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### **► To cite this version:**

Ayman Chouayakh, Aurelien Bechler, Isabel Amigo, Loutfi Nuaymi, Patrick Maillé. Auction mechanisms for Licensed Shared Access: reserve prices and revenue-fairness tradeoffs. IFIP WG PERFORMANCE 2018 - 36th International Symposium on Computer Performance, Modeling, Measurements and Evaluation 2018, Dec 2018, Toulouse, France. hal-01883697

**HAL Id: hal-01883697**

**<https://hal.archives-ouvertes.fr/hal-01883697>**

Submitted on 28 Sep 2018

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# Auction mechanisms for Licensed Shared Access: reserve prices and revenue-fairness tradeoffs

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## ABSTRACT

Licensed shared access (LSA) is a new approach that allows Mobile Network Operators to use a portion of the spectrum initially licensed to another incumbent user, by obtaining a license from the regulator via an auction mechanism. In this context, different truthful auction mechanisms have been proposed, and differ in terms of allocation (who gets the spectrum) but also on revenue. Since those mechanisms could generate an extremely low revenue, we extend them by introducing a reserve price per bidder which represents the minimum amount that each winning bidder should pay. Since this may be at the expense of the allocation fairness, for each mechanism we find by simulation the reserve price that optimizes a trade-off between expected fairness and expected revenue. Also, for each mechanism, we analytically express the expected revenue when valuations of operators for the spectrum are independent and identically distributed from a uniform distribution.

## 1 INTRODUCTION

Mobile Internet traffic continues to increase exponentially. By 2020, there will be nearly eight times more mobile Internet traffic than in 2016 [1]. To satisfy that growth of data traffic through a more efficient usage of the radio spectrum, spectrum sharing has been proposed. Traditionally, spectrum sharing refers to the situation where a secondary user like a Mobile Network Operator (MNO) uses the bandwidth of a primary user and has to release it whenever the primary user wants it; TV White Space spectrum sharing [5] is an example where MNOs are able to use TV bands without obtaining a license. However, the sharing duration is not defined and the access to the bandwidth is not guaranteed, which is not desirable from the point of view of MNOs. To solve those problems, in November 2011, the Radio Spectrum Policy group (RSPG) proposed a new sharing concept called Licensed Shared Access (LSA) [2]. LSA is a new approach, which is technically achievable [11], that allows MNOs to obtain access to additional bands of an incumbent. This sharing is supervised by the regulator and can be carried out on specified frequencies of the licensed spectrum, time periods, and geographical regions [10].

In that context, allocating LSA spectrum to MNOs via auction mechanisms is a natural approach. An auction mechanism is composed by a bidding format, an allocation rule and a payment rule. In general, an allocation rule  $\alpha$  is a function of the bids, it indicates

how much resource each bidder gets. A payment rule  $P$  indicates how much each player has to pay. A desirable propriety of an auction mechanism is *truthfulness*: a truthful or incentive compatible mechanism incentivizes all bidders to voluntarily reveal their true valuation, hence preventing market manipulation through insincere bids. Some truthful [3, 4, 16] and non-truthful [13] mechanisms have been proposed as candidates to allocate spectrum in the LSA context. Other mechanisms, such as the well-known Vickrey-Clarke-Groves (VCG) mechanism [8], which is also truthful, can be adapted to the LSA context.

While truthfulness is important, the revenue generated by the auction is also an important criterion. In this paper, we propose to ameliorate the revenue of those mechanisms while preserving truthfulness. The idea is to introduce a reserve price  $R$  per bidder such that no winning player pays a unit price below  $R$ , an approach initially proposed by Hartline and Roughgarden [6], and differing from the more classical seller-oriented reserve price [8]. Introducing this reserve price may exclude some MNOs, which is not desirable from the point of view of the allocation's fairness. Hence we propose a trade-off between those two metrics, that we model and analyze in this paper, in order to suggest which mechanism to use and with which reserve price  $R$ .

The rest of the paper is organized as follows: Section 2 presents the system model. In Section 3, we present the LSA candidate mechanisms. The extension of those mechanisms to include a reserve price is given in Section 4, and Section 5 provides an analytical analysis of the corresponding average revenues. Results on the trade-off between fairness and revenue are shown in Section 6, and we conclude and suggest directions for future work in Section 7.

## 2 SYSTEM MODEL

### 2.1 Interference and spectrum reusability

Each base station acts as a player, that is, an operator wishing to use some LSA spectrum. We consider  $N$  base stations of different operators, so we will use the terms operator, base station and bidder interchangeably in the paper. In reality, each base station is not necessarily in direct competition with all the others: when two base stations do not interfere, they can use the same LSA spectrum simultaneously. Hence, a well-designed spectrum mechanism has to take spectrum re-usability in consideration to make the most out of the spectrum.

A way to exploit the re-usability is to transform the competition between the  $N$  base stations into a competition between  $M$  groups, in such a way that any two base stations in the same group do not interfere: one can then allocate the same spectrum to all the base stations of the same group. That same approach is taken in [3, 4, 13, 16]. It can be captured in a model by using a so-called interference graph. Figure 1 shows an example of an interference graph built from the overlapping of the different coverage areas: base stations are represented by vertices, an edge between two vertices meaning that those base stations interfere. In our example, base stations in the set  $\{1,3,5\}$  can use the same spectrum simultaneously. An example of groups constitution for the instance of Figure 1 is:  $g_1 = \{1,3,5\}$ ,  $g_2 = \{2,5\}$  and  $g_3 = \{1,4\}$ . Notice that groups can be formed in different ways. In this paper, we suppose that groups are formed by the auctioneer before the actual auction and are advertised to bidders before any bids are submitted. Additionally, since the mechanisms in [3, 16] are truthful only when each base station belongs to only one group, we assume the groups formed by the auctioneer satisfy that constraint.

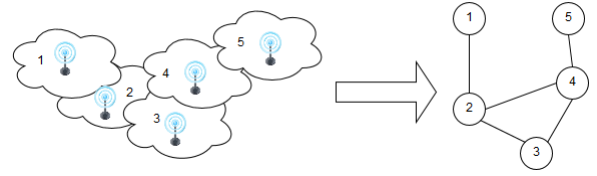


Figure 1: Interference graph example

## 2.2 Players preferences

We suppose that each bidder (operator)  $i = 1, \dots, N$  has a constant marginal valuation  $v_i$  for spectrum and a quasilinear utility function: for a given mechanism Mec, if it obtains a fraction  $\alpha_i^{\text{Mec}} > 0$  of all the available bandwidth and pays  $p_i^{\text{Mec}}$ ,  $i$ 's utility then is:

$$u_i(\alpha_i^{\text{Mec}}, p_i^{\text{Mec}}) = \alpha_i^{\text{Mec}} v_i - p_i^{\text{Mec}}.$$

Otherwise its utility is zero. Notice that we have assumed indistinguishable channel properties [14, 15], i.e., operators are only sensitive to the amount of bandwidth –and not to the specific bands– they can use.

## 2.3 Fairness and regulator's utility

The utility of the regulator depends on the revenue of the mechanism  $\text{Rev}^{\text{Mec}}$ , which is equal to  $\sum_{i=1}^N p_i^{\text{Mec}}$ , but also on the fairness of the allocation. That second criterion needs to be quantified: several definitions are used to quantify fairness [7, 9], we decide to use Jain's fairness index  $J$ , which is given for an allocation vector  $\alpha$ , by

$$J(\alpha) = \frac{(\sum_{i=1}^N \alpha_i)^2}{N \sum_{i=1}^N \alpha_i^2}. \quad (1)$$

This index is a continuous function of the allocation, and measures its equity: if for any two base stations  $i$  and  $j$ ,  $\alpha_i = \alpha_j$  then  $J$  is maximum and equal to 1, and if the bandwidth is allocated to only one base station then  $J$  is minimum and equal to  $\frac{1}{N}$ . Note that in our case, we can have  $\sum_i \alpha_i > 1$  due to spectrum being possibly used by several non-interfering base stations.

In this paper, we assume the regulator is sensitive both to the revenue from the auction and the allocation fairness. More specifically, we suppose that, given a mechanism, the normalized utility of the regulator  $U_{\text{Reg}}^{\text{Mec}}$  is of the form

$$U_{\text{Reg}}^{\text{Mec}} = \beta J(\alpha^{\text{Mec}}) + (1 - \beta) \frac{\text{Rev}^{\text{Mec}}}{\text{Rev}^{\text{max}}}, \quad (2)$$

where  $\beta \in [0, 1]$  is the weight that the regulator puts on fairness relative to revenue, and  $\text{Rev}^{\text{max}}$  is the maximum revenue over the set of candidates mechanisms that we use to normalize the revenue criterion in (2).

## 3 CANDIDATE LSA AUCTION MECHANISMS

In this section, we briefly review the auction mechanisms that have been proposed in the context of LSA spectrum allocation, and that we modify (adding a per-bidder reserve price) and compare in this paper.

