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ON BOUNDED PSEUDODIFFERENTIAL OPERATORS IN A HIGH-DIMENSIONAL SETTING

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Dedicated to the memory of Bernard Lascar

Abstract. This work is concerned with extending the results of Calderón and Vaillancourt proving the boundedness of Weyl pseudodifferential operators $Op_h^{Weyl}(F)$ in $L^2(\mathbb{R}^n)$. We state conditions under which the norm of such operators has an upper bound independent of n. To this aim, we apply a decomposition of the identity to the symbol F, thus obtaining a sum of operators of a hybrid type, each of them behaving as a Weyl operator with respect to some of the variables and as an anti-Wick operator with respect to the other ones. Then we establish upper bounds for these auxiliary operators, using suitably adapted classical methods like coherent states.

1. Introduction

Since the work of Calderón and Vaillancourt [2], it is well known that, if a function F, defined on $\mathbb{R}^{2n}$, is smooth and has bounded derivatives, it is possible to associate with it a pseudodifferential operator, depending on a parameter $h > 0$, which is bounded on $L^2(\mathbb{R}^n)$ (see also [7, **?**, 20, 22]. This operator is formally defined by:

(1.1)
$$(Op_h^{Weyl}(F)f)(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\cdot\xi} F\left(\frac{x+y}{2}, \xi\right) f(y)dyd\xi \qquad x \in \mathbb{R}^n$$

for f belonging to $L^2(\mathbb{R}^n)$. (When $h = 1$ the subscript h will be omitted). Moreover, its norm is bounded above by

$$\|Op_h^{Weyl}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C \sum_{|\alpha+\beta|\leq N} \|\partial_x^\alpha \partial_\xi^\beta F\|_{L^\infty(\mathbb{R}^{2n})} \tag{1.2}$$

where N and C depend on the dimension n.

The aim of this work is to prove that, under certain conditions, the constants appearing in the upper bound do not depend on the dimension. The set of derivation multi-indices which are used depends on the dimension in a way that will be precisely stated.

We shall thus be able to give examples where the dimension goes to infinity and the norm, nevertheless, remains bounded.

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In a later work we shall study pseudodifferential operators where the configuration space $\mathbb{R}^n$ will be replaced by an infinite dimensional Hilbert space, by a method differing from Bernard Lascar's (see [9]-[18]). These results have been announced in a preprint [1] in September 2012.

We first recall an example in which the constant appearing in the upper bound on the norm does not depend on the dimension. This is the case when the function F is the Fourier transform of a function G belonging to $L^1(\mathbb{R}^n)$:

$$F(x,\xi) = (2\pi h)^{-2n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(a\cdot x + b\cdot \xi)} G(a,b) da db.$$

Since the Weyl operator associated with the function

$$(x,\xi) \to E_{a,b,h}(x,\xi) = e^{\frac{i}{h}(a\cdot x + b\cdot \xi)}$$

is the operator $W_{a,b,h}$ defined by

$$(Op_h^{Weyl}(E_{a,b,h})f)(u) = (W_{a,b,h}f)(u) = e^{\frac{i}{h}a\cdot u + \frac{i}{2h}a\cdot b} f(u+b),$$

the equality (1.1) may be rewritten in the form

$$Op_h^{Weyl}(F) = (2\pi h)^{-2n} \int_{\mathbb{R}^{2n}} G(a,b) W_{a,b,h} da db.$$

Since $W_{a,b,h}$ is unitary,

$$\|Op_h^{Weyl}(F)\| \leq (2\pi h)^{-2n} \int_{\mathbb{R}^{2n}} |G(a,b)| \; da db.$$

Situations of this kind have been considered by B. Lascar ([9]-[18]) in an infinite dimensional setting, but the L^2 boundedness was not the main motivation of these works.

Our approach is different, in that we aim at extending the bound (1.2). Let us specify the set of multi-indices which will be used. Cordes [5], Coifman Meyer [3], Hwang [8] noticed that one does not need all the multi-indices to state (1.2) but only the (α,β) satisfying $0 \leq \alpha_j \leq 1$ and $0 \leq \beta_j \leq 1$ for each j. In this paper we shall use the multi-indices (α,β) such that $0 \leq \alpha_j \leq 2$ and $0 \leq \beta_j \leq 2$ for each j. We now can state the hypotheses on the function F.

