# A belief combination rule for a large number of sources

Kuang Zhou, Arnaud Martin, Quan Pan

# A belief combination rule for a large number of sources

Kuang Zhou[a], Arnaud Martin[b], and Quan Pan[a]
a. Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China.
b. DRUID, IRISA, University of Rennes 1, Rue E. Branly, 22300 Lannion, France

***Abstract*—The theory of belief functions is widely used for data from multiple sources. Different evidence combination rules have been proposed in this framework according to the properties of the sources to combine. However, most of these combination rules are not efficient when there are a large number of sources. This is due to either the complexity or the existence of an absorbing element such as the total conflict mass function for the conjunctive based rules when applied on unreliable evidence. In this paper, based on the assumption that the majority of sources are reliable, a combination rule for a large number of sources is proposed using a simple idea: the more common ideas the sources share, the more reliable these sources are supposed to be. This rule is adaptable for aggregating a large number of sources which may not all be reliable. It will keep the spirit of the conjunctive rule to reinforce the belief on the focal elements with which the sources are in agreement. The mass on the empty set will be kept as an indicator of the conflict.**

**The proposed rule, called LNS-CR (Conjunctive combination Rule for a Large Number of Sources), is evaluated on synthetic mass functions. The experimental results verify that the rule can be effectively used to combine a large number of mass functions and to elicit the major opinion.**

***Index Terms*—Theory of belief functions, big data, combination, large number of sources, reliability**

## I. Introduction

In recent years, Dempster–Shafer Theory (DST), also called the theory of belief functions, has gained increasing attention in the scientific community as it allows to the deal with the imprecise and uncertain information. It has been applied in various domains, such as data classification [2, 3], data clustering [4, 5], social network analysis [6], etc. In complex environment, multiple stake-holders attempt to reach a decision by combining several sources of information and aggregating their points of view by stressing common agreement. The theory of belief functions, which has provided many rules to combine information represented by mass functions [7], are widely used for decision making. In real applications, there are usually a large number of sources. Most of the existing combination rules are not applicable in this case, and cannot be used to find the major opinion from many participants.

One of the most famous combination rule in belief function framework is the Dempster's rule [7]. Smets [8] proposed a modification of Dempster's rule, often called "conjunctive rule", where the empty set can be assigned with a non-null mass under the Transferable Belief Model (TBM) [9]. In fact, the conjunctive rule is equivalent to the Dempster rule without the normalization process. It has a fast and clear convergence towards a solution. But this rule has a strong assumption that all the sources are reliable. In real applications, it is difficult to be either satisfied or verified. Moreover, the more sources there are, the more chance that there is some unreliable evidence.

Smets [8] reasoned that the mass on the empty set can play the role of alarm. When the global conflict (the mass assigned to the empty set) is high, it indicates that there is strong disagreement among the sources of mass functions to combine. However, as observed in [10, 11, 12], the mass on the empty set is not sufficient to exactly describe the conflict since it includes an amount of auto-conflict [13]. Sometimes when there is only a small amount of concordant evidence, the total conflict mass function, *i.e.* $m(\emptyset) = 1$ will be an absorbing element. Consequently, when combining a large number of (incompatible) mass functions using the conjunctive rule, the global conflict may tend to 1. This makes it impossible to reveal the cause of high global conflict. We do not know whether it is due to the sources to fuse or caused by the absorption power of the empty set [10, 14]. In other words, even the combined mass function by the conjunctive rule is $m(\emptyset) \approx 1$, the proposition that the sources are highly conflicting may be incorrect.

In order to rectify the drawbacks of the classical Dempster's rule and Smets' conjunctive rule, many approaches have been made through the modification of the combination rule. Some authors tried to find alternative repartitions of the conflict. A plethora of combination rules have been brought forward in this way. For example, Yager [15] and Dubois and Prade [16] suggested assigning the highly conflicting mass to the whole set or a particular set. The Proportional Conflict Redistribution (PCR) rule, which can distribute the partial conflicts among the involved focal elements rather than to their union, is developed in [13, 17]. Apart from these approaches working directly on the combination rule, some studies manage the conflict through evidence discounting, where the reliability of sources is automatically and adaptively taken into account [10, 16, 18, 19].

Most of the existing combination rules are not efficient when applied on a large number of sources due to the ineffective way to handle conflict or the high complexity of the computation. Orponen [20] proved that the complexity of the conjunctive rule is NP-hard, but the complexity depends on the way to program the belief functions [21]. Some rules can manage efficiently the conflict but have large complexity [13, 16, 22, 23], making them infeasible when applied to combine a large

This paper is an extension and revision of [1].

number of mass functions.

In this paper, a conjunctive-based combination rule, named LNS-CR (Large Number of Sources), is proposed to aggregate a large number of mass functions. Our perspective on belief function combination is that combining mass functions from different sources is similar to combining opinions from multiple stake-holders in group decision-making [24], *i.e.* the more one's opinion is consistent with the other experts, the more reliable the source is. We assume that all the mass functions available are separable mass functions, which means they can be expressed by a group of simple support mass functions. In many applications, the mass assignments are directly in the form of Simple Support Functions (SSF) [25]. The advantage of SSFs is that we can group the mass functions in such a way that sources in the same group share the same viewpoint. Mass functions in each small group are first fused and then discounted according to the proportions. After that the number of mass functions participating the next global combination process is independent of the number of sources, but only depends on the number of classes. As a result, the problem brought by the absorbing element (the empty set) using the conjunctive rule can be avoided. Moreover, an approximation method when the number of mass functions is large enough is presented. The main contributions of this paper are as follows:

- A new conjunctive-based combination rule, named LNS-CR rule, is brought froward. The property to reinforce the belief on the focal elements with which most of the sources agree is preserved in the proposed rule;
- The assumption of the LNS-CR rule on the reliability of the sources is more relaxed, as it does not require all the sources are reliable, but only at least half of them are reliable.
- LNS-CR can be used to combine mass functions from a large number of sources, especially can be used to elicit the major opinion;
- Derivation that the LNS-CR rule is within acceptable complexity.

The rest of this paper is organized as follows. In Section 2, some basic knowledge of belief function theory is briefly introduced. The proposed evidence combination approach is presented in detail in Section 3. Numerical examples are employed to compare different combination rules and show the effectiveness of LNS-CR rule in Section 4. Finally, Section 5 concludes the paper.

## II. BACKGROUND

### A. Basic knowledge of belief function theory

Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ be the discernment frame. A mass function is defined on the power set $2^\Theta = \{A : A \subseteq \Theta\}$. The mass function $m : 2^\Theta \to [0, 1]$ is said to be a Basic Belief Assignment (bba) on $2^\Theta$, if it satisfies:

$$\sum_{A \subseteq \Theta} m(A) = 1. \tag{1}$$

Every $A \in 2^\Theta$ such that $m(A) > 0$ is called a focal element, and the set of focal elements is denoted by $\mathcal{F}$. In a practical way of programming, the element of $2^\Theta$ can be arranged by natural order [26]: $\theta_1, \theta_2, \{\theta_1, \theta_2\}, \theta_3, \cdots, \{\theta_1, \theta_2, \theta_3\}, \theta_4, \cdots, \Theta$.

The frame of discernment can also be a focal element. If $\Theta$ is a focal element, the mass function is called non-dogmatic. The mass assigned to the frame of discernment, $m(\Theta)$, is interpreted as a degree of ignorance. In the case of total ignorance, $m(\Theta) = 1$. This type of mass assignment is vacuous. If there is only one focal element, *i.e.* $m(A) = 1, A \subset \Theta$, the mass function is categorical. Another special case of assignment is named consonant mass functions, where the focal elements include each other as a subset, *i.e.* if $A, B \in \mathcal{F}, A \subset B$ or $B \subset A$.

The credibility and plausibility functions are derived from a bba $m$ as in Eqs. (2) and (3):

$$Bel(A) = \sum_{B \subseteq A, B \neq \emptyset} m(B), \quad \forall A \subseteq \Theta, \tag{2}$$

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \quad \forall A \subseteq \Theta. \tag{3}$$

Each quantity $Bel(A)$ measures the minimal belief on $A$ justified by available information on $B (B \subseteq A)$ , while $Pl(A)$ is the maximal belief on $A$ justified by information on $B$ which are not contradictory with $A$ ($A \cap B \neq \emptyset$). The commonality function $q$ and the implicability function $b$ are defined respectively as

$$q(A) = \sum_{A \subseteq B} m(B), \quad \forall A \subseteq \Theta \tag{4}$$

and

$$b(A) = Bel(A) + m(\emptyset), \quad \forall A \subseteq \Theta. \tag{5}$$

A bba $m$ can be recovered from any of these functions. For instance,

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \quad \forall A \subseteq \Theta \tag{6}$$

and

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} b(B), \quad \forall A \subseteq \Theta. \tag{7}$$

Belief functions can be transformed into a probability function by Smets' method [27], where each mass of belief $m(A)$ is equally distributed among the elements of $A$. This leads to the concept of pignistic probability, $\mathrm{BetP}$. For all $\theta_i \in \Theta$, we have

$$\mathrm{BetP}(\theta_i) = \sum_{A \subseteq \Theta | \theta_i \in A} \frac{m(A)}{|A|(1 - m(\emptyset))}, \tag{8}$$

where $|A|$ is the cardinality of set $A$ (number of elements of $\Theta$ in $A$). Pignistic probabilities can help make a decision.

### B. Consistency of mass assignments

The consistency between two bbas can be defined in two different ways. Suppose the sets of focal elements for $m_1$ and $m_2$ are $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. Mass functions $m_1$ and $m_2$ are called strong consistent if and only if

$$\cap_{E \in \{\mathcal{F}_1 \cup \mathcal{F}_2\}} \neq \emptyset. \tag{9}$$

Meanwhile, bbas $m_1$ and $m_2$ are called weak consistent if and only if

$$\forall A \in \mathcal{F}_1, B \in \mathcal{F}_2, A \cap B \neq \emptyset. \tag{10}$$

Strong consistent evidence means that there is at least one element that is common to all subsets [28]. It is easy to see that, when $m_1$ and $m_2$ are strong consistent, they are sure to be weak consistent. This is the definition of consistency between belief functions. The inconsistency within an individual mass assignment can be defined similarly [12].

### *C. Reliability-based discounting*

When the sources of evidence are not completely reliable, the discounting operation proposed by Shafer [25] and justified by Smets [29] could be applied. Denote the reliability degree of mass function $m$ by $\alpha \in [0,1]$, then the discounting operation can be defined as:

$$m^{'}(A) = \begin{cases} \alpha \times m(A) & \forall A \subset \Theta, \\ 1-\alpha+\alpha \times m(\Theta) & \text{if } A = \Theta. \end{cases} \tag{11}$$

If $\alpha = 1$, the evidence is completely reliable and the bba will remain unchanged. On the contrary, if $\alpha = 0$, the evidence is completely unreliable. In this case the so-called vacuous belief function, $m(\Theta) = 1$, could be got. It describes the total ignorance.

Before evoking the discounting process, the reliability of each sources should be known. One possible way to estimate the reliability is to use confusion matrices [30]. Generally, the goal of discounting is to reduce global conflict before combination. One can assume that the conflict comes from the unreliability of the sources. Therefore, the source reliability estimation is to some extent linked to the estimation of conflict between sources.

