A rigidity result for the Holm-Staley b-family of equations with application to the asymptotic stability of the Degasperis-Procesi peakon

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A RIGIDITY RESULT FOR THE HOLM-STALEY b-FAMILY OF EQUATIONS WITH APPLICATION TO THE ASYMPTOTIC STABILITY OF THE DEGASPERIS-PROCESI PEAKON

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Abstract. We prove that the peakons are asymptotically H^1-stable, under the flow of the Degasperis-Procesi equation, in the class of functions with a momentum density that belongs to $\mathcal{M}_+(\mathbb{R})$. The key argument is a rigidity result for uniformly in time exponentially decaying global solutions that is shared by the Holm-Staley b-family of equations for $b \geq 1$. This extends previous results obtained for the Camassa-Holm equation ($b = 2$).

1. Introduction

The Degasperis-Procesi equation (D-P) reads

$$u_t - u_{txx} = -4uu_x + 3u_x u_{xx} + uu_{xxx}, \quad (t, x) \in \mathbb{R}^2, \tag{1.1}$$

It is a particular case of the more general family of equations called b-family equation that reads

$$u_t - u_{txx} = -(b + 1)uu_x + bu_x u_{xx} + uu_{xxx}, (t, x) \in \mathbb{R}^2, \tag{1.2}$$

with $b \in \mathbb{R}$. This family of equations was introduced by Holm and Staley ([29],[30]) to study the exchange of stability in the dynamics of solitary wave solutions under changes in the nonlinear balance in a one dimensional shallow water waves equation. It reduces for $b = 3$ to the DP equation and for $b = 2$ to the famous Camassa-Holm equation (C-H)

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, (t, x) \in \mathbb{R}^2 . \tag{1.3}$$

Even if, as noticed by Holm and Staley, the behavior of the solutions do change at $b = 0, \mp 1, \mp 2, \mp 3$, the b-family equations share the same peaked solitary waves given by

$$u(t, x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{-|x-ct|}, \ c \in \mathbb{R}.$$

They are called peakon whenever $c > 0$ and antipeakon whenever $c < 0$. Note that the initial value problem associated with (1.2) has to be rewriten as

$$\begin{cases} u_t + uu_x + (1 - \partial_x^2)^{-1}\partial_x(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2) = 0 \\ u(0) = u_0, \end{cases} \tag{1.4}$$

to give a meaning to these solutions. It is also worth noticing that the b-family equation (1.2) can be rewritted as

$$y_t + uy_x + bu_x y = 0 \tag{1.5}$$

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which is a transport equation for the momentum density $y = u - u_{xx}$. As a consequence an initial data with a signed initial momentum density gives rise to a solution that keeps this property. This is one of the main point to prove that smooth initial data with integrable signed initial momentum density give rise to global solutions (see [25]).

Both the Camassa and the Degasperis-Procesi equation can be derived as a model for the propagation of unidirectional shallow water waves over a flat bottom ([7], [31]), [1] and [13]. They are also known to be completely integrable (see [7],[8], [18], [19]) and to be bi-Hamiltonian. They both can be written in Hamiltonian form as

$$\partial_t E'(u) = -\partial_x F'(u) \quad . \tag{1.6}$$

For the Camassa-Holm equation it holds

$$E(u) = \int_{\mathbb{R}} u^2 + u_x^2 \text{ and } F(u) = \int_{\mathbb{R}} u^3 + u u_x^2 \tag{1.7}$$

whereas for the Degasperis-Procesi equation it holds

$$E(u) = \mathcal{H}(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} 5v^2 + 4v_x^2 + v_{xx}^2 \text{ and } F(u) = \int_{\mathbb{R}} u^3 \tag{1.8}$$

with $v = (4 - \partial_x^2)^{-1} u$.

It is worth noticing that they are the only elements of the b-family that enjoy such Hamiltonian structure and thus for which results on the orbital stability of the peakon do exist[1]. In a pioneer work [16], Constantin and Strauss proved the orbital stability in $H^1(\mathbb{R})$ of the peakon for the Camassa-Holm equation. This approach has been then adapted in [36] for the Degasperis-Procesi equation but only in the case of a non negative density momentum. Note also that a great simplification of the proof has been done in [33]. However, as far as the author knows, there is no available orbital stability result for the peakon of the DP equation with respect to solutions with non signed momentum density.

In [42], the author proved that the peakons are asymptotically stable under the CH-flow in the class of solutions emanating from initial data that belong to $H^1(\mathbb{R})$ with a density momentum that is a non negative finite measure. The main point was the proof of a rigidity result for uniformly almost localized solutions of the CH equation in this class. The asymptotic stability result being then obtained by following the approach developed by Martel and Merle ([39], [40]).

In this paper we first prove that this rigidity property can be extended to the b-family equations whenever $b \geq 1$. This emphasizes that this property is not directly linked to the integrability structure of the equation since it was proven in [18] that (1.4) is not integrable for $b \notin \{2, 3\}$. In a second part, we establish the asymptotic stability of the peakons under the DP-flow in the class of solutions emanating from initial data that belong to $H^1(\mathbb{R})$ with a density momentum that is a non negative finite measure.

Before stating our results let us introduce the function space where our initial data will take place. Following [15], we introduce the following space of functions

$$Y = \{u \in H^1(\mathbb{R}) \text{ such that } u - u_{xx} \in \mathcal{M}(\mathbb{R})\} \ . \tag{1.9}$$

We denote by Y_+ the closed subset of Y defined by $Y_+ = \{u \in Y \,/\, u - u_{xx} \in \mathcal{M}_+\}$ where $\mathcal{M}_+$ is the set of non negative finite Radon measures on $\mathbb{R}$. Note that since we

[1] However, numerical simulations ([29]) seems to indicate that the peakons are stable under the flow of (1.4) as soon as $b > 1$

are not aware of available uniqueness, global existence and continuity with respect to initial data results of all the b-family equations for initial data in this class, we establish such results in Section 2.

Let $C_b(\mathbb{R})$ be the set of bounded continuous functions on $\mathbb{R}$, $C_0(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$ that tends to 0 at infinity and let $I \subset \mathbb{R}$ be an interval. A sequence $\{\nu_n\} \subset \mathcal{M}$ is said to converge tightly (resp. weakly) towards $\nu \in \mathcal{M}$ if for any $\phi \in C_b(\mathbb{R})$ (resp. $C_0(\mathbb{R})$), $\langle \nu_n, \phi\rangle \to \langle \nu, \phi\rangle$. We will then write $\nu_n \rightharpoonup\ast\, \nu$ tightly in $\mathcal{M}$ (resp. $\nu_n \rightharpoonup\ast\, \nu$ in $\mathcal{M}$).

Throughout this paper, $y \in C_{ti}(I;\mathcal{M})$ (resp. $y \in C_w(I;\mathcal{M})$) will signify that for any $\phi \in C_b(\mathbb{R})$ (resp. $\phi \in C_0(\mathbb{R})$) , $t \mapsto \Big\langle y(t), \phi \Big\rangle$ is continuous on I and $y_n \rightharpoonup\ast\, y$ in $C_{ti}(I;\mathcal{M})$ (resp. $y_n \rightharpoonup\ast\, y$ in $C_w(I;\mathcal{M})$) will signify that for any $\phi \in C_b(\mathbb{R})$ (resp. $C_0(\mathbb{R})$), $\Big\langle y_n(\cdot), \phi \Big\rangle \to \Big\langle y(\cdot), \phi \Big\rangle$ in $C(I)$.

Definition 1.1. We say that a solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$ of (1.4) is Y-almost localized if there exist $c > 0$ and a C^1-function $x(\cdot)$, with $x_t \geq c > 0$, for which for any $\varepsilon > 0$, there exists $R_\varepsilon, > 0$ such that for all $t \in \mathbb{R}$ and all $\Phi \in C(\mathbb{R})$ with $0 \leq \Phi \leq 1$ and $\operatorname{supp} \Phi \subset [-R_\varepsilon, R_\varepsilon]^c$.

$$\Big\langle \Phi(\cdot - x(t)), u(t) - u_{xx}(t) \Big\rangle \leq \varepsilon \ . \tag{1.10}$$

We will say that such solution is uniformly in time exponentially decaying (up to translation) if there exist $a_1, a_2 > 0$ such that for all $(t, x) \in \mathbb{R}^2$

$$|u(t,x)| \leq a_1 e^{-a_2|x - x(t)|} \tag{1.11}$$

Remark 1.1. In [42] we replaced (1.10) by

$$\int_{\mathbb{R}} (u^2(t) + u_x^2(t)) \Phi(\cdot - x(t))\, dx + \Big\langle \Phi(\cdot - x(t)), u(t) - u_{xx}(t) \Big\rangle \leq \varepsilon \ . \tag{1.12}$$

in the definition of Y-almost localized solutions. This characterization was natural in the context fo the Camassa-Holm equation since the H^1-norm is a conservation law. For the b-family this is not the case and this is why it seems more appropriate to give the characterization (1.10) that is however equivalent to (1.12) (see the beginning of Section 2 for the proof of this equivalence).

Theorem 1.1. *Let $b \geq 1$ and $u \in C(\mathbb{R}; H^1(\mathbb{R}))$, with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$, be a Y-almost localized solution of the b-family equation* (1.4) *that is not identically vanishing. Assume moreover that u is uniformly in time exponentially decaying in the case $b \notin \{1, 2, 3\}$. Then there exists $c^* > 0$ and $x_0 \in \mathbb{R}$ such that*

$$u(t) = c^*\, \varphi(\cdot - x_0 - c^* t), \quad \forall t \in \mathbb{R}\,.$$

As a consequence we get the asymptotic stablity of the peakons to the DP equation ($b = 3$) that reads in the weak form as

$$u_t + u u_x + \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1} u^2 = 0 \ . \tag{1.13}$$

Theorem 1.2. *Let $c > 0$ be fixed. There exists a constant $0 < \eta \ll 1$ such that for any $0 < \theta < 1$ and any $u_0 \in Y_+$ satisfying*

$$\mathcal{H}(u_0 - \varphi_c) \leq \eta\, \theta^8 \ , \tag{1.14}$$

there exists $c^*>0$ with $|c-c^*|\ll c$ and a C^1-function $x:\mathbb{R}\to\mathbb{R}$ with $\lim_{t\to\infty}\dot{x}=c^*$ such that

$$u(t,\cdot+x(t)) \underset{t\to+\infty}{\rightharpoonup} \varphi_{c^*} \text{ in } H^1(\mathbb{R}) \;, \tag{1.15}$$

where $u\in C(\mathbb{R};H^1)$ is the solution of the DP equation ($b=3$) emanating from u_0. Moreover, for any $z\in\mathbb{R}$,

$$\lim_{t\to+\infty}\|u(t)-\varphi_{c^*}(\cdot-x(t))\|_{H^1(]-\infty,z[\cup]\theta t,+\infty[)}=0 \;. \tag{1.16}$$

Remark 1.2. Using that (1.13) is invariant by the change of unknown $u(t,x)\mapsto -u(t,-x)$, we obtain as well the asymptotic stability of the antipeakon profile $c\varphi$ with $c<0$ in the class of H^1-function with a momentum density that belongs to $\mathcal{M}_-(\mathbb{R})$.

Remark 1.3. We choose to write the convergence results (1.15)-(1.16) in $H^1(\mathbb{R})$ by sake of simplicity. Actually these convergence results hold in any $H^s(\mathbb{R})$ for $0\le s<3/2$.

This paper is organized as follows : the next section is devoted to the proof of the well-posedness results of the b-family equations in the class of solutions we will work with. In Section 3, we prove the rigidity result for Y-almost localized global solutions of the b-family equations with $b\ge 1$ with the additional uniform exponential decay condition as soon as $b>1$. Finally, in Section 4 we prove Theorem 1.2 as well as an asymptotic stability result for train of peakons to the DP equation. Note that we postponed to the appendix the proof of an almost monotonicity result for the DP equation that leads to the uniform exponential decay result for Y-almost localized global solutions.

2. Global well-posedness results

We first recall some obvious estimates that will be useful in the sequel of this paper. Noticing that $p(x)=\frac{1}{2}e^{-|x|}$ satisfies $p*y=(1-\partial_x^2)^{-1}y$ for any $y\in H^{-1}(\mathbb{R})$ we easily get

$$\|u\|_{W^{1,1}}=\|p*(u-u_{xx})\|_{W^{1,1}}\lesssim\|u-u_{xx}\|_{\mathcal{M}}$$

and

$$\|u_{xx}\|_{\mathcal{M}}\le\|u\|_{L^1}+\|u-u_{xx}\|_{\mathcal{M}}$$

which ensures that

$$Y\hookrightarrow\{u\in W^{1,1}(\mathbb{R}) \text{ with } u_x\in\mathcal{B}V(\mathbb{R})\} \;. \tag{2.1}$$

Moreover, Young's convolution inequalities lead to

$$\max(\|u\|_{L^2},\|u\|_{L^\infty},\|u_x\|_{L^2},\|u_x\|_{L^\infty})\le\|u-u_{xx}\|_{\mathcal{M}} \;. \tag{2.2}$$

It is also worth noticing that since for $u\in C_0^\infty(\mathbb{R})$,

$$u(x)=\frac{1}{2}\int_{-\infty}^{x}e^{x'-x}(u-u_{xx})(x')dx'+\frac{1}{2}\int_{x}^{+\infty}e^{x-x'}(u-u_{xx})(x')dx'$$

and

$$u_x(x)=-\frac{1}{2}\int_{-\infty}^{x}e^{x'-x}(u-u_{xx})(x')dx'+\frac{1}{2}\int_{x}^{+\infty}e^{x-x'}(u-u_{xx})(x')dx' \;,$$

we get $u_x^2 \le u^2$ as soon as $u - u_{xx} \ge 0$ on $\mathbb{R}$. By the density of $C_0^\infty(\mathbb{R})$ in Y, we deduce that

$$|u_x| \le u \text{ for any } u \in Y_+ \; . \tag{2.3}$$

In this paper we will often use that the Y-almost localization of a global solution u leads to a L^p-almost localization of u for $p \in [1, +\infty]$ and even to a H^1-almost localization. Indeed, for u satisfying Definition 1.1, taking $\tilde{\Phi} \in C(\mathbb{R})$ with $0 \le \tilde{\Phi} \le 1$, $\tilde{\Phi} \equiv 0$ on $[-R_{\varepsilon/8}, R_{\varepsilon/8}]$ and $\tilde{\Phi} \equiv 1$ on $[-2R_{\varepsilon/8}, 2R_{\varepsilon/8}]^c$, we get for $p \in [1, +\infty]$ and any $R > 2R_{\varepsilon/8}$ that

$$\|u(t, \cdot + x(t))\|_{L^p(|x|>R)} \le \frac{1}{2}\|e^{-|\cdot|} * (y(t, \cdot + x(t))\tilde{\Phi})\|_{L^p(|x|>R)} + \frac{1}{2}\Big\|e^{-|\cdot|} * (y(t, \cdot + x(t))(1-\tilde{\Phi}))\Big\|_{L^p(|x|>R)} .$$

with

$$\|e^{-|\cdot|} * (y(t, \cdot + x(t))\tilde{\Phi})\|_{L^p(|x|>R)} \le \|e^{-|\cdot|}\|_{L^p} \|y(t, \cdot + x(t))\tilde{\Phi}\|_{\mathcal{M}} \le \varepsilon/4$$

and for $p \neq +\infty$,

$$\begin{aligned}\Big\|e^{-|\cdot|} * \Big(y(t, \cdot + x(t))(1 - \tilde{\Phi})\Big)\Big\|_{L^p(|x|>R)} &= \Big(\int_{|x|>R} \Big|\langle e^{-|x-\cdot|}(1-\tilde{\Phi}), y(t, \cdot + x(t))\rangle\Big|^p\Big)^{1/p} \\ &\le e^{2R_{\varepsilon/8}} M(u) \|e^{-|\cdot|}\|_{L^p(|x|>R)} \le 2e^{-R+2R_{\varepsilon/8}} M(u) \; .\end{aligned}$$

with obvious modifications for $p = +\infty$. Taking $R > 2R_{\varepsilon/8}$ such that $2e^{-R+2R_{\varepsilon/8}} M(u) < \varepsilon/2$, this ensures that

$$\|u(\cdot + x(t)\|_{L^p(|x|>R)} \le \varepsilon$$

and applying this estimate for $p = 2$ with (2.3) in hands, we infer that (1.12) holds for $R'_\varepsilon > 2R_{\varepsilon/8}$ with $e^{-R'_\varepsilon} M(u) < R_{\varepsilon/8}\, \varepsilon/8$.