LSAA [13] was the first auction mechanism proposed specifically for the LSA context. To evaluate the performance of that mechanism in terms of revenue, its authors compare LSAA with TAMES [3] and TRUST [16], two other applicable auction schemes. The classical VCG [8] scheme can be applied to LSA. Finally, another mechanism called Proportional Allocation Mechanism (PAM) [4] has been recently proposed as a candidate mechanism for LSA spectrum allocation and pricing. Contrary to the previous mechanisms which allocate the whole bandwidth to one and only one group, PAM divides the bandwidth among groups in proportion to their *groupbids* (a value summarizing the bids of a group). In addition to allocation fairness and revenue, some interesting properties of an auction mechanism include:

- **Truthfulness:** For every bidder  $i$  and every fixed set of bids from the other bidders, proposing a bid  $b_i = v_i$  maximizes  $i$ 's utility. Truthfulness ensures that operators will not bid strategically, since their best option is simply to reveal their true valuation.
- **Individual rationality:** Every bidder has an interest to participate in the auction implying that truthful bidders are guaranteed non-negative utility by the mechanism.

We are interested in TAMES, TRUST, VCG and PAM because they verify the previous two proprieties (while LSAA is not truthful). Following [6], we extend the previous mechanisms by introducing a reserve price per bidder  $R$ . Notice that PAM contains  $R$  by definition. In the following, we explain those mechanisms, before introducing the reserve price  $R$ . All used notations are summarized in Table 1.

$R$	reserve price per bidder, set by the auctioneer
$M$	number of groups
$N$	total number of players (operators)
$g_k$	set of players in group $k$
$n_k$	number of players in group $k$
$v_i$	true valuation of player $i$
$b_i$	bid of player $i$
$b_{(j)}^k$	$j$ th minimum bid within group $k$
$(b_{(j)}^k)^{-i}$	$j$ th minimum bid within group $k$ excluding $i$
$b_{-i}$	bids of all players except $i$
$B_{\text{Tot}}$	sum of all bids of all groups
$B_g^{-i}$	sum of bids of the group to which $i$ belongs to, ignoring $i$ 's bid
$B_{\text{Tot}}^{-i}$	sum of the total bids of all groups ignoring $i$ 's bid

**Table 1: Notations**

### 3.1 TAMES

TAMES [3] defines the groupbid of group  $k$  as:  $b_{(1)}^k (n_k - 1)$ . Then players of the group with the highest groupbid are winners (i.e., each one can use the whole auctioned spectrum), except the player with the lowest bid  $b_{(1)}^{\text{win}}$ , where  $\text{win}$  is the winning group. Each winning player pays that price  $b_{(1)}^{\text{win}}$ .

### 3.2 TRUST

TRUST [16] computes the groupbid of group  $k$  as:  $b_{(1)}^k n_k$ . All players of the group with the highest groupbid are winners (they can use the whole spectrum) and each one pays  $\frac{B_{\text{second}}^{\text{TRUST}}}{n_{\text{win}}}$ , where  $B_{\text{second}}^{\text{TRUST}}$  denotes the second-highest groupbid and  $n_{\text{win}}$  the cardinal of the winning group.

### 3.3 VCG

VCG [8] computes the groupbid of group  $k$  as  $\sum_{i=1}^N b_i \mathbb{1}_{i \in g_k}$  and allocates the whole spectrum to the group  $g_{\text{win}}$  with the highest groupbid. Players should pay the ‘‘damage’’ in term of efficiency they impose i.e., each player pays his/her ‘‘social cost’’ (how much her presence hurts the others). We denote by  $B_{\text{win}}^{\text{VCG}}$  the groupbid of the winning group and by  $B_{\text{second}}^{\text{VCG}}$  the second highest groupbid. If a player belongs to a losing group, she pays 0 because whether being present or not the winning group is unchanged. If a player belongs to the winning group then we can distinguish two cases: if her presence does not change the outcome i.e.,  $B_{\text{win}}^{\text{VCG}} - b_i \geq B_{\text{second}}^{\text{VCG}}$  then he/she pays 0, otherwise he/she pays  $B_{\text{second}}^{\text{VCG}} - (B_{\text{win}}^{\text{VCG}} - b_i)$ . To summarize, the price paid by player  $i$  submitting bid  $b_i$  is given by:

$$p_i^{\text{VCG}} = [B_{\text{second}}^{\text{VCG}} - (B_{\text{win}}^{\text{VCG}} - b_i)]^+ \mathbb{1}_{i \in g_{\text{win}}}. \quad (3)$$

### 3.4 Proportional Allocation Mechanism (PAM)

To each group  $k$ , PAM [4] allocates a fraction  $\alpha_k$  of the bandwidth in proportion to the bids submitted by players belonging to that group

i.e.,  $\alpha_i = \frac{\sum_{i=1}^N b_i \mathbb{1}_{i \in g_k}}{B_{\text{Tot}}}$  and each player pays an amount computed to

ensure incentive compatibility, given by [4]:

$$p_i^{\text{PAM}} = \frac{b_i + B_g^{-i}}{b_i + B_{\text{Tot}}^{-i}} R + \left( B_{\text{Tot}}^{-i} - B_g^{-i} \right) \left( \ln \frac{b_i + B_{\text{Tot}}^{-i}}{R + B_{\text{Tot}}^{-i}} + \frac{R + B_{\text{Tot}}^{-i}}{b_i + B_{\text{Tot}}^{-i}} - 1 \right). \quad (4)$$

### 3.5 Introducing a per-bidder reserve price

Without introducing a reserve price, all those mechanisms may generate an extremely low revenue. For TAMES and TRUST, if the minimum valuation in each group is low then the revenue will be low. For VCG, suppose we have two groups such that the first group is composed by two players with valuations respectively 2 and 3 and the second group is composed by one player with valuation equal to 1; then in this situation group one wins the auction and each player pays zero. Traditionally, to avoid those situations, the seller fixes a reserve price in such a way that his revenue will be at least that fixed amount which will be paid by the winning group. This is usually simple to implement: the seller submits a bid on its own, whose value is the reserve price. Then, mechanisms are unmodified and allocate the resource to the seller if the groupbids are below the reserve price, which with classical mechanisms yields the wanted property, i.e., a selling price below the reserve price. But in our case, that method does not work. We illustrate that by the following example.

Consider a situation of two groups composed by four and one players respectively, with bids  $\{1, 1, 1, 1\}$  and  $\{2\}$ , for which we apply VCG with a reserve price of 2. If we directly apply VCG with an extra bid (from the seller) of value 2, then group one is the winning group and each player of group one pays zero, hence a revenue lower than the reserve price. On the other hand, if we force players of group one to pay 2 altogether then each player has to pay 0.5 since bids are equal. However, for each player proposing a bid lower than 0.5 leads to a strictly higher utility, hence some incentive issues that arise. To summarize, introducing a seller-centered reserve price is not easily doable in our context.

Hence we prefer to introduce a reserve price *per bidder*, that is, a minimum unit price that each winner will pay. Notice that after introducing the reserve price per bidder, there is no guarantee on the seller revenue since the number of users paying that reserve price is unknown a priori, but getting some guarantees on what each individual winner will pay can also be desirable from the regulator point of view, since it reflects the seriousness of the candidates for spectrum usage. Note also that such a per-bidder reserve price has already been proposed, for auctions in other contexts [6].

## 4 ENHANCED MECHANISMS WITH RESERVE PRICES

In this section, we explain how to introduce a reserve price  $R$  per bidder in each mechanism as explained in the previous subsection. We then prove that all the mechanisms keep their incentive properties.

Note that the per-bidder reserve price was already included in the construction of PAM, so in this section we focus on the three other candidate mechanisms, namely TRUST, TAMES, and VCG.

#### 4.1 Implementation of the per-bidder reserve price

We propose here a generic way to modify the existing mechanisms, so as to take into account a per-bidder reserve price  $R$  set by the auctioneer.

For each mechanism, we apply two changes with respect to the initial version:

- each bidder with bid below  $R$  is simply ignored;
- we then apply the mechanism allocation and payment rules on the remaining bidders, but possibly affect the unit price by taking the maximum of  $R$  and the one given by the mechanism.

In the rest of the paper, we use a “bar” for the notations in Table 1 to represent the first modification, i.e., the removal of bidders with bids below  $R$ . As an example,  $\bar{g}_k$  denotes  $g_k$  without bids below  $R$ . Expressing mathematically the second change, we can then write that winning player pays a unit price equal to  $\max\{R, p_i^{\text{Mec}}\}$ .

#### 4.2 Incentive properties of the enhanced mechanisms

In this subsection, we prove that the modified mechanisms maintain their incentive properties.

**PROPOSITION 4.1.** *The modified version of TAMES with any per-bidder reserve price  $R$  is still incentive compatible*

**PROOF.** We consider a player  $i$  in a group  $k$ , and distinguish two cases:

*Case 1:  $v_i < R$ .* Player  $i$  cannot do better than bidding truthfully (and not getting any resource) since she is sure to be charged more than she is willing to pay if she obtained some resource.