Hence, Martin et al. [10] proposed to use a conflict measure to evaluate the relative reliability of experts. Once the degree of conflict is computed, the relative reliability of the source can be computed accordingly. Suppose there are $S$ sources, $\mathcal{S} = \{s_1, s_2, \cdots, s_S\}$, the reliability discounting factor $\alpha_j$ of source $s_j$ can be defined as follows:

$$\alpha_j = f\left(\mathrm{Conf}\left(s_j, \mathcal{S}\right)\right), \tag{12}$$

where $\mathrm{Conf}\left(s_j, \mathcal{S}\right)$ quantifies the degree that source $s_j$ conflicts with the other sources in $\mathcal{S}$, and $f$ is a decreasing function. The following function is suggested by the authors:

$$\alpha_j = \left(1 - \mathrm{Conf}\left(s_j, \mathcal{S}\right)^{\lambda}\right)^{\frac{1}{\lambda}}, \tag{13}$$

where $\lambda > 0$.

In [31], the authors considered to use those two possible conflict origins, extrinsic measure and intrinsic measure, to estimate reliability. In their opinion, conflict may not only come from the source's contradiction (extrinsic measure), but also from the confusion rate of a source (intrinsic measure). The reliability discounting factor, called Generic Discounting Factor (GDF), is then suggested to be a weighted sum of the two items:

$$\alpha = \frac{k\delta + l\beta}{k+l}, \tag{14}$$

where $k > 0, l > 0$ are the weight factors. In the above equation, $\delta$ denotes the internal conflict measure of the treated source indicating its confusion rate while $\beta$ is the average distance between the treated sources $s_i$ and $s_j$ where $j \in \mathcal{S}, j \neq i$. Different intrinsic and extrinsic conflict measures can be adopted here.

There are some other methods to estimate the reliability. In [32], the authors proposed to estimate the reliability of sources based on a degree of falsity. The bbas are sequentially and incrementally discounted until the mass assigned to the empty set is smaller than a given threshold $k$. After that the discounted mass functions can be combined using the conjunctive rule since there is little global conflict at this time. In [33], the source reliability is obtained by minimizing the distance between the pignistic probabilities computed from the discounted beliefs and the actual value of the data. In Samet et al. [34], the authors proposed two different versions of generic discounting approaches: weighted GDA and exponent GDA. A new degree of disagreement is proposed by Yang et al. [35], where the reliability discounting factor can be generated. Klein and Colot [36] viewed the degree of conflict as a function of discounting rates and introduced a new criterion assessing bbas' reliability. These reliability estimation methods either consider the distance (or dissimilarity) between each pair of bbas, or the mass assigned to the empty set after the conjunctive combination. However, these methods are of high complexity and not suitable for large data applications.

### *D. Simple support function*

Suppose $m$ is a bba defined on the frame of discernment $\Theta$. If there exists a subset $A \subseteq \Theta$ such that $m$ could be expressed in the following form:

$$m(X) = \begin{cases} w & X = \Theta, \\ 1-w & X = A, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

where $w \in [0,1]$, then the belief function related to bba $m$ is called a Simple Support Function (SSF) (also called simple mass function) [25] focused on $A$. Such a SSF can be denoted by $A^w(\cdot)$ where the exponent $w$ of the focal element $A$ is the basic belief mass (bbm) given to the frame of discernment $\Theta$, $m(\Theta)$. The complement of $w$ to 1, *i.e.* $1-w$, is the bbm allocated to $A$ [37]. If $w = 1$ the mass function represents the total ignorance, if $w = 0$ the mass function is a categorical bba on $A$.

A belief function is separable if it is a SSF or if it is the conjunctive combination of some SSFs [38]. In the work of [38], this kind of separable masses is called u-separable where "u" stands for "unnormalized", indicating the conjunctive rule is the unnormalized version of Dempster-Shafer rule. The set of separable mass functions is not obvious to obtain. It is easy to see consonant mass functions (the focal element are nested) are separable [39]. Smets [37] defined the Generalized Simple Support Function (GSSF) by relaxing the weight $w$ to $[0, \infty)$. Those GSSFs with $w \in (1, \infty)$ are called Inverse Simple Support Functions (ISSF). Smets proved all non-dogmatic mass functions are separable if one uses GSSFs. For any non-dogmatic belief function $m_0$, the canonical decomposition method proposed by Smets is as follows. First, calculate the commonality number for all focal elements, which is given by

$$Q_0(X) = \sum_{B \supseteq X} m_0(B). \tag{16}$$

Secondly for any $A \subseteq \Theta$, calculate $w_A$ value as follows:

$$w_A = \prod_{X \supseteq A} Q_0(X)^{(-1)^{|X|-|A|+1}}. \tag{17}$$

Then the belief function $m_0$ can be represented by the conjunctive combination of all the functions $A_{w_A}$, *i.e.*

$$m_0 = \bigcirc\!\!\!\!\cap_{A \subseteq \Theta} A^{w_A}, \tag{18}$$

where $\bigcirc\!\!\!\!\cap$ denotes the conjunctive combination rule. For fast computation, the Fast Möbius Transform (FMT) method [40] can be evoked.

### E. Some combination rules

How to combine efficiently several bbas coming from distinct sources is a major information fusion problem in the belief function framework. Many rules have been proposed for such a task. Here we just briefly recall how some most popular rules are mathematically defined.

When information sources are reliable, the used fusion operators can be based on the conjunctive combination. If bbas $m_j, j = 1, 2, \cdots, S$ describing $S$ distinct items of evidence on $\Theta$, the included result of the **conjunctive rule** [9] is defined as

$$m_{\text{conj}}(X) = (\bigcirc\!\!\!\!\cap_{j=1,\cdots,S} m_j)(X) = \sum_{Y_1 \cap \cdots \cap Y_S = X} \prod_{j=1}^{S} m_j(Y_j), \tag{19}$$

where $m_j(Y_j)$ is the mass allocated to $Y_j$ by expert $j$. To apply this rule, the sources are assumed reliable and cognitively independent.

Another kind of conjunctive combination is **Dempster's rule** [41]. Assuming that $m_{\text{conj}}(\emptyset) \neq 1$, the result of the combination by Dempster's rule is

$$m_{\text{Dempster}}(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ \frac{m_{\text{conj}}(X)}{1 - m_{\text{conj}}(\emptyset)} & \text{otherwise.} \end{cases} \tag{20}$$

The item

$$\kappa \triangleq m_{\text{conj}}(\emptyset) = \sum_{Y_1 \cap \cdots \cap Y_S = \emptyset} \prod_{j=1}^{S} m_j(Y_j)$$

is generally called Dempster's degree of conflict of the combination or the inconsistency of the combination. As the conjunctive rule is not idempotent, $m_{\text{conj}}(\emptyset)$ includes an amount of auto-conflict [42], and it is called global conflict to make the difference.

The conjunctive rule can be applied only if all the experts are reliable. In the other case, the **disjunctive rule** [43], which only assumes that at least one of the sources is reliable, can be used. The disjunctive combination of $S$ sources can be defined as

$$m_{\text{disj}}(X) = \left(\bigcirc\!\!\!\!\cup_{j=1,\cdots,S} m_j\right)(X) = \sum_{Y_1 \cup \cdots \cup Y_S = X} \prod_{j=1}^{S} m_j(Y_j). \tag{21}$$

The conjunctive and disjunctive rules can be conveniently expressed by means of the commonality function $q$ (Eq. (4)) and the implacability function $b$ (Eq. (5)) [43]. Let $q_i$ and $b_i$ be the commonality function and implacability function respectively (associated with $m_i$), then the commonality function of the conjunctive combination of $S$ bbas is

$$q_{\text{conj}}(A) = \prod_{i=1}^{S} q_i(A), \quad \forall A \subseteq \Theta \tag{22}$$

while the implacability function of the disjunctive combination of $S$ bbas is

$$b_{\text{disj}}(A) = \prod_{i=1}^{S} b_i(A), \quad \forall A \subseteq \Theta. \tag{23}$$

Since functions $m$, $q$ and $b$ (as well as $bel$ and $pl$) are equivalent representations, the mass function $m$ can be recovered using the Fast Möbius Transform (FMT) method given the functions $q$ and $b$. The conversion can be done in time proportional to $n2^n$ [44]*. For the conjunctive combination of $S$ sources, the $S$ bbas should be converted into commonality functions first. After calculating the product of $S$ commonality functions, another transformation from $m$ to $q$ should be evoked. Overall the total complexity is $O(Sn2^n + S2^n + n2^n)$, and the time needed is proportional to $Sn2^n$ [44, 45].

The conflict could be redistributed on partial ignorance like in the Dubois and Prade rule (**DP rule**) [16], which can be seen as a mixed conjunctive and disjunctive rule. For all $X \subseteq \Theta, X \neq \emptyset$:

$$\begin{aligned} m_{\text{DP}}(X) = & \sum_{Y_1 \cap \cdots \cap Y_S = X} \prod_{j=1}^{S} m_j(Y_j) + \\ & \sum_{\substack{Y_1 \cup \cdots \cup Y_S = X \\ Y_1 \cap \cdots \cap Y_S = \emptyset}} \prod_{j=1}^{S} m_j(Y_j), \end{aligned} \tag{24}$$

where $m_j$ is the mass function delivered by expert $j$. In a general case, this rule cannot be programmed with the Fast Möbius Transform method because all the partial conflict must be considered. If the implementation is made like that in Ref. [46], it takes much more time than the conjunctive rule.

Denœux [38] proposed a family of conjunctive and disjunctive rules using triangular norms. The **cautious rule** [47, 48] belongs to that family and could be used to combine mass functions for which independence assumption is not verified. Cautious combination of $S$ non-dogmatic mass functions

*This is based on the assumption that the mass functions are arranged in natural order. If not, the complexity is proportional to $n^2 2^n$. The complexity analysis in this work all assumes that the bbas to be combined are encoded using the natural order.

$m_j, j=1,2,\cdots,S$ is defined by the bba with the following weight function:

$$w(A) = \overset{S}{\underset{j=1}{\wedge}} w_j(A), \quad A \in 2^\Theta \setminus \Theta. \tag{25}$$

We thus have

$$m_{\text{Cautious}}(X) = \underset{A \subsetneq \Theta}{\bigcirc\!\!\!\!\cap} A^{\overset{S}{\underset{j=1}{\wedge}} w_j(A)}, \tag{26}$$

where $A^{w_j(A)}$ is the simple support function focused on $A$ with weight function $w_j(A)$ issued from the canonical decomposition of $m_j$. Note also that $\wedge$ is the min operator. The time consumption of the cautious rule includes the canonical decomposition of non-dogmatic mass functions and is therefore bigger than the conjunctive rule. If this rule is implemented in Fast Möbius Transform method, the complexity is proportional to $Sn2^n$.

Murphy [49] presented the **average combination rule** and proposed to utilize the mean of the basic belief assignments as the fusion of evidence. Therefore, for each focal element $X \in 2^\Theta$ of $S$ mass functions, the combined one is defined as follows:

$$m_{\text{Ave}}(X) = \frac{1}{S}\sum_{j=1}^{S} m_j(X), \forall X \subseteq \Theta. \tag{27}$$

The complexity of the average is proportional to $S2^n$.