Finally, throughout this paper, we will denote $\{\rho_n\}_{n\ge 1}$ the mollifiers defined by

$$\rho_n = \Big(\int_{\mathbb{R}} \rho(\xi)\, d\xi\Big)^{-1} n\rho(n\cdot) \text{ with } \rho(x) = \begin{cases} e^{1/(x^2-1)} & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \ge 1 \end{cases} \tag{2.4}$$

In [25] the global well-posedness result for smooth solutions with a non negative momentum density of the Camassa-Holm equation (see [11]) is adapted to (1.4). This result can be summarized in the following proposition

Proposition 2.1. *(Strong solutions* [25]*)*
Let $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Then the initial value problem (1.4) *has a unique solution $u \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1})$ where $T = T(\|u_0\|_{H^{\frac{3}{2}+}}) > 0$ and, for any $r > 0$, the map $u_0 \to u$ is continuous from $B(0, r)_{H^s}$ into $C([0, T(r); H^s(\mathbb{R}))$ Moreover, let $T^* > 0$ be the maximal time of existence of u in $H^s(\mathbb{R})$ then*

$$T^* < +\infty \quad \Leftrightarrow \quad \liminf_{t \nearrow T} u_x = -\infty \tag{2.5}$$

and if $y_0 = u_0 - u_{0,xx} \ge 0$ with $y_0 \in L^1(\mathbb{R})$ then $T^* = +\infty$ and

$$\|y(t)\|_{L^1} = \|y(0)\|_{L^1}, \forall t \in \mathbb{R}_+ \; . \tag{2.6}$$

Unfortunately, the peakons do not enter in this framework since their profiles do not belong even to $H^{\frac{3}{2}}(\mathbb{R})$. In [15] an existence and uniqueness result of global solutions to Camassa-Holm in a class of functions that contains the peakon is proved. This result was shown to hold also for the DP equation in [37]. We check below that it can be actually extended to the whole b-family and thus do not require the hamiltonian structure of the equation.

Proposition 2.2. *Let $u_0 \in Y_+$ be given.*

1. Existence and uniqueness : *Then the IVP* (1.4) *has a global solution $u \in C^1(\mathbb{R}; L^1(\mathbb{R}) \cap L^2(\mathbb{R})) \cap C(\mathbb{R}; W^{1,1} \cap H^1(\mathbb{R}))$ such that $y = (1-\partial_x^2)u \in C_w(\mathbb{R}; \mathcal{M})$ Moreover, this solution is unique in the class*

$$\{f \in C(\mathbb{R}_+; H^1(\mathbb{R})) \cap \{f - f_{xx} \in L^\infty(\mathbb{R}+; \mathcal{M}_+)\} \ . \tag{2.7}$$

2. Continuity with respect to initial data in $W^{1,1}(\mathbb{R})$*: For any sequence $\{u_{0,n}\}$ bounded in Y_+ such that $u_{0,n} \to u_0$ in $W^{1,1}(\mathbb{R})$, the emanating sequence of solution $\{u_n\} \subset C^1(\mathbb{R}; L^1(\mathbb{R}) \cap L^2(\mathbb{R})) \cap C(\mathbb{R}; W^{1,1} \cap H^1(\mathbb{R}))$ satisfies for any $T > 0$*

$$u_n \to u \text{ in } C([-T,T]; W^{1,1} \cap H^1(\mathbb{R})) \tag{2.8}$$

and

$$(1-\partial_x^2)u_n \rightharpoonup\ast\ y \text{ in } C_{ti}([-T,T], \mathcal{M}) \ . \tag{2.9}$$

3. Continuity with respect to initial data in Y equipped with its weak topology *: For any sequence $\{u_{0,n}\} \subset Y_+$ such that*[2] *$u_{0,n} \rightharpoonup\ast\ u_0$ in Y, the emanating sequence of solution $\{u_n\} \subset C^1(\mathbb{R}; L^1(\mathbb{R}) \cap L^2(\mathbb{R})) \cap C(\mathbb{R}; W^{1,1} \cap H^1(\mathbb{R}))$ satisfies for any $T > 0$,*

$$u_n \underset{n\to\infty}{\rightharpoonup} u \text{ in } C_w([-T,T]; H^1(\mathbb{R})) \ , \tag{2.10}$$

and (2.9).

Proof. **1.** First note that in [25] the results are stated only for positive times but, since the equation is invariant by the change of unknown $u(t,x) \mapsto u(-t,-x)$, it is direct to check that the results hold as well for negative times.

Uniqueness The uniqueness follows the same lines as for the Camassa-Holm equation (see [15]) Let u and v be two solutions of (1.4) within the class (2.7). At this stage it is worth noticing that (1.4) then ensures that $u, v \in C^1(\mathbb{R}; L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$. We set $w = u - v$ and

$$M = \sup_{t\geq 0} \|u(t,\cdot) - u_{xx}(t,\cdot)\|_{\mathcal{M}} + \|v(t,\cdot) - v_{xx}(t,\cdot)\|_{\mathcal{M}} \quad .$$

One can easily check (see [15]) that

$$\sup_{(t,x)\in\mathbb{R}^2} (|u(t,x)|, |v(t,x)|, |u_x(t,x)|, |v_x(t,x)|) \leq \frac{1}{2}\ M$$

and

$$\sup_{t\in\mathbb{R}} (\|u(t)\|_{L^1}, \|v(t)\|_{L^1}, \|u_x(t)\|_{L^1}, \|v_x(t)\|_{L^1}) \leq M \ .$$

We proceed exactly as in [15], using exterior regularization. It holds

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w| \quad &= \quad \int_{\mathbb{R}} (\rho_n * w_t)\,\mathrm{sgn}\,(\rho_n * w) \\ &\leq \quad \frac{1+|b|}{2} M \int_{\mathbb{R}} |\rho_n * w\| + \frac{1+|3-b|}{2} M \int_{\mathbb{R}} |\rho_n * w_x\| + R_n(t) \end{aligned}$$

[2]By this we mean that $u_{0,n} \rightharpoonup u_0$ in $H^1(\mathbb{R})$ and $u_{0,n} \rightharpoonup\ast\ u_0$ in $\mathcal{M}$

and

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}|\rho_n * w_x\| &= \int_{\mathbb{R}}(\rho_n * w_{tx})\,\mathrm{sgn}\,(\rho_n * w_x)\\ &\le \frac{4+|b|}{2}M\int_{\mathbb{R}}|\rho_n * w\| + \frac{5+|3-b|}{2}M\int_{\mathbb{R}}|\rho_n * w_x\| + R_n(t)\end{aligned}$$

with

$$R_n(t)\to 0 \text{ as } n\to+\infty \text{ and } |R_n(t)|\lesssim 1,\ n\ge 1, t\in\mathbb{R}.$$

Gathering these two inequalities and using Gronwall lemma, we thus get

$$\int_{\mathbb{R}}(|\rho_n*w|+|\rho_n*w_x|)(t,x) \le \int_0^t e^{c(b)(t-s)}R_n(s)ds+\Big[\int_{\mathbb{R}}(|\rho_n*w|+|\rho_n*w_x|)(0,x)\Big]e^{c(b)t}$$

with $c(b)=$. Fixing $t\in\mathbb{R}$ and letting $n\to+\infty$ this leads to

$$\|w(t)\|_{W^{1,1}} \le e^{c(b)t}\|w_0\|_{W^{1,1}}, \quad \forall t\in\mathbb{R}\ . \tag{2.11}$$

This yields the uniqueness in the class (2.7).
Existence : Let $u_0\in Y_+$. According to Proposition 2.1 for any $n\ge 1$, $\rho_n * u_0$ gives rise to a global smooth solution u^n of (1.4) and (2.6) ensures that $(u^n)_{n\ge1}$ is bounded in $C(\mathbb{R};Y)$. As $\rho_n * u_0 \to u_0$ in $W^{1,1}$, fixing $T>0$, we deduce from (2.11) that $(u^n)_{n\ge1}$ is a Cauchy sequence in $C([-T,T]\ ;W^{1,1}))$ and thus in $C([-T,T]\ ;H^1(\mathbb{R}))$ in view of (2.3). By a diagonal process we thus construct a function $u\in C(\mathbb{R};W^{1,1}\cap H^1)$ that satisfies (1.4) in $L^2(\mathbb{R})$ for all $t\in\mathbb{R}$. Moreover, (2.6) ensures that u belongs to the uniqueness class (2.7) and thus to $C^1(\mathbb{R};L^1(\mathbb{R})\cap L^2(\mathbb{R}))$ as noticed in the uniquess proof. It remains to prove that

$$(1-\partial_x^2)u^n \rightharpoonup\ast\ y=u-u_{xx} \text{ in } C_{ti}([-T,T],\mathcal{M})\ . \tag{2.12}$$

Indeed, this will force $M(u)$ to be a conserved quantity since $M(u_n)$ is a conserved quantity. To do this, we notice that for any $v\in BV(\mathbb{R})$ and any $\phi\in C_b^1(\mathbb{R})$, it holds

$$\langle v',\phi\rangle = -\int v\phi'\ .$$

Therefore, setting $y^n=u^n-u^n_{xx}$, The convergence of u^n towards u in $C([-T,T]\ ;W^{1,1}))$, $T>0$, ensures that, for any $t\in\mathbb{R}$,

$$\int_{\mathbb{R}} y^n(t)\phi = \int_{\mathbb{R}}(u^n(t)\phi+u^n_x(t)\phi') \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}}(u(t)\phi+u_x(t)\phi') = \langle y(t),\phi\rangle$$

and thus $y^n(t)\rightharpoonup\ast\ y(t)$ tightly in $\mathcal{M}$. Using that the equation (1.4) forces $\{\partial_t u^n\}$ to be bounded in $L^\infty(0,T;L^1)$, Arzela-Ascoli theorem leads then to (2.12). Indeed, for any $\phi\in C_b^2(\mathbb{R})$ and any $T>0$, we observe that the sequence of C^1 functions $\{t\mapsto\langle y^n(t),\phi\rangle\}$ is uniformly equi-continuous on $[0,T]$ since

$$|\frac{d}{dt}\langle y^n(t),\phi\rangle| = |\int_{\mathbb{R}} u^n_t(\phi-\phi_{xx})| \le 2\|u^n_t\|_{L^\infty(0,T;L^1)}\|\phi\|_{C^2}\ .$$

2. It is clear that (2.8) is a direct consequence of the $W^{1,1}$-Lipschitz bound (2.11) and (2.9) follows then exactly as above.
3. According to Banach-Steinhaus Theorem, $\{u_{0,n}\}$ is bounded in Y_+. Therefore, the sequence of emanating solution $\{u_n\}$ is bounded in $C(\mathbb{R}_+;W^{1,1}\cap H^1(\mathbb{R}))$ with $\{u_{n,x}\}$ bounded in $L^\infty(\mathbb{R};\mathcal{B}V(\mathbb{R}))$. Hence, there exists $v\in L^\infty(\mathbb{R}_+;H^1(\mathbb{R}))$ with $(1-\partial_x^2)v\in L^\infty(\mathbb{R};\mathcal{M}_+(\mathbb{R}))$ such that, for any $T>0$,

$$u_n \underset{n\to\infty}{\rightharpoonup} v\in L^\infty([-T,T];H^1(\mathbb{R})) \text{ and } (1-\partial_x^2)u_n \underset{n\to\infty}{\rightharpoonup\ast}(1-\partial_x^2)v \text{ in } L^\infty(]-T,T[;\mathcal{M}_+(\mathbb{R}))$$

But, using that $\{\partial_t u_n\}$ is bounded in $L^\infty(\mathbb{R};L^2(\mathbb{R})\cap L^1(\mathbb{R}))$, Helly's, Aubin-Lions compactness and Arzela-Ascoli theorems then ensure that v is a solution to (1.4) that belongs to $C_w([-T,T];H^1(\mathbb{R}))$ with $v(0)=u_0$ and that (2.9) holds. In particular, $v_t\in L^\infty(]-T,T[;L^2(\mathbb{R}))$ and thus $v\in C([-T,T];L^2(\mathbb{R}))$. Since $v\in L^\infty(]-T,T[;H^{\frac{3}{2}-}(\mathbb{R}))$, this actually implies that $v\in C([-T,T];H^{\frac{3}{2}-}(\mathbb{R}))$. Therefore, v belongs to the uniqueness class which ensures that $v=u$. □

Remark 2.1. As noticed by Danchin in [17] for the Camassa-Holm equation, (1.5) ensures that smooth solution of (1.4) emanating from an initial data with an integrable initial momentum density satisfies

$$\frac{d}{dt}\int_{\mathbb{R}}|y| \ = -\int_{\mathbb{R}} u\partial_x|y| \ - b\int_{\mathbb{R}} u_x|y| \ = (1-b)\int_{\mathbb{R}} u_x|y| \ .$$

Therefore by (2.2) and Hölder inequality we get

$$\frac{d}{dt}\|y(t)\|_{L^1}\le |b-1|\|y(t)\|_{L^1}^2$$

that leads to the a priori estimate

$$\|y(t)\|_{L^1}\le \frac{\|y_0\|_{L^1}}{1-|(b-1)t|\|y_0\|_{L^1}};.$$

This shows that in the case $b=1$ we can actually replace the requirement $u_0\in Y_+$ by $u_0\in Y$ without changing the conclusion. Whereas in the case $b\neq 1$, replacing the requirement $u_0\in Y_+$ by $u_0\in Y$, exactly the same approach as the one to prove Proposition 2.2 leads to the existence and uniqueness of a local solution $u\in C^1([-T,T];L^2(\mathbb{R}))\cap C([-T,T];W^{1,1}\cap H^1(\mathbb{R}))$ such that $y=(1-\partial_x^2)u\in C_w([-T,T];\mathcal{M})$ to (1.4) with $T=T(\|y_0\|_{\mathcal{M}})>0$.

3. A rigidity result for Y-exponentially localized solution of the b-family moving to the right

This section is devoted to the proof of Theorem 1.1. We will need the following lemma (see for instance [26])

Lemma 3.1. *Let μ be a finite nonnegative measure on $\mathbb{R}$. Then μ is the sum of a nonnegative non atomic measure ν and a countable sum of positive Dirac measures (the discrete part of μ). Moreover, for all $\varepsilon>0$ there exists $\delta>0$ such that, if I is an interval of length less than δ , then $\nu(I)\le\varepsilon$.*

3.1. **Boundedness from above of the momentum density support.**

Proposition 3.2. *Let $b\in\mathbb{R}$ and let $u\in C(\mathbb{R};Y_+)$ be a Y-localized global solution with $x_t\ge c_0>0$ of the b-family equation* (1.4) *which is moreover uniformly exponentially decaying if $b\neq 1$. Assume furthermore that* $\inf_{t\in\mathbb{R}}\|u(t)\|_{L^2}\ge\gamma_0>0$. *There exists $r_0>0$ such that for all $t\in\mathbb{R}$, it holds*

$$\operatorname{supp} y(t,\cdot+x(t))\subset]-\infty,r_0], \tag{3.1}$$

and

$$u(t,x(t)+r_0)=-u_x(t,x(t)+r_0)\ge\frac{e^{-2r_0}}{4r_0}M(u)\,. \tag{3.2}$$

Proof. Clearly, it suffices to prove the result for $t = 0$. Let $u \in C(\mathbb{R}; H^1)$,with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$, be a Y-uniformly exponentially decaying solution to (1.4) and let $\phi \in C^\infty(\mathbb{R})$ with $\phi \equiv 0$ on $]-\infty, -1]$, $\phi' \geq 0$ and $\phi \equiv 1$ on $\mathbb{R}_+$. We claim that

$$\Big\langle y(0), \phi(\cdot - (x(0) + r_0))\Big\rangle = 0 \tag{3.3}$$

which proves the result since $y \in \mathcal{M}_+$.