*Case 2:  $v_i \geq R$ .*

- If  $v_i > \overline{(b_{(1)}^k)^{-i}}$ :
  - If  $\overline{(b_{(1)}^k)^{-i}(\bar{n}_k - 1)} > \overline{B_{\text{second}}^{\text{TAMES}}}$ : bidding truthfully leads to a utility equal to  $v_i - \overline{(b_{(1)}^k)^{-i}} > 0$ , while any other bid  $b_i > \overline{\frac{B_{\text{second}}^{\text{TAMES}}}{(\bar{n}_k - 1)}}$  leads to the same utility and any bid  $b_i \leq \overline{\frac{B_{\text{second}}^{\text{TAMES}}}{(\bar{n}_k - 1)}}$  leads to a null utility.
  - $\overline{(b_{(1)}^k)^{-i}(\bar{n}_k - 1)} > \overline{B_{\text{second}}^{\text{TAMES}}}$ : bidding truthfully leads to a null utility because group  $k$  loses the auction. Player  $i$  could not change the outcome by proposing  $b_i \geq \overline{(b_{(1)}^k)^{-i}}$  because the groupbid is still the same, on the other hand, if she proposes a bid lower than  $\overline{(b_{(1)}^k)^{-i}}$  then the groupbid will be lower than the previous one. Hence, in this situation any bid results in a utility equal to zero.
- if  $v_i < \overline{(b_{(1)}^k)^{-i}}$ : bidding truthfully leads to a null utility. We can distinguish the following cases:
  - by proposing a bid  $b_i < \overline{(b_{(1)}^k)^{-i}}$  player  $i$  is still a losing player.
  - by proposing a bid  $b_i > \overline{(b_{(1)}^k)^{-i}}$ , group  $k$  may win the auction, however player  $i$  will pay  $\overline{(b_{(1)}^k)^{-i}}$  leading to a negative utility.

Hence, in all possible cases, bidding truthfully maximizes the utility.  $\square$

**PROPOSITION 4.2.** *The modified version of TRUST with any per-bidder reserve price  $R$  is still incentive compatible.*

**PROOF.** The proof follows steps similar to the one for TAMES, the details are given in Appendix A.  $\square$

**PROPOSITION 4.3.** *The modified version of VCG with any per-bidder reserve price  $R$  is still incentive compatible.*

**PROOF.** We can distinguish two cases:

*Case 1, player with  $v_i < R$ :* this player has no interest to propose a bid  $\geq R$  because if he wins he will pay at least  $R$  leading to a negative utility.

*Case 2, player with  $v_i \geq R$  who belongs to the group  $k$ :* we can distinguish two cases:

- $\overline{(B_k^{\text{VCG}})^{-i}} > \overline{B_{\text{second}}^{\text{VCG}}}$ : group  $k$  is the winning, any bid  $b_i > R$  (in particular  $b_i = v_i$ ) leads to a strictly positive utility  $v_i - R$
- $\overline{(B_k^{\text{VCG}})^{-i}} < \overline{B_{\text{second}}^{\text{VCG}}}$ 
  - If  $\overline{(B_k^{\text{VCG}})^{-i}} + v_i > \overline{B_{\text{second}}^{\text{VCG}}}$ , by proposing  $b_i = v_i$  player  $i$  wins the auction and pays  $p_i^{\text{VCG}} = \max\{R, \overline{B_{\text{second}}^{\text{VCG}}} - \overline{(B_k^{\text{VCG}})^{-i}}\}$  which leads to a strictly positive utility  $v_i - p_i^{\text{VCG}}$ . Any other bid  $b_i > \overline{B_{\text{second}}^{\text{VCG}}} - \overline{(B_k^{\text{VCG}})^{-i}}$  leads to the same utility and otherwise i.e.,  $b_i \leq \overline{B_{\text{second}}^{\text{VCG}}} - \overline{(B_k^{\text{VCG}})^{-i}}$  player  $i$  loses the auction.
  - $\overline{(B_k^{\text{VCG}})^{-i}} + v_i \leq \overline{B_{\text{second}}^{\text{VCG}}}$  if player  $i$  proposes  $b_i = v_i$  then he loses the auction, he wins the auction if and only if he proposed  $b_i \geq \overline{B_{\text{second}}^{\text{VCG}}} - \overline{(B_k^{\text{VCG}})^{-i}}$  leading to a negative utility.

To conclude, bidding truthfully maximizes player’s utility in all possible cases.  $\square$

## 5 ANALYTICAL EXPRESSION OF AVERAGE REVENUE

In the following, we provide analytical expressions of the average revenues of the mechanisms, under two assumptions:

- Each player belongs to one and only one group.
- Valuations of players are drawn from the uniform distribution on the interval  $[a, b]$ .

### 5.1 TAMES

After introducing a reserve price  $R$ , the groupbid of  $g_k$  under TAMES is

$$\begin{aligned} \overline{B_k^{\text{TAMES}}} &= (n_k - 1)b_{(1)}^k \mathbb{1}_{b_{(1)}^k \geq R^+} \\ &+ \sum_{i=2}^{n_k-1} \mathbb{1}_{(b_{(i-1)}^k < R)} \mathbb{1}_{(b_{(i)}^k \geq R)} (n_k - i)b_{(i)}^k. \end{aligned}$$

We denote by  $\overline{B_{\text{max}}^{\text{TAMES}}} = \max\{\overline{B_1^{\text{TAMES}}}, \dots, \overline{B_M^{\text{TAMES}}}\}$  The revenue of TAMES is equal to  $\overline{B_{\text{max}}^{\text{TAMES}}}$ , Hence the average revenue  $\overline{\text{Rev}}_{\text{TAMES}}$

is given by:

$$\begin{aligned}\overline{\text{Rev}}_{\text{TAMES}} &= \int_0^\infty (1 - \mathbb{P}(\overline{B}_{\max}^{\text{TAMES}} \leq x)) \\ &= \int_0^\infty (1 - \mathbb{P}(B_{\max}^{\text{TAMES}} \leq x)) \\ &= \int_0^\infty (1 - \prod_{i=1}^M \mathbb{P}(B_i^{\text{TAMES}} \leq x)).\end{aligned}$$

Notice that for each  $1 \leq i \leq M$ ,  $\mathbb{P}(B_i^{\text{TAMES}} \leq x)$  is given in Appendix A (12).

$$\mathbb{P}(B_i^{\text{TAMES}} \leq x) = \mathbb{P}(S_2^{n_i} \leq x).$$

## 5.2 TRUST

After introducing  $R$ , the group bid of  $g_k$  under TRUST is

$$\begin{aligned}\overline{B}_k^{\text{TRUST}} &= n_k b_{(1)}^k \mathbb{1}_{b_{(1)}^k \geq R} + \\ &\quad \sum_{i=2}^{n_k} \mathbb{1}_{(b_{(i-1)}^k < R)} \mathbb{1}_{(b_{(i)}^k \geq R)} (n_k - i + 1) b_{(i)}^k.\end{aligned}$$

We denote by  $\overline{B}_{\max}^k = \max\{\overline{B}_1^{\text{TRUST}}, \overline{B}_{k-1}^{\text{TRUST}}, \overline{B}_{k+1}^{\text{TRUST}}, \dots, \overline{B}_M^{\text{TRUST}}\}$ . The winning group which is composed by  $\overline{n}_{\text{win}}$  players will not pay always the second highest group bid. In fact, we can distinguish two cases: if  $R \times \overline{n}_{\text{win}} \geq \overline{B}_{\max}^k$  then each player of the winning group pays  $R$  i.e., the revenue equal to  $R \times \overline{n}_{\text{win}}$  otherwise each player of the winning group pays  $\frac{\overline{B}_{\max}^k}{\overline{n}_{\text{win}}}$  i.e., the revenue equal to  $\overline{B}_{\max}^k$ . Let us compute the payment  $\overline{P}_k^{\text{TRUST}}$  of the group  $k$ .

$$\begin{aligned}\overline{P}_k^{\text{TRUST}} &= \mathbb{1}_{b_{(1)}^k \geq R} \mathbb{1}_{n_k b_{(1)}^k \geq \overline{B}_{\max}^k} (\overline{B}_{\max}^k \mathbb{1}_{\overline{B}_{\max}^k \geq n_k R} + \\ &\quad n_k R \mathbb{1}_{\overline{B}_{\max}^k < n_k R}) + \sum_{i=1}^{n_k-1} \mathbb{1}_{b_{(i)}^k < R} \mathbb{1}_{b_{(i+1)}^k \geq R} \mathbb{1}_{(n_k-i)b_{(i+1)}^k \geq \overline{B}_{\max}^k} \\ &\quad (\overline{B}_{\max}^k \mathbb{1}_{\overline{B}_{\max}^k \geq (n_k-i)R} + (n_k-i)R \mathbb{1}_{\overline{B}_{\max}^k < (n_k-i)R})\end{aligned}$$

Therefore, the average payment  $\overline{P}_k^{\text{TRUST}}$  of the group  $k$  is given by:

$$\begin{aligned}\overline{P}_k^{\text{TRUST}} &= \int_R^b \int_{n_k R}^{n_k b_{(1)}^k} f_{b_{(1)}^k} \overline{B}_{\max}^k \overline{B}_{\max}^k \overline{dB}_{\max}^k db_{(1)}^k + \\ &\quad n_k R \int_R^b \int_0^{n_k R} f_{b_{(1)}^k} \overline{B}_{\max}^k \overline{dB}_{\max}^k db_{(1)}^k + \\ &\quad \sum_{i=1}^{n_k-1} \int_0^R \int_R^b \left( \int_{(n_k-i)R}^{(n_k-i)b_{(i+1)}^k} \overline{B}_{\max}^k f_{\overline{B}_{\max}^k} \overline{dB}_{\max}^k + \right. \\ &\quad \left. \int_0^{(n_k-i)R} (n_k-i)R f_{\overline{B}_{\max}^k} \overline{dB}_{\max}^k \right) f_{b_{(i)}^k} b_{(i+1)}^k db_{(i)}^k db_{(i+1)}^k\end{aligned}$$