Let $(\rho_j)_{1\leq j\leq n}$ and $(\delta_j)_{1\leq j\leq n}$ be two sequences satisfying $\rho_j \geq 0$ and $\delta_j \geq 0$ for every $j \leq n$, let M be a nonnegative real number. Suppose that

(H) for every multi-index (α,β) such that $0 \leq \alpha_j \leq 2$ and $0 \leq \beta_j \leq 2$ for every $j \leq n$, the partial derivative $\partial_x^\alpha \partial_\xi^\beta F$ exists, is continuous, bounded and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta F(x,\xi)| \leq M \prod_{j=1}^{n} \rho_j^{\alpha_j} \delta_j^{\beta_j}. \tag{1.3}$$

If $\rho_j = 0$ and $\alpha_j = 0$, we set that $\rho_j^{\alpha_j} = 1$.

Our main result is the following Theorem.

Theorem 1.1. *If a function F defined on $\mathbb{R}^{2n}$ satisfies hypothesis (H), then the operator $Op_h^{Weyl}(F)$, defined formally by (1.1), is bounded in $L^2(\mathbb{R}^n)$ and satisfies*

$$\|Op_h^{Weyl}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq M \prod_{j=1}^{n}(1+81\pi h\rho_j\delta_j) \tag{1.4}$$

if $0 < h\rho_j\delta_j \leq 1$ for every $j \leq n$.

Example 1.2. Let $V \geq 0$ be a real-valued bounded function in $C^\infty(\mathbb{R})$, whose derivatives are all bounded. For all integer $n \geq 1$, set

$$H_n(x,\xi) = \sum_{j\leq n}\xi_j^2 + \sum_{\substack{j\leq n, k\leq n\\ |j-k|=1}} g_j g_k V(x_j - x_k)$$

where (g_j) is a sequence of positive numbers such that, for some $C_0 > 0$, we have $g_j \leq C_0 g_k$ if $|j-k| \leq 1$. Set:

$$P_n(x,\xi) = e^{-H_n(x,\xi)} \tag{1.5}$$

We shall see that Hypothesis (H) is satisfied, with

$$M = 1, \qquad \delta_j = C_1 \qquad \rho_j = C_1\lambda_j \tag{1.6}$$

$$\lambda_j = \max_{1\leq\nu\leq 4}(g_j^2\|V^{(\nu)}\|_{L^\infty})^{1/\nu} \tag{1.7}$$

where C_1 is a real constant, to be determined, depending only on C_0. Let us set:

$$W(x) = -\sum_{\substack{j\leq n, k\leq n\\ |j-k|=1}} g_j g_k V(x_j - x_k),$$

and

$$A_n = \{(j,k), |j-k| = 1, \quad 1 \leq j,k \leq n\}.$$

We shall estimate $\partial^\alpha e^W$ when $\alpha_j \leq 2$ for every j. For each function $f \in C^4(\mathbb{R})$ with bounded derivatives, set:

$$M(f) = \max_{1\leq\nu\leq 4}(\|f^{(\nu)}\|_{L^\infty})^{1/\nu}.$$

We notice that:

$$e^{-f}\partial^\nu e^f = (\partial + f')^\nu \cdot 1.$$

From this and a simple computation follows that:

$$|e^{-f}\partial^\nu e^f| \leq (2M(f))^\nu. \tag{1.8}$$

In order to apply this inequality we divide W into two parts. Set:

$$U_j = U_j(x_1,...,x_n) = -g_j g_{j+1}U(x_j - x_{j+1}), \qquad U(y) = V(y) + V(-y)$$

and

$$W_e = \sum_{j\ even} U_j, \qquad\qquad W_o = \sum_{j\ odd} U_j.$$

Then $W = W_e + W_o$ and we notice that the variable x_ν occurs only once in the W_e and W_o. Also, since

$$\partial^\alpha e^{W_e} = \prod_{j\ even,\ j<n}(\partial_j^{\alpha_j}\partial_{j+1}^{\alpha_{j+1}}e^{U_j})$$

we have the estimate

$$|\partial^\alpha e^{W_e}| \leq e^{W_e} \prod_{j\ even,\ j<n} T_j \tag{1.9}$$

where T_j is the L^∞ norm of the function:

$$e^{f_j}\partial^{\alpha_j+\alpha_{j+1}}e^{-f_j}, \qquad f_j = g_j g_{j+1} U(x).$$