A family of fusion rules based on new Proportional Conflict Redistributions (PCR) for the combination of uncertainty and conflicting information have been developed in Dezert–Smarandache Theory (DSmT) framework [50]. Among them, the fusion rule called PCR6 proposed by Martin and Osswald [13] is one of the most popular one among the PCR rules. For the combination of $S > 2$ sources, the fused mass is given by $m_{\text{PCR6}}(\emptyset) = 0$, and for $X \neq \emptyset$ in $2^\Theta$

$$m_{\text{PCR6}}(X) = m_{\text{conj}}(X) + \sum_{i=1}^{S} \Bigg\{ (m_i(X))^2 \times \sum_{\substack{\bigcap_{k=1}^{S-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\cdots,Y_{\sigma_i(S-1)}) \in (2^\Theta)^{S-1}}} \left( \frac{\prod_{j=1}^{S-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{S-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right) \Bigg\}, \tag{28}$$

where $\sigma_i$ counts from 1 to $S$ avoiding $i$:

$$\begin{cases} \sigma_i(j) = j & \text{if} \quad j < i, \\ \sigma_i(j) = j+1 & \text{if} \quad j \geq i. \end{cases} \tag{29}$$

As $Y_i$ is a focal element of expert/source $i$, we have $m(Y_i) > 0$. Then

$$m_i(X) + \sum_{j=1}^{S-1} m_{\sigma_i(j)}\left(Y_{\sigma_i(j)}\right) \neq 0.$$

In Eq. (28), $m_{\text{conj}}$ is the conjunctive rule given by Eq. (19). Here again, the Fast Möbius Transform method to program the belief functions is not generally the best way. If the implementation is made like that in Ref. [46], the time consumption is very high.

# III. A COMBINATION RULE FOR A LARGE NUMBER OF MASS FUNCTIONS

The main idea of the conjunctive combination rule is to reinforce the belief on the focal elements with which most of the sources agree. Martin et al. [10] showed that the mass on the empty set, which is an absorbing element, tends quickly to 1 with the number of sources when combining inconsistent bbas. Consequently, when using Dempster rule (Eq. (20)), the gap between $\kappa$ and 1 may rapidly exceed machine precision, even if the combination is valid theoretically. In that case the fused bba by the conjunctive rules (normalized or not) and the pignistic probability are inefficient. Moreover, the assumption that all the sources are reliable for the conjunctive combination rule is difficult to reach in real applications. The more sources there are, the less chance that this assumption is valid.

The principle of the conjunctive rule with the reinforcement of belief and the role of the empty set as an alarm are essential in the theory of belief functions. In order to propose a rule which can be adapted to the combination of a large number of mass functions and keep the previous behavior, the following assumptions are made:

- The majority of sources are reliable;
- The larger extent one source is consistent with others, the more reliable the source is;
- The sources are cognitively independent [43].

These assumptions seem reasonable if we consider combing mass functions as some kind of group decision making problems. As a result, the proposed rule will give more importance to the groups of mass functions that are in a domain, and it is without auto-conflict [13, 14]. In order to take into account this effect, this rule will discount the mass functions according to the number of sources giving bbas with the same focal elements. The discounting factor is directly given by the proportion of mass functions with the same focal elements. This procedure is for the elicitation of the majority opinion.

The simple support mass functions are considered here. In this case, the mass functions can be grouped in the light of their focal elements (except the frame $\Theta$). To make the rule applicable on separable mass functions, the decomposition process should be performed to decompose each bba into simple support mass functions. In most of applications, the basic belief can be defined using separable mass functions, such as simple support functions [2] and consonant mass functions [51, 52].

Hereafter we describe the proposed LNS-CR rule for simple support functions, and then an approximation calculation method of LNS-CR rule is suggested.

## A. LNS-CR rule for simple support functions

Suppose that each evidence is represented by a SSF. Then all the bbas can be divided into at most $2^n$ groups (where $n = |\Theta|$). It is easy to see that there is no conflict at all in each group because of consistency. The focal elements of the SSF are singletons and $\Theta$ itself. For the combination of bbas inside each group, the conjunctive rule can be employed directly. Then the fused bbas are discounted according to the number of mass functions in each group. Finally, the global combination of the bbas of different groups is preformed also using the conjunctive rule. Suppose that all bbas are defined on the frame of discernment $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, and denoted by $m_j = (A_i)^{w_j}, j = 1, \cdots, S$ and $i = 1, 2, \cdots, c$, where $c \leq 2^n$. The detailed process of the combination is listed as follows. Our proposed rule called LNS-CR for Large Number of Sources rule is composed of the four following steps:

1) Cluster the simple bbas into $c$ groups based on their focal element $A_i$. For the convenience, each class is labeled by its corresponding focal element.
2) Combine the bbas in the same group. Denote the combined bba in group $A_k$ by SSF

$$\hat{m}_k = (A_k)^{\hat{w}_k}, k = 1, 2, \cdots, c.$$

Let the number of bbas in group $A_k$ is $s_k$. If the conjunctive rule is adopted, we have

$$\hat{m}_k = \bigcirc\!\!\!\!\cap_{j=1,\cdots,s_k} m_j = (A_k)^{\prod_{j=1}^{s_k} w_j}. \tag{30}$$

3) Reliability-based discounting. Suppose the fused bba of all the mass functions in $A_k$ is $\hat{m}_k$. At this time, each group can be regarded as a source, and there are $c$ sources in total. The reliability of one source can be estimated as compared to a group of sources. In our opinion, the reliability of source $A_k$ is related to the proportion of bbas in this group. The larger the number of bbas in group $A_k$ is, the more reliable $A_k$ is. Then the reliability discounting factor of $\hat{m}_k$ can be defined as:

$$\alpha_k = \frac{s_k}{\sum_{i=1}^{c} s_i}. \tag{31}$$

In order to keep the mass function representing total ignorance as a neutral element of the rule, in Eq. (31) we let $a_k = 0$ for the group with $A_k = \Theta$. Another version of the discounting can be given by a factor taking into account the precision of the group by:

$$\alpha_k = \frac{\beta_k^{\eta} s_k}{\sum_{i=1}^{c} \beta_i^{\eta} s_i}, \tag{32}$$

where

$$\beta_k = \frac{|\Theta|}{|A_k|}. \tag{33}$$

Parameter $\eta$ can be used to adjust the precision of the combination results. The larger the value of $\eta$ is, the less imprecise the resulting bba is. The discounted bba of $\hat{m}_k$ can be denoted by SSF $\hat{m}_k^{'} = (A_k)^{\hat{w}_k^{'}}$ with $\hat{w}_k^{'} = 1 - \alpha_k + \alpha_k \hat{w}_k$. As we can see, when the number of bbas in one group is larger, $\alpha$ is closer to 1. That is to say, the fused mass in this group is more reliable.
4) Global combine the fused bbas in different groups using the conjunctive rule:

$$m_{\text{LNS-CR}} = \bigcirc\!\!\!\!\cap_{k=1,\cdots,c} \hat{m}_k' = \bigcirc\!\!\!\!\cap_{k=1,\cdots,c} (A_k)^{\hat{w}_k^{'}}. \tag{34}$$

**Remarks:**

- The reliability estimation method proposed here is very simple compared with the previous mentioned methods in Section II-C, where usually the distance between bbas should be calculated or a special learning process is required. In the LNS-CR rule, to evaluate the reliability discounting factor, we only need to count the number of SSFs in each group. Note that other reliability estimation methods can also be used here.
- In the last step of combination, as the number of mass functions that take part in the global combination is small (at most $2^n$), other combination rules such as DP rule and PCR rules are also possible in practice instead of Eq. (34).

### *B. LNSa-CR rule for the approximated combination*

If there is a large number of mass functions in each group, an approximation method is suggested here to calculate the combined mass in the given group. Suppose the mass functions in group with focal element $A_k$ ($k = 1, 2, \cdots, c$) are:

$$m_j(A) = \begin{cases} 1 - w_j & A = A_k, \\ w_j & A = \Theta, \quad 0 \leq w_j < 1, j = 1, 2, \cdots, s_k. \\ 0 & \text{otherwise}, \end{cases} \tag{35}$$

The combination of the masses in this group using the conjunctive rule is

$$\hat{m}_k(A) = \begin{cases} 1 - \prod_{j=1}^{s_k} w_j & A = A_k, \\ \prod_{j=1}^{s_k} w_j & A = \Theta, \\ 0 & \text{otherwise}. \end{cases} \tag{36}$$

It is easy to get

$$\lim_{s_k \to \infty} \hat{m}_k(A) = \begin{cases} 1 & A = A_k, \\ 0 & A = \Theta, \\ 0 & \text{otherwise}. \end{cases} \tag{37}$$

This is an illustration of the conjunctive property. After the discounting with factor $\alpha_k$, the fused bba using for the global combination is

$$\lim_{n_k \to \infty} \hat{m}_k^{'}(A) = \begin{cases} \alpha_k & A = A_k, \\ 1 - \alpha_k & A = \Theta, \\ 0 & \text{otherwise}. \end{cases} \tag{38}$$

It can be represented by SSF

$$\hat{m}_k^{'} = (A_k)^{1-\alpha_k}, \tag{39}$$

where $\alpha_k$ is shown in Eq. (31) or (32). If the conjunctive rule is adopted for the global combination at step 4, the final bba we get is

$$m_{\text{LNSa-CR}} = \bigcirc\!\!\!\!\cap (A_k)^{1-\alpha_k}. \tag{40}$$

In this approximate rule for the large number of sources, the initial mass functions is no longer considered, and the combination process of the bbas inside each group is not required any more. This can accelerate the algorithm to a large extent. The LNS-CR and LNSa-CR rule provide different results when the number of sources is small. However, when the number of sources is large enough, they can be regarded as equivalent.

### C. Properties

The proposed rule is commutative, but not associative. The rule is not idempotent, but there is no absorbing element. The vacuous mass function is a neutral element of the LNS-CR rule.

There are four steps when applying LNS-CR rule†: decomposition (not necessary for simple support mass functions), inner-group combination, discounting and global combination. The LNS-CR rule has the same memory complexity as some other rules such as conjunctive, Dempster and cautious rules if all the rules are combined globally using FMT method. Only DP and PCR6 rules have higher memory complexity because of the partial conflict to manage. Suppose the number of mass functions to combine is $S$, and the number of elements in the frame of discernment is $n$. The complexity for decomposing‡ mass functions to SSFs is $O(Sn2^n)$. For combining the mass functions in each group, due to the structure of the simple support mass functions, we only need to calculate the product of the masses on only one focal element $\Theta$. Thus the complexity is $O(S)$. The complexity of the discounting is $O(2^n)$. In the process of global combination, the bbas are all SSFs. If we use the Fast Möbius Transform method, the complexity is $O(n2^n)$. And there are at most $2^n$ mass functions participating the following discounting and global conjunctive combination processes. Since in most application cases with a large number of mass functions, we have $2^n \ll S$, the last two steps are not very time-consuming. The total complexity of LNS-CR is $O(Sn2^n + S + 2^n + n2^n)$ and so is approximately equivalent to $O(Sn2^n)$.

For the approximate method, we can also save the time for inner combination and the discounting. The fused mass in each group is calculated by the proportions, and the complexity is also $O(S)$. Although the approximate method does not reduce the complexity, in the experimental part, we will show that it will save some running time in applications when $S$ is quite large.

We remark here that one of the assumptions of LNS-CR rule is that the majority of sources are reliable. However, this condition is not always satisfied in every applicative context. Consider here an example with two sensor technologies: TA and TB. The system has two TA-sensors ($S_1$ and $S_2$), and one TB-sensor $S_3$. Suppose also a parasite signal causes TA sensors to malfunction. In this situation, the majority of sensors are unreliable. And we could not get a good result if the LNS-CR rule is used directly as LNS-CR$(S_1, S_2, S_3)$ at this time. Actually there is an underlying hierarchy in the sources of information, LNS-CR rule could be evoked according to the hierarchy, such as LNS-CR(LNS-CR$(S_1, S_2), S_3)$. We will study that more in the future work.

†The source code for LNS-CR rule can be found in R package *ibelief* [53].

‡In the decomposing process, the Fast Möbius Transform method is used.

## IV. Experiments

In this section, several experiments will be conducted to illustrate the behavior of the proposed combination rule LNS-CR and to compare with other classical rules. Some different types of randomly generated mass functions will be used. The function *RandomMass* in R package *ibelief* [53] is adopted to generate random mass functions [54].