Notice that the CDF of  $\overline{B}_{\max}^k$  is given by:

$$\mathbb{P}(\overline{B}_{\max}^k \leq x) = \prod_{i=1, i \neq k}^M \mathbb{P}(B_i^{\text{TRUST}} \leq x) \quad (5)$$

With  $\mathbb{P}(B_i^{\text{TRUST}} \leq x) = \mathbb{P}(S_1^{n_i} \leq x)$  (see Appendix B) and the joint CDF of  $(b_i)^k, b_{i+1}^k$  is given in Appendix B (10) and by replacing  $n$

with  $n_k$ . Hence the average revenue  $\overline{\text{Rev}}_{\text{TRUST}}$  is given by:

$$\overline{\text{Rev}}_{\text{TRUST}} = \sum_{k=1}^M \overline{P}_k^{\text{TRUST}}.$$

## 5.3 VCG

Under VCG the groupbid of a group  $k$  is the sum of bids of its members.  $\overline{B}_k = \sum_{i \in g_m} 1_{b_i \geq R}$ . We denote by  $\overline{M}^{-k} = \max\{\overline{B}_1, \dots, \overline{B}_{k-1}, \overline{B}_{k+1}, \dots, \overline{B}_M\}$ . To win the auction, group  $k$  has to propose a bid  $\overline{B}_k^{\text{VCG}}$  greater than  $\overline{M}^{-k}$ . After introducing the minimum amount, the revenue from a player  $i$  is:

$$\begin{cases} R, & \text{if } \overline{B}_k^{-i} \leq \overline{M}^{-k} \leq R \leq b_i \\ \overline{M}^{-k}, & \text{if } \overline{B}_k^{-i} < R \leq \overline{M}^{-k} < b_i \\ \max\{R, \overline{M}^{-k} - \overline{B}_k^{-i}\}, & \text{if } R \leq \overline{B}_k^{-i}, R < \overline{M}^{-k} < \overline{B}_k^{-i}. \end{cases} \quad (6)$$

Hence the average revenue from a player  $i$  is:

$$\begin{aligned}\overline{p}_i^{\text{VCG}} &= \int_R^b \int_0^R \int_0^R R f_{\overline{M}^{-k}} f_{\overline{B}_k^{-i}} f_{\overline{B}_k^{-i}} d\overline{M}^{-k} d\overline{B}_k^{-i} db_i + \\ &\quad \int_R^b \int_0^R \int_R^{b_i} \overline{M}^{-k} f_{\overline{M}^{-k}} f_{\overline{B}_k^{-i}} f_{\overline{B}_k^{-i}} d\overline{M}^{-k} d\overline{B}_k^{-i} db_i + \\ &\quad \int_R^b \int_R^{(n_m-1)b} \int_{\overline{B}_k^{-i}+R}^{\overline{B}_k^{-i}+b_i} (\overline{M}^{-k} - \overline{B}_k^{-i}) \\ &\quad f_{\overline{M}^{-k}} f_{\overline{B}_k^{-i}} f_{\overline{B}_k^{-i}} d\overline{M}^{-k} d\overline{B}_k^{-i} db_i + \\ &\quad \int_R^b \int_R^{(n_m-1)b} \int_0^{\overline{B}_k^{-i}+R} R \\ &\quad f_{\overline{M}^{-k}} f_{\overline{B}_k^{-i}} f_{\overline{B}_k^{-i}} d\overline{M}^{-k} d\overline{B}_k^{-i} db_i\end{aligned}$$

where:

- $f_{\overline{B}_k^{-i}}$ : PDF of  $\overline{B}_k^{-i}$  which is computed in Appendix C (18) and by replacing  $n$  with  $n_k - 1$
- $f_{\overline{M}^{-k}}$ : PDF of  $\overline{M}^{-k}$  which is given by  $\prod_{i=1, i \neq k}^M f_{n_i}^R$  where  $f_{n_i}^R$  is given in Appendix C (18) and by replacing  $n$  with  $n_i$

Finally  $\overline{\text{Rev}}^{\text{VCG}} = \sum_{i=1}^N \overline{p}_i^{\text{VCG}}$ .

## 5.4 PAM's average revenue

The average payment of a player  $i$  is given by:

$$\overline{p}_i^{\text{PAM}} = \mathbb{E}_{v_1, \dots, v_N} (p_i^{\text{PAM}}) \quad (7)$$

Before computing (7), let us introduce the following notations:

- $f_i$  probability density function of valuation of player  $i$ .
- $F_i$  cumulative density function of valuation of player  $i$ .
- $\varphi_i(v_i)$ : virtual valuation of player  $i$ ,  $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ .

We will use Rougharden's formula [12] for the expected revenue of an auction. This formula can be illustrated as follows: if the allocation rule is monotone and the cumulative density function of

each player  $F_i$  is regular, i.e. the virtual valuation is an increasing function of  $v_i$  then we have

$$\widehat{p}_i^{\text{PAM}} = \mathbb{E}_{v_1, \dots, v_N} \left( (\alpha_i(v_i) \varphi(v_i)) \right) \quad (8)$$

Notice that there is no need to compute the revenue generated from each player: hence players are iid, the average revenue generated by players of the same group is the same.

Let us compute  $\widehat{p}_1^{\text{PAM}}$  the average payment of player 1. Without loss of generality, we suppose that

- Player 1 belongs to  $g_1$ .
- $g_1$  is composed by the first  $n_1$  players.

Using (8) we get

$$\widehat{p}_1^{\text{PAM}} = \frac{1}{(b-a)^N} \int_a^b \underbrace{\varphi(v_1)}_N \frac{v_1 + B_g^{-1}}{v_1 + B_g^{-1} + B_{Tot}^{-1}} \mathbb{1}_{v_1 \geq R} dV \quad (9)$$

Using Appendix D we get:  $\widehat{p}_1^{\text{PAM}} = \frac{1}{(b-a)^N} I_3(n_1, N)$  Finally the average revenue of PAM  $\text{Rev}_{\text{PAM}}$  is:  $\text{Rev}_{\text{PAM}} = \sum_{i=1}^N \widehat{p}_i^{\text{PAM}}$ .

## 6 WHAT MECHANISM TO CHOOSE?

In this section we numerically compare the different mechanisms, by performing simulations for fairness and evaluating our previously deduced analytical expressions of average revenue in a given scenario.

### 6.1 Estimating fairness: simulation setting

We have fixed 100 players ( $N = 100$ ) distributed among five groups such that  $n_1 = 25$ ,  $n_2 = 30$ ,  $n_3 = 15$ ,  $n_4 = 10$  and  $n_5 = 20$ . Valuations are drawn from the uniform distribution over the interval  $[0; 50]$ . For each reserve price per bidder  $R$ , we compute the average fairness –using Jain’s index as introduced in (1)–, generated by those mechanisms over 1000 independent draws. The normalized utility is computed using (2), where  $\text{Rev}^{\max}$  is the maximum revenue which could be obtained over the set of candidate mechanisms for all possible values of  $R$ .

### 6.2 Revenue-fairness tradeoff

In terms of fairness, as shown in Figure 2, PAM is the best for all reserve prices. In terms of revenue, Figure 3 suggests that VCG can generate the highest revenue if the reserve price is set optimally. The trade-off between those criteria is illustrated in Figure 4 when  $\beta = 0.5$ : the auctioneer can then maximize his average utility by choosing PAM and fixing  $R \approx 16$ . Generally, with our parameter values, when  $\beta < 0.42$  the regulator should choose VCG to maximize the utility, while for  $\beta \geq 0.42$  he should choose PAM. Table 2 shows the optimal mechanisms for some given values of  $\beta$ , together with the best choice of the reserve price, and the resulting utilities. Notice that other structures of groups may lead to different outcomes.

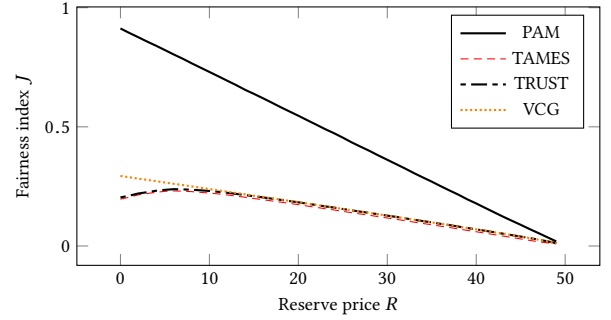


Figure 2: Average Fairness as a function of the reserve price.