Let λ_j be defined by (1.7), and set:

$$M_j = \max_{1\leq\nu\leq 4}(\|g_j g_{j+1} U^{(\nu)}\|_{L^\infty})^{1/\nu}.$$

Then

$$M_j \leq 2C_0\lambda_j, \qquad M_j \leq 2C_0\lambda_{j+1} \tag{1.10}$$

where C_0 is defined so that $g_j g_{j+1} \leq C_0 \min(g_j^2, g_{j+1}^2)$. It follows from (1.8) and (1.10) that:

$$T_j \leq (2M_j)^{\alpha_j+\alpha_{j+1}} \leq (4C_0\lambda_j)^{\alpha_j}(4C_0\lambda_{j+1})^{\alpha_{j+1}}.$$

Then (1.9) gives:

$$|\partial^\alpha e^{W_e}| \leq e^{W_e} \prod_{j\ even, j<n} (4C_0\lambda_j)^{\alpha_j}(4C_0\lambda_{j+1})^{\alpha_{j+1}}.$$

In a similar way one gets the estimate:

$$|\partial^\alpha e^{W_o}| \leq e^{W_o} \prod_{j\ odd, j<n} (4C_0\lambda_j)^{\alpha_j}(4C_0\lambda_{j+1})^{\alpha_{j+1}}.$$

Then we write:

$$\partial^\alpha e^W = \partial^\alpha e^{W_e} e^{W_o} = \sum{}' \tbinom{\alpha}{\beta} (\partial^{\alpha-\beta} e^{W_e})(\partial^\beta e^{W_o})$$

where the prime indicates that one only takes the summation over terms with $\beta_1 = \alpha_1$ and $\beta_n = \alpha_n$ if n is even, and with $\beta_1 = \alpha_1$ and $\beta_n = 0$ if n is odd. We get the estimate:

$$|\partial^\alpha e^W| \leq e^W \sum{}' \tbinom{\alpha}{\beta} \Big(\prod_{j\ even,\ j<n} (4C_0\lambda_j)^{\alpha_j-\beta_j}(4C_0\lambda_{j+1})^{\alpha_{j+1}-\beta_{j+1}} \Big) \cdot$$

$$\cdot \Big(\prod_{j\ odd,\ j<n} (4C_0\lambda_j)^{\beta_j}(4C_0\lambda_{j+1})^{\beta_{j+1}} \Big)$$

$$= e^W \sum{}' \tbinom{\alpha}{\beta} \prod_{j=1}^n (4C_0\lambda_j)^{\alpha_j} \leq 2^{|\alpha|} e^W \prod_{j=1}^n (4C_0\lambda_j)^{\alpha_j}.$$

We have proved that:

$$|\partial^\alpha e^W| \leq e^W \prod_{j=1}^n (8C_0\lambda_j)^{\alpha_j}.$$

Therefore, hypothesis (H) is satisfied with the choice (1.6). We may apply Theorem 1.1 if $h\rho_j\delta_j \leq 1$ for all j. It follows that $\|Op_h^{Weyl}(P_n)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ is bounded independently of n if the sum $\sum_{j\geq 1} g_j^{1/2}$ converges, and if h is small enough. If $g_j = 1$, the norm is not bounded, but estimated, with some constant $C > 0$, independent of the dimension, by:

$$\|Op_h^{Weyl}(P_n)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq e^{Chn}. \tag{1.11}$$

Example 1.3. The mean-field approximation uses hamiltonians of the form

$$H_n(x,\xi) = \sum_{j\leq n} \xi_j^2 + \frac{1}{n} \sum_{j\leq n, k\leq n} V(x_j - x_k)$$

where V is as in Example 1.2. Let P_n be the function defined as in (1.5). Then hypothesis (H) is satisfied with $M = 1$ and $\rho_j = \delta_j = C_1$, where C_1 does not depend on n. In this case, Theorem 1.1 shows that, provided C_1 is small enough, we have also (1.11) for some constant C which is independent of n.

We express our thanks to the referee for his helpful suggestions, which allowed us in particular to gain on the number of derivatives and to simplify the proofs.