**Experiment 1** (Elicitation of the majority opinion). In some applications, the elicitation of the majority opinion is very important. In this experiment, it is assumed that reliable sources can provide some imprecise and uncertain information, which is assumed to be in the form of the mass functions $m_j$ $(j = 1, 2, \cdots, 6)$ over the same discernment frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$:

$$
\begin{aligned}
m_1 &: m_1(\{\theta_1\}) = 0.12, \ m_1(\Theta) = 0.88,\\
m_2 &: m_2(\{\theta_1\}) = 0.16, \ m_2(\Theta) = 0.84,\\
m_3 &: m_3(\{\theta_1\}) = 0.15, \ m_3(\Theta) = 0.85,\\
m_4 &: m_4(\{\theta_1\}) = 0.11, \ m_4(\Theta) = 0.89,\\
m_5 &: m_5(\{\theta_1\}) = 0.14, \ m_5(\Theta) = 0.86,\\
m_6 &: m_6(\{\theta_2\}) = 0.95, \ m_6(\Theta) = 0.05.
\end{aligned}
$$

As can be seen, the first five sources share similar belief (supporting $\{\theta_1\}$) whereas the sixth one delivers a mass function strongly committed to another solution (supporting $\{\theta_2\}$). These six mass functions cannot be regarded as conflicting, because the majority of evidence shows the preference of $\{\theta_1\}$. Here, source 6, is assumed not reliable since it contradicts with all the other sources.

The combination results by conjunctive rule, Dempster rule, disjunctive rule, DP rule, PCR6 rule, cautious rule, average rule and the proposed LNS-CR rule§ are depicted in Table I. As can be observed, the conjunctive rule assigns most of the belief to the empty set, regarding the sources as highly conflictual. Dempster rule, DP rule, PCR6 rule and average rule redistribute all the global conflict to other focal elements. The disjunctive rule gives the total ignorance mass functions. The cautious rule and the proposed LNS-CR rule keep some of the conflict and redistribute the remaining. But the belief given to $\{\theta_2\}$ is more than that to $\{\theta_1\}$ when using Dempster, DP, PCR6, cautious and the average rules, which indicates that these rules are not robust to the unreliable evidence. The obtained fused bba by the proposed rule assigns the largest mass to focal element $\{\theta_1\}$, which is consistent with the intuition. It keeps a certain level of global conflict, and at the same time reflects the superiority of $\{\theta_1\}$ compared with $\{\theta_2\}$. From the results we can see that only the LNS-CR rule can correctly elicit the major opinion.

The LNS-CR rule is a conjunctive based combination rule for mass functions with different reliability degrees. As mentioned before, the principle of the LNS-CR rule is similar that of Schubert's method [32]. Table II lists the results by Schubert's combination method with different values of $k$. As can be seen, the result by the use of the LNS-CR rule is similar to that by Schubert's method with a small value of

§As the focal elements are singletons except $\Theta$, parameter $\eta$ has no effects on the final results when using LNS-CR rule.

TABLE I
THE COMBINATION OF SIX MASSES. FOR THE NAMES OF COLUMNS, $\theta_{ij}$ IS USED TO DENOTE $\{\theta_i, \theta_j\}$.

| | Conjunctive | Dempster | Disjunctive | DP | PCR6 | Cautious | Average | LNS-CR |
|---|---|---|---|---|---|---|---|---|
| $\emptyset$ | 0.49313 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.15200 | 0.00000 | 0.06849 |
| $\{\theta_1\}$ | 0.02595 | 0.05120 | 0.00000 | 0.02595 | 0.04783 | 0.00800 | 0.11333 | 0.36408 |
| $\{\theta_2\}$ | 0.45687 | 0.90136 | 0.00000 | 0.45687 | 0.56639 | 0.79800 | 0.15833 | 0.08984 |
| $\{\theta_1, \theta_2\}$ | 0.00000 | 0.00000 | 0.00004 | 0.49313 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_1, \theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_2, \theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\Theta$ | 0.02405 | 0.04744 | 0.99996 | 0.02405 | 0.38578 | 0.04200 | 0.72833 | 0.47759 |

threshold $k$. When $k$ is set small, the discounting process in Schubert's method needs more steps. And in each step, the conjunctive rule should be evoked to calculate the falsity. It is more complex compared with the reliability estimation process of the LNS-CR rule in that sense.

TABLE II
THE COMBINATION OF SIX MASSES BY SCHUBERT'S METHOD WITH DIFFERENT VALUES OF $k$.

| $k$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| $\emptyset$ | 0.09776 | 0.19471 | 0.28680 | 0.37803 | 0.46444 |
| $\{\theta_1\}$ | 0.32187 | 0.26219 | 0.19350 | 0.12081 | 0.04980 |
| $\{\theta_2\}$ | 0.13521 | 0.23145 | 0.31033 | 0.37979 | 0.43871 |
| $\{\theta_1, \theta 2\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_1, \theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_2, \theta 3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\Theta$ | 0.44516 | 0.31165 | 0.20937 | 0.12137 | 0.04704 |

We also compare with another reliability discounting based combination method proposed by Martin et al. [10]. Same as Schubert's method, after the reliability degree of each source is estimated, the bbas are discounted following with a conjunctive combination. There is a parameter $\lambda$ in the method to adjust the discounting factor. The results varying with different values of $\lambda$ are shown in Table III. We can see this rule is similar to LNS-CR rule when $\lambda$ is set to be around 1. When $\lambda$ is not well set, the results are not good. Moreover, in this method, the distance between bbas should be calculated first. Consequently, it increases the complexity and makes the method not feasible for combining a large number of sources.

TABLE III
THE COMBINATION OF SIX MASSES BY MARTIN'S METHOD WITH DIFFERENT VALUES OF $\lambda$.

| $\lambda$ | 0.1 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $\emptyset$ | 0.00000 | 0.00350 | 0.10485 | 0.23330 | 0.31956 |
| $\{\theta_1\}$ | 0.00000 | 0.21206 | 0.34700 | 0.26789 | 0.19410 |
| $\{\theta_2\}$ | 0.00000 | 0.01272 | 0.12719 | 0.23219 | 0.30256 |
| $\{\theta_1, \theta_2\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_1, \theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\{\theta_2, \theta_3\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\Theta$ | 1.00000 | 0.77172 | 0.42096 | 0.26661 | 0.18378 |

**Experiment 2** (The discounting mechanism). In this experiment, we will discuss the reliability discounting mechanism of the LNS-CR rule. Two reliability discounting methods proposed by Schubert [32] and Martin et al. [10] will be used to compare. Same as the LNS-CR rule, after the discounting process by these two methods, the conjunctive rule is adopted to combine the new mass functions. For simplicity, here we call the combination rule, where the Schubert's discounting method (or Martin's discounting method) is first evoked and then the conjunctive combination rule is used, "Schubert's method" (Martin's method, correspondingly). A set of $3 * x$ bbas on a frame of discernment $\Theta = \{\theta_1, \theta_2\}$ are generated, $x$ of them are unreliable while $2 * x$ are reliable. The reliable sources assign a large mass to the singleton $\{\theta_1\}$. The unreliable sources assign a large mass to the singleton $\{\theta_2\}$. The gain factor for sequential discounting in Schubert's method is set to be 0.1 here. Schubert and Martin's methods are evoked with different values of $k$ and $\lambda$ respectively. Let $x = 10$, the fused bbas by the use of different rules are listed in Table IV.

From the table we can see, the behavior of Martin's discounting method is similar to that of LNS-CR rule when $\lambda$ is set around 0.4. The conjunctive combination based on Schubert's discounting does not give any belief to $\{\theta_2\}$ and $\Theta = \{\theta_1, \theta_2\}$ at all although there are $1/3$ of sources supporting $\{\theta_2\}$. Moreover, when $k$ is larger, most of the mass is assigned to the empty set in this rule. From these results we can see that only LNS-CR rule can give more belief on $\{\theta_1\}$ which can be regarded as the major opinion. The time elapsed for Schubert's method with different values of threshold $k$ is listed in Table V. The smaller the value of $k$ is, the more discounting steps are required in Schubert's method. Consequently, the time consumption becomes larger. The running time for both LNS-CR rule and Martin's method is less than one second. Schubert's method is much more time-consuming.

We have also tested the combination methods based on the discounting factors proposed by Schubert [32] and Martin et al. [10] on some simple support mass functions with arbitrary focal elements. The results are not shown here as we can get similar conclusions from the results: The reliability estimation process of these methods takes more time compared with that of LNS-CR rule. The behavior of these two methods is similar to that of LNS-CR rule when the parameter $k$ or $\lambda$ is set to be in a fixed range. But they are much more time-consuming compared with LNS-CR rule. This confirms that the reliability discounting method in LNS-CR rule is effective for the following conjunctive combination.

**Experiment 3** (The influence of parameter $\eta$). We test here the

TABLE IV
THE COMBINATION RESULTS BY DIFFERENT RULES.

| | Schubert's method | | | | Martin's method | | | | LNS-CR |
|---|---|---|---|---|---|---|---|---|---|
| | $k=0.2$ | $k=0.3$ | $k=0.5$ | $k=0.7$ | $\lambda=0.3$ | $\lambda=0.4$ | $\lambda=0.6$ | $\lambda=1$ | |
| $\emptyset$ | 0.19949 | 0.29860 | 0.49704 | 0.69306 | 0.00248 | 0.10019 | 0.60681 | 0.98649 | 0.15060 |
| $\{\theta_1\}$ | 0.80051 | 0.70140 | 0.50296 | 0.30694 | 0.16901 | 0.56713 | 0.38729 | 0.01351 | 0.48612 |
| $\{\theta_2\}$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.01200 | 0.04995 | 0.00360 | 0.00000 | 0.08593 |
| $\Theta$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.81650 | 0.28274 | 0.00230 | 0.00000 | 0.27735 |

TABLE V
TIME ELAPSED FOR SCHUBERT'S METHOD WITH DIFFERENT VALUES OF $k$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| Time Elapsed (s) | 46.81 | 21.64 | 13.46 | 9.28 | 6.64 | 4.88 | 3.67 | 2.73 | 1.79 |

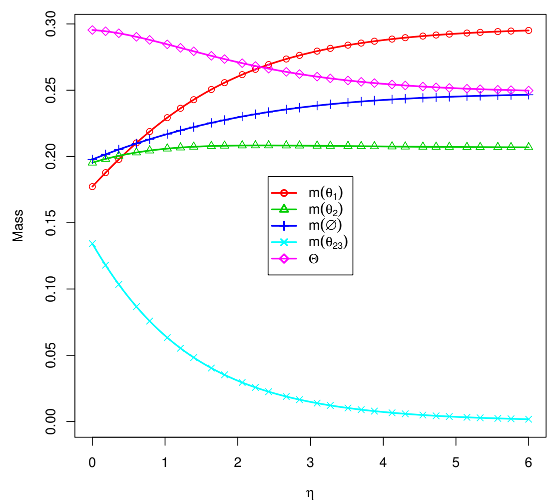
influence of parameter $\eta$ in the LNS-CR rule. Simple support mass functions are utilized in this experiment. Suppose that the discernment frame under consideration is $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Three types of SSFs are adopted. First $s_1 = 60$ and $s_2 = 50$ SSFs with focal elements $\{\theta_1\}$ and $\{\theta_2\}$ respectively (the other focal element is $\Theta$) are uniformly generated, and then $s_3 = 50$ SSFs with focal element $\theta_{23} \triangleq \{\theta_2, \theta_3\}$ are generated. The value of masses are randomly generated. Different values of $\eta$ (see Eq. (32)) ranging from 0 to 6 are used to test. The mass values in the fused bba by LNS-CR varying with $\eta$ are displayed in Figure 1.a, and the corresponding pignistic probabilities are shown in Figure 1.b.