We prove (3.3) by contradiction. We approximate $u_0 = u(0)$ by the sequence of smooth functions $u_{0,n} = \rho_n * u_0$ that belongs to $H^\infty(\mathbb{R}) \cap Y_+$ so that (2.8)-(2.9) hold for any $T > 0$. We denote by u_n the solution to (1.4) emanating from $u_{0,n}$ and by $y_n = u_n - u_{n,xx}$ its momentum density. Let us recall that Proposition 2.1 ensures that $u_n \in C(\mathbb{R}; H^\infty(\mathbb{R}))$ and $y_n \in C_w((\mathbb{R}; L^1(\mathbb{R}))$. We fix $T > 0$ and we take $n_0 \in \mathbb{N}$ large enough so that for all $n \geq n_0$,

$$\|u_n - u\|_{L^\infty(]-T,T[;H^1)} < \frac{1}{10}\min(c_0, M(u)) \tag{3.4}$$

and

$$\|y_{0,n} - y_0\|_{\mathcal{M}} < \frac{\varepsilon_0}{2} \quad . \tag{3.5}$$

As explain in the beginning of Section 2, the Y-almost localization of u actually forces an almost localization in all L^p for $p \in [1, +\infty]$. Therefore there exists $r_0 > 0$ such that

(3.6)
$$\|u(t, \cdot + x(t))\|_{L^1(]-r_0,r_0[^c)} + \|u(t, \cdot + x(t))\|_{L^\infty(]-r_0,r_0[^c)} + u(t, x(t)+x) \leq \frac{1}{10}\min(c_0, M(u)), \quad \forall t \in \mathbb{R}\,.$$

Combining this estimate with (3.4) we infer that for $n \geq n_0$,

$$u_n(t, x(t) + x) \leq \frac{1}{5}\min(c_0, M(u)), \forall (|x|, t) \in [r_0, +\infty[\times[-T, T]\,. \tag{3.7}$$

Now, we introduce the flow q_n associated with u_n defined by

$$\begin{cases} q_{n,t}(t, x) &= u_n(t, q_n(t, x)) \quad , (t, x) \in \mathbb{R}^2 \\ q_n(0, x) &= x \quad , x \in \mathbb{R} \end{cases} . \tag{3.8}$$

Following [9], we know that for any $t \in \mathbb{R}$,

$$y_n(0, x) = y_n(t, q_n(t, x)) q_{n,x}(t, x)^b \tag{3.9}$$

Indeed, on one hand, (1.5) clearly ensures that

$$\frac{\partial}{\partial t}\Big(y_n(t, q_n(t, x)) e^{b\int_0^t u_{n,x}(s, q_n(s,x))\, ds}\Big) = 0$$

and, on the other hand, (3.8) ensures that $q_{n,x}(0, x) = 1$, $\forall x \in \mathbb{R}$, and

(3.10)
$$\frac{\partial}{\partial t} q_{n,x}(t, x) = u_{n,x}(t, q_n(t, x)) q_{n,x}(t, x) \Rightarrow q_{n,x}(t, x) = \exp\Big(\int_0^t u_{n,x}(s, q_n(s, x))\, ds\Big)\,.$$

We claim that for all $n \geq n_0$ and $t \in [-T, 0]$,

$$q_n(t, x(0) + r_0) - x(t) \geq r_0 + \frac{c_0}{2}|t|\,. \tag{3.11}$$

Indeed, fixing $n \geq n_0$, in view of (3.7) and the continuity of u_n there exists $t_0 \in [-T, 0[$ such that for all $t \in [t_0, 0]$,

$$u_n(t, q_n(t, x(0) + r_0)) \leq \frac{c_0}{4}$$

and thus according to (3.8), for all $t\in[t_0,0]$,

$$\frac{d}{dt}q_n(t,x(0)+r_0)\le\frac{c_0}{4}$$

which leads to

$$q_n(t,x(0)+r_0)-x(t)\ge r_0+\frac{c_0}{2}|t|,\quad t\in[t_0,0]\,.$$

This proves (3.11) by a continuity argument.

Now, in the case $b\neq 1$, we thus deduce from the uniform exponential decay of u that for all $t\in[-T,0]$ and all $x\ge 0$,

$$u(t,q_n(t,x(0)+r_0+x)\le C\exp\Bigl(-\beta(r_0+c_0|t|/2)\Bigr) \tag{3.12}$$

Therefore, in view of (2.8) and (2.3), there exists $n_1\ge n_0$ such that for all $t\in[-T,0]$ and all $x\ge 0$,

$$u_n(t,q_n(t,x(0)+r_0+x)+|u_{n,x}(t,q_n(t,x(0)+r_0+x)|\le 4C\exp\Bigl(-\beta(r_0+c_0|t|/2)\Bigr) \tag{3.13}$$

The formula (see (3.10))

$$q_{n,x}(t,x)=\exp\Bigl(-\int_t^0 u_{n,x}(s,q_n(s,x))\,ds\Bigr) \tag{3.14}$$

thus ensures that $\forall t\in[-T,0]$, $\forall x\ge 0$ and $\forall n\ge n_0$,

$$\exp\Bigl(-4C\int_{-T}^0 e^{-\beta(r_0+c_0|s|/2)}\,ds\Bigr)\le q_{n,x}(t,x(0)+r_0+x)\le\exp\Bigl(4C\int_{-T}^0 e^{-\beta(r_0+c_0|s|/2)}\,ds\Bigr)$$

Setting $C_0:=e^{\frac{8Ce^{-\beta r_0}}{\beta c_0}}$ this leads, in the case $b\neq 1$, to

$$\frac{1}{C_0}\le q_{n,x}(t,x(0)+r_0+x)\le C_0\,,\ \forall t\in[-T,0]\,. \tag{3.15}$$

Now, we claim that for any $b\in\mathbb{R}$ and any $n\ge n_1$ it holds

$$\int_{x(0)+r_0}^{+\infty}y_n(0,x)\,dx\le C_0^{b-1}\int_{x(t)+r_0+c_0|t|/2}^{+\infty}y_n(t,z)\,dz\,,\ \forall t\le[-T,0]\,. \tag{3.16}$$

Letting $n\to+\infty$ using (2.9) and then letting $T\to\infty$, this ensures that

$$\Bigl\langle y(0),\phi(\cdot-x(t)-r_0)\Bigr\rangle\le C_0^{b-1}\Bigl\langle y(t),\phi(\cdot-x(t)-r_0-c_0|t|/2+1)\Bigr\rangle,\ \forall t\le 0$$

which proves (3.3) since the Y-uniform localization of u forces the right-hand side member to goes to 0 as $t\to-\infty$. Therefore, to complete the proof of (3.1), it remains to prove (3.16). First, it follows from (3.9) that for any $t\le 0$ and any $r_0'>r_0$,

$$\int_{x(0)+r_0}^{x(0)+r_0'}y_n(0,x)\,dx=\int_{x(0)+r_0}^{x(0)+r_0'}y_n(t,q_n(t,x))q_{n,x}(t,x)^b\,dx$$

and (3.15) leads for any $b\in\mathbb{R}$ to

$$\int_{x(0)+r_0}^{x(0)+r_0'}y_n(0,x)\,dx\le C_0^{b-1}\int_{x(0)+r_0}^{x(0)+r_0'}y_n(t,q_n(t,x))q_{n,x}(t,x)\,dx$$

The change of variables $z = q_n(t,x)$ then yields

$$\int_{x(0)+r_0}^{x(0)+r_0'} y_n(0,x)\,dx \le C_0^{b-1} \int_{q_n(t,x(0)+r_0)}^{q_n(t,x(0)+r_0')} y_n(t,z)\,dz$$

and (3.16) then follows from (3.11) by letting r_0' tend to $+\infty$.

Let us now prove (3.2). We first notice that thanks to (3.6) and the conservation of $M(u) = \langle y, 1\rangle = \int_{\mathbb{R}} u$, it holds

$$\max_{[-r_0,r_0]} u(t,\cdot + x(t)) \ge \frac{1}{2r_0}\|u(t,\cdot + x(t))\|_{L^1(]-r_0,r_0[)} \ge \frac{M(u)}{4r_0}\,. \tag{3.17}$$

But, since $u_x \ge -u$ on $\mathbb{R}^2$, for any $(t,x_0) \in \mathbb{R}^2$ it holds

$$u(t,x) \le u(t,x_0)e^{-x+x_0}, \quad \forall x \le x_0\,.$$

Applying this estimate with $x_0 = x(t) + r_0$ we obtain that

$$u(t,x(t)+r_0) \ge \max_{[-r_0,r_0]} u(t,\cdot + x(t))e^{-2r_0}$$

which, combined with (3.17) yields (3.2). □

Remark 3.1. It is worth pointing out that $b = 1$ is a particular case here. Indeed, in this case we do not need the Y-uniform exponential localization of u but only a Y-uniform almost localization since we do not need (3.15) anymore.

3.2. **Study of the supremum of the support of y.** We define

$$x_+(t) = \inf\{x \in \mathbb{R}, \operatorname{supp} y(t) \subset]-\infty, x(t)+x]\}$$

In the sequel we set

$$\alpha_0 := \frac{e^{-2r_0}}{4r_0} M(u)$$

to simplify the expressions. According to Proposition 3.2, $t \mapsto x_+(t)$ is well defined with values in $]-\infty, r_0]$ and

$$u(t,x(t)+x_+(t)) = -u_x(t,x(t)+x_+(t)) \ge \alpha_0\,. \tag{3.18}$$

The following lemma proved in [42] ensures that $t \mapsto x(t)+x_+(t)$ is an integral line of u.

Lemma 3.3. *For all $t \in \mathbb{R}$, it holds*

$$x(t) + x_+(t) = q(t, x(0) + x_+(0))\,. \tag{3.19}$$

where $q(\cdot,\cdot)$ is defined by

$$\left\{\begin{array}{lcll} q_t(t,x) & = & u(t,q(t,x)) & ,\ (t,x) \in \mathbb{R}^2 \\ q(0,x) & = & x & ,\ x \in \mathbb{R} \end{array}\right. . \tag{3.20}$$

In the sequel we define $q^* : \mathbb{R} \to \mathbb{R}$ by

$$q^*(t) = q(t, x(0)+x_+(0)) = x(t) + x_+(t), \quad \forall t \in \mathbb{R}\,. \tag{3.21}$$

Proposition 3.4. *Assume that u satisfies the hypotheses of Proposition 3.2 with $b \ge 1$. Let $a : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$a(t) = u_x(t, q^*(t)-) - u_x(t, q^*(t)+), \quad \forall t \in \mathbb{R}\,. \tag{3.22}$$

Then $a(\cdot)$ is a bounded non decreasing derivable function on $\mathbb{R}$ with values in $[\frac{\alpha_0}{8}, M(u)]$ such that

$$a'(t) = \frac{1}{2}(u^2 - u_x^2)(t, q^*(t)-),\ \forall t \in \mathbb{R}. \tag{3.23}$$

Proof. First, the fact that $a(t) \le M(u)$ follows from the conservation of M together with Young convolution inequalities since $u = \frac{1}{2}e^{-|x|} * y$. To prove that $a(t) \ge \frac{\alpha_0}{8}$, we proceed by contradiction. So let us assume that there exists $t_0 \in \mathbb{R}$ such that $a(t_0) < \alpha_0/8$. Since $y(t_0) \in \mathcal{M}_+$ with $\operatorname{supp} y(t_0) \subset]-\infty, q^*(t_0)]$, according to Lemma 3.1 we must have

$$\lim_{z \nearrow q^*(t_0)} \|y(t_0)\|_{\mathcal{M}(]z,+\infty[)} < \frac{\alpha_0}{8} .$$

Without loss of generality we can assume that $t_0 = 0$ and thus there exists $\beta_0 > 0$ such that

$$\|y(0)\|_{\mathcal{M}(]q^*(0))-\beta_0,+\infty[)} < \frac{\alpha_0}{8} . \tag{3.24}$$

By convoluting u_0 by ρ_n (see (2.4)), for some $n \ge 0$, we can approach u_0 by a smooth function $\tilde{u}_0 \in Y_+ \cap H^\infty(\mathbb{R})$. Taking n large enough, we may assume that there exists $\tilde{x}_+ > x_+(0)$ close to $x_+(0)$, such that

$$\tilde{y}_0 = (1 - \partial_x^2)\tilde{u}_0 \equiv 0 \text{ on } [x(0) + \tilde{x}_+, +\infty[\tag{3.25}$$

and

$$\|\tilde{y}_0\|_{L^1(]x(0)+\tilde{x}_+-\beta_0,+\infty[)} \le \frac{\alpha_0}{8} + \frac{\alpha_0}{2^6} , \tag{3.26}$$

where $\tilde{y}_0 = \tilde{u}_0 - \tilde{u}_{0,xx}$. Moreover, defining $\tilde{q}_2 : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{q}_2(t) = \tilde{q}(t, x(0) + \tilde{x}_+)$$

where $\tilde{q}(\cdot, \cdot)$ is defined by (3.20) with u replaced by $\tilde{u}$, (2.8) enables us to assume that the emanating solution $\tilde{u}$ satisfies

$$\|u(t) - \tilde{u}(t)\|_{H^1} \le \frac{\alpha_0}{2^6} \tag{3.27}$$

and

$$|q^*(t) - \tilde{q}_2(t)| < \frac{\alpha_0}{2^6 M(u)} \tag{3.28}$$

for all $t \in [-t_1, t_1]$ with $t_1 > 0$ to specified later. It is worth noticing that (3.27)-(3.28), (3.25), (3.9), (3.2) and the mean-value theorem - recall that by Young inequalities $|u_x| \le \frac{1}{2}M(u)$ - ensure that

$$-\tilde{u}_x(t, \tilde{q}_2(t)) = \tilde{u}(t, \tilde{q}_2(t)) \ge (1 - 2^{-5})\alpha_0 \quad \forall t \in [-t_1, t_1] . \tag{3.29}$$

We claim that for all $t \in [-t_1, 0]$ it holds

$$\tilde{u}_x(t, x) \le -\frac{3\alpha_0}{4} \quad \text{on} \quad [\tilde{q}_1(t), \tilde{q}_2(t)] , \tag{3.30}$$

where $\tilde{q}_1(t)$ is defined by $\tilde{q}_1(t) = \tilde{q}(t, x(0) + \tilde{x}_+ - \beta_0)$.
To see this, for $\gamma > 0$, we set

$$A_\gamma = \{t \in \mathbb{R}_- \,/\, \forall \tau \in [t, 0],\ u_x(\tau, x) < -\gamma \text{ on } [\tilde{q}_1(\tau), \tilde{q}_2(\tau)]\,\} .$$

Recalling (3.2), (3.26), (3.29) and that $\tilde{u} \ge 0$, we get for $0 \le \beta \le \beta_0$,

$$\begin{aligned}\tilde{u}_x(0, x(0) + \tilde{x}_+ - \beta) &\le \tilde{u}_x(0, x(0) + \tilde{x}_+) + \|\tilde{y}_0\|_{L^1(]x(0)+\tilde{x}_+-\beta_0,+\infty[)} \\ &\le -\alpha_0 + \frac{\alpha_0}{2^5} + \frac{\alpha_0}{8} + \frac{\alpha_0}{2^5} < -\frac{3\alpha_0}{4} \ ,\end{aligned}$$

which ensures that $A_{\frac{3\alpha_0}{4}}$ is non empty. By a continuity argument, it thus suffices to prove that $A_{\frac{\alpha_0}{2}} \subset A_{\frac{3\alpha_0}{4}}$. First we notice that for any $t \in A_{\frac{\alpha_0}{2}}$ and any $x \in [\tilde{q}_1(t), \tilde{q}_2(t)]$, the definition of $A_{\frac{\alpha_0}{2}}$ ensures that

$$\tilde{q}_x(t, x) = \exp\Big(-\int_t^0 \tilde{u}_x(\tau, \tilde{q}(\tau, x))\, d\tau\Big) \ge 1 \ ,$$

where $\tilde{q}(\cdot, \cdot)$ is the flow associated to $\tilde{u}$ by (3.20). Therefore, $\tilde{u} \ge 0$, $\tilde{y} \ge 0$, a change of variables, (3.9) and (3.26) ensure that for any $x \in [\tilde{q}_1(t), \tilde{q}_2(t)]$,

$$\int_x^{\tilde{q}_2(t)} \tilde{u}_{xx}(t, s)\, ds \ge -\int_x^{\tilde{q}_2(t)} \tilde{y}(t, s)\, ds \ge -\int_{\tilde{q}_1(t)}^{\tilde{q}_2(t)} \tilde{y}(t, s)\, ds = -\int_{x(0)+\tilde{x}_+-\beta_0}^{x(0)+\tilde{x}_+} \tilde{y}(t, \tilde{q}(t, s))\tilde{q}_x(t, s)\, ds \ .$$

Using that $b \ge 1$ this leads to

$$\int_x^{\tilde{q}_2(t)} \tilde{u}_{xx}(t, s)\, ds \ge -\int_{x(0)+\tilde{x}_+-\beta_0}^{x(0)+\tilde{x}_+} \tilde{y}(t, \tilde{q}(t, s))\tilde{q}_x(t, s)^b\, ds = -\int_{x(0)+\tilde{x}_+-\beta_0}^{x(0)+\tilde{x}_+} \tilde{y}_0(s)\, ds \ge -\frac{\alpha_0}{8} - \frac{\alpha_0}{2^6}$$

and (3.29) yields

$$\tilde{u}_x(t, x) = \tilde{u}_x(t, q_2(t)) - \int_x^{q_2(t)} \tilde{u}_{xx}(t, s)\, ds \le -\alpha_0 + \frac{\alpha_0}{8} + \frac{\alpha_0}{2^4} < -\frac{3\alpha_0}{4} \ ,$$

which proves the desired result.