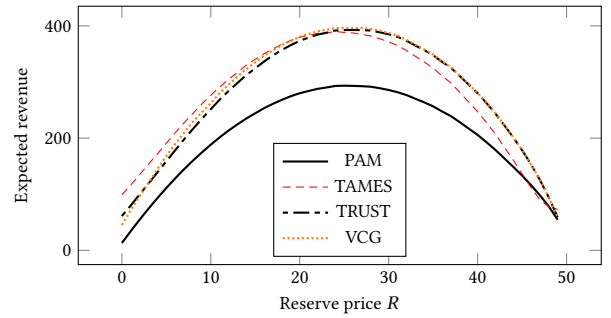


Figure 3: Average Revenue as a function of the reserve price.

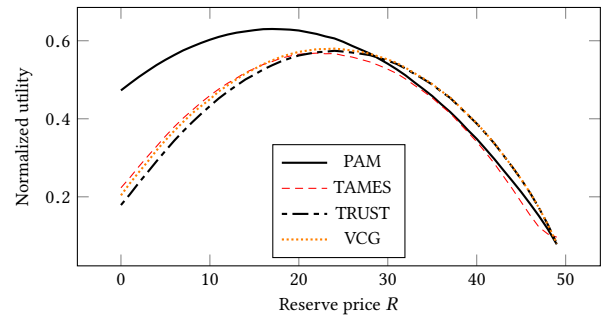


Figure 4: Average normalized utility of the regulator as a function of the reserve price for  $\beta = 0.5$

$\beta$	Optimal $R$	Optimal Mechanism	Average utility
0	$\approx 26$	VCG	1
0.4	$\approx 24$	VCG	0.66
0.5	$\approx 16$	PAM	0.62
1	0	PAM	0.91

Table 2: Optimal mechanisms and reserve prices for some specific values of  $\beta$ .

## 7 CONCLUSION

In this paper, we have considered four possible auction mechanisms for allocating and pricing spectrum in the context of LSA, which all have good incentive properties.

Since the revenues from those mechanisms can be very low, we have shown how to enhance them by introducing a per-bidder reserve price while maintaining their incentive compatibility. We have also conducted an analytical study of the expected revenue from those auction schemes under some specific assumptions, but numerical methods can also be applied in any setting.

We have finally shown how a regulator could trade-off the allocation fairness and the auction revenue, and how it could select the best-performing mechanism once the relative weights on those criteria are set.

As directions for future works, we would like to relax some of the assumptions made. In particular we want to treat the cases when one base station can be in several groups, and when one player (operator) controls several base stations, which complicates the auction analysis since that player could coordinate several bids. Finally, we intend to focus on the grouping process itself—which was out of the scope of this paper—and its impact on the auction outcome

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## APPENDIX A PROOF OF TRUTHFULNESS OF THE MODIFIED VERSION OF TRUST

PROOF. Recall that with TRUST, as for TAMES, only one group wins the whole auctioned spectrum, whose quantity is normalized to 1. Consider a player  $i$  who belongs to a group  $k$ .

*Case 1:  $v_i < R$ .* The player has no interest to propose a bid above  $R$  because if she wins she would pay at least  $R$ , hence a strictly negative utility. Proposing  $b_i = v_i$  (or any other bid below  $R$ ) maximizes her utility, which is zero in this situation.

*Case 2:  $v_i \geq R$ .* We consider the following situations:

- if  $v_i > \overline{(b_{(1)}^k)^{-i}}$  i.e.,  $i$  was not the lowest bidder of her group, then:
  - If  $\overline{(b_{(1)}^k)^{-i} \bar{n}_k} > \overline{B_{\text{second}}^{\text{TRUST}}}$  then bidding truthfully ensures a positive utility (because group  $k$  is the winning group). By proposing any other bid  $b_i > \frac{\overline{B_{\text{second}}^{\text{TRUST}}}}{\bar{n}_k}$   $i$  is still a winning player, and bid  $b_i \leq \frac{\overline{B_{\text{second}}^{\text{TRUST}}}}{\bar{n}_k}$  would make group  $k$  lose the auction and bidder  $i$  gets null utility. Therefore, proposing a bid  $b_i = v_i$  maximizes her utility.
  - $\overline{(b_{(1)}^k)^{-i} \bar{n}_k} \leq \overline{B_{\text{second}}^{\text{TRUST}}}$ : bidding truthfully leads to a null utility, player  $i$  could not change the outcome by changing her bid (setting  $b_i \geq \overline{(b_{(1)}^k)^{-i}}$  has no impact on the groupbid, and setting  $b_i < \overline{(b_{(1)}^k)^{-i}}$  lowers the groupbid and group  $k$  is still a losing group). Thus, any bid  $b_i$  generates a utility equal to zero.
- if  $v_i < \overline{(b_{(1)}^k)^{-i}}$ , then:
  - if  $v_i \bar{n}_k > \overline{B_{\text{second}}^{\text{TRUST}}}$ , if player  $i$  proposes a bid  $b_i = v = i$  then group  $k$  is a winning group. Any other bid  $b_i > \frac{\overline{B_{\text{second}}^{\text{TRUST}}}}{\bar{n}_k}$  generates the same utility because group  $k$  is still the winning group, and bids below that value make the group lose the auction, yielding utility 0.
  - if  $v_i \bar{n}_k \leq \overline{B_{\text{second}}^{\text{TRUST}}}$  then if  $i$  proposes a bid  $b_i = v_i$  then group  $k$  is a losing group, by proposing  $b_i \leq \frac{\overline{B_{\text{second}}^{\text{TRUST}}}}{\bar{n}_k}$  group  $k$  is still a losing group and by proposing  $b_i > \frac{\overline{B_{\text{second}}^{\text{TRUST}}}}{\bar{n}_k}$  group  $k$  could be a winning a group (depending on the other bids in her group), however, if it is the case then  $i$  would pay at least  $\frac{\overline{B_{\text{second}}^{\text{TRUST}}}}{\bar{n}_k}$  leading to a strictly negative utility.

Hence, in all possible scenarios, bidding truthfully maximizes the utility.  $\square$

## APPENDIX B CALCULATIONS RELATED TO TAMES AND TRUST

We denote by  $B = \{b_1, b_2, \dots, b_n\}$   $n$  independent and identically distributed random variables drawn from a distribution with PDF and CDF  $f$  and  $F$  respectively. We denote by  $\{b_{(1)}, b_{(2)}, \dots, b_{(n)}\}$  the order statistics i.e.,  $b_{(1)} = \min\{b_1, b_2, \dots, b_n\}$  and  $b_{(n)} = \max\{b_1, b_2, \dots, b_n\}$ . Let  $R$  be a constant. We denote by  $\bar{B}$  the set  $B$  after excluding variables below  $R$ . The objectives of this chapter are as follows:

- (1) To compute the joint CDF of  $(b_{(j)}, b_{(j+1)})$  for  $j \in \{1, \dots, n-1\}$ .
- (2) To compute the CDF of  $S_1^n = \min\{\bar{B}\}|\bar{B}|$ .
- (3) To compute the CDF of  $S_2^n = \min\{\bar{B}\}(|\bar{B}| - 1)$ .

The first two points are needed to compute the revenue of TRUST, the third point is needed for TAMES.

- (1) For computing the joint CDF of  $(b_{(j)}, b_{(j+1)})$ , we can distinguish two cases, if  $x \leq y$  then this event happens either if we have exactly  $j$  variables lower than  $x$ , and all the remaining  $n-j$  variables must be greater than  $x$  but not all greater than  $y$  or when we have at least  $j+1$  variables lower than  $x$ . On the other hand, if  $y < x$  then this event happens when we have at least  $j+1$  variables lower than  $y$ .

Hence,  $\mathbb{P}(b_{(j)} < x, b_{(j+1)} < y) =$

$$\begin{cases} \binom{j}{n} F(x)^j ((1-F(x))^{n-j} - (1-F(y))^{n-j}) + \sum_{i=j+1}^n \binom{i}{n} F(x)^i (1-F(x))^{n-i}, & \text{if } x \leq y \\ \sum_{i=j+1}^n \binom{i}{n} F(y)^i (1-F(y))^{n-i}, & \text{otherwise} \end{cases} \quad (10)$$

- (2) To derive  $\mathbb{P}(S_1^n \leq x)$ , we can distinguish the following cases:

- $x < R$ : the event  $S_1^n < x$  happens when all variables are lower than  $R$  i.e.,  $b_{(n)} < R$ .
- $jR < x < (j+1)R$  where  $j \in \{1, \dots, (n-1)\}$ , the event  $S_1 < x$  is the union of the following disjoint events:
  - All variables are lower than  $R$
  - $(b_{(n-i)} < R \leq b_{(n-i+1)} \leq \frac{x}{i})$  for  $i \in \{1, \dots, j\}$
- $nR \leq x \leq nb$ , the event  $S_1 < x$  is the union of the following disjoint events:
  - $R \leq b_{(1)} \leq \frac{x}{n}$
  - $b_{(i-1)} < R \leq b_{(i)} \leq \frac{x}{n-i+1}$
  - $b_{(n)} \leq R$