From these figures, we can see that $\eta$ can have some effects on the final decision. Figure 1.a shows that with the increasing of $\eta$, the mass assigned to the singleton focal elements increases. On the contrary, the mass given to the focal element whose cardinality is bigger than one decreases. In fact parameter $\eta$ in LNS-CR aims at weakening the imprecise evidence which gives only positive mass to focal elements with high cardinality, and the exponent $\eta$ allows to control the degree of discounting. If $\eta$ is larger, more weight is given to the sources of evidence whose focal elements are more specific, and more discount will be committed to the imprecise evidence. As a result, in the experiment when $\eta$ is larger than 1.2, $\mathrm{BetP}(\theta_1) > \mathrm{BetP}(\theta_2)$ (Figure 1.b). At this time the mass functions with focal element $\{\theta_2, \theta_3\}$ make little contribution to the fusion process, while the final decision mainly depends on the other two types of simple support mass functions with singletons as focal elements.

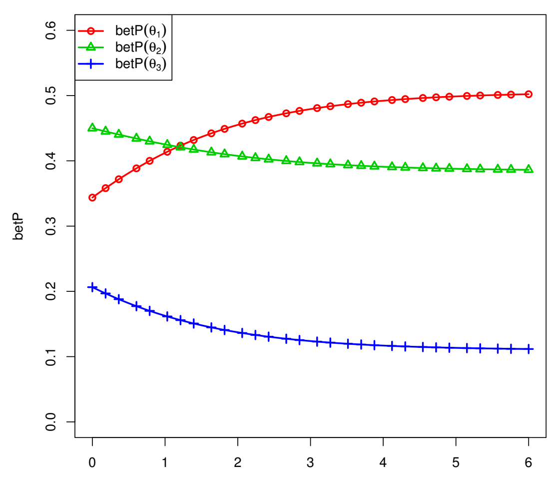
In real applications, $\eta$ could be determined based on specific requirement. This work is not specially focusing on how to determine $\eta$, thus in the following experiment we will set $\eta = 1$ as default.

**Experiment 4** (The principle for the global conflict). The goal of this experiment is to show how Dempster's degree of conflict is dealt with by most of rules when combining a large number of conflicting sources.

In this experiment, the frame of discernment is set to $\Theta = \{\theta_1, \theta_2\}$. Assume that there are only 2 focal elements on each bba. One is the whole frame $\Theta$, and the other is any of the singletons ($\{\theta_1\}$ or $\{\theta_2\}$). The number of bbas which




a. bba




b. Pignistic probability

Fig. 1. Combination results for three types of SSFs using LNS-CR rule. The mass functions are generated randomly, and LNS-CR rule is evoked with different values of $\eta$ ranging from 0 to 6.

have the focal element $\{\theta_1\}$ is denoted by $s_1$, while that with $\{\theta_2\}$ is $s_2$. We first fix the value of $s_2$, and let $s_1 = t * s_2$, with $t$ a positive integer. We generate $S = s_1 + s_2$ such kind of bbas randomly, but only withholding the bbas for which the mass value assigned to $\{\theta_1\}$ or $\{\theta_2\}$ is greater than 0.5.

Four values of $t$ are considered here: $t = 1, 2, 3, 4$. If $t = 1$, $s_1 = s_2 = S/2$. If $t = 2$, the number of mass functions supporting $\{\theta_1\}$ is two times of that supporting $\{\theta_2\}$, and so on. The global conflict (mass given to the empty set) after the combination with different values of $s_2$ for the four cases is displayed in Figures 2– 5 respectively. The mass assigned to the focal element $\{\theta_1\}$ with different combination approaches is shown in Figures 6 – 9.

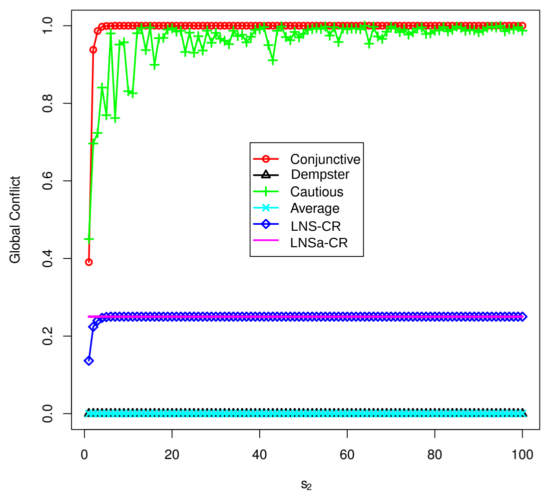



Fig. 2. The global conflict after the combination with $s_2$ ranging from [0,100] and $s_1 = s_2$.

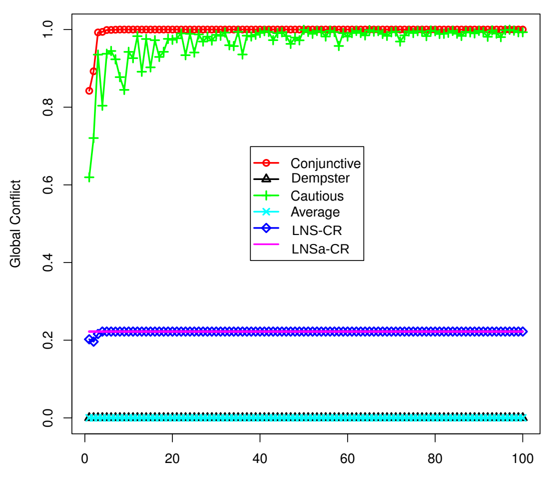



Fig. 3. The global conflict after the combination with $s_2$ ranging from [0,100] and $s_1 = 2 * s_2$.

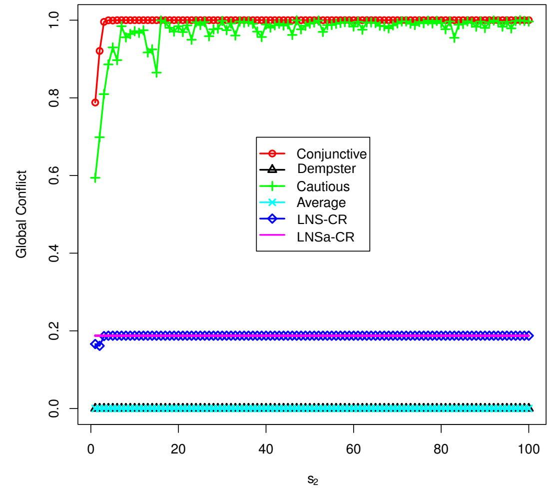



Fig. 4. The global conflict after the combination with $s_2$ ranging from [0,100] and $s_1 = 3 * s_2$.

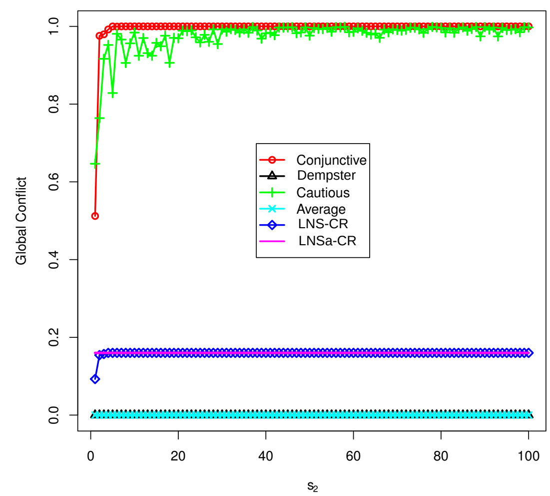



Fig. 5. The global conflict after the combination with $s_2$ ranging from [0,100] and $s_1 = 4 * s_2$.

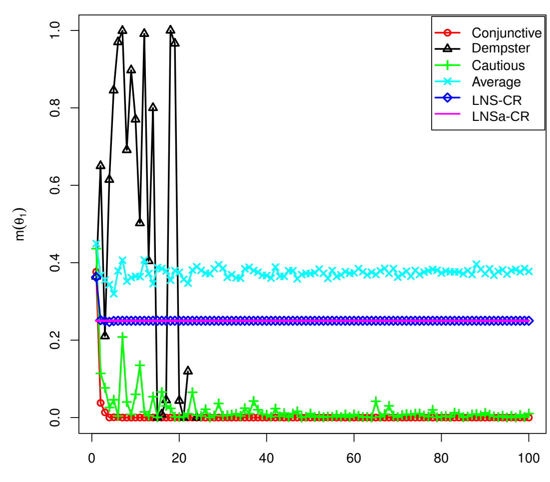



Fig. 6. The mass on $\{\theta_1\}$ after the combination with $s_2$ ranging from [0,100] and $s_1 = s_2$.

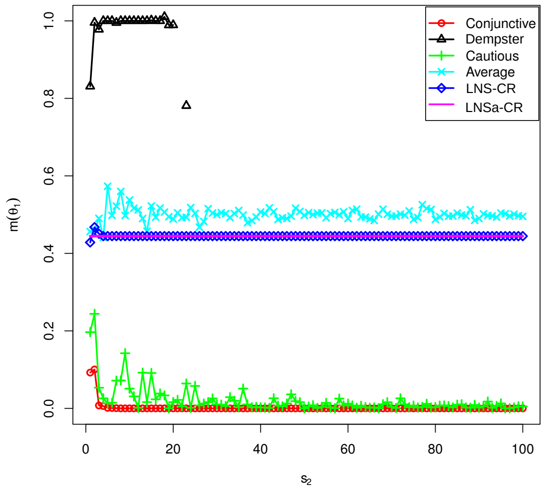



Fig. 7. The mass on $\{\theta_1\}$ after the combination with $s_2$ ranging from [0,100] and $s_1 = 2 * s_2$.

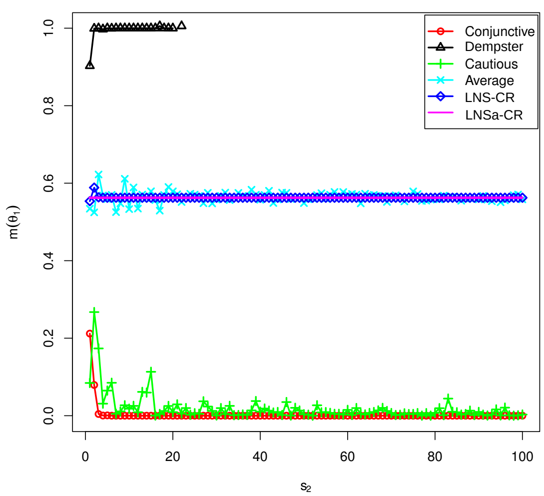



Fig. 8. The mass on $\{\theta_1\}$ after the combination with $s_2$ ranging from [0,100] and $s_1 = 3 * s_2$.

It is intuitive that when $t$ becomes larger, the global conflict should be smaller and we should give more belief to the focal element $\{\theta_1\}$. From Figures 2 – 9 we can see that only the results by LNS-CR rule are in accordance with this common sense. The simple average rule assigns larger bba to $\{\theta_1\}$, but it does not keep any conflict. In Figures 6 – 9, the mass given to $\{\theta_1\}$ by Dempster rule cannot be displayed when $S$ is large (and also for some small $S$), because in these cases the global conflict is 1 and the normalization could not be processed. As we can see, Dempster rule could not work at all when $s_2$ is larger than 20. Although the conjunctive rule and cautious rule could work when combining a larger number of mass functions, the obtained fused mass function is $m(\emptyset) \approx 1$, which is useless for decision in practical situations.