We deduce from (3.30) that $\forall t \in [-t_1, 0]$,

$$\begin{aligned}\frac{d}{dt}(\tilde{q}_2(t) - \tilde{q}_1(t)) &= \tilde{u}(\tilde{q}_2(t)) - \tilde{u}(\tilde{q}_1(t)) \\ &= \int_{\tilde{q}_1(t)}^{\tilde{q}_2(t)} \tilde{u}_x(t, s)\, ds \\ &\le -\frac{\alpha_0}{2}(\tilde{q}_2(t) - \tilde{q}_1(t)) \ .\end{aligned}$$

Therefore,

$$(\tilde{q}_2 - \tilde{q}_1)(t) \ge (\tilde{q}_2 - \tilde{q}_1)(0) e^{-\frac{\alpha_0}{2} t} = \beta e^{-\frac{\alpha_0}{2} t} \ .$$

On the other hand, since according to (3.29) and (3.30), $\tilde{u}(t, \tilde{q}_2(t)) \ge 2\alpha_0/3$ and $\tilde{u}_x \le 0$ on $]\tilde{q}_1(t), \tilde{q}_2(t)[$, we deduce that

$$\tilde{u}(t, \tilde{q}_1(t)) \ge \tilde{u}(t, \tilde{q}_2(t)) \ge 2\alpha_0/3, \qquad \text{on } [-t_1, 0] \ .$$

Coming back to the solution u emanating from u_0, it follows from (3.27) that

$$\min\Big(u(t, \tilde{q}_1(t_1)), u(t, \tilde{q}_2(t_1))\Big) \ge \frac{\alpha_0}{2} \text{ with } (\tilde{q}_2 - \tilde{q}_1)(t_1) \ge \beta e^{-\frac{\alpha_0}{2} t} \ , \quad \forall t \in [-t_1, 0] \, .$$

Taking $t_1 > 0$ large enough, this contradicts the Y-almost localization of u which proves that $a(t) \ge \frac{\alpha_0}{8}$ and thus $u_x(t, \cdot)$ has got a jump at $x(t) + x_+(t)$.

It is worth noticing that,according Lemma 3.1, this ensures that for all $t \in \mathbb{R}$, one can decompose $y(t)$ as

$$y(t) = \nu(t) + a(t)\delta_{x(t)+x_+(t)} + \sum_{i=1}^{\infty} a_i(t)\delta_{x_i(t)} \tag{3.31}$$

where $\nu(t)$ is a non negative non atomic measure with $\nu(t)\equiv 0$ on $]x(t)+x_+(t),+\infty[$, $\{a_i\}_{n\geq 1}\subset(\mathbb{R}_+)^{\mathbb{N}}$ with $\sum_{i=1}^{\infty}a_i(t)<\infty$ and $x_i(t)<x(t)+x_+(t)$ for all $i\in\mathbb{N}^*$. It remains to prove that for all couple of real numbers (t_1,t_2) with $t_1<t_2$,

$$a(t_2)-a(t_1)=\frac{1}{2}\int_{t_1}^{t_2}(u^2-u_x^2)(\tau,q^*(\tau)-)\,d\tau \tag{3.32}$$

Indeed, since $|u_x|\leq u$ and $u\in L^\infty(\mathbb{R}^2)$ this will force a to be non decreasing and derivable on $\mathbb{R}$. Let $\phi:\mathbb{R}\to\mathbb{R}_+$ be a non decreasing C^∞-function such that $\operatorname{supp}\phi\subset[-1,+\infty[$ and $\phi\equiv 1$ on $\mathbb{R}_+$. We set $\phi_\varepsilon=\phi(\frac{\cdot}{\varepsilon})$. Since u is continuous and $y(t,\cdot)=0$ on $]x(t)+x_+(t),+\infty[$ it follows from (3.31) that for all $t\in\mathbb{R}$,

$$a(t)=\lim_{\varepsilon\searrow 0}\langle y(t),\phi_\varepsilon(\cdot-q^*(t))\rangle\ .$$

Without loss of generality, it suffices to prove (3.32) for $t_1=0$ and $t_2=t\in]0,1[$. Let $\beta>0$ be fixed, we claim that there exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$,

$$\Big|\langle y(t),\phi_\varepsilon(\cdot-q_*(t))\rangle-\langle y(0),\phi_\varepsilon(\cdot-q^*(0))\rangle-\frac{1}{2}\int_0^t\int_{\mathbb{R}}(u^2-u_x^2)(\tau,q^*(t)+\varepsilon z)\phi'(z)\,dz\,d\tau\Big|\leq\beta,\quad\forall t\in]0,1[\tag{3.33}$$

Passing to the limit as ε tends to 0, this leads to the desired result. Indeed, since $(u^2-u_x^2)(\tau,\cdot)\in BV(\mathbb{R})$ and $\phi'\equiv 0$ on $\mathbb{R}_+$, for any fixed (τ,z), it is clear that

$$(u^2-u_x^2)(\tau,q^*(\tau)+\varepsilon z)\phi'(z)\underset{\varepsilon\to 0}{\longrightarrow}(u^2-u_x^2)(\tau,q^*(\tau)-)\phi'(z)$$

and, since it is dominated by $2M(u)^2\phi'$, the dominated convergence theorem leads to

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}(u^2-u_x^2)(\tau,q^*(t)+\varepsilon z)\phi'(z))\,dz\,d\tau\quad\underset{\varepsilon\to 0}{\longrightarrow}\quad&\int_0^t\int_{\mathbb{R}}(u^2-u_x^2)(\tau,q^*(\tau)-)\phi'(z))\,dz\,d\tau\\ &=\int_0^t(u^2-u_x^2)(\tau,q^*(\tau)-)\,d\tau\ .\end{aligned}$$

To prove (3.33) we first notice that according to (3.31) for any $\alpha>0$ there exists $\gamma(\alpha)>0$ such that

$$\|y\|_{\mathcal{M}(]q^*(0)-\gamma(\alpha),q^*(0)[)}<\alpha\ . \tag{3.34}$$

We approximate again $u(0)$ by a sequence $\{u_{0,n}\}\subset H^\infty(\mathbb{R})\cap Y_+$ such that $M(u_{0,n})\leq 2M(u)$ and we ask that

$$\|y(0)-y_{0,n}\|_{\mathcal{M}(\mathbb{R})}\leq e^{(1-b)M(u)}\beta/4\ . \tag{3.35}$$

where $y_{0,n}=u_{0,n}-\partial_x^2u_{0,n}$. We again denote respectively by u_n and y_n, the solution to (1.4) emanating from $u_{0,n}$ and its momentum density $u_n-u_{n,xx}$. Let now $q_n^*:\mathbb{R}\to\mathbb{R}$ the integral line of u_n defined by $q_n^*(t)=q_n(t,q^*(0))$ where q_n is defined in (3.8). On account of (1.5), it holds

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} y_n\phi_\varepsilon(\cdot-q_n^*(t)) &= -u_n(t,q_n^*(t))\int_{\mathbb{R}} y_n\phi'_\varepsilon-\int_{\mathbb{R}}\partial_x(y_nu_n)\phi_\varepsilon-(b-1)\int_{\mathbb{R}} y_nu_{n,x}\phi_\varepsilon\\
&= \int_{\mathbb{R}}\Big[u_n(t,\cdot)-u_n(t,q^*(t))\Big]y_n(t,\cdot)\phi'_\varepsilon+\frac{b-1}{2}\int_{\mathbb{R}}(u_n^2(t,\cdot)-u_{n,x}^2(t,\cdot))\phi'_\varepsilon\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}}\Big[u_n(t,\cdot)-u_n(t,q^*(t))\Big]y_n(t,\cdot)\phi'\Big(\frac{\cdot-q_n^*(t)}{\varepsilon}\Big)\\
&\quad+\frac{b-1}{2}\int_{\mathbb{R}}(u_n^2-u_{n,x}^2)(t,q_n^*(t)+\varepsilon z)\phi'(z)\,dz\\
&= I_t^{\varepsilon,n}+II_t^{\varepsilon,n}\ . \qquad (3.36)
\end{aligned}$$

Since, according to (2.2), $|u_{n,x}|\le\frac{1}{2}M(u)$,

$$\begin{aligned}
|I_t^{\varepsilon,n}| &\le \frac{M(u)}{\varepsilon}\int_{\mathbb{R}}|x-q_n^*(t)|y_n(t,x)\phi'(\frac{x-q_n^*(t)}{\varepsilon})\,dx\\
&\le M(u)\int_{\mathbb{R}}y_n(t,x)\phi'(\frac{x-q_n^*(t)}{\varepsilon})\,dx
\end{aligned}$$

Now, in view of (3.14) we easily get

$$e^{-M(u)}\le q_{n,x}(t,z)\le e^{M(u)},\quad \forall(t,z)\in]-1,1[\times\mathbb{R} \tag{3.37}$$

and the change of variables $x=q_n(t,z)$ together with the identity (3.9) lead to

$$\begin{aligned}
\int_{\mathbb{R}}y_n(t,x)\phi'(\frac{x-q_n^*(t)}{\varepsilon})\,dx &= \int_{\mathbb{R}}y_n(t,q_n(t,z))q_{n,x}(t,z)\phi'_\varepsilon(q_n(t,z)-q_n^*(t))\,dx\\
&\le e^{(b-1)M(u)}\int_{\mathbb{R}}y_n(t,q_n(t,z))(q_{n,x}(t,z))^b\phi'_\varepsilon(q_n(t,z)-q_n^*(t))\,dz\\
&\le e^{(b-1)M(u)}\int_{\mathbb{R}}y_n(0,z)\phi'_\varepsilon(q_n(t,z)-q_n^*(t))\,dz\ .
\end{aligned}$$

But, making use of the mean value theorem, (3.37) and the definition of ϕ_ε, we obtain that, for any $t\in[0,1]$, $z\mapsto\phi'_\varepsilon(q_n(t,z)-q_n^*(t))$ is supported in an interval of length at most $\varepsilon e^{M(u)}$. Therefore, according to (3.34) and (3.35), setting $\varepsilon_0=e^{-M(u)}\gamma(\frac{\beta}{2}e^{(1-b)\mathcal{M}(u)})$, it follows that for all $0<\varepsilon<\varepsilon_0$ and all $n\in\mathbb{N}$,

$$\int_0^t|I_\tau^{\varepsilon,n}|d\tau\le 3\beta/4\ . \tag{3.38}$$

To estimate the contribution of $II_t^{\varepsilon,n}$ we first notice that thanks (2.8) it holds

$$u_{n,x}\to u_x \text{ in } C([-1,1];L^2(\mathbb{R}))$$

and for all $t\in[-1,1]$, Helly's theorem ensures that

$$u_{n,x}(t,\cdot)\to u_x(t,\cdot)\ \text{ a.e. on } \mathbb{R}\ .$$

Hence, for any fixed $t\in[-1,1]$ there exists a set $\Omega_t\subset\mathbb{R}$ of Lebesgue measure zero such that $u_x(t)$ is continuous at every point $x\in\mathbb{R}/\Omega_t$ and

$$u_{n,x}(t,x)\to u_x(t,x)\ ,\quad \forall x\in\mathbb{R}/\Omega_t\ .$$

Since $q_n^*(t)\to q^*(t)$, it follows that

$$u_{n,x}(t,q_n^*(t)+x)\to u_x(t,q^*(t)+x)\ ,\quad \forall x\in\mathbb{R}/\tau_{q^*(t)}(\Omega_t)\ .$$

where for any set $\Lambda\subset\mathbb{R}$ and any $a\in\mathbb{R}$, $\tau_a(\Lambda)=\{x-a, a\in\Lambda\}$.
Since the integrand in $II_t^{\varepsilon,n}$ is bounded by $M(u)\phi'\in L^1(\mathbb{R})$, it follows from Lebesgue dominated convergence theorem that for any $t\in[-1,1]$,

$$II_t^{\varepsilon,n} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\int_{\mathbb{R}}(u^2-u_x^2)(t,q^*(t)+\varepsilon z)\phi'(z)\,dz\ .$$

Therefore, invoking again Lebesgue dominated convergence theorem, but on $]0,t[$, keeping in mind that $\{|u_n|\}|$ and $\{|u_{n,x}|\}$ are uniformly bounded on $\mathbb{R}^2$ by $M(u)$, we finally deduce that for any fixed $t\in]0,1[$,

$$\int_0^t II_\tau^{\varepsilon,n}\,d\tau \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\int_0^t\int_{\mathbb{R}}(u^2-u_x^2)(\tau,q^*(t)+\varepsilon z)\phi'(z))\,dz\,d\tau \tag{3.39}$$

Now, we fix $t\in]0,1[$ and $\varepsilon\in]0,\varepsilon_0[$. According to the convergence result (2.9), for n large enough it holds

$$|\langle y_n(t)-y(t),\phi_\varepsilon(\cdot-q_*(t))\rangle|+|\langle y_n(0-y(0),\phi_\varepsilon(\cdot-q_*(0))\rangle|\le\beta/4\ .$$

which together with (3.36) and (3.38)-(3.39) prove the claim (3.33). □

Lemma 3.5. *There exists $(a_-,a_+)\in[\frac{\alpha_0}{8},M(u)]^2$, with $a_-\le a_+$ such that*

$$\lim_{t\to+\infty}u(t,x(t)+x_+(t)) = \lim_{t\to+\infty}a(t)/2=a_+/2\ , \tag{3.40}$$

$$\lim_{t\to-\infty}u(t,x(t)+x_+(t)) = \lim_{t\to-\infty}a(t)/2=a_-/2\ , \tag{3.41}$$

Proof. The existence of the limits at $\mp\infty$ for $a(\cdot)$ follows from the monotonicity of $a(\cdot)$. Now, in view of Proposition 3.4, for all $t\in\mathbb{R}$,

$$\begin{aligned}0\le a'(t)=\frac{1}{2}(u^2-u_x^2)(t,x(t)+x_+(t)-) &= \frac{a(t)}{2}(u-u_x)(t,x(t)+x_+(t)-)\\ &= \frac{a(t)}{2}(2u(t,x(t)+x_+(t))-a(t))\ .\end{aligned} \tag{3.42}$$

Therefore, since a takes values in $[\alpha_0/8,M(u)]$, it remains to prove that $a'(t)\to0$ as $t\to\pm\infty$. Since

$$\int_{\mathbb{R}}a'(\tau)\,d\tau<\infty\ ,$$

the desired result will follow if a' is Lipschitz on $\mathbb{R}$. But this is not too hard to check. Indeed, first from (3.23) we have for all $t\in\mathbb{R}$, $|a(t)-a(0)|\le 2t\|u_0\|_{H^1}$ and thus $t\mapsto a(t)$ is clearly Lipschitz on $\mathbb{R}$. Second, since $x(t)+x_+(t)=q^*(t)$, it holds

$$\frac{d}{dt}u(t,x(t)+x_+(t))=u(t,q^*(t))u_x(t,q^*(t))+u_t(t,q^*(t))\ .$$

But, $\sup_{(t,x)\in\mathbb{R}^2}|uu_x|\le M(u)^2$ and

$$\begin{aligned}\sup_{(t,x)\in\mathbb{R}^2}|u_t| &\le \sup_{(t,x)\in\mathbb{R}^2}|uu_x|+\sup_{(t,x)\in\mathbb{R}^2}\Big|(1-\partial_x^2)^{-1}\partial_x(\frac{b}{2}u^2+\frac{3-b}{2}u_x^2)\Big|\\ &\lesssim M(u)^2+\sup_{t\in\mathbb{R}}\|\frac{b}{2}u^2(t)+\frac{3-b}{2}u_x^2(t)\|_{L^1_x}\\ &\lesssim M(u)^2\ .\end{aligned}$$

Therefore $t\mapsto u(t,x(t)+x_+(t))$ is also Lipschitz on $\mathbb{R}$ which achieves the proof thanks to (3.42). □

3.3. **End of the proof ot Theorem 1.1.** In this subsection, we conclude by proving that the jump of $u_x(0,\cdot)$ at $x(0)+x_+(0)$ is equal to $-2u(0,x(0)+x_+(0))$. This saturates for all $v\in Y_+$, the relation between the jump of v_x and the value of v at a point $\xi\in\mathbb{R}$ and forces $u(0,\cdot)$ to be equal to $u(0,x(0)+x_+(0))\varphi(\cdot-x(0)+x_+(0))$.