- $nb \leq x$ , the event  $S_1 < x$  happens always.
- Hence,  $\mathbb{P}(S_1^n \leq x) =$

$$\begin{cases} \mathbb{P}(b_{(n)} \leq R), & \text{if } x \leq R \\ \mathbb{P}(b_{(n)} \leq R) + \sum_{i=1}^j \mathbb{P}(b_{(n-i)} < R \leq b_{(n-i+1)} \leq \frac{x}{i}), & \text{if } jR \leq x \leq (j+1)R, \\ & j \in \{1, \dots, n-1\} \\ \mathbb{P}(b_{(n)} \leq R) + \sum_{i=2}^n \mathbb{P}(b_{(i-1)} < R \leq b_{(i)} \leq \frac{x}{n-i+1} + \mathbb{P}(R \leq b_{(1)} \leq \frac{x}{n}), & \text{if } nR \leq x \leq nb \\ 1 & \text{otherwise} \end{cases} \quad (11)$$

- (3) By using an analogous reasoning, we can derive the distribution of  $S_2^n$  which is given by:  
 $\mathbb{P}(S_2^n \leq x) =$

$$\begin{cases} \mathbb{P}(b_{(n-1)} \leq R), & \text{if } x \leq R \\ \mathbb{P}(b_{(n-1)} \leq R) + \sum_{i=1}^j \mathbb{P}(b_{(n-i-1)} < R \leq b_{(n-i)} \leq \frac{x}{i}), & \text{if } jR \leq x \leq (j+1)R, \\ & j \in \{1, \dots, n-2\} \\ \mathbb{P}(b_{(n-1)} \leq R) + \sum_{i=2}^{n-1} \mathbb{P}(b_{(i-1)} < R \leq b_{(i)} \leq \frac{x}{n-i} + \mathbb{P}(R \leq b_{(1)} \leq \frac{x}{n-1}), & \text{if } (n-1)R \leq x \leq (n-1)b \\ 1 & \text{otherwise} \end{cases} \quad (12)$$

Finally, to evaluate  $S_1^n$  and  $S_2^n$ , we have to compute the following probabilities:

- $\mathbb{P}(b_{(j)} \leq y)$ : the event  $b_{(j)} \leq y$  happens when at least  $j$  among  $n$  variables are lower than  $y$ , this is given by:

$$G_j(y) = \mathbb{P}(b_{(j)} < y) = \sum_{i=j}^n \binom{i}{j} F(y)^i (1-F(y))^{n-i} \quad (13)$$

- $\mathbb{P}(b_{(j)} < y_1 < b_{(j+1)} < y_2)$  where  $(j \leq n-1)$  and  $(y_2 > y_1)$ , this event happens when we have exactly  $j$  variables lower than  $y_1$ , and all the remaining  $n-j$  variables must be greater than  $y_1$  but not all greater than  $y_2$ :

$$\mathbb{P}(b_{(j)} < y_1 < b_{(j+1)} < y_2) = \binom{j}{n} F(y_1)^j \left( (1-F(y_1))^{n-j} - (1-F(y_2))^{n-j} \right) \quad (14)$$

## APPENDIX C CALCULATIONS RELATED TO VCG

Let  $y$  be a random variable drawn from the uniform distribution  $[a, b]$ . Let  $\bar{y}$  be a random variable constructed from  $y$  such that  $\bar{y} = y \mathbb{1}_{y \geq R}$ .

Let  $(\bar{y}_1, \dots, \bar{y}_n)$  be  $n$  independent random variables drawn from the same distribution as  $\bar{y}$ . Let  $\bar{Y}$  be the sum of those variables.  $\bar{Y} = \sum_{i=1}^n \bar{y}_i$

Let  $f_n^R$  denotes the PDF of  $\bar{Y}$ . The objective of this chapter is to compute  $f_n^R$ .

The CDF of  $\bar{y}$  is given by :  $p(\bar{y} \leq x) =$

$$\begin{cases} \frac{R-a}{b-a}, & \text{if } 0 \leq x \leq R \\ \frac{x-a}{b-a} & \text{if } R \leq x \leq b \end{cases} \quad (15)$$

Hence the PDF of  $\bar{y}$ ,  $f^R$  is given by:

$$\frac{R-a}{b-a} \delta(x) + \frac{1}{b-a} \mathbb{1}_{x \in [R, b]} \quad (16)$$

Hence

$$f_n^R = \underbrace{f^R \otimes f^R \dots \otimes f^R}_n \quad (17)$$

Where  $\otimes$  is the convolution product.

PROPOSITION C.1. The PDF of  $f_n^R$  is given by

$$f_n^R(x) = \frac{1}{2(b-a)^n} \sum_{k=1}^n \sum_{j=0}^k (-1)^j \binom{k}{n} \binom{j}{k} \frac{(R-a)^{n-k}}{(k-1)!} (x+Rj-bj-Rk)^{k-1} \text{sign}(x+Rj-bj-Rk) + \frac{(R-a)^n}{(b-a)^n} \delta(x) \quad (18)$$

where  $x \in [0, nb]$  and  $\text{sign}(x) =$

$$\begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (19)$$

PROOF. We denote by  $TF$  the Fourier transform.

$$\begin{aligned} f_n^R &= \underbrace{f^R \otimes f^R \dots \otimes f^R}_n = TF^{-1} \circ TF(\underbrace{f^R \otimes f^R \dots \otimes f^R}_n) \\ TF(\underbrace{f^R \otimes f^R \dots \otimes f^R}_n) &= TF(f^R)^n \\ &= \frac{1}{(b-a)^n} \left( \int_{-\infty}^{\infty} ((R-a)\delta(x)e^{-i2\pi vx} + e^{-i2\pi vx} \mathbf{1}_{x \in [R, b]}) dx \right)^n \\ &= \frac{1}{(b-a)^n} \left( (R-a) + \frac{e^{-i2\pi vR} - e^{-i2\pi vb}}{2\pi i v} \right)^n \\ &= \frac{1}{(b-a)^n} \left( \sum_{k=0}^n \binom{k}{n} (R-a)^{n-k} \frac{(e^{-i2\pi vR} - e^{-i2\pi vb})^k}{(2\pi i v)^k} \right) \\ &= \frac{1}{(b-a)^n} \left( \sum_{k=0}^n \binom{k}{n} \frac{(R-a)^{n-k}}{(2\pi i v)^k} \sum_{j=0}^k \binom{j}{k} (-1)^j (e^{-i2\pi vR(k-j)} e^{-i2\pi vbj}) \right) \\ &= \frac{1}{(b-a)^n} \left( \sum_{k=0}^n \sum_{j=0}^k \binom{k}{n} \binom{j}{k} \frac{(R-a)^{n-k}}{(2\pi i v)^k} (-1)^j (e^{i2\pi v(Rj-Rk-jb)}) \right) \\ TF^{-1} \circ (TF(f^R))^n &= \frac{1}{(b-a)^n} \left( \sum_{k=0}^n \sum_{j=0}^k \binom{k}{n} \binom{j}{k} (R-a)^{n-k} (-1)^j \int_{-\infty}^{\infty} \frac{e^{i2\pi v(x+Rj-Rk-jb)}}{(2\pi i v)^k} dv \right) \\ &= \frac{1}{(b-a)^n} \left( \sum_{k=1}^n \sum_{j=0}^k \binom{k}{n} \binom{j}{k} (R-a)^{n-k} (-1)^j (x+Rj-Rk-jb)^k \int_{-\infty}^{\infty} \frac{e^{i2\pi v(x+Rj-Rk-jb)}}{(2\pi i v(x+Rj-Rk-jb))^k} dv \right) + \frac{(R-a)^n}{(b-a)^n} \delta(x) \\ &= \frac{1}{(b-a)^n} \left( \sum_{k=1}^n \sum_{j=0}^k \binom{k}{n} \binom{j}{k} (R-a)^{n-k} (-1)^j (x+Rj-Rk-jb)^{k-1} \text{sign}(x+Rj-Rk-jb) \int_{-\infty}^{\infty} \frac{e^{i2\pi v}}{(2\pi i v)^k} dv \right) + \frac{(R-a)^n}{(b-a)^n} \delta(x) \\ &= \frac{1}{2(b-a)^n} \left( \sum_{k=1}^n \sum_{j=0}^k (-1)^j \binom{k}{n} \binom{j}{k} \frac{(R-a)^{n-k}}{(k-1)!} (x+Rj-bj-Rk)^{k-1} \text{sign}(x+Rj-bj-Rk) \right) + \frac{(R-a)^n}{(b-a)^n} \delta(x) \end{aligned}$$

□

In the previous demonstration we have used  $\int_{-\infty}^{\infty} \frac{e^{i2\pi V}}{(2\pi i V)^k} dV = \frac{1}{2(k-1)!}$ . Hence the CDF of  $\bar{Y}$ ,  $F_n^R$  is given by:

$$P(F_n^R < y) = \frac{1}{2(b-a)^n} \left( \sum_{k=1}^n \sum_{j=0}^k (-1)^j \binom{k}{n} \binom{j}{k} \frac{(R-a)^{n-k}}{(k)!} \right. \\ \left. ((-1)^{\mathbb{1}_{y \leq bj+Rk-Rj}} (y+Rj-bj-Rk)^k + (Rj-bj-Rk)^k) \right) + \frac{(R-a)^n}{(b-a)^n} \quad (20)$$

Notice that we have used:

$$k \int_0^y (x-t)^{k-1} \text{sign}(x-t) dx = (-1)^{\mathbb{1}_{y \leq t}} (y-t)^k + (-t)^k$$

## APPENDIX D CALCULATIONS RELATED TO PAM

Let  $(c_1, \dots, c_n)$  be  $n$  iid random variables drawn from the uniform distribution  $[a, b]$ . Let  $k$  and  $m$  be two constants such that  $k < n$  and  $m = n - k$ . The objective of this chapter is to compute:

$$I_3(k, n) = \int_a^b \frac{(2c_1 - b)(c_1 + \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R})}{c_1 + \sum_{i=2}^n c_i \mathbb{1}_{c_i \geq R}} \mathbb{1}_{c_1 \geq R} dc_1 \dots dc_n \quad (21)$$

$I_3$  can be written as:

$$I_3 = \int_a^b 2(c_1 - b) \underbrace{\int_a^b}_{k-1} (c_1 + \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R}) \underbrace{\int_a^b}_m \frac{1}{c_1 + \sum_{i=2}^n c_i \mathbb{1}_{c_i \geq R}} dC_{k+1}^n dC_2^k \mathbb{1}_{c_1 \geq R} dc_1$$

where  $dC_i^k = \prod_{j=i}^k dc_j$ .