2. Hybrid Weyl anti-Wick quantization.

In order to prove Theorem 1.1, we may as well assume that $\rho_j = \delta_j$ for every $j \leq n$ and that $h = 1$. Indeed, if a function F satisfies hypothesis (H) with two sequences (ρ_j) and (δ_j) of positive real numbers, then the function $\widetilde{F}$ defined by

$$\widetilde{F}(x,\xi) = F\left(x_1\sqrt{h}\lambda_1, ..., x_n\sqrt{h}\lambda_n, \frac{\xi_1\sqrt{h}}{\lambda_1}, ..., \frac{\xi_n\sqrt{h}}{\lambda_n}\right) \qquad \lambda_j = \sqrt{\frac{\delta_j}{\rho_j}}$$

satisfies (H) with ρ_j and δ_j replaced by $\varepsilon_j = \sqrt{h\rho_j\delta_j}$. If Theorem 1.1 is valid for Op_1^{Weyl}, then we get that, if $\varepsilon_j^2 \leq 1$:

$$\|Op_1^{Weyl}(\widetilde{F})\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq M \prod_{j=1}^{n}(1 + 81\pi\varepsilon_j^2).$$

Since $Op_h^{Weyl}(F) = T^{-1}Op_1^{Weyl}(\widetilde{F})T$, where T is a unitary operator acting in $L^2(\mathbb{R}^n)$, Theorem 1.1 for $Op_h^{Weyl}(F)$ holds true. This follows by continuity if some of the ρ_j or δ_j are equal to 0.

Consequently, we shall assume from now on that $\rho_j = \delta_j \leq 1$ for every $j \leq n$ and that $h = 1$.

In order to prove Theorem 1.1 we shall split the operator into a sum of operators which will behave as Weyl operators with respect to a first subset of the variables (meaning the operators will be defined by a formula analogous to (1.1) in which only these variables appear) and as anti-Wick operators with respect to the other variables.

We first need to recall the anti-Wick quantization. The definition uses the coherent states, which is the family of functions Ψ_X indexed by $X = (x,\xi) \in \mathbb{R}^{2n}$, defined by

(2.1)

$$\Psi_X(u) = \pi^{-n/4} e^{-\frac{|u-x|^2}{2}} e^{iu.\xi - \frac{i}{2}x.\xi} \qquad X = (x,\xi) \in \mathbb{R}^{2n} \qquad u \in \mathbb{R}^n.$$

Recall that

$$< f, g >= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} < f, \Psi_X > < \Psi_X, g > dX. \tag{2.2}$$

If F is a function in $L^\infty(\mathbb{R}^{2n})$, one can associate with it an (anti-Wick) operator $Op^{AW}(F)$ such that, for all f and g in $L^2(\mathbb{R}^n)$:

$$< Op^{AW}(F)f, g >= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} F(X) < f, \Psi_X > < \Psi_X, g > \ dX. \tag{2.3}$$

We then have

$$\|Op^{AW}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|F\|_{L^\infty(\mathbb{R}^{2n})}. \tag{2.4}$$

The relationship between Weyl and anti-Wick quantizations is given, for every F in $L^\infty(\mathbb{R}^{2n})$, by :

$$Op^{AW}(F) = Op^{Weyl}\left(e^{\frac{1}{4}\Delta}F\right) \tag{2.5}$$

where

$$\Delta = \sum_{j\leq n} \Delta_j \qquad\qquad \Delta_j = \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2}. \tag{2.6}$$

This fact is classical (see Folland [6]). One has an identity decomposition in $L^\infty(\mathbb{R}^{2n})$:

$$I = \sum_{E\subseteq\{1,\ldots,n\}} T(E)e^{\frac{1}{4}\Delta_{E^c}} \qquad\qquad T(E) = \prod_{j\in E}(I - e^{\frac{1}{4}\Delta_j}) \tag{2.7}$$

$$\Delta_{E^c} = \sum_{j\in E^c} \Delta_j. \tag{2.8}$$

For every subset $E \subseteq \{1, ..., n\}$ and every symbol F, we define an operator $Op^{hyb,E}(F)$ by :

$$Op^{hyb,E}(F) = Op^{Weyl}\left(e^{\frac{1}{4}\Delta_{E^c}}F\right) \tag{2.9}$$

This operator behaves as a Weyl operator with respect to the variables x_j $(j \in E)$ and as an anti-Wick operator with respect to the variables x_j $(j \in E^c)$. If $E = \emptyset$, it is the anti-Wick operator and conversely if $E = \{1, ..., n\}$, it is the Weyl operator.