We use the invariance of the (CH) equation under the transformation $(t,x)\mapsto(-t,-x)$. This invariance ensures that $v(t,x)=u(-t,-x)$ is also a solution of the (C-H) equation that belongs to $C(\mathbb{R};H^1(\mathbb{R})$, with $u-u_{xx}\in C_w(\mathbb{R};\mathcal{M}_+)$ and shares the property of Y-almost localization with $x(\cdot)$ replaced by $-x(-\cdot)$ and the same fonction $\varepsilon\mapsto R_\varepsilon$ (See Definition 1.1). Therefore, by applying Propositions 3.2, 3.4 and Lemma 3.3 for v we infer that there exists a C^1-function $x_-:\mathbb{R}\mapsto]-\infty,r_0]$ and a derivable non decreasing function $\tilde{a}:\mathbb{R}\to[\alpha_0/8,M(u)]$ with $\lim_{t\to\mp\infty}\tilde{a}(t)=\tilde{a}_\mp$ such that

$$\tilde{a}(t)=v_x(t,(-x(-t)+x_+(t))+)-v_x(t,(-x(-t)+x_+(t))-),\quad\forall t\in\mathbb{R}\,. \tag{3.43}$$

Moreover,

$$\lim_{t\to\mp\infty}v(t,-x(-t)+x_+(t))=\lim_{t\to\mp\infty}\tilde{a}(t)/2=\tilde{a}_\mp/2\,.$$

Coming back to u this ensures that

$$\lim_{t\to+\infty}u(t,x(t)-x_-(-t))\;=\;\lim_{t\to-\infty}\tilde{a}(t)/2=\tilde{a}_-/2\,, \tag{3.44}$$

$$\lim_{t\to-\infty}u(t,x(t)-x_-(-t))\;=\;\lim_{t\to+\infty}\tilde{a}(t)/2=\tilde{a}_+/2\,, \tag{3.45}$$

At this stage let us underline that since

$$x_-(-t)=\sup\{x\in\mathbb{R},\,\operatorname{supp}y(-t)\in[x(t)-x(-t),+\infty[\}$$

and $u\not\equiv0$ we must have $x(-t)+x(t)\ge0$ for all $t\in\mathbb{R}$. We claim that this forces

$$\tilde{a}_-=\tilde{a}_+=a_-=a_+\,. \tag{3.46}$$

Note first that since $\tilde{a}_-\le\tilde{a}_+$ and $a_-\le a_+$, it suffices to prove that $\tilde{a}_-\ge a_+$ and $\tilde{a}_+\le a_-$. This follows easily by a contradiction argument. Indeed, assume for instance that $\tilde{a}_-<a_+$.Then, there exists $t_0\in\mathbb{R}$ and $\varepsilon>0$ such that $u(t,x(t)-x_-(-t))<u(t,x(t)+x_+(t))-\varepsilon$ for all $t\ge t_0$. Since $x(t)-x_-(-t)=q(t-t_0,x(t_0)-x_-(-t_0))$ and $x(t)+x_+(t)=q(t-t_0,x(t_0)+x_+(t_0))$, it follows from (3.20) that

$$x_+(t)+x_-(-t))\ge\varepsilon(t-t_0)\underset{t\to+\infty}{\longrightarrow}+\infty$$

which contradicts that $(x_+(t),x_-(t))\in]-\infty,r_0]^2$. Exactly the same argument but with $t\to-\infty$ ensures that $\tilde{a}_+\le a_-$ and completes the proof of the claim (3.46).

We deduce from (3.46) that $a(t)=a+$ for all $t\in\mathbb{R}$ and thus (3.42), (3.19) and (3.22) force

$$u(t,x(0)+x_+(0)+\frac{a_+}{2}t)=\frac{a_+}{2},\quad\forall t\in\mathbb{R}$$

and

$$u_x\Big(t,(x(0)+x_+(0)+\frac{a_+}{2}t)-\Big)-u_x\Big(t,(x(0)+x_+(0)+\frac{a_+}{2}t)+\Big)=a_+,\quad\forall t\in\mathbb{R}\,.$$

In particular, in view of (3.31),

$$u(0,x(0)+x_+(0))=\frac{a_+}{2}\text{ and }y(0)=a_+\delta_{x(0)+x_+(0)}+\mu$$

with $\mu\in\mathcal{M}_+(\mathbb{R})$. But this forces $\mu=0$ since

$$(1-\partial_x^2)^{-1}(a_+\,\delta_{x(0)+x_+(0)})=\frac{a_+}{2}\exp\Big(-|\cdot-(x(0)+x_+(0))|\Big)$$

and for any $\mu \in \mathcal{M}_+(\mathbb{R})$, with $\mu \neq 0$, it holds

$$(1-\partial_x^2)^{-1}\nu = \frac{1}{2}e^{-|x|} * \nu > 0 \text{ on } \mathbb{R}\ .$$

We thus conclude that $y(0) = a_+\delta_{x(0)+x_+(0)}$ which leads to

$$u(t,x) = \frac{a_+}{2}\exp\Bigl(-\Bigl|x - x(0) - x_+(0) - \frac{a_+}{2}t\Bigr|\Bigr)$$

4. Asymptotic stability of the DP peakon

We now focus on the case $b = 3$ that corresponds to the Degasperis-Procesi equation. Recall that in this case (1.4) becomes (1.13). The following proposition proven in the appendix ensures that a Y- almost localized global solutions to the DP equation enjoys actually a uniform exponential decay.

Proposition 4.1. *Let $u \in C(\mathbb{R}; L^2(\mathbb{R}))$ with $y = (1-\partial_x^2)u \in C_w(\mathbb{R}; \mathcal{M}+)$ be a Y-almost localized solution of (1.13) with $\inf_{\mathbb{R}} \dot{x} \geq c_0 > 0$. Then there exists $C > 0$ such that for all $t \in \mathbb{R}$, all $R > 0$ and all $\Phi \in C(\mathbb{R})$ with $0 \leq \Phi \leq 1$ and $\operatorname{supp}\Phi \subset [-R,R]^c$.*

(4.1)
$$\int_{\mathbb{R}}\Bigl(4v^2(t) + 5v_x^2(t) + v_{xx}^2(t)\Bigr)\Phi(\cdot - x(t))\,dx + \Bigl\langle \Phi(\cdot - x(t)), y(t)\Bigr\rangle \leq C\,\exp(-R/6)\ .$$

In particular, u is uniformly in time exponentially decaying.

This proposition is a direct consequence of an almost monotonicity result for $\mathcal{H}(u) + c_0 M(u)$ at the right of an almost localized solution that is contained in the following lemma (see the appendix for a sketch of the proof). At this stage it is important to notice that direct calculations lead to

$$\frac{1}{4}\|w\|_{L^2}^2 \leq \mathcal{H}(w) \leq \|w\|_{L^2}^2, \quad \forall u \in L^2(\mathbb{R})\ . \tag{4.2}$$

As in [40], we introduce the C^∞-function Ψ defined on $\mathbb{R}$ by

$$\Psi(x) = \frac{2}{\pi}\arctan\Bigl(\exp(x/6)\Bigr) \tag{4.3}$$

Lemma 4.2. *Let $0 < \alpha < 1$ and let $u \in C(\mathbb{R}; L^2(\mathbb{R}))$, with $y = (1-\partial_x^2)u \in C_w(\mathbb{R}; \mathcal{M}_+)$, be a solution of (1.13) such that there exist $x : \mathbb{R} \to \mathbb{R}$ of class C^1 with $\inf_{\mathbb{R}} \dot{x} \geq c_0 > 0$ and $R_0 > 0$ with*

$$\|u(t)\|_{L^\infty(|x-x(t)|>R_0)} \leq \frac{(1-\alpha)c_0}{2^6}\ ,\ \forall t \in \mathbb{R}. \tag{4.4}$$

For $0 < \beta \leq \alpha$, $0 \leq \gamma \leq \frac{1}{8}(1-\alpha)c_0$, $R > 0$, $t_0 \in \mathbb{R}$ and any C^1-function

$$z : \mathbb{R} \to \mathbb{R} \text{ with } (1-\alpha)\dot{x}(t) \leq \dot{z}(t) \leq (1-\beta)\dot{x}(t), \quad \forall t \in \mathbb{R}, \tag{4.5}$$

setting

$$I_{t_0}^{\mp R}(t) = \Bigl\langle 5v^2(t) + 4v_x^2(t) + v_{xx}^2(t) + \gamma y(t), \Psi\Bigl(\cdot - z_{t_0}^{\mp R}(t)\Bigr)\Bigr\rangle \tag{4.6}$$

where

$$z_{t_0}^{\mp R}(t) = x(t_0) \mp R + z(t) - z(t_0)$$

we have

$$I_{t_0}^{+R}(t_0) - I_{t_0}^{+R}(t) \leq K_0 e^{-R/6}, \quad \forall t \leq t_0 \tag{4.7}$$

and

$$I_{t_0}^{-R}(t) - I_{t_0}^{-R}(t_0) \le K_0 e^{-R/6}, \quad \forall t \ge t_0 \quad , \tag{4.8}$$

for some constant $K_0 > 0$ that only depends on $\mathcal{H}(u)$, $M(u)$, c_0, R_0 and β.

According to [36] and [33], for any speed $c > 0$ there exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that for any $u_0 \in Y_+$ with

$$\|u_0 - c\varphi\|_{\mathcal{H}} < \varepsilon^4 \,, \quad 0 < \varepsilon < \varepsilon_0, \tag{4.9}$$

it holds

$$\sup_{t\in\mathbb{R}} \|u(t) - c\varphi(\cdot - \xi(t))\|_{\mathcal{H}} < C_0\varepsilon \,,$$

where $u \in C(\mathbb{R}; H^1)$ is the solution emanating from u_0 and $\xi(t) \in \mathbb{R}$ is unique point where the function $v(t,\cdot) = (4 - \partial_x^2)^{-1} u(t,\cdot)$ reaches its maximum. According to [32], by the implicit function theorem, one can prove that there exists $\varepsilon_0' > 0$ and $K > 1$ such that if a solution $u \in C(\mathbb{R}; \mathcal{H})$ to (1.13) satisfies

$$\sup_{t\in\mathbb{R}} \inf_{y\in\mathbb{R}} \|u(t) - c\varphi(\cdot - y)\|_{\mathcal{H}} < \varepsilon \tag{4.10}$$

for some $0 < \varepsilon \le \varepsilon_0'$ then there exists a uniquely determined C^1-function $x : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \|u(t) - c\varphi(\cdot - x(t))\|_{\mathcal{H}} \le K\varepsilon \tag{4.11}$$

and

$$\int_{\mathbb{R}} v(t)\rho'(\cdot - x(t)) = 0, \quad \forall t \in \mathbb{R} \,, \tag{4.12}$$

where $v = (4 - \partial_x)^{-1} u$ and $\rho = (4 - \partial_x)^{-1}\varphi$. Moreover, for all $t \in \mathbb{R}$, it holds

$$|\dot{x}(t) - c| \le K\varepsilon \,. \tag{4.13}$$

At this stage, we fix $0 < \theta < c$ and we take

$$\varepsilon = \min\Big(\frac{2^{-10}\theta}{KC_0}, \varepsilon_0, \frac{\varepsilon_0'}{C_0}\Big) \tag{4.14}$$

so that (4.9) ensures that (4.11) and (4.13) hold with $K\varepsilon \le \frac{\theta}{2^{10}} \le \frac{c}{2^{10}}$. It follows that $\dot{x} \ge \frac{3}{4}c$ on $\mathbb{R}$. Moreover, combining (4.11), (4.2) and (2.3) we infer that there exists $R > 0$ such that

$$\|u(t, \cdot + x(t))\|_{H^1(]-R,R[^c)} \le 2^{-9}\theta \,.$$

Hence, u satisfies the hypotheses of Lemma 4.2 for any $0 < \alpha < 1$ such that

$$(1 - \alpha) \ge \frac{\theta}{4c} \tag{4.15}$$

and any $0 \le \gamma \le \frac{1}{12}(1-\alpha)c$. In particular, u satisfies the hypotheses of Lemma 4.2 for $\alpha = 1/3$. Note that the hypothesis (1.14) with

$$\eta = \min\Big(\frac{2^{-10}\theta}{KC_0}, \varepsilon_0, \frac{\varepsilon_0'}{C_0}\Big)^8$$

implies that (4.9) holds with ε given by (4.14).

In the sequel we will make use of the following functionals that measure the quantity $\mathcal{H}(u)+\gamma M(u)$ at the right and at the left of u. For $0\le\gamma\le\frac{c}{12}$, $u\in Y$ and $R>0$ we set

$$J^R_{\gamma,r}(w)=\Big\langle 5v^2+4v_x^2+v_{xx}^2+\gamma(u-u_{xx}),\Psi(\cdot-R)\Big\rangle\ . \tag{4.16}$$

and

$$J^R_{\gamma,l}(w)=\Big\langle 5v^2+4v_x^2+v_{xx}^2+\gamma(u-u_{xx}),(1-\Psi(\cdot+R))\Big\rangle \tag{4.17}$$

where $v=(4-\partial_x^2)^{-1}u$.

Let $t_0\in\mathbb{R}$ be fixed. Fixing $\alpha=\beta=1/3$ and taking $z(\cdot)=(1-\alpha)x(\cdot)$, $z(\cdot)$ clearly satisfies (4.5). Moreover, we have $J^R_{\gamma,r}(u(t_0,\cdot+x(t_0))=I^{+R}_{t_0}(t_0)$ where $I^{+R}_{t_0}$ is defined in (4.6). Since obviously,

$$J^R_{\gamma,r}\Big(u(t,\cdot+x(t))\Big)\ge I^{+R}_{t_0}(t)\ ,\quad \forall t\le t_0,$$

we deduce from (4.7) that

$$J^R_{\gamma,r}\Big(u(t_0,\cdot+x(t_0))\Big)\le J^R_{\gamma,r}\Big(u(t,\cdot+x(t))\Big)+K_0e^{-R/6}\ ,\quad \forall t\le t_0, \tag{4.18}$$

where K_0 is the constant appearing in (4.7). Now, let us define

$$\begin{aligned}\tilde I^R_{t_0}(t) &= \Big\langle 5v^2(t)+4v_x^2(t)+v_{xx}^2(t)+\frac{c}{12}y(t),1-\Psi(\cdot-x(t)+R+\alpha(x(t_0)-x(t)))\Big\rangle\\ &= E(u(t))+cM(u(t))-I^{-R}_{t_0}(t)\ ,\end{aligned}$$

where we take again $z(\cdot)=(1-\alpha)x(\cdot)$. Since $M(\cdot)$ and $E(\cdot)$ are conservation laws, (4.8) leads to

$$\tilde I^R_{t_0}(t)\ge \tilde I^R_{t_0}(t_0)-Ce^{-R/6},\ \forall t\ge t_0\ .$$

We thus deduce as above that $\forall t\ge t_0$,

$$J^R_{\gamma,l}\Big(u(t,\cdot+x(t))\Big)\ge J^R_{\gamma,l}\Big(x(t_0,\cdot+x(t_0))\Big)-K_0e^{-R/6}\ . \tag{4.19}$$

The following proposition ensures that, for ε small enough, the ω-limit set for the weak H^1-topology of the orbit of u_0 is constituted by initial data of Y-almost localized solutions. The crucial tools in the proof are the almost monotonicity properties (4.18) and (4.19). We omit the proof since it is exactly the same that the proof of Proposition 5.2 in [42].