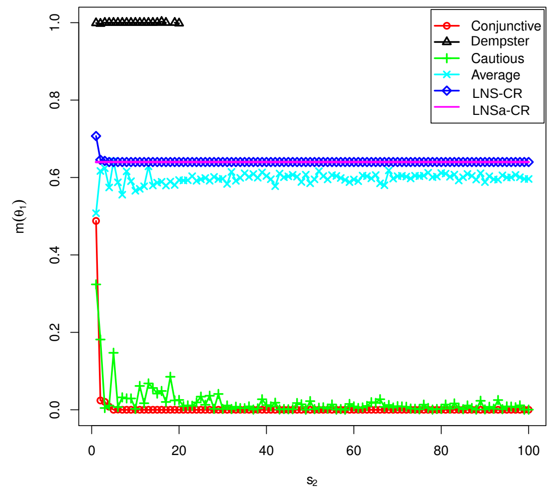



Fig. 9. The mass on $\{\theta_1\}$ after the combination with $s_2$ ranging from [0,100] and $s_1 = 4 * s_2$.

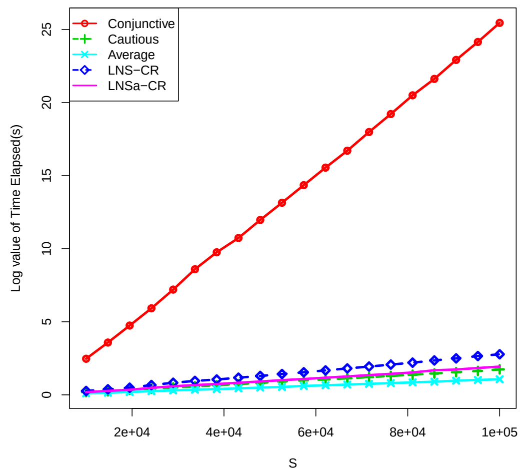
The results also confirm the equivalent of the LNS-CR rule and LNSa-CR rule when the number of sources is large, although the results provided by the two rules are not the same when there are not many mass functions to combine. From Figures 2 – 5 we can see a kind of limit of the global conflict for the LNS-CR rule. In fact, the mass on the empty set for this rule depends on the size of the frame of discernment and more directly on the number of groups created in the first step of the rule. The limit value of the global conflict will tend to 1 with the increase of the size of discernment when considering only categorical bbas on different singletons.

**Experiment 5** (The complexity). In this experiment, the complexity of LNS-CR rule will be compared with other combination rules in terms of time consumption. Simple support mass functions defined on a frame of discernment with eight elements are considered first. The focal elements of each bba are set to be a random subset of $\Theta$ and $\Theta$ itself. The time elapsed (and also the $\log$ value of the time elapsed) with the number of sources $S$ varying from 10,000 to 100,000 is shown in Figure 10¶. We can see that the running time of LNS-CR is much smaller than that of the conjunctive rule. LNSa-CR rule takes almost the same time as cautious rule. Average rule is the best among the five rules. As $S$ increases, the application of LNSa-CR rule can save more time compared with the use of LNS-CR rule. The increment of time consumption with respect to $S$ is moderate. This tends to show that LNS-CR rule is suitable for combining a large number of SSFs. Remark that the decomposition process is not required when the cautious rule or LNS-CR(a) rule is adopted for combining SSFs.

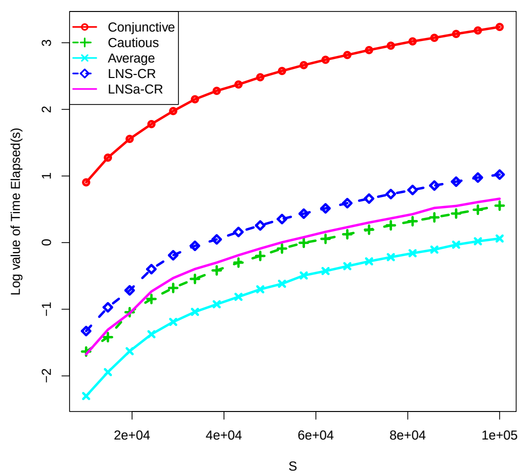
As mentioned before, for the combination of general separable mass functions (not SSFs), LNS-CR needs four steps: decomposition, inner-group combination, discounting and global combination. The difference between the combination of any kind of separable bbas and of SSFs is the decomposition process, which is not necessary for the latter. We have designed another experiment on consonant bbas‖ over a frame of

¶The result of Dempster rule is the same as that of conjunctive rule.

‖All consonant bbas are separable.

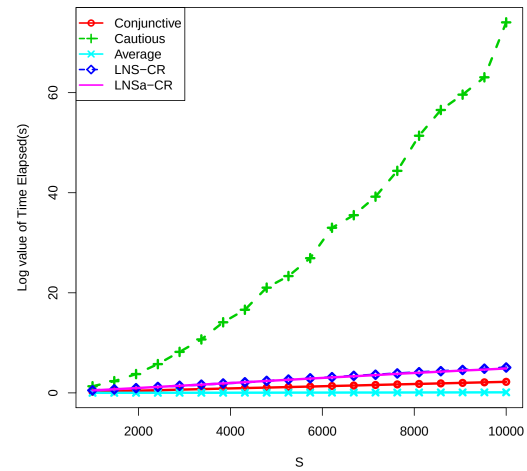



a. Time lapse by five different rules

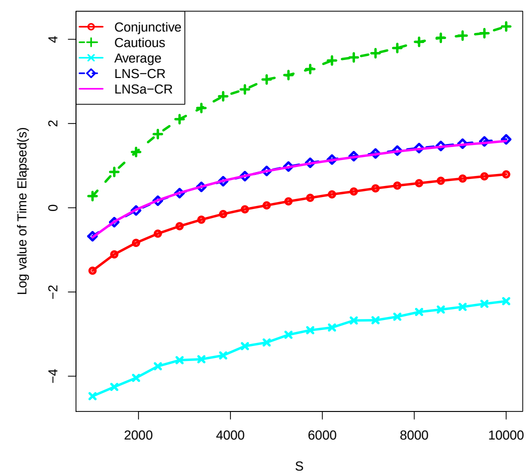



b. The log value of Time lapse by five different rules

Fig. 10. Time lapse for combining SSFs.




a. Time lapse by five different rules




b. The log value of Time lapse by five different rules

Fig. 11. Time lapse for combining consonant bbas.

discernment with eight elements, and the number of focal elements is set to 5. The focal elements are randomly set to five nested subsets of $\Theta$, and the mass values are generated uniformly. The average running time (and the $\log$ value of the running time) of 10 trials by the use of different combination rules with different number of sources $S$ is displayed in Figure 11.a (and Figure 11.b)**. In order to show the complexity of LNS-CR rule more clearly, the elapsed time in each of the four steps is shown in Figure 12.

As we can see from these figures, the time consumption of LNS-CR is significantly smaller than the cautious rule, but a little worse than the conjunctive rule and the average rule. Although the complexity of cautious rule is the same as LNS-CR rule and both of them require a decomposition process, it takes more running time than LNS-CR rule. The reason may be the different combination approach for the mass functions in the same group. The complexity of that process by cautious rule is $O(S2^n)$ (The calculation is to find the minimum of each row in a $S \times 2^n$ matrix), while for LNS-CR is $O(S)$. LNSa-CR is faster than LNS-CR when $S$ is large. Figure 12 shows that the most time-consuming step in

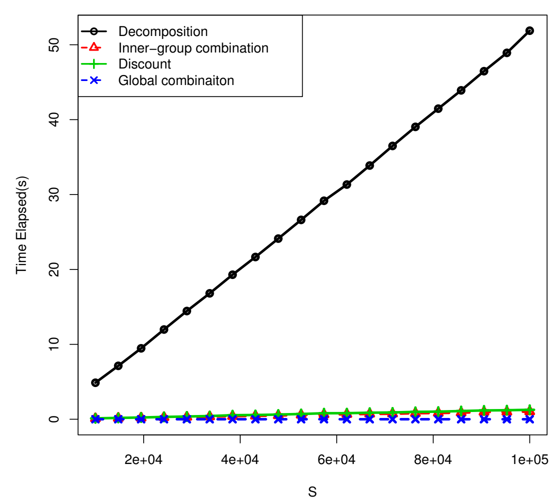



Fig. 12. Time lapse of each step using LNS combination rule with $S$ varying from 10,000 to 100,000.

**The result of cautious rule is not displayed for large $S$, as it has been already shown that cautious rule is significantly worse than the other rules in terms of time consumption when $S$ is small.

LNS-CR rule is the decomposition. Moreover as $S$ increases, the increase of time lapse for the inner-group combination, discount, and global combination is limited. This is compliant with the complexity analysis of each step for LNS-CR rule in Section III-C. In many applications the mass functions are directly SSFs in which case there is no need to perform the decomposition, and LNS-CR is the best choice to fuse a large number of bbas.

## V. PERSPECTIVE ON APPLICATIONS

Pattern recognition is a class of problems where the theory of belief functions has proved to allow increased performances [2]. In such problems we can be facing many bbas to combine. Denœux [2] proposed Evidential KNN method (EKNN) as an extension of KNN in the framework of the theory of belief functions to better model the uncertainty in neighbor point interactions. The Dempster rule is adopted to combine the mass evidence from $K$ neighbors in EKNN.

The problem considered here is to classify an input pattern $\boldsymbol{x}$ into $n$ categories or classes, denoted by $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$. The available information is assumed to consist of a training set $\mathcal{L} = \left\{(\boldsymbol{x}^{(1)}, \theta^{(1)}), (\boldsymbol{x}^{(2)}, \theta^{(2)}), \cdots, (\boldsymbol{x}^{(N)}, \theta^{(N)})\right\}$ of $N$ patterns $\boldsymbol{x}^{(i)}$ $i = 1, 2, \cdots, N$ with known class labels $\theta^{(i)} \in \Theta$. To classify pattern $\boldsymbol{x}$, each pair $(\boldsymbol{x}^{(i)}, \theta^{(i)})$ constitutes a distinct item of evidence regarding the class membership of $\boldsymbol{x}$. If the $K$ nearest neighbors according to the distance measure are considered, $K$ items of evidence can be obtained. These bbas can be constructed according to a relevant metric between pattern $\boldsymbol{x}$ and its $j^{\text{th}}$ neighbor $\boldsymbol{x}^{(i)}$

$$\begin{aligned} &m_i(\{\theta_q\}) = \alpha\phi(d^{(i)}), \\ &m_i(\Theta) = 1 - \alpha\phi(d^{(i)}), \\ &m_i(A) = 0 \quad \forall A \in 2^{\Theta} \setminus \{\{\theta_q\}, \Theta\}, \end{aligned} \tag{41}$$

where $d^{(i)}$ is the (Euclidean) distance between $\boldsymbol{x}$ and its $j^{\text{th}}$ neighbor $\boldsymbol{x}^{(i)}$ with class label $\theta^{(i)} = \theta_q$, $\alpha$ is a discounting parameter and $\phi(\cdot)$ is a decreasing function on $\mathbb{R}^+$ defined as

$$\phi(d^{(i)}) = \exp\left(-\gamma_q \left(d^{(i)}\right)^2\right) \tag{42}$$

with $\gamma_q$ being a positive parameter associated to class $\theta_q$. It can be heuristically set to the inverse of the mean Euclidean distance between training data belonging to class $\theta_q$. In EKNN, the $K$ bbas for each neighbor are aggregated using the Dempster rule to form a resulting bba. A decision has to be made regarding the assignment of sample $\boldsymbol{x}$ to one individual class. The maximum of pignistic probability can be used for decision-making.