One derives a decomposition of the Weyl operator $Op^{Weyl}(F)$:

$$Op^{weyl}(F) = \sum_{E\subseteq\{1,\ldots,n\}} Op^{hyb,E}(T(E)F). \tag{2.10}$$

We shall now prove an upper bound on the norm of a hybrid operator $Op^{hyb,E}(G)$, where the function G is bounded on $\mathbb{R}^{2n}$. The only derivatives of G which will play a role are the derivatives with respect to x_j or ξ_j with $j \in E$. For every integer m we introduce the set of multi-indices

(2.11)

$$I_m(E) = \{(\alpha, \beta), \quad \alpha_j \leq m, \quad \beta_j \leq m, \ \ (1 \leq j \leq n) \qquad \alpha_j = \beta_j = 0 \quad \text{ if } \ j \notin E\}.$$

We shall prove the following Lemma in Section 3, by adapting classical methods (Unterberger [22]).

Lemma 2.1. *If F satisfies hypothesis (H) and if $E \neq \emptyset$, then*

$$\|Op^{hyb,E}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \left(\frac{9\pi}{2}\right)^{|E|} \sum_{(\alpha,\beta)\in I_2(E)} \|\partial_x^\alpha \partial_\xi^\beta F\|_{L^\infty(\mathbb{R}^{2n})} \tag{2.12}$$

We shall establish the following Lemma in Section 4.

Lemma 2.2. *If F satisfies hypothesis (H) with $\rho_j = \delta_j \leq 1$ for every $j \leq n$, and if $E \neq \emptyset$, the function $T(E)F$ satisfies*

$$\sum_{(\alpha,\beta)\in I_2(E)} \|\partial_x^\alpha \partial_\xi^\beta T(E)F\|_{L^\infty(\mathbb{R}^{2n})} \leq M 18^{|E|} \prod_{j\in E} \rho_j^2 \tag{2.13}$$

where $T(E)$ is defined in (2.7).

End of the proof of Theorem 1.1. We assume that $h = 1$ and that $\rho_j = \delta_j \leq 1$ for every $j \leq n$. According to (2.10), we have

$$\|Op^{Weyl}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \sum_{E\subseteq\{1,\dots,n\}} \|Op^{hyb,E}(T(E)F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.$$

By Lemma 2.1:

$$\|Op^{Weyl}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \sum_{E\subseteq\{1,\dots,n\}} \left(\frac{9\pi}{2}\right)^{|E|} \sum_{(\alpha,\beta)\in I_2(E)} \|\partial_x^\alpha \partial_\xi^\beta T(E)F\|_{L^\infty(\mathbb{R}^{2n})}.$$

With the same hypotheses, Lemma 2.2 shows that:

$$\|Op^{Weyl}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq M \sum_{E\subseteq\{1,\dots,n\}} \left(\frac{9\pi}{2} \times 18\right)^{|E|} \prod_{j\in E} \rho_j^2.$$

It follows easily that

$$\|Op^{Weyl}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq M \prod_{j\leq n} (1 + 81\pi\rho_j^2).$$

The theorem is proved in the case when $h = 1$ and $\rho_j = \delta_j \leq 1$ for all $j \leq n$. In the general case, the announced result follows as we saw.

□

3. Proof of Lemma 2.1.

We shall use the results of Unterberger [21, 22] concerning the upper bound of $< A\Psi_X, \Psi_Y >$, where A is a pseudodifferential operator and the Ψ_X are the coherent states defined by (2.1). We first recall the integral expression of this scalar product and give an analogous statement for hybrid operators.

Proposition 3.1. *Let F be a function defined on $\mathbb{R}^{2n}$ and satisfying hypothesis (H). Then we get, for every X and Y in $\mathbb{R}^{2n}$:*

$$< Op^{Weyl}(F)\Psi_X, \Psi_Y >= \pi^{-n} \int_{\mathbb{R}^{2n}} F(Z)\ \Phi_n(X, Y, Z)dZ \tag{3.1}$$

with

$$\Phi_n(X, Y, Z) = e^{-|Z-\frac{X+Y}{2}|^2 - i\sigma(Z,X-Y) - \frac{i}{2}\sigma(X,Y)} \tag{3.2}$$

where the symplectic form σ is given by $\sigma(X,Y) = y\cdot\xi - x\cdot\eta$ for all $X=(x,\xi)$ and $Y=(y,\eta)$ in $\mathbb{R}^{2n}$,