Proposition 4.3. *Let $u_0\in Y_+$ satisfying (4.9) with ε defined as in (4.14) and let $u\in C(\mathbb{R};H^1(\mathbb{R}))$ the emanating solution of (1.13). For any sequence $t_n\nearrow+\infty$ there exists a subsequence $\{t_{n_k}\}\subset\{t_n\}$ and $\tilde u_0\in Y_+$ such that*

$$u(t_{n_k},\cdot+x(t_{n_k}))\ \underset{n_k\to+\infty}{\rightharpoonup}\ \tilde u_0\ \textit{in}\ H^1(\mathbb{R}) \tag{4.20}$$

and

$$u(t_{n_k},\cdot+x(t_{n_k}))\ \underset{n_k\to+\infty}{\longrightarrow}\ \tilde u_0\ \textit{in}\ H^1_{loc}(\mathbb{R}) \tag{4.21}$$

where $x(\cdot)$ is the C^1-function uniquely determined by (4.11)-(4.12). Moreover, the solution of (1.13) emanating from $\tilde u_0$ is Y-almost localized.

So, let $u_0 \in Y_+$ satisfying (4.9) with ε defined as in (4.14) and let $t_n \nearrow +\infty$ be a sequence of positive real numbers. According to the above proposition, (4.20)-(4.21) hold for some subsequence $\{t_{n_k}\} \subset \{t_n\}$ and $\tilde{u}_0 \in Y_+$ such that the solution of (1.13) emanating from $\tilde{u}_0$ is Y-almost localized. Theorem 1.1 then forces

$$\tilde{u}_0 = c_0\varphi(\cdot - x_0)$$

for some $x_0 \in \mathbb{R}$ and c_0 such that $|c - c_0| \le K\varepsilon \le c/2^9$. Since (4.20) implies that

$$v(t_{n_k}, \cdot + x(t_{n_k})) \underset{n_k \to +\infty}{\rightharpoonup} \tilde{v}_0 \text{ in } H^3(\mathbb{R})$$

with $v_n = (4 - \partial_x^2)^{-1}u$ and $\tilde{v}_0 = (4 - \partial_x^2)^{-1}\tilde{u}_0$, we infer that $\tilde{v}_0$ satisfies the orthogonality condition (4.12) and thus we must have $x_0 = 0$. On the other hand, (4.21) and (4.11) ensure that $c_0 = \lim\limits_{n\to+\infty} \max\limits_{\mathbb{R}} u(t_{n_k})$ and thus

$$u(t_{n_k}, \cdot + x(t_{n_k})) - \lambda(t_{n_k})\varphi \underset{k \to +\infty}{\rightharpoonup} 0 \text{ in } H^1(\mathbb{R})$$

where we set $\lambda(t) := \max_{\mathbb{R}} u(t), \quad \forall t \in \mathbb{R}$. Since this is the only possible limit, it follows that

$$u(t, \cdot + x(t)) - \lambda(t)\varphi \underset{t \to +\infty}{\rightharpoonup} 0 \text{ in } H^1(\mathbb{R}) \; .$$

and thus

$$u(t, \cdot + x(t)) - \lambda(t)\varphi \underset{t \to 0}{\longrightarrow} 0 \text{ in } H^1_{loc}(\mathbb{R}) \tag{4.22}$$

and

$$v(t, \cdot + x(t)) - \lambda(t)\rho \underset{t \to 0}{\longrightarrow} 0 \text{ in } H^3_{loc}(\mathbb{R}) \tag{4.23}$$

4.1. **Convergence in** $H^1(]-A, +\infty[)$ **for any** $A > 0$**.** Let $\delta > 0$ be fixed. Choosing $R > 0$ such that $J^R_{0,r}(u(0), \cdot + x(0)) < \delta$ and $K_0 e^{-R/6} \le \delta$, where K_0 is the constant that appears in (4.18). We deduce from (4.18) that $J^R_{0,r}\Big(u(t, \cdot + x(t))\Big) < 2\delta$ for all $t \ge 0$. This fact together with the local strong convergence (4.23) clearly ensure that

$$v(t, \cdot + x(t)) - \lambda(t)\rho \underset{t \to +\infty}{\longrightarrow} 0 \text{ in } H^2(]-A, +\infty[) \text{ for any } A > 0 \tag{4.24}$$

and thus

$$u(t, \cdot + x(t)) - \lambda(t)\varphi \underset{t \to +\infty}{\longrightarrow} 0 \text{ in } L^2(]-A, +\infty[) \text{ for any } A > 0 \; .$$

Since $\{u(t), t \in \mathbb{R}\}$ is bounded in Y, this leads to

$$u(t, \cdot + x(t)) - \lambda(t)\varphi \underset{t \to +\infty}{\longrightarrow} 0 \text{ in } H^1(]-A, +\infty[) \text{ for any } A > 0 \; . \tag{4.25}$$

4.2. **Convergence of the scaling parameter.** We claim that

$$\lambda(t) \underset{t \to +\infty}{\longrightarrow} c_0 \; . \tag{4.26}$$

Let us fix again $\delta > 0$ and take $R > 0$ such that $K_0 e^{-R/6} < \delta$. (4.19) with $\gamma = 0$ together with the conservation of $E(u)$ ensure that, for any couple $(t, t') \in \mathbb{R}^2$ with $t > t'$ it holds

$$\int_{\mathbb{R}} (5v^2 + 4v_x^2 + v_{xx}^2)(t, x)\Psi(x - x(t) + R)\, dx \le \int_{\mathbb{R}} (5v^2 + 4v_x^2 + v_{xx}^2)(t', x)\Psi(x - x(t') + R)\, dx + \delta$$

On the other hand, by the strong convergence (4.24) and the exponential localization of φ, φ' and Ψ, there exists $T>0$ such that for all $t \geq T$,

$$\Big|\int_{\mathbb{R}} (5v^2+4v_x^2+v_{xx}^2)(t,x)\Psi(x-x(t)+R)\,dx - \lambda^2(t)E(\varphi)\Big| \leq \delta\ .$$

It thus follows that

$$\lambda^2(t)E(\varphi) \leq \lambda^2(t')E(\varphi)+3\delta, \quad \forall t>t'>T\ .$$

Since $\delta>0$ is arbitrary, this forces λ to have a limit at $+\infty$ and completes the proof of the claim.

4.3. **Convergence of $\dot{x}$.** We set $W(t,\cdot) := c_0\rho(\cdot - x(t))$ and $\eta(t) = v(t) - c_0\rho(\cdot - x(t)) = v(t) - W(t)$ for all $t \geq 0$. Differentiating (4.12) with respect to time and using that $4\rho - \rho'' = \varphi$, we get

$$\int_{\mathbb{R}} \eta_t \partial_x W = \dot{x}\int_{\mathbb{R}} \eta \partial_x^2 W = -c_0\dot{x}\int_{\mathbb{R}} \eta\varphi(\cdot - x(t)) + 4\dot{x}\int_{\mathbb{R}} \eta W,$$

and thus

$$\Big|\int_{\mathbb{R}} \eta_t\partial_x W\Big| \leq (c_0|\dot{x}-c_0|+c_0^2|)\int_{\mathbb{R}} \eta\varphi(\cdot - x(t))| + 4c_0|\int_{\mathbb{R}}\eta W|\ . \tag{4.27}$$

We notice that $v=(4-\partial_x^2)^{-1}$ is solution of

$$v_t = -2\partial_x v^2 - \frac{1}{2}\partial_x(1-\partial_x^2)^{-1}(12v^2+8v_x^2+v_{xx}^2),\ (t,x)\in\mathbb{R}_+\times\mathbb{R}. \tag{4.28}$$

Substituting v by $\eta+W$ in (4.28) and using the equation satisfied by W, we obtain the following equation satisfied by η :

$$\eta_t - (\dot{x}-c_0)\partial_x W = -4\partial_x\eta W - (1-\partial_x^2)^{-1}\partial_x\Big(8\eta W + 16\eta_x W_x + \eta_{xx}W_{xx}\Big)\ . \tag{4.29}$$

At this stage it is worth noticing that (4.24)-(4.26) ensures that

$$|\int_{\mathbb{R}} \eta\varphi(\cdot - x(t))| + |\int_{\mathbb{R}}\eta W| + \sum_{i=0}^{2}\|\partial_x^i\eta(t)\partial_x^i W(t)\|_{L^2} \underset{t\to+\infty}{\longrightarrow} 0\ . \tag{4.30}$$

Taking the L^2-scalar product with $\partial_x W$ with (4.29), integrating by parts, using that $\|\partial_x W\|_{L^2}^2 = \frac{7}{54}c_0^2$ and the decay of ρ and its first derivative, (4.27), (4.30), (4.11) lead to

$$|\dot{x}(t)-c_0| \underset{t\to\infty}{\longrightarrow} 0\ .$$

4.4. **Strong H^1-convergence on $]\theta t, +\infty[$.** We deduce from (4.26) that

$$v(t,\cdot) - c_0\rho(\cdot - x(t)) \underset{t\to+\infty}{\rightharpoonup} 0 \text{ in } H^1(\mathbb{R})$$

and

$$v(t,\cdot + x(t)) - c_0\rho \underset{t\to+\infty}{\longrightarrow} 0 \text{ in } H^1(]-A,+\infty[) \text{ for any } A>0\ . \tag{4.31}$$

(1.16) will follow by combining these convergence results with the almost non increasing property (4.7). Indeed, let us fix $\delta>0$ and take $R\gg 1$ such that

$$\|\rho\|_{H^2(]-\infty,-R/2[}^2 < \delta \quad \text{and} \quad \|\Psi - 1\|_{L^\infty(]R/2,+\infty[)} < \delta \tag{4.32}$$

where Ψ is defined in (6.4). According to the above convergence result there exists $t_0>0$ such that $x(t_0)>R$ and for all $t\ge t_0$,

$$\int_{-R/2}^{+\infty}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t,\cdot+x(t))<\delta\,,$$

where we set $\eta=v(t)-c_0\rho(\cdot-x(t))$. In particular, (4.32) ensures that

(4.33)
$$\Big|E(\varphi)-\int_{\mathbb{R}}\Big(5v(t,\cdot+x(t))\rho+4v_x(t,\cdot+x(t))\rho_x+v_{xx}(t,\cdot+x(t))\rho_{xx}(t,\cdot+x(t))\Big)\Psi(\cdot+y)\Big|\lesssim\delta,\quad\forall y\ge R,\,\forall t\ge t_0\,,$$

We set $z(t)=\frac{\theta}{2}t$ and notice that (4.15) ensures that (4.5) is satisfied with $1-\alpha=\frac{\theta}{4c}$ and $\beta=1/4$. Moreover, as noticing in the beginning of this section (see (4.15)), u satisfies the hypotheses of Lemma (4.2) for such α. According to (4.8) with $\gamma=0$, we thus get for all $t\ge t_0$,

$$\int_{\mathbb{R}}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t,\cdot)\Psi(\cdot-x(t_0)-\frac{\theta}{2}(t-t_0)+R)\le\int_{\mathbb{R}}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t_0,\cdot)\Psi(x-x(t_0)+R)+K_0(\alpha)e^{-R/6}$$

which leads to

$$\begin{aligned}
\int_{\mathbb{R}}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t,\cdot)\Psi\Big(\cdot-x(t_0)-\frac{\theta}{2}(t-t_0)+R\Big)&=\int_{\mathbb{R}}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t,\cdot)\Psi\Big(\cdot-x(t_0)-\frac{\theta}{2}(t-t_0)+x_0\Big)\\
&-2c_0\int_{\mathbb{R}}\Big(5v(t)\rho(\cdot-x(t))+v_x(t)\rho_x(\cdot-x(t))+v_{xx}(t)\rho_{xx}(\cdot-x(t))\Big)\Psi\Big(\cdot-x(t_0)-\frac{\theta}{2}(t-t_0)+R\Big)\\
&+c_0^2\int_{\mathbb{R}}(5\rho^2+4\rho_x^2+\rho_{xx}^2)(t,\cdot-x(t))\Psi\Big(\cdot-x(t_0)-\frac{\theta}{2}(t-t_0)+R\Big)\\
&\le\int_{\mathbb{R}}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t_0,\cdot)\Psi(\cdot-x(t_0)+R)+K_0(\alpha)e^{-R/6}\\
&-2c_0\int_{\mathbb{R}}(5v(t_0)\rho(\cdot-x(t_0))+4v_x(t_0)\rho_x(\cdot-x(t_0))+v_{xx}(t_0)\rho_{xx}(\cdot-x(t_0))\Psi(\cdot-x(t_0)+R)+C\,\delta\\
&+c_0^2\int_{\mathbb{R}}\Big(5\rho^2+4\rho_x^2+\rho_{xx}^2)(t_0,\cdot-x(t_0))\Big)\Psi(\cdot-x(t_0)+R)+Ce^{-R/6}\\
&\lesssim\int_{\mathbb{R}}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t_0,\cdot)\Psi(\cdot-x(t_0)+R)+C(e^{-R/6}+\delta)\\
&\lesssim\delta+e^{-R/6}
\end{aligned}$$

where in the next to the last step we used that ρ decays exponentially fast and (4.33) since $x(t)-x(t_0)-\frac{\theta}{2}(t-t_0)+R\ge R$ for all $t\ge t_0$. Taking R large enough and $t_1>t_0$ such that $\theta t_1\ge x(t_0)+\frac{\theta}{2}(t_1-t_0)-R$, it follows that for $t\ge t_1$,

$$\int_{\mathbb{R}}(5\eta^2+4\eta_x^2+\eta_{xx}^2)(t,\cdot)\Psi(\cdot-\theta t)\lesssim\delta$$

which completes the proof of the strong H^2 convergence of $v(t,\cdot+x(t))$, on $]\theta t,+\infty[$. The strong H^1-convergence of $u(t,\cdot+x(t))$ follows by using that $u=(4-\partial_x^2)v$ and that u is uniformly in time bounded in $H^{\frac{3}{2}+}$.

4.5. **Strong H^1-convergence at the left of any given point.** In this subsection we complete the proof of Theorem 1.2 by proving the lemma below. As for the C-H equation, the main observation to prove this lemma is that all the energy of the solutions to the DP equation that have a non negative density momentum, is traveling to the right. This property should be shared by most Hamiltonian CH-type equation because of the absence of linear part.

