Big(\frac{2\Lambda}{\Gamma_n}\sum_{k=1}^{n-1}\big[D_j^{k,\varphi}(X_{k-1},U_k,Z_k)-C\gamma_k^2\big(1+\frac{2\Lambda}{\Gamma_n}\big)\sqrt{V}(X_{k-1})-\frac{2C\Lambda^2}{\Gamma_n^2}\gamma_k^2|U_k|^2\big]\Big)\Big] \\
&\overset{(\mathbf{GC})}{\leq}\exp\Big(\frac{2\Lambda}{\Gamma_n}\gamma_n^{1+\frac{3+\beta}{2}}\frac{2[\varphi^{(3)}]_\beta\|\sigma\|_\infty^{(3+\beta)}\mathbb{E}\big[|U_1|^{3+\beta}\big]}{(1+\beta)(2+\beta)(3+\beta)}+\frac{4\Lambda^2}{\Gamma_n^2}4\gamma_n^3\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2 \\
&+\mathbb{E}\Big[\exp\Big(\frac{2\Lambda}{\Gamma_n}\sum_{k=1}^{n-1}\big[D_j^{k,\varphi}(X_{k-1},U_k,Z_k)-C\gamma_k^2\big(1+\frac{2\Lambda}{\Gamma_n}\big)\sqrt{V}(X_{k-1})-\frac{C2\Lambda^2}{\Gamma_n^2}\gamma_k^2|U_k|^2\big]\Big)\Big].
\end{aligned}$$

The last inequality is a consequence of the bound (4.8) in the proof of Lemma 5 and the Lipschitz modulus control of $\psi_n^{\bar\varphi}(X_{n-1},\cdot)$ in Lemma 1. Hence, we iterate this procedure and with some positive constants $\tilde C_1,\tilde C_2$ not depending on Λ neither n but only on the assumptions we get :

$$\Upsilon_1 \leq \exp\Big(\frac{\tilde C_1\Lambda\Gamma_n^{(\frac{5+\beta}{2})}}{\Gamma_n}+\frac{\tilde C_2\Lambda^2\Gamma_n^{(3)}}{\Gamma_n^2}\Big)=\exp\Big(\big(\frac{\Lambda}{\sqrt{\Gamma_n}}+\frac{\Lambda^2}{\Gamma_n}\big)e_n\Big), \tag{4.21}$$

where, using $\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}\to 0$ for $\theta>\frac{1}{3}$. Eventually, inequalities (4.20) and (4.21) yields that:

$$\mathbb{E}\Big[\exp\big(\frac{\Lambda}{\Gamma_n}\sum_{k=1}^{n}D_j^k(X_{k-1},U_k,Z_k)\big)\Big]\leq\exp\Big(\big(\frac{\Lambda}{\sqrt{\Gamma_n}}+\frac{\Lambda^2}{\Gamma_n}\big)e_n\Big).$$

□

5. Numerical Results

This section is a numerical illustration of the deviations results of the empirical measure ν_n from Theorem 2. We consider the mono-dimensional case, $d=r=1$. The innovations $(U_i)_{i\geq1}$ and X_0 are Gaussian variables. Also, a difficulty is to approximate the jump part of generator $\mathcal{A}\varphi$, namely $\pi(\varphi(x+\kappa(x)\cdot)-\varphi(x))$ for $x\in\mathbb{R}$. To avoid this problem, we choose $(Y_k)_{k\geq0}$ to be Bernoulli variables, hence we directly get $\pi(\varphi(x+\kappa(x)\cdot)-\varphi(x))=\frac{1}{2}\big(\varphi(x+\kappa(x))+\varphi(x-\kappa(x))\big)-\varphi(x)$. We consider for the coefficients and the test function $b(x)=-\frac{x}{2}$, and $\sigma(x)=\kappa(x)=\varphi(x)=\cos(x)$ in (E). Note, in particular, that we have picked a degenerate framework. Thanks to Theorem 2, for $(\gamma_k)_{k\geq1}=(k^{-\theta})_{k\geq1},\ \theta\in[1/3,1]$ (corresponding to $\beta=1$ therein) the function

$$a\in\mathbb{R}^+\mapsto g_n(a):=\log\big(\mathbb{P}[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)|\geq a]\big)$$

is such that for

$$g_n(a)\leq -c_n\frac{a^2}{2\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2+4\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2}+\log(2C_n).$$

In Figure 1, we plot the the curves of g_n for $\theta=\frac{1}{3}+10^{-3}$ We perform the simulations for $n=5\times10^4$ in Figure 1, the probability is estimated by Monte Carlo simulation with $MC=10^4$ realizations of the random variable $|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)|$ in the unbiased case. Let us also introduce the function

$$S_\nu(a) := -\frac{a^2}{2\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2 + 4\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2},$$

such that $g_n(a) \leq c_n S_\nu(a) + \log(2C_n)$.

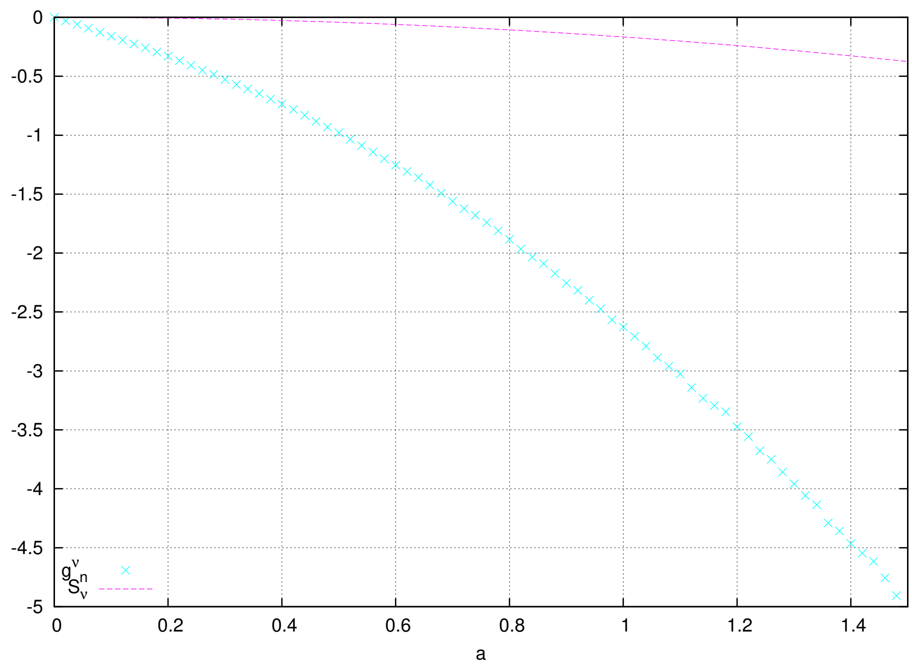

FIGURE 1. Plot of $a \mapsto g_n(a)$, for $\theta = \frac{1}{3}$, with $\varphi(x) = \sigma(x) = \cos(x)$.

The Figure 1 enhance the fact that $g_n(a)$ is indeed under a quadratic form in a. Nevertheless, we see s that the result of Theorem 2 is not sharp, to obtain such a result we have to avoid the dimension dependency and a sharp inequality of Proposition 1.

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