### A. A small data set with noisy training sample

Figure 13 illustrates a simple two-class (red circle and green triangle) data set, where there are seven objects in each class. The pattern $\boldsymbol{x}$ marked by blue star is the sample data to be classified. The $K$ bbas using the distance to its neighbor could be constructed by Eq. (41), and the five nearest neighbors are denoted by $N_i$ orderly in the figure. Set $\alpha = 0.95$ and $\gamma_i$ is the inverse of the average distance between the points in class $\theta_i$, $i = 1, 2$. The fused mass function by different combination rules with $K = 4$ and $K = 5$ are listed in Table VI and VII respectively.

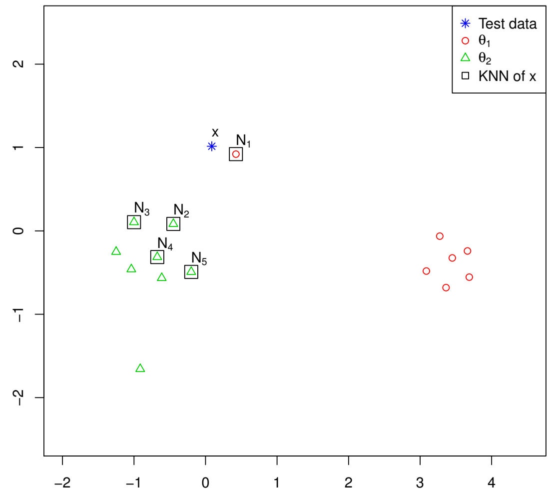



Fig. 13. A small data set.

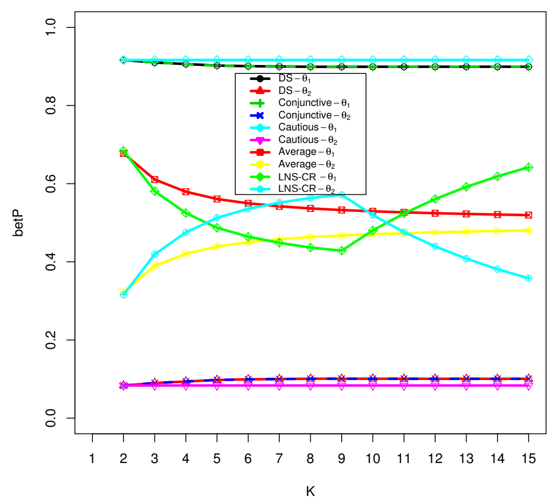



Fig. 14. Pignistic probability.

As we can see from Figure 13, pattern $\boldsymbol{x}$ is closer to class $\theta_2$. Among pattern $\boldsymbol{x}$'s five nearest neighbor $N_j, j = 1, 2, \cdots, 5$, four belong to class $\theta_2$ while only 1 to class $\theta_1$. The real class of object $N_1$ is $\theta_1$, but it is located in the boundary of the class and far from the other data points in the class. It may be a noisy item of $\theta_1$. The standard KNN rule can correctly classify object $x$ to $\theta_2$ when $K > 3$. However, if the evidential KNN model is applied, due to the existence of a such neighbor, the behavior of the combination rules has been affected. From Table VI we can see, when $K = 4$, the fused bbas by all combination rules all assign more mass to $\theta_1$ than to $\theta_2$. Consequently, pattern $\boldsymbol{x}$ will be classified into class $\theta_1$ if the pignistic probability is considered for making decision. The same phenomenon also occurs when $K$ is smaller than 4 (see Figure 14). When $K = 5$ (Table VII), only the LNS-CR rule could partition pattern $\boldsymbol{x}$ into class $\theta_2$, which seems more reasonable. The pignistic probabilities (Figure 14) by the Dempster, conjunctive, cautious and average rules for class $\theta_1$ are significantly higher than those for class $\theta_2$, even when $K$ is large. These rules are not robust to the noisy training data. Pattern $\boldsymbol{x}$ could be correctly classified to $\theta_2$ by LNS-CR rule when $K$ is between 5 and 10.

It is indicated that when there are some noisy data in the training data set, the performance of the combination rule may become worse with small $K$. We should increase $K$ moderately to improve the performance of the classifier. But as we analyzed before, the existing combination rules do not work well for aggregating a large number of mass functions. This is a limit of the use of evidential classifier.

TABLE VI
THE FUSED BBA BY DIFFERENT COMBINATION RULES ($K = 4$).

| | Conjunctive | Dempster | Cautious | Average | LNS-CR |
|---|---|---|---|---|---|
| $\emptyset$ | 0.2009 | 0.0000 | 0.1473 | 0.0000 | 0.0377 |
| $\{\theta_1\}$ | 0.6771 | 0.8473 | 0.7307 | 0.2195 | 0.1818 |
| $\{\theta_2\}$ | 0.0279 | 0.0349 | 0.0205 | 0.0606 | 0.1339 |
| $\Theta$ | 0.0941 | 0.1177 | 0.1015 | 0.7199 | 0.6466 |

TABLE VII
THE FUSED BBA BY DIFFERENT COMBINATION RULES ($K = 5$).

| | Conjunctive | Dempster | Cautious | Average | LNS-CR |
|---|---|---|---|---|---|
| $\emptyset$ | 0.2198 | 0.0000 | 0.1473 | 0.0000 | 0.0352 |
| $\{\theta_1\}$ | 0.6582 | 0.8436 | 0.7307 | 0.1756 | 0.1404 |
| $\{\theta_2\}$ | 0.0305 | 0.0391 | 0.0205 | 0.0541 | 0.1651 |
| $\Theta$ | 0.0915 | 0.1172 | 0.1015 | 0.7703 | 0.6593 |

### B. Real data sets

In this section, we consider some well known real data sets from the UCI repository[††] summarized in Table VIII. The classification rates by using different combination rules in evidential KNN model are displayed in Figure 15. Note that the "leave-one-out" method is adopted here to test the classifier.

TABLE VIII
A SUMMARY OF UCI DATA SETS.

| Data set | No. of objects | No. of cluster | No. of attributes |
|---|---|---|---|
| Iris | 150 | 3 | 4 |
| Yeast | 1484 | 10 | 8 |
| Digits | 5620 | 10 | 64 |

As we can see from Figure 15, for all the three data sets, the performance is almost the same for the two combination rules, LNS-CR and DS, in terms of classification rates. But there is a little improvement by the use of LNS-CR rule when $K$ is large. To make it clear, we specially depict the results on Digits data set in Figure 16. It is shown that when $K > 12$, the classification rates by the use LNS-CR rule are a little larger than those through DS rule. We show the mass given to the empty set (global conflict) after the combination using conjunctive rule and LNS-CR rule with different values of $K$ in Figure 17. The $y$-axis is the maximal assignment to $\emptyset$ among all the mass functions for the test data. As we can see, the global conflict tends to 1 quickly as $K$ increases, while LNS-CR rule keeps a moderate degree of global conflict. As DS rule is a normalized conjunctive rule, there is not sense to normalize a mass assignment with high global conflict.

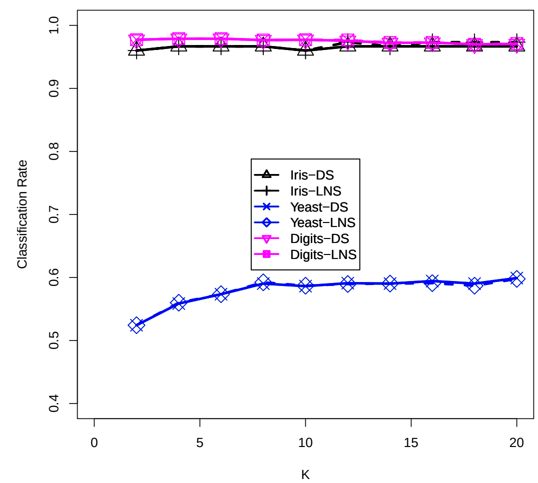



Fig. 15. Classification results with different values of $K$ on UCI data set. In the figure, the legend "Iris-DS" means it is the classification rates on Iris data set using DS combination rule. Same as the other legends.

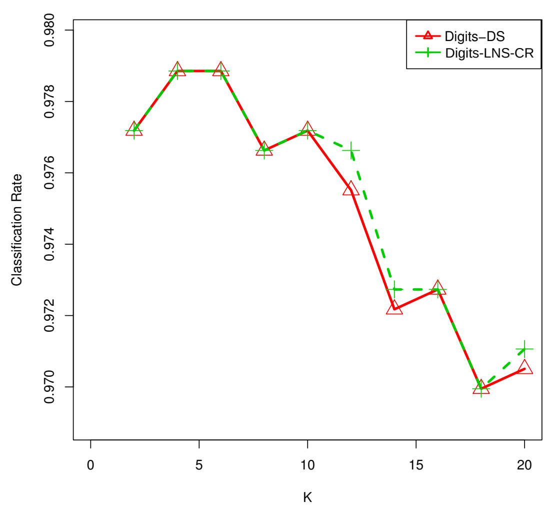



Fig. 16. Classification rates on Digits data set.

### C. Perspective

The above two examples are just two perspectives on the application of LNS-CR rule. In the first example, there are some special noisy data in the training data set. At this time, the sources should not be considered with equal reliability.

[††]http://archive.ics.uci.edu/ml/datasets.html

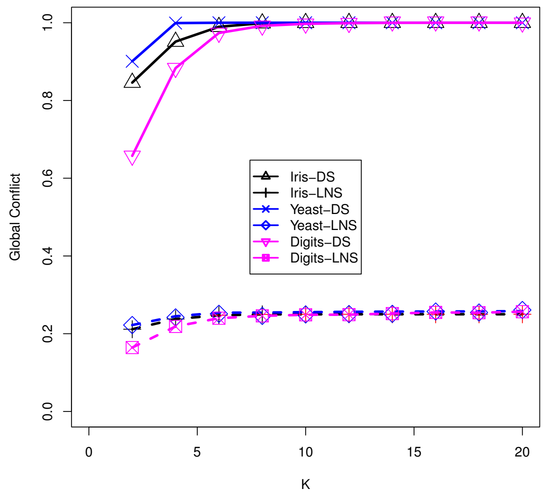



Fig. 17. Global conflict using conjunctive rule and LNS-CR rule varying with different values of $K$. In the figure, the legend "Iris-DS" means it is the conflict on Iris data set using DS combination rule. Same as the other legends.

In this situation, using the DS rule or the conjunctive rule in EKNN model could not get good results. In the second example, it is shown that the global conflict may tend to one quickly as $K$ increases. Sometimes we even could not do the normalization process for DS rule because of the machine precision.

In real world social networks, the available information can be uncertain, or even noisy. At this time, if we want to do a classification task such as for recommendation, the conjunctive rule could not be applied as the sources are not all reliable. Even if the sources are reliable, the global conflict may tend to 1 quickly if the bbas are not consistent. At this time, LNS-CR rule can be an alternative choice. In the future work, we will study how Dempster's degree of conflict is distributed in the feature space, and to study what special information contained in the moderate degree of global conflict kept by LNS-CR rule.

## VI. Conclusion

Uncertainty in big data applications has attracted more and more attention. The theory of belief functions is one of the uncertainty theories allowing a model to deal with imprecise and uncertain information. This theory is also well designed for information fusion. However, despite that a lot of combination rules have been proposed in recent years in this framework, they are not able to combine a large number of sources because of the complexity or the absorbing element.

In this paper, a new combination rule, named LNS-CR rule, preserving the principle of the conjunctive rule is proposed. This rule considers the mass functions given by the sources and groups them according to their set of focal elements (without auto-conflict). The mass functions of each group can be summarized by one mass function after combination. The reliability of the source is estimated by the proportion of bbas in one group. Therefore, after discounting the mass function of each group by the reliability factor, the final combination can be proceeded by the conjunctive rule (or another rule according to the application). If the number of sources in each group is high enough, an approximation method is presented.