Lemma 4.4. *For any $u_0 \in Y_+$ and any $z \in \mathbb{R}$, denoting by $u \in C(\mathbb{R}; H^1)$ solution of* (1.13) *emanating from u_0 it holds*

$$\lim_{t\to+\infty} \|u(t)\|_{H^1(]-\infty,z[)} = 0 \; . \tag{4.34}$$

Proof. Let $0 < \gamma < \|u_0\|^2_{\mathcal{H}}$ and let $x_\gamma : \mathbb{R} \to \mathbb{R}$ be defined by

$$\int_{\mathbb{R}} (5v^2 + 4v_x^2 + v_{xx}^2)(t)\Psi(\cdot - x_\gamma(t)) = \gamma \tag{4.35}$$

with Ψ defined in (4.3). Note that $x_\gamma(\cdot)$ is well-defined since $u_0 \in Y_+$ forces $u > 0$ on $\mathbb{R}^2$ and thus for any fixed $t \in \mathbb{R}$, $z \mapsto \int_{\mathbb{R}}(5v^2+4v_x^2+v_{xx}^2)(t)\Psi(\cdot - z)$ is a decreasing continuous bijection from $\mathbb{R}$ to $]0, \|u_0\|^2_{\mathcal{H}}[$. Moreover, $u \in C(\mathbb{R}; H^1)$ ensures that $v \in C(\mathbb{R}; H^3)$ and thus $x_\gamma(\cdot)$ is a continuous function. (4.34) is clearly a direct consequence of the fact that

$$\lim_{t\to+\infty} x_\gamma(t) = +\infty \; . \tag{4.36}$$

To prove (4.36) we first claim that for any $t \in \mathbb{R}$ and any $\Delta > 0$ it holds

$$x_\gamma(t+\Delta) - x_\gamma(t) \geq \frac{10}{27}\Big(\int_t^{t+\Delta} \int_{x_\gamma(t)}^{x_\gamma(t)+2} u^2(\tau, s)\, ds\, d\tau\Big)^{1/2} > 0 \; . \tag{4.37}$$

Let us prove this claim. First we notice that by continuity with respect to initial data, it suffices to prove (4.37) for $u \in C^\infty(\mathbb{R}; H^\infty) \cap L^\infty(\mathbb{R}; Y_+)$. Then a simple application of the implicit function theorem ensures that $t \mapsto x_\gamma(t)$ is of class C^1. Indeed,

$$\psi : (z, v) \mapsto \int_{\mathbb{R}} (5v^2 + 4v_x^2 + v_{xx}^2)\Psi(\cdot - z)$$

is of class C^1 from $\mathbb{R} \times H^2(\mathbb{R})$ into $\mathbb{R}$ and for any $(z_0, v) \in \mathbb{R} \times H^3(\mathbb{R})/\{0\}$, $\partial_z\psi(z_0, v) = \int_{\mathbb{R}}(5v^2 + 4v_x^2 + v_{xx}^2)\Psi'(\cdot - z_0) > 0$. Now we need the two following Lemmas proved in the appendix :

Lemma 4.5. *Let $u \in C(]-T, T[; H^\infty(\mathbb{R}))$, with $0 < T \leq +\infty$, be a solution of equation* (1.13)*. For any smooth space function $g : \mathbb{R} \mapsto \mathbb{R}$, it holds*

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right)(t)g \\ &= \frac{2}{3}\int_{\mathbb{R}} u^3(t)g' + 5\int_{\mathbb{R}} v(t)h(t)g' - 4\int_{\mathbb{R}} vu^2(t)g' + \int_{\mathbb{R}} v_x(t)h_x(t)g', \quad \forall t \in]-T, T[\end{aligned} \tag{4.38}$$

where $h = (1 - \partial_x^2)^{-1}u^2$.

Lemma 4.6. *Let $u \in Y_+$ and $v = (4 - \partial_x^2)^{-1}u$. Then the following estimates hold :*

$$3v \leq u \leq 6v, \quad |v_x| \leq 2v \quad \text{and } |v_{xx}| \leq \frac{4}{3}u \tag{4.39}$$

Integrating by parts the last term of the right-hand side member of (4.38) and using that $h_{xx} = -u^2 + h$, we infer that

$$\begin{aligned}
\dot{x}_\gamma(t)\int_{\mathbb{R}}(5v^2+4v_x^2+v_{xx}^2)\Psi'(\cdot-x_\gamma(t)) &= \frac{2}{3}\int_{\mathbb{R}} u^3(t)\Psi'(\cdot-x_\gamma(t)) \\
&+4\int_{\mathbb{R}} v(t)h(t)\Psi'(\cdot-x_\gamma(t)) - 3\int_{\mathbb{R}} v(t)u^2(t)\Psi'(\cdot-x_\gamma(t)) + \int_{\mathbb{R}} v(t)h_x(t)\Psi''(\cdot-x_\gamma(t))
\end{aligned} \tag{4.40}$$

Observe that by using integration by parts and $|u_x| \le u$ we get

$$\begin{aligned}
h(x) &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\eta}u^2(\eta) + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\eta}u^2(\eta) \\
&= \frac{1}{2}u^2(x) - e^{-x}\int_{-\infty}^{x} e^{\eta}uu_x(\eta) + \frac{1}{2}u^2(x) + e^{x}\int_{x}^{+\infty} e^{-\eta}uu_x(\eta) \\
&\ge u^2(x) - \int_{\mathbb{R}} e^{-|x-\eta|}u^2(\eta) \\
&\ge u^2(x) - 2h(x)
\end{aligned}$$

and thus $h(x) \ge \frac{1}{3}\, u^2(x)$. Combining this estimate with $|h_x| \le h$ and (4.39), using that by direct calculations $|\Psi''| \le \Psi'/6$, we infer that

$$3vu^2\Psi' + v|h_x||\Psi''| \le (3u^2 + \frac{1}{6}v|h_x|)\Psi' \le (4vh + \frac{31}{54}u^3)\Psi'$$

Injecting this last inequality in (4.40) we eventually get

$$\dot{x}_\gamma(t)\int_{\mathbb{R}}(5v^2+4v_x^2+v_{xx}^2)\Psi'(\cdot-x_\gamma(t)) \ge \frac{5}{54}\int_{\mathbb{R}} u^3(t)\Psi'(\cdot-x_\gamma(t))\ .$$

Noticing that by (4.39), $5v^2 + 4v_x^2 + v_{xx}^2 \le \frac{37}{9}u^2$, Hölder inequality together with (4.39) and the fact that Ψ' is a non negative function of total mass 1 lead to

$$\dot{x}_\gamma(t) \ge \frac{1}{50}\Big(\int_{\mathbb{R}} u^2\Psi'(\cdot-x_\gamma(t))\Big)^{1/2}\ . \tag{4.41}$$

Integrating this inequality between t and $t+\Delta$ yields (4.37) that obviously implies that $x_\gamma(\cdot)$ is an increasing function. In particular there exists $x_\gamma^\infty \in \mathbb{R}\cap\{+\infty\}$ such that $x_\gamma(t) \nearrow x_\gamma^\infty$ as $t \to +\infty$ and it remains to prove that $x_\gamma^\infty = +\infty$. Assuming the contrary, we first notice that (4.39) and $u \le \|y_0\|_{\mathcal{M}}$ on $\mathbb{R}^2$ ensure that (4.35) leads to

$$\lim_{t\to+\infty}\int_{\mathbb{R}}(5v^2+4v_x^2+v_{xx}^2)\Psi'(\cdot-x_\gamma(t)) = \lim_{t\to+\infty}\int_{\mathbb{R}}(5v^2+4v_x^2+v_{xx}^2)\Psi'(\cdot-x_\gamma^\infty) = \gamma\ . \tag{4.42}$$

Now, taking $\Delta = 1$, (4.37) forces

$$\lim_{t\to+\infty}\int_t^{t+1}\int_{x_\gamma(t)}^{x_\gamma(t)+2} u^2(\tau,s)\,ds\,d\tau = 0$$

which, recalling that $x_\gamma(t) \to x_\gamma^\infty$, leads to

$$\lim_{t\to+\infty}\int_t^{t+1}\int_{x_\gamma^\infty}^{x_\gamma^\infty+2} u^2(\tau,s)\,ds\,d\tau = 0\ .$$

In particular there exists a sequence $(t_n, x_n)_{n\geq 1} \subset \mathbb{R}\times[x_\gamma^\infty, x_\gamma^\infty+2]$ with $t_n \nearrow +\infty$ such that $u(t_n, x_n) \to 0$ as $n\to\infty$. Therefore, making use of the fact that $|u_x| \leq u$ on $\mathbb{R}^2$ forces, for any $(t, x_0) \in \mathbb{R}^2$, that

$$u(t,x) \leq e^{|x_0-x|} u(t,x_0), \quad \forall x \in \mathbb{R}\ , \tag{4.43}$$

we infer that for any $A > 0$,

$$\lim_{n\to\infty} \sup_{x\in[x_\gamma^\infty-A, x_\gamma^\infty+A]} u(t_n, x) = 0\ .$$

and (4.39) then yields

$$\lim_{n\to\infty} \sup_{x\in[x_\gamma^\infty-A, x_\gamma^\infty+A]} [5v^2(t_n,x) + 4v_x^2(t_n,x) + v_{xx}^2(t_n,x)] = 0\ . \tag{4.44}$$

Finally, taking $A > 0$ such that $x_\gamma^\infty - A < x_{\gamma'}(0)$ with $\gamma < \gamma' < \|u_0\|_{H^1}^2$, we infer from (4.44) and the monotonicity of $t \mapsto x_{\gamma'}(t)$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}} (5v^2 + 4v_x^2 + v_{xx}^2)(t_n, \cdot)\Psi(\cdot - x_\gamma^\infty) = \gamma'\ .$$

This contradicts (4.42) and concludes the proof of the lemma. □

5. Asymptotic stability of train of peakons

In [32] the orbital stability in $L^2(\mathbb{R})$ of well ordered trains of peakons is established. More precisely, the following theorem is proved :

Theorem 5.1 ([32])**.** *Let be given N velocities $c_1, .., c_N$ such that $0 < c_1 < c_2 < .. < c_N$. There exist $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $u \in C(\mathbb{R}; H^1)$ is the global solution of (C-H) emanating from $u_0 \in Y_+$, with*

$$\|u_0 - \sum_{j=1}^N \varphi_{c_j}(\cdot - z_j^0)\|_{\mathcal{H}} \leq \varepsilon^2 \tag{5.1}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$, with $L > L_0$, then there exist N C^1-functions $t \mapsto x_1(t), .., t \mapsto x_N(t)$ uniquely determined such that

$$\sup_{t\in\mathbb{R}+} \|u(t,\cdot) - \sum_{j=1}^N \varphi_{c_j}(\cdot - x_j(t))\|_{\mathcal{H}} \leq A\sqrt{\sqrt{\varepsilon} + L^{-1/8}} \tag{5.2}$$

and

$$\int_{\mathbb{R}} \Big(u(t,\cdot) - \sum_{j=1}^N \varphi_{c_j}(\cdot - x_j(t))\Big)\partial_x\varphi_{c_i}(\cdot - x_j(t))\, dx = 0\ , \quad i \in \{1, .., N\}. \tag{5.3}$$

Moreover, for $i = 1, .., N$

$$|\dot{x}_i - c_i| \leq A\sqrt{\sqrt{\varepsilon} + L^{-1/8}}, \quad \forall t \in \mathbb{R}_+\ . \tag{5.4}$$

This result combined with the asymptotic stability of a single peakon established in the preceding section, yields the asymptotic stability of a train of well ordered peakons by following the general strategy developped in [38] (see also [21]). We do not give the proof but refer the reader to [42] for a detailed proof in the case of the Camassa-Holm equation.

Theorem 5.2. *Let be given N velocities $c_1,..,c_N$ such that $0<c_1<c_2<..<c_N$ and $0<\theta_0<c_1/4$. There exist $L_0>0$ and $\varepsilon_0>0$ such that if $u\in C(\mathbb{R};H^1)$ is the solution of (C-H) emanating from $u_0\in Y_+$, with*

$$\|u_0-\sum_{j=1}^N \varphi_{c_j}(\cdot - z_j^0)\|_{\mathcal{H}}\le \varepsilon_0^2 \quad \text{and} \quad z_j^0-z_{j-1}^0\ge L_0, \tag{5.5}$$

then there exist $0<c_1^*<..<c_N^*$ and C^1-functions $t\mapsto x_1(t),..,t\mapsto x_N(t)$, with $\dot{x}_j(t)\to c_j^*$ as $t\to+\infty$, such that,

$$u(\cdot+x_j(t)) \underset{t\to+\infty}{\rightharpoonup} \varphi_{c_j^*} \text{ in } H^1(\mathbb{R}),\ \forall j\in\{1,..,N\}\ . \tag{5.6}$$

Moreover, for any $z\in\mathbb{R}$,

$$u-\sum_{j=1}^N \varphi_{c_j^*}(\cdot - x_j(t)) \underset{t\to+\infty}{\longrightarrow} 0 \text{ in } H^1(]-\infty,z[\cup]\theta_0 t,+\infty[)\ . \tag{5.7}$$

6. Appendix

6.1. **Proof of the Lemma 4.5.** First we notice that applying the operator $(4-\partial_x^2)^{-1}$ to the two members of (1.13) and using that

$$(4-\partial_x^2)^{-1}(1-\partial_x^2)^{-1}=\frac{1}{3}(1-\partial_x^2)^{-1}-\frac{1}{3}(4-\partial_x^2)^{-1}\ , \tag{6.1}$$

we infer that v satisfies

$$v_t=-\frac{1}{2}h_x\ . \tag{6.2}$$

With this identity in hand it is easy to check that

$$4\frac{d}{dt}\int_{\mathbb{R}} v^2 g = 8\int_{\mathbb{R}} v v_t g=-4\int_{\mathbb{R}} v h_x g\ ,$$

and

$$\begin{aligned}5\frac{d}{dt}\int_{\mathbb{R}} v_x^2 g &= 10\int_{\mathbb{R}} v_x v_{xt} g=-5\int_{\mathbb{R}} v_x(1-\partial_x^2)^{-1}\partial_x^2 u^2 g = 5\int_{\mathbb{R}} v_x u^2 g - 5\int_{\mathbb{R}} v_x h g\\ &= 5\int_{\mathbb{R}} v_x u^2 g + 5\int_{\mathbb{R}} v h_x g + 5\int_{\mathbb{R}} v h g'\end{aligned}$$

In the same way we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} v_{xx}^2 g = 2\int_{\mathbb{R}} v_{xx}v_{xxt} g &= -\int_{\mathbb{R}} v_{xx}\partial_x(1-\partial_x^2)^{-1}\partial_x^2 u^2 g\\ &= \int_{\mathbb{R}} v_{xx}\partial_x(u^2) g - \int_{\mathbb{R}} v_{xx}(1-\partial_x^2)^{-1}\partial_x(u^2) g\\ &= A+B\end{aligned}$$

where it holds

$$\begin{aligned}A &= \int_{\mathbb{R}} (4-\partial_x^2)^{-1}\partial_x^2 u\,\partial_x(u^2) g = -\int_{\mathbb{R}} u\partial_x(u^2) g + 4\int_{\mathbb{R}} v\partial_x(u^2) g\\ &= \frac{2}{3}\int_{\mathbb{R}} u^3 g' - 4\int_{\mathbb{R}} v_x u^2 g - 4\int_{\mathbb{R}} v u^2 g'\end{aligned}$$

and

$$\begin{aligned} B &= \int_{\mathbb{R}} v_x (1-\partial_x^2)^{-1}\partial_x^2(u^2) g + \int_{\mathbb{R}} v_x h_x g' \\ &= -\int_{\mathbb{R}} v_x u^2 g + \int_{\mathbb{R}} v_x h g + \int_{\mathbb{R}} v_x h_x g' \ . \end{aligned}$$

Gathering the above identities, (4.38) follows.

6.2. **Proof of Lemma 4.6.** According to (6.1) it holds

$$v = (4-\partial_x^2)^{-1}(1-\partial_x^2)^{-1} y = \frac{1}{3} u - \frac{1}{3}(4-\partial_x^2)^{-1} y \tag{6.3}$$

which proves that $v \le \frac{1}{3} u$ since $y \ge 0$. On the other hand, (6.1) also leads to

$$\begin{aligned} 6v - u &= (1-\partial_x^2)^{-1} y - 2(4-\partial_x^2)^{-1} y \\ &= \frac{1}{2} e^{-|\cdot|} * y - \frac{1}{2} e^{-2|\cdot|} * y \\ &= \frac{1}{2}(e^{-|\cdot|} - e^{-2|\cdot|}) * y \ge 0 \end{aligned}$$

which proves that $u \le 6v$. Now, the identities

$$v(x) = \frac{e^{-2x}}{4} \int_{-\infty}^{x} e^{2x'} u(x') dx' + \frac{e^{2x}}{4} \int_{x}^{+\infty} e^{-2x'} u(x') dx'$$

and

$$v_x(x) = -\frac{e^{-2x}}{2} \int_{-\infty}^{x} e^{2x'} u(x') dx' + \frac{e^{2x}}{2} \int_{x}^{+\infty} e^{-2x'} u(x') dx' \ ,$$

ensure that $|v_x| \le 2v$. Finally, combining the previous estimates with $v_{xx} = 4v - u$, we eventually get that $|v_{xx}| \le \frac{4}{3} u$.