To compute  $I_3$  we start by evaluating  $I_1$  which is given by:

$$I_1 = \int_a^b \frac{1}{\underbrace{c_1 + \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R}}_m + \sum_{i=k+1}^n c_i \mathbb{1}_{c_i \geq R}} dC_{k+1}^n$$

PROPOSITION D.1. Let  $c$  be a constant,  $m \geq 1$ . Then,

$$\underbrace{\int_a^b}_m \frac{1}{c + \sum_{i=1}^m c_i \mathbb{1}_{c_i \geq R}} dC_1^m = \sum_{j=1}^m \binom{j}{m} (R-a)^{m-j} \underbrace{\int_R^b}_j \frac{1}{c + \sum_{i=1}^j c_i} dC_1^j + \frac{(R-a)^m}{c} \quad (22)$$

PROOF. By induction on  $m$ . For  $m = 1$ :

$$\int_a^b \frac{1}{c + c_1 \mathbb{1}_{c_1 \geq R}} = \frac{R-a}{c} + \int_R^b \frac{1}{c + c_1} \quad (\text{true})$$

we assume the induction hypothesis, that is, we assume that

$$\underbrace{\int_a^b}_m \frac{1}{c + \sum_{i=1}^m c_i \mathbb{1}_{c_i \geq R}} dC_1^m = \sum_{j=1}^m \binom{j}{m} (R-a)^{m-j} \underbrace{\int_R^b}_j \frac{1}{c + \sum_{i=1}^j c_i} dC_1^j + \frac{(R-a)^m}{c}$$

Now we have

$$\begin{aligned}
I_1(m+1) &= \int_a^b \sum_{j=1}^m \binom{j}{k} (R-a)^{m-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i + c_{m+1} \mathbb{1}_{c_{m+1} \geq R}}}_{j} dC_1^j + \frac{(R-a)^m}{c + c_{m+1} \mathbb{1}_{c_{m+1} \geq R}} dc_{m+1} \\
&= \sum_{j=1}^m \binom{j}{m} (R-a)^{m+1-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i}}_j dC_1^j + \sum_{j=0}^m \binom{j}{m} (R-a)^{m-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^{j+1} c_i}}_{j+1} dC_1^{j+1} + \frac{(R-a)^{(m+1)}}{c} \\
&= \sum_{j=1}^m \binom{j}{m} (R-a)^{m+1-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i}}_j dC_1^j + \sum_{j=1}^{m+1} \binom{j-1}{m} (R-a)^{m+1-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i}}_j dC_1^j + \frac{(R-a)^{m+1}}{c} \\
&= \frac{(R-a)^{m+1}}{c} + \sum_{j=1}^m \left( \binom{j}{m} + \binom{j-1}{m} \right) (R-a)^{m+1-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i}}_j dC_1^j + \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^{m+1} c_i}}_{m+1} dC_1^{m+1} \\
&= \frac{(R-a)^{m+1}}{c} + \sum_{j=1}^m \binom{j}{m+1} (R-a)^{m+1-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i}}_j dC_1^j + \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^{m+1} c_i}}_{m+1} dC_1^{m+1} \\
&= \sum_{j=1}^{m+1} \binom{j}{m+1} (R-a)^{m+1-j} \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i}}_j dC_1^j + \frac{(R-a)^{m+1}}{c}
\end{aligned}$$

□

PROPOSITION D.2. Let  $c$  be a constant,  $j \geq 1$ . Then

$$A(j) = \underbrace{\int_R^b \frac{1}{c + \sum_{i=1}^j c_i}}_j dC_1^j = \sum_{i=0}^j \frac{(-1)^i}{(j-1)!} \binom{i}{j} (c + iR + (j-i)b)^{j-1} (\ln(c + iR + (j-i)b) - \sum_{t=1}^{j-1} \frac{1}{t})$$

PROOF. For  $j = 1$ :

$$\int_R^b \frac{1}{c + c_1} = \ln(c + b) - \ln(c + R) \text{ (true)}$$

We assume the induction hypothesis, we have

$$\begin{aligned}
A(j+1) &= \int_R^b \sum_{i=0}^j \frac{(-1)^i}{(j-1)!} \binom{i}{j} (c + c_{j+1} + iR + (k-i)b)^{j-1} (\ln(c + c_{j+1} + iR + (j-i)b) - \sum_{t=1}^{j-1} \frac{1}{t}) dc_{j+1} \\
&= \sum_{i=0}^j \frac{(-1)^i}{(j-1)!} \binom{i}{j} \frac{1}{j} \left[ (c + c_{j+1} + iR + (j-i)b)^j \left( \ln(c + c_{j+1} + iR + (j-i)b) - \sum_{t=1}^{j-1} \frac{1}{t} \right) \right]_R^b
\end{aligned}$$

$$\begin{aligned}
A(j+1) &= \sum_{i=0}^j \frac{(-1)^i}{(j-1)!} \binom{i}{j} \frac{1}{j} \left( c + iR + (j+1-i)b \right)^j \left( \ln(c + iR + (j+1-i)b) - \sum_{t=1}^j \frac{1}{t} \right) - \\
&\quad \sum_{i=0}^j \frac{(-1)^i}{(j-1)!} \binom{i}{j} \frac{1}{j} \left( c + (i+1)R + (j-i)b \right)^j \left( \ln(c + (i+1)R + (j-i)b) - \sum_{t=1}^j \frac{1}{t} \right) \\
&= \sum_{i=0}^j \frac{(-1)^i}{(j-1)!} \binom{i}{j} \frac{1}{j} \left( c + iR + (j+1-i)b \right)^j \left( \ln(c + iR + (j+1-i)b) - \sum_{t=1}^j \frac{1}{t} \right) + \\
&\quad \sum_{i=1}^{j+1} \frac{(-1)^i}{(j-1)!} \binom{i-1}{j} \frac{1}{j} \left( c + iR + (j+1-i)b \right)^j \left( \ln(c + iR + (j+1-i)b) - \sum_{t=1}^j \frac{1}{t} \right) \\
&= \sum_{i=1}^{j+1} \frac{(-1)^i}{(j-1)!} \left( \binom{i-1}{j} + \binom{i}{j} \right) \frac{1}{j} \left( c + iR + (j+1-i)b \right)^j \left( \ln(c + iR + (j+1-i)b) - \sum_{t=1}^j \frac{1}{t} \right) + \frac{(-1)^{j+1}}{(j)!} \binom{j}{j} \\
&\quad (c + (j+1)R)^j \left( \ln(c + (j+1)R) - \sum_{t=1}^j \frac{1}{t} \right) + \frac{1}{(j)!} \binom{0}{j} (c + (j+1)b)^j \left( \ln(c + (j+1)b) - \sum_{t=1}^j \frac{1}{t} \right) \\
&= \sum_{i=0}^{j+1} \frac{(-1)^i}{(j)!} \binom{i}{j+1} \left( c + iR + (j+1-i)b \right)^j \left( \ln(c + iR + (j+1-i)b) - \sum_{t=1}^j \frac{1}{t} \right)
\end{aligned}$$

□

PROPOSITION D.3. *The first integral  $I_1$  is given by:*

$$\begin{aligned}
I_1 &= \sum_{j=1}^m \binom{j}{m} (R-a)^{m-j} \sum_{i=0}^j \frac{(-1)^i}{(j-1)!} \binom{i}{j} \left( c_1 + \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R} + iR + (j-i)b \right)^{j-1} \\
&\quad \left( \ln \left( c_1 + \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R} + iR + (j-i)b \right) - \sum_{t=1}^{j-1} \frac{1}{t} \right) + \frac{(R-a)^m}{c_1 + \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R}}
\end{aligned}$$

PROOF. Direct application of proposition D.1 and D.2 and by replacing  $c$  with  $c_1 + \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R}$