The LNS-CR rule is able to combine a large number of sources. The only existing method allowing to combine a large number of mass functions is the average rule. However, that rule may give more importance to few sources with a high belief (even if the source is not reliable) and cannot capture the conflict between the sources. The proposed rule with a reasonable complexity (lower than the DP and PCR6 rules) can provide good combination results.

Overall, this work provides a perspective for the application of belief functions on big data. We will study how to apply LNS-CR rule on the problems of social network and crowdsourcing in the future research work.

## Acknowledgements

This work was supported by the National Natural Science Foundation of China (Nos.61701409, 61135001, 61403310, 61672431), the Natural Science Basic Research Plan in Shaanxi Province of China (No.2018JQ6005), and the Fundamental Research Funds for the Central Universities of China (No.3102016QD088).

## References

[1] K. Zhou, A. Martin, and Q. Pan, "Evidence combination for a large number of sources," in *20th International Conference on Information Fusion*. IEEE, 2017, pp. 1–8.

[2] T. Denœux, "A $k$–nearest neighbor classification rule based on dempster-shafer theory," *Systems, Man and Cybernetics, IEEE Transactions on*, vol. 25, no. 5, pp. 804–813, 1995.

[3] X. Deng, Q. Liu, Y. Deng, and S. Mahadevan, "An improved method to construct basic probability assignment based on the confusion matrix for classification problem," *Information Sciences*, vol. 340, pp. 250–261, 2016.

[4] M.-H. Masson and T. Denœux, "ECM: An evidential version of the fuzzy $c$-means algorithm," *Pattern Recognition*, vol. 41, no. 4, pp. 1384–1397, 2008.

[5] K. Zhou, A. Martin, Q. Pan, and Z.-G. Liu, "Ecmdd: Evidential $c$-medoids clustering with multiple prototypes," *Pattern Recognition*, vol. 60, pp. 239 – 257, 2016.

[6] K. Zhou, A. Martin, Q. Pan, and Z.-g. Liu, "Median evidential $c$-means algorithm and its application to community detection," *Knowledge-Based Systems*, vol. 74, pp. 69–88, 2015.

[7] P. Smets, "Analyzing the combination of conflicting belief functions," *Information Fusion*, vol. 8, pp. 387–412, 2007.

[8] ——, "The combination of evidence in the transferable belief model," *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, vol. 12, no. 5, pp. 447–458, 1990.

[9] P. Smets and R. Kennes, "The transferable belief model," *Artificial intelligence*, vol. 66, no. 2, pp. 191–234, 1994.

[10] A. Martin, A.-L. Jousselme, and C. Osswald, "Conflict measure for the discounting operation on belief functions," in *Information Fusion, 2008 11th International Conference on*. IEEE, 2008, pp. 1–8.

[11] W. Liu, "Analyzing the degree of conflict among belief functions," *Artificial Intelligence*, vol. 170, no. 11, pp. 909–924, 2006.
[12] S. Destercke and T. Burger, "Toward an axiomatic definition of conflict between belief functions," *Cybernetics, IEEE Transactions on*, vol. 43, no. 2, pp. 585–596, 2013.
[13] A. Martin and C. Osswald, "Human experts fusion for image classification," *Information & Security: An International Journal, Special issue on Fusing Uncertain, Imprecise and Paradoxist Information (DSmT)*, vol. 20, pp. 122–143, 2006.
[14] E. Lefèvre and Z. Elouedi, "How to preserve the conflict as an alarm in the combination of belief functions," *Decision Support Systems*, vol. 56, pp. 326–333, 2013.
[15] R. R. Yager, "On the dempster-shafer framework and new combination rules," *Information sciences*, vol. 41, no. 2, pp. 93–137, 1987.
[16] D. Dubois and H. Prade, "Representation and combination of uncertainty with belief functions and possibility measures," *Computational Intelligence*, vol. 4, no. 3, pp. 244–264, 1988.
[17] R. Ilin and E. Blasch, "Information fusion with belief functions: A comparison of proportional conflict redistribution PCR5 and PCR6 rules for networked sensors," in *18th International Conference on Information Fusion*. IEEE, 2015, pp. 2084–2091.
[18] A. Martin, C. Osswald, J. Dezert, and F. Smarandache, "General combination rules for qualitative and quantitative beliefs," *Journal of Advances in Information Fusion*, vol. 3, no. 2, pp. 67–89, 2008.
[19] Y. Zhao, R. Jia, and P. Shi, "A novel combination method for conflicting evidence based on inconsistent measurements," *Information Sciences*, vol. 367–368, pp. 125–142, 2016.
[20] P. Orponen, "Dempster's rule of combination is #$P$-complete," *Artificial Intelligence*, vol. 44, pp. 245–253, 1990.
[21] W. T. Da Silva and R. L. Milidiú, "Algorithms for combining belief functions," *International Journal of Approximate Reasoning*, vol. 7, no. 1-2, pp. 73 – 94, 1992.
[22] A. Martin and C. Osswald, "Toward a combination rule to deal with partial conflict and specificity in belief functions theory," in *10th International Conference on Information Fusion*. IEEE, 2007, pp. 1–8.
[23] P. Smets, "The $\alpha$-junctions: the commutative and associative non interactive combination operators applicable to belief function," in *1st International Joint Conference on Qualitative and Quantitative Practical Reasoning*, 1997, pp. 131–153.
[24] Y. Leung, N.-N. Ji, and J.-H. Ma, "An integrated information fusion approach based on the theory of evidence and group decision-making," *Information Fusion*, vol. 14, no. 4, pp. 410–422, 2013.
[25] G. Shafer, *A mathematical theory of evidence*. Princeton University Press, 1976.
[26] P. Smets, "The application of the matrix calculus to belief functions," *International Journal of Approximate Reasoning*, vol. 31, no. 1, pp. 1–30, 2002.
[27] ——, "Decision making in the TBM: the necessity of the pignistic transformation," *International Journal of Approximate Reasoning*, vol. 38, no. 2, pp. 133–147, 2005.
[28] K. Sentz and S. Ferson, "Combination of evidence in dempster-shafer theory," SAndia National Laboratorie, Tech. Rep., 2002.
[29] P. Smets, "Belief Functions: the Disjunctive Rule of Combination and the Generalized Bayesian Theorem," *International Journal of Approximate Reasoning*, vol. 9, pp. 1–35, 1993.
[30] A. Martin, "Comparative study of information fusion methods for sonar images classification," in *Information Fusion, 2005 8th International Conference on*, vol. 2. IEEE, 2005, pp. 7–pp.
[31] A. Samet, E. Lefevre, and S. Ben Yahia, "Reliability estimation with extrinsic and intrinsic measure in belief function theory," in *5th International Conference on Modeling, Simulation and Applied Optimization*. IEEE, 2013, pp. 1–6.
[32] J. Schubert, "Conflict management in dempster–shafer theory using the degree of falsity," *International Journal of Approximate Reasoning*, vol. 52, no. 3, pp. 449–460, 2011.
[33] Z. Elouedi, K. Mellouli, and P. Smets, "The evaluation of sensors' reliability and their tuning for multisensor data fusion within the transferable belief model," in *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*. Springer, 2001, pp. 350–361.
[34] A. Samet, E. Lefèvre, I. Hammami, and S. Ben Yahia, "Reliability estimation measure: Generic discounting approach," *International Journal of Pattern Recognition and Artificial Intelligence*, vol. 29, no. 07, p. 1559011, 2015.
[35] Y. Yang, D. Han, and C. Han, "Discounted combination of unreliable evidence using degree of disagreement," *International Journal of Approximate Reasoning*, vol. 54, no. 8, pp. 1197–1216, 2013.
[36] J. Klein and O. Colot, "Singular sources mining using evidential conflict analysis," *International Journal of Approximate Reasoning*, vol. 52, no. 9, pp. 1433–1451, 2011.
[37] P. Smets, "The canonical decomposition of a weighted belief," in *14th International Joint Conference on Artificial Intelligence*, vol. 95, 1995, pp. 1896–1901.
[38] T. Denœux, "Conjunctive and disjunctive combination of belief functions induced by nondistinct bodies of evidence," *Artificial Intelligence*, vol. 172, no. 2, pp. 234–264, 2008.
[39] X. Ke, L. Ma, and Y. Wang, "Some notes on canonical decomposition and separability of a belief function," in *Belief Functions: Theory and Applications*, ser. Lecture Notes in Computer Science, F. Cuzzolin, Ed., vol. 8764. Springer International Publishing, 2014, pp. 153–160.
[40] R. Kennes, "Computational aspects of the möbius transformation of graphs," *Systems, Man and Cybernetics, IEEE Transactions on*, vol. 22, no. 2, pp. 201–223, 1992.

[41] A. P. Dempster, "Upper and lower probabilities induced by a multivalued mapping," *The annals of mathematical statistics*, pp. 325–339, 1967.

[42] C. Osswald and A. Martin, "Understanding the large family of dempster-shafer theory's fusion operators-a decision-based measure," in *9th International Conference on Information Fusion*. IEEE, 2006, pp. 1–7.

[43] P. Smets, "Belief functions: the disjunctive rule of combination and the generalized bayesian theorem," *International Journal of approximate reasoning*, vol. 9, no. 1, pp. 1–35, 1993.

[44] N. Wilson, "Algorithms for Dempster-Shafer theory," in *Hanbook of defeasible reasoning and uncertainty management*, D. Gabbay and P. Smets, Eds. Boston: Kluwer Academic Publisher, 2000, vol. 5: Algorithms for uncertainty and Defeasible Reasoning, pp. 421–475.

[45] T. Denœux and A. Ben Yaghlane, "Approximating the combination of belief functions using the fast Möbius transform in a coarsened frame," *International Journal of Approximate Reasoning*, vol. 30, no. 1-2, pp. 77–101, 2002.

[46] A. Martin, "Implementing general belief function framework with a practical codification for low complexity," in *Advances and Applications of DSmT for Information Fusion*, F. Smarandache and J. Dezert, Eds. American Research Press Rehoboth, 2009, vol. 3, ch. 7, pp. 217–274.

[47] T. Denœux, "The cautious rule of combination for belief functions and some extensions," in *9th International Conference on Information Fusion*. IEEE, 2006, pp. 1–8.

[48] K.-S. Chin and C. Fu, "Weighted cautious conjunctive rule for belief functions combination," *Information Sciences*, vol. 325, pp. 70–86, 2015.

[49] C. K. Murphy, "Combining belief functions when evidence conflicts," *Decision support systems*, vol. 29, no. 1, pp. 1–9, 2000.

[50] F. Smarandache and J. Dezert, *Advances and Applications of DSmT for Information Fusion*. American Research Press, Rehoboth, 2004–2009, vol. 1–3.

[51] D. Dubois and H. Prade, "Consonant approximation of belief functions," *International Journal of Approximate Reasoning*, vol. 4, no. 5-6, pp. 419–449, 1990.

[52] A. Aregui and T. Denœux, "Constructing consonant belief functions from sample data using confidence sets of pignistic probabilities," *International Journal of Approximate Reasoning*, vol. 49, no. 3, pp. 575 – 594, 2008.

[53] K. Zhou and A. Martin, *ibelief: Belief Function Implementation*, 2015, r package version 1.2. [Online]. Available: http://CRAN.R-project.org/package=ibelief

[54] T. Burger and S. Destercke, "How to randomly generate mass functions," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 21, no. 05, pp. 645–673, 2013.