6.3. **Proof of Lemma 4.2.** Let us first notice that $\Psi(-\cdot) = 1 - \Psi$ on $\mathbb{R}$, Ψ' is a positive even function and that there exists $C > 0$ such that $\forall x \le 0$,

$$|\Psi(x)| + |\Psi'(x)| \le C \exp(x/6) \ . \tag{6.4}$$

Moreover, by direct calculations, it is easy to check that

$$|\Psi'''| \le \frac{1}{2} \Psi' \text{ on } \mathbb{R} \tag{6.5}$$

and that

$$\Psi'(x) \ge \Psi'(2) = \frac{1}{3\pi} \frac{e^{1/3}}{1+e^{2/3}}, \quad \forall x \in [0,2] \ . \tag{6.6}$$

We first approximate $u(t_0)$ by the sequence of smooth functions $u_{0,n} = \rho_n * u(t_0)$, with $\{\rho_n\}$ defined in (2.4), that belongs to $H^\infty(\mathbb{R}) \cap Y_+$ and converges to $u(t_0)$ in Y. According to Propositions 2.1 and 2.2, the sequence of solutions $\{u_n\}$ to (1.13) with $u_n(t_0) = u_{0,n}$ belongs to $C(\mathbb{R}; H^\infty(\mathbb{R}))$ and for any fixed $T > 0$ it holds

$$u_n \to u \text{ in } C([t_0 - T, t_0 + T]; H^1) \tag{6.7}$$

$$v_n \to v \text{ in } C([t_0 - T, t_0 + T]; H^3) \tag{6.8}$$

$$y_n \rightharpoonup * \ y \text{ in } C_{ti}(]t_0 - T, t_0 + T[; \mathcal{M}) \tag{6.9}$$

where $v_n=(4-\partial_x^2)^{-1}u_n$ and $y_n=u_n-\partial_x^2 u_n$. In particular, for any fixed $T>0$, there exists $n_0=n_0(T)\geq 0$ such that for any $n\geq n_0$,

$$\|u-u_n\|_{L^\infty(]t_0-T,t_0+T[\times\mathbb{R})}<\frac{(1-\alpha)c_0}{2^6}\,,$$

which together with (4.4) force

$$\sup_{t\in]t_0-T,t_0+T[}\|u_n(t)\|_{L^\infty(|x-x(t)|>R_0)}<\frac{(1-\alpha)c_0}{2^5}\,. \tag{6.10}$$

At this stage it is worth noticing that (4.39) then ensure that it also holds

$$\sup_{t\in]t_0-T,t_0+T[}\|v_n(t)+|v_x(t)|\|_{L^\infty(|x-x(t)|>R_0)}<\frac{(1-\alpha)c_0}{2^5}\,. \tag{6.11}$$

We first prove that (4.7) holds on $[t_0-T,t_0]$ with u replaced by u_n for $n\geq n_0$. The following computations hold for u_n with $n\geq n_0$ but , to simplify the notation, we drop the index n. For any function $g\in C^1(\mathbb{R})$ it is not too hard to check that (1.5) with $b=3$ leads to

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} yg\,dx &= -\int_{\mathbb{R}}\partial_x(yu)g-2\int_{\mathbb{R}} yu_x g\\ &= \int_{\mathbb{R}} yug'-2\int_{\mathbb{R}}(u-u_{xx})u_x g\\ &= \int_{\mathbb{R}} yug'+\int_{\mathbb{R}}(u^2-u_x^2)g' \end{aligned}\tag{6.12}$$

Applying (4.38) and (6.12) with $g(t,x)=\Psi(x-z_{t_0}^R(t))$ we get

$$\begin{aligned}\frac{d}{dt}I_{t_0}^{+R}(t) &= -\dot z(t)\int_{\mathbb{R}}\Psi'\Big[4v^2+5v_x^2+v_{xx}^2+\gamma y\Big]+\gamma\int_{\mathbb{R}}(u^2-u_x^2)\Psi'\\ &\quad+\int_{\mathbb{R}} u\,(\frac{2}{3}u^2-4uv+\gamma y)\Psi'+5\int_{\mathbb{R}} vh\Psi'+\int_{\mathbb{R}} v_x h_x\Psi'\\ &\leq -\dot z(t)\int_{\mathbb{R}}\Psi'\Big[4v^2+5v_x^2+v_{xx}^2+\gamma y\Big]+\gamma\int_{\mathbb{R}} u^2\Psi'\\ &\quad+\int_{\mathbb{R}} u\,(\frac{2}{3}u^2-4uv+\gamma y)\Psi'+\int_{\mathbb{R}}(5v+|v_x|)h\Psi'\\ &\leq -\dot z(t)\int_{\mathbb{R}}\Psi'\Big[4v^2+5v_x^2+v_{xx}^2+\gamma y\Big]+\gamma\int_{\mathbb{R}} u^2\Psi'+J_1+J_2 \end{aligned}\tag{6.13}$$

where from the first to the second step we used that $\Psi'\geq 0$ and that $h=(1-\partial_x^2)^{-1}u^2$ ensures that $|h_x|\leq h$(see the proof of (2.3)).
We observe that

$$\int_{\mathbb{R}}(u^2-u_x^2)\Psi'\leq\int_{\mathbb{R}} u^2\Psi'=\int_{\mathbb{R}}(4v-v_{xx})^2\Psi'\leq 2\int_{\mathbb{R}}(16v^2+v_{xx}^2)\Psi'\,. \tag{6.14}$$

so that, for $0\leq\gamma\leq\frac{1}{8}(1-\alpha)c_0$, it holds

$$-\dot z(t)\int_{\mathbb{R}}\Psi'\Big[4v^2+5v_x^2+v_{xx}^2+\gamma y\Big]+\gamma\int_{\mathbb{R}}(u^2-u_x^2)\Psi'\leq-\frac{\dot z(t)}{2}\int_{\mathbb{R}}\Psi'\Big[4v^2+5v_x^2+v_{xx}^2+\gamma y\Big]$$

To estimate J_1 we divide $\mathbb{R}$ into two regions relating to the size of $|u|$ as follows

$$\begin{aligned} J_1(t) &= \int_{|x-x(t)|<R_0} u\,(\frac{2}{3}u^2-4uv+\gamma y)\Psi' + \int_{|x-x(t)|>R_0} u\,(\frac{2}{3}u^2-4uv+\gamma y)\Psi' \\ &= J_{11}+J_{12} \quad . \end{aligned} \tag{6.15}$$

Observe that (4.5) ensures that $\dot{x}(t)-\dot{z}(t)\ge \beta c_0$ for all $t\in\mathbb{R}$ and thus, for $|x-x(t)|<R_0$,

$$x-z_{t_0}^R(t)=x-x(t)-R+(x(t)-z(t))-(x(t_0)-z(t_0))\le R_0-R-\beta c_0(t_0-t) \tag{6.16}$$

and thus the decay properties of Ψ' lead to

$$\begin{aligned} J_{11}(t) &\lesssim \Bigl[\|u(t)\|_{L^\infty}(\|u(t)\|_{L^2}^2+\|v\|_{L^2}^2+c_0\|y(t)\|_{L^1})\Bigr]e^{R_0/6}e^{-R/6}e^{-\frac{\beta}{6}c_0(t_0-t)} \\ &\lesssim \|u_0\|_{\mathcal{H}}(\|u_0\|_{\mathcal{H}}^2+c_0\|y_0\|_{L^1})e^{R_0/6}e^{-R/6}e^{-\frac{\beta}{6}c_0(t_0-t)} \quad . \end{aligned} \tag{6.17}$$

where we used that $\|v\|_{L^2}\lesssim\|u\|_{\mathcal{H}}$ and that $u-u_{xx}\ge 0$ ensures that

$$\|u\|_{L^\infty}^2\le\|u\|_{H^1}^2\le 2\|u\|_{L^2}^2\lesssim\|u\|_{\mathcal{H}}^2 \; .$$

On the other hand, (6.10) ,Young's inequality and (6.14) lead for all $t\in[t_0-T,t_0]$ to

$$\begin{aligned} J_{12} &\le 4\|u\|_{L^\infty(|x-x(t)|>R_0)}\int_{|x-x(t)|>R_0}(u^2+v^2+\gamma y)\Psi' \\ &\le \frac{(1-\alpha)c_0}{8}\int_{|x-x(t)|>R_0}\Bigl[4v^2+5v_x^2+v_{xx}^2+\gamma y\Bigr]\Psi' \quad . \end{aligned} \tag{6.18}$$

It thus remains to estimate $J_2(t)$. For this, we decompose again $\mathbb{R}$ into two regions relating to the size of $\max(v,|v_x|)$. First proceeding as in (6.17) we easily check that

$$\begin{aligned} &\int_{|x-x(t)|<R_0}(5v+|v_x|)\Psi'(1-\partial_x^2)^{-1}(u^2) \\ &\le \frac{5}{2}\|v+|v_x|\|_{L^\infty}\sup_{|x-x(t)|<R_0}|\Psi'(x-z_{t_0}^R(t))|\int_{\mathbb{R}}e^{-|x|}*u^2\,dx \\ &\le C\|u_0\|_{\mathcal{H}}^3\,e^{R_0/6}e^{-R/6}e^{-\frac{\beta}{6}c_0(t-t_0)} \end{aligned} \tag{6.19}$$

since $\|v+|v_x|\|_{L^\infty}\lesssim\|v\|_{H^2}\lesssim\|u\|_{\mathcal{H}}$ and

$$\forall f\in L^1(\mathbb{R}),\quad (1-\partial_x^2)^{-1}f=\frac{1}{2}e^{-|x|}*f \quad . \tag{6.20}$$

Now in the region $|x-x(t)|>R_0$, noticing that Ψ' and u^2 are non-negative, we get

$$\begin{aligned} &\int_{|x-x(t)|>R_0}(5v+|v_x|)\Psi'(1-\partial_x^2)^{-1}(u^2) \\ \le\;& 5\|v(t)+|v_x(t)|\|_{L^\infty(|x-x(t)|>R_0)}\int_{|x-x(t)|>R_0}\Psi'((1-\partial_x^2)^{-1}(u^2) \\ \le\;& 5\|v(t)+|v_x(t)|\|_{L^\infty(|x-x(t)|>R_0)}\int_{\mathbb{R}}(u^2)(1-\partial_x^2)^{-1}\Psi' \end{aligned} \tag{6.21}$$

On the other hand, from (6.5) and (6.20) we infer that

$$(1-\partial_x^2)\Psi'\ge\frac{1}{2}\Psi'\Rightarrow(1-\partial_x^2)^{-1}\Psi'\le 2\Psi' \; .$$

Therefore, on account of (6.11) and (6.14),

$$\begin{aligned}&\int_{|x-x(t)|>R_0}(5v+|v_x|)\Psi'(1-\partial_x^2)^{-1}(u^2)\\&\le 10\|v(t)+|v_x(t)|\|_{L^\infty(|x-x(t)|>R_0)}\int_{\mathbb{R}}u^2\Psi'\\&\le \frac{(1-\alpha)c_0}{8}\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi' \qquad (6.22)\end{aligned}$$

Gathering (6.15), (6.17), (6.18), (6.19) and (6.22) we conclude that there exists C only depending on R_0, M(u) and $\mathcal{H}(u)$ such that for $R\ge R_0$ and $t\in[-T+t_0,t_0]$ it holds

$$\frac{d}{dt}I^{+R}_{t_0}(t)\le Ce^{-R/6}e^{-\frac{\beta}{6}(t_0-t)}\ . \qquad (6.23)$$

Integrating between t and t_0 we obtain (4.7) for any $t\in[t_0-T,t_0]$ and u replaced by u_n with $n\ge n_0$. Note that the constant appearing in front of the exponential now also depends on β. The convergence results (6.7)-(6.9) then ensure that (4.7) holds also for u and $t\in[t_0-T,t_0]$ and the result for $t\le t_0$ follows since $T>0$ is arbitrary. Finally, (4.8) can be proven in exactly the same way by noticing that for $|x-x(t)|<R_0$ it holds

(6.24)

$$x-z^{-R}_{t_0}(t)=x-x(t)+R+(x(t)-z(t))-(x(t_0)-z(t_0))\ge -R_0+R+\beta c_0(t-t_0)\ .$$

□

6.4. **Proof of Proposition 4.1.** First, as explained in Remark 1.1, the Y-almost localization of u implies that u is H^1-almost localized and since $v=(4-\partial_x^2)^{-1}u$ the same type arguments show that v is H^3-almost localized. Therefore, it is clear that u satisfies the hypotheses of Lemma 4.2 for $\alpha=1/3$ and $R_0>0$ big enough. We fix $\alpha=1/3$ and take $\beta=1/3$, $\gamma=\frac{c_0}{12}$ and $z(\cdot)=\frac{2}{3}x(\cdot)$ which clearly satisfy (4.5). Let us show that $I^{+R}_{t_0}(t)\underset{t\to-\infty}{\longrightarrow}0$ which together with (4.7) will clearly lead to

$$I^{+R}_{t_0}(t_0)\le Ce^{-R/6}\quad . \qquad (6.25)$$

For $R_\varepsilon>0$ to be specified later we decompose $I^{+R}_{t_0}$ into

$$\begin{aligned}I^{+R}_{t_0}(t)&=\Big\langle 4v^2(t)+5v_x^2(t)+v_{xx}^2(t)+\frac{c_0}{12}y(t),\Psi(\cdot-z^R_{t_0}(t))\Big(1-\phi(\frac{\cdot-x(t)}{R_\varepsilon})\Big)\Big\rangle\\&\quad+\Big\langle 4v^2(t)+5v_x^2(t)+v_{xx}^2(t)+\frac{c_0}{12}y(t),\Psi(\cdot-z^R_{t_0}(t))\phi(\frac{\cdot-x(t)}{R_\varepsilon})\Big\rangle\\&=I_1(t)+I_2(t)\quad .\end{aligned}$$

where $\phi\in C^\infty(\mathbb{R})$ is supported in $[-1,1]$ with $0\le\phi\le1$ on $[-1,1]$ and $\phi\equiv1$ on $[-1/2,1/2]$. From the Y-almost localization of u and the $H^2(\mathbb{R})$-almost localization of v, for any $\varepsilon>0$ there exists $R_\varepsilon>0$ such that $I_1(t)\le\varepsilon/2$. On the other hand, we observe that

$$I_2(t)\le(\|u_0\|^2_{\mathcal{H}}+c_0\|y_0\|_{\mathcal{M}})\Psi\Big(R_\varepsilon-R-\frac{1}{3}(x(t_0)-x(t))\Big)\quad .$$

But $\dot{x} > c_0 > 0$ obviously imply that, for $|x - x(t)| \le R_\varepsilon$,

$$x - z_{t_0}^{+R}(t) = x - x(t) - R - \frac{1}{3}(x(t_0) - x(t)) \le R_\varepsilon - R - \frac{1}{3}c_0(t_0 - t) \underset{t\to-\infty}{\longrightarrow} -\infty$$

which proves our claim since $\lim\limits_{x\to-\infty} \Psi(x) = 0$.
It follows from (6.25) that for all $t \in \mathbb{R}$, all $x_0 > 0$ and all $\Phi \in C(\mathbb{R})$ with $0 \le \Phi \le 1$ and $\operatorname{supp} \Phi \subset [x_0, +\infty[$.

$$\int_{\mathbb{R}} (4v^2(t) + 5v_x^2(t) + v_{xx}^2(t))\Phi(\cdot - x(t))\,dx + \frac{c_0}{12}\Big\langle \Phi(\cdot - x(t)), y(t) \Big\rangle \le C \exp(-x_0/6)\ .$$

The invariance of (C-H) under the transformation $(t, x) \mapsto (-t, -x)$ yields the result for $\operatorname{supp}\Phi \subset]-\infty, -x_0]$. Finally, the identity $u = (4 - \partial_x^2)v$ together with (2.3) ensure that

$$\int_{\mathbb{R}} (u^2(t) + u_x^2(t))\Phi(\cdot - x(t))\,dx \le C' \exp(-x_0/6)\ .$$

and the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ enables to conclude that u is uniformly exponentially decaying.

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