□

We denote by  $S_2^k = \sum_{i=2}^k c_i \mathbb{1}_{c_i \geq R}$  and by  $I_2 = \underbrace{\int_a^b (c_1 + S_2^k) I_1}_{k-1} dC_2^k$

$$A_1 = \underbrace{\int_a^b}_{k-1} (c_1 + S_2^k + iR + (j-i)b)^j \left( \ln(c_1 + S_2^k + iR + (j-i)b) - \sum_{t=1}^{j-1} \frac{1}{t} \right) dC_2^k$$

$$A_2 = \underbrace{\int_R^b}_{z} (c_1 + S_2^k + iR + (j-i)b)^j \left( \ln(c_1 + S_2^k + iR + (j-i)b) - \sum_{t=1}^{j-1} \frac{1}{t} \right) dC_2^k$$

To compute the second integral, we will use the following propositions, (the proof is by induction)

PROPOSITION D.4.

$$A_1 = \sum_{z=0}^{k-1} \binom{z}{k-1} (R-a)^{k-1-z} \underbrace{\int_R^b}_{z} (c_1 + S_2^k + iR + (j-i)b)^j \left( \ln(c_1 + S_2^k + iR + (j-i)b) - \sum_{t=1}^{j-1} \frac{1}{t} \right) dC_2^k$$

PROPOSITION D.5.

$$A_2 = \sum_{h=0}^z \frac{j!}{(j+z)!} (-1)^h \binom{h}{z} (c_1 + (i+h)R + (j-i)b + (z-h)b)^{j+z} \left( \ln(c_1 + (i+h)R + (j-i)b + (z-h)b) - \sum_{t=1}^{j+z} \frac{1}{t} + \frac{1}{j} \right)$$

$$\begin{aligned}
I_2 &= \underbrace{\int_a^b (c_1 + S_2^k) I_1 \, dC_2^k}_{k-1} \\
&= \sum_{j=1}^m \sum_{i=0}^j \binom{j}{m} \binom{i}{j} (R-a)^{m-j} \frac{(-1)^i}{(j-1)!} \underbrace{\int_a^b (c_1 + S_2^k + iR + (j-i)b)^j}_{k-1} \\
&\quad \left( \ln(c_1 + S_2^k + iR + (j-i)b) - \sum_{t=1}^{k-1} \frac{1}{t} \right) dC_2^k - \sum_{j=1}^m \sum_{i=0}^j \binom{j}{m} \binom{i}{j} (R-a)^{m-j} \frac{(-1)^i}{(j-1)!} \\
&\quad \underbrace{\int_a^b (c_1 + S_2^k + iR + (j-i)b)^{j-1} \left( \ln(c_1 + S_2^k + iR + (j-i)b) - \sum_{t=1}^{k-1} \frac{1}{t} \right)}_{k-1} dC_2^k + (R-a)^m (b-a)^{k-1} \\
&= I_2^1 + I_2^2 + I_2^3 \\
I_2^1 &= \sum_{j=1}^m \sum_{i=0}^j \binom{j}{m} \binom{i}{j} (R-a)^{m-j} \frac{(-1)^i}{(j-1)!} \underbrace{\int_a^b (c_1 + S_2^k + iR + (j-i)b)^j}_{k-1} \\
&\quad \left( \ln(c_1 + S_2^k + iR + (j-i)b) - \sum_{t=1}^{k-1} \frac{1}{t} \right) dC_2^k \\
&= \sum_{j=1}^m \sum_{i=0}^j \sum_{z=0}^{k-1} \sum_{h=0}^z \left[ \binom{j}{m} \binom{i}{j} \binom{z}{k-1} \binom{h}{z} (R-a)^{m+k-(j+1+z)} (-1)^i \right. \\
&\quad \left. \frac{j}{(j+z)!} (-1)^h (c_1 + (i+h)R + (j-i)b + (z-h)b)^{j+z} \right. \\
&\quad \left. \left( \ln(c_1 + (i+h)R + (j-i)b + (z-h)b) - \sum_{t=1}^{j+z} \frac{1}{t} + \frac{1}{j} \right) \right] \\
I_2^2 &= \sum_{j=1}^m \sum_{i=0}^j \binom{j}{m} \binom{i}{j} (R-a)^{m-j} \frac{(-1)^i}{(j-1)!} \\
&\quad \underbrace{\int_a^b (c_1 + S_2^k + iR + (j-i)b)^{j-1} \left( \ln(c_1 + S_2^k + iR + (j-i)b) - \sum_{t=1}^{k-1} \frac{1}{t} \right)}_{k-1} dC_2^k \\
&= \sum_{j=1}^m \sum_{i=0}^j \sum_{z=0}^{k-1} \sum_{h=0}^z \left[ \binom{j}{m} \binom{i}{j} \binom{z}{k-1} \binom{h}{z} (R-a)^{m+k-(j+1+z)} (-1)^i \right. \\
&\quad \left. (iR + (j-i)b) \frac{(-1)^h}{(j+z-1)!} (c_1 + (i+h)R + (j-i)b + (z-h)b)^{j+z-1} \right. \\
&\quad \left. \left( \ln(c_1 + (i+h)R + (j-i)b + (z-h)b) - \sum_{t=1}^{j+z-1} \frac{1}{t} \right) \right] \\
I_2^3 &= (R-a)^m (b-a)^{k-1}
\end{aligned}$$

Once  $I_2$  is evaluated, we can derive the expression of  $I_3$ .

$$I_3 = \int_R^b (2c_1 - b) I_2(c_1) \, dc_1 \quad (23)$$

(24)

To simplify the expression of  $I_3$ , let us introduce the following notations:

- $\phi = (i, h, j, z, R, b)$
- $\theta = (\phi, m, k, a)$
- $C_1(\theta) = \binom{j}{m} \binom{i}{j} \binom{z}{k-1} \binom{h}{z} (R-a)^{m+k-(j+1+z)} (-1)^i \frac{j}{(j+z)!} (-1)^h$
- $C_2(\theta) = \binom{j}{m} \binom{i}{j} \binom{z}{k-1} \binom{h}{z} (R-a)^{m+k-(j+1+z)} (-1)^i (iR + (j-i)b) \frac{(-1)^h}{(j+z-1)!}$
- $ind_1(\phi) = (i+h)R + (j+z+1-h-i)b$
- $ind_2(\phi) = (i+h+1)R + (j+z-h-i)b$
- $ind_3(\phi) = (i+h)R + (j+z-h-i)b$

$$\begin{aligned}
I_3^1 &= \int_R^b (2c_1 - b) I_2^1 dc_1 \\
&= \sum_{j=1}^m \sum_{i=0}^j \sum_{z=0}^{k-1} \sum_{h=0}^z C_1(\theta) \left[ \left( \frac{2}{j+z+2} ind_1(\phi)^{j+z+2} (\ln(ind_1(\phi))) - \sum_{t=1}^{j+z+2} \frac{1}{t} + \frac{1}{j} + \frac{1}{j+z+1} \right) - \right. \\
&\quad \left( \frac{2}{j+z+2} ind_2(\phi)^{j+z+2} (\ln(ind_2(\phi))) - \sum_{t=1}^{j+z+2} \frac{1}{t} + \frac{1}{j} + \frac{1}{j+z+1} \right) - \\
&\quad \left. \left( \frac{(2ind_3(\phi) + b)}{j+z+1} ind_1(\phi)^{j+z+1} (\ln(ind_1(\phi))) - \sum_{t=1}^{j+z+1} \frac{1}{t} + \frac{1}{j} \right) + \left( ind_2(\phi)^{j+z+1} \frac{(2ind_3(\phi) + b)}{j+z+1} (\ln(ind_2(\phi))) - \sum_{t=1}^{j+z+1} \frac{1}{t} + \frac{1}{j} \right) \right]
\end{aligned}$$

$$\begin{aligned}
I_3^2 &= \int_R^b (2c_1 - b) I_2 dc_1 \\
&= \sum_{j=1}^m \sum_{i=0}^j \sum_{z=0}^{k-1} \sum_{h=0}^z C_2(\theta) \left[ \left( \frac{2}{j+z+1} ind_1(\phi)^{j+z+1} (\ln(ind_1(\phi))) - \sum_{t=1}^{j+z+1} \frac{1}{t} + \frac{1}{j+z} \right) - \right. \\
&\quad \left( \frac{2}{j+z+1} ind_2(\phi)^{j+z+1} (\ln(ind_2(\phi))) - \sum_{t=1}^{j+z+1} \frac{1}{t} + \frac{1}{j+z} \right) - \\
&\quad \left. \left( \frac{(2ind_3(\phi) + b)}{j+z} ind_1^{j+z} (\ln(ind_1(\phi))) - \sum_{t=1}^{j+z} \frac{1}{t} \right) + \left( ind_2(\phi)^{j+z} \frac{(2ind_3(\phi) + b)}{j+z} (\ln(ind_2(\phi))) - \sum_{t=1}^{j+z} \frac{1}{t} \right) \right]
\end{aligned}$$

$$I_3^3 = \int_R^b (2c_1 - b) I_2^3 dc_1 = (bR - R^2)(b-a)^{k-1}(R-a)^m$$

Finally

$$I_3 = I_3^1 + I_3^2 + I_3^3. \tag{25}$$