Proof. For all functions f and g belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, one defines the Wigner function $H(f,g,Z)$ $(Z\in\mathbb{R}^{2n})$ by :

(3.3)
$$H(f,g,Z)=\int_{\mathbb{R}^n} e^{-it\cdot\zeta} f\left(z+\frac{t}{2}\right)\overline{g\left(z-\frac{t}{2}\right)}dt \qquad\qquad Z=(z,\zeta)\in\mathbb{R}^{2n}$$

(cf Unterberger [22], or Lerner [19], sections 2.1.1 et 2.1.2, or Combescure Robert [4], section 2.2). The following equality is proved in [22] or [19] or [4], for all f and g in $\mathcal{S}(\mathbb{R}^n)$ and every Borel function F which is bounded on $\mathbb{R}^{2n}$:

$$< Op^{Weyl}(F)f,g >= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} F(Z)H(f,g,Z)dZ. \tag{3.4}$$

An explicit computation using the coherent spaces Ψ_X defined by (2.1) shows that

$$H(\Psi_X,\Psi_Y,Z)=2^n\Phi_n(X,Y,Z), \tag{3.5}$$

which implies (3.1). □

Let $n'<n$ and $n''=n-n'$. We denote by $X=(X',X'')$ the variable in $\mathbb{R}^{2n}$, with $X'=(X_1,...,X_{n'})$ and $X''=(X_{n'+1},...,X_n)$. Set

$$\Delta''=\sum_{j=n'+1}^{n}(\partial^2_{x_j}+\partial^2_{\xi_j}).$$

Proposition 3.2. *For all f and g in $\mathcal{S}(\mathbb{R}^n)$, we have :*

$$< Op^{Weyl}(e^{\frac{1}{4}\Delta''}F)f,g >= ... \tag{3.6}$$

$$= C(n',n'')\int_{\mathbb{R}^{6n'+2n''}} F(Z',T'')\Phi_{n'}(X',Y',Z')< f,\Psi_{X',T''}>< \Psi_{Y',T''},g > dX'dY'dZ'dT''$$

where $C(n',n'')=2^{n'}(2\pi)^{-3n'-n''}$, $\Phi_{n'}$ being the function defined by (3.2) with n',X' instead of n,X.

Proof. Proposition 3.1 and the integral expression for the heat operator $e^{\frac{1}{4}\Delta''}$ give:

(3.7)
$$< Op^{Weyl}(e^{\frac{1}{4}\Delta''}F)\Psi_X,\Psi_Y >= \pi^{-n-n''}\int_{\mathbb{R}^{2(n+n'')}} e^{-|Z''-T''|^2}F(Z',T'')\Phi_n(X,Y,Z)dZdT''$$

An explicit computation yields

$$\pi^{-n''}\int_{\mathbb{R}^{2n''}} e^{-|Z''-T''|^2}\Phi_{n''}(X'',Y'',Z'')dZ''=2^{-n''}< \Psi_{X''},\Psi_{T''} >_{n''}< \Psi_{T''},\Psi_{Y''} >_{n''}$$

where $< \cdot,\cdot >_{n''}$ is the scalar product of $L^2(\mathbb{R}^{n''})$. Combining (3.7) with this equality one sees that

$$< Op^{Weyl}(e^{\frac{1}{4}\Delta''}F)\Psi_X,\Psi_Y >= ... \tag{3.8}$$

$$= 2^{-n''}\pi^{-n}\int_{\mathbb{R}^{2n}} F(Z',T'')\Phi_{n'}(X',Y',Z')< \Psi_{X''},\Psi_{T''} >_{n''}< \Psi_{T''},\Psi_{Y''} >_{n''} dZ'dT''.$$

For all f and g in $\mathcal{S}(\mathbb{R}^n)$, one gets, applying (2.2) twice :

$$< Op^{Weyl}(e^{\frac{1}{4}\Delta''}F)f,g >= ... \tag{3.9}$$

$$(2\pi)^{-2n}\int_{\mathbb{R}^{4n}} < f,\Psi_X >< Op^{Weyl}(e^{\frac{1}{4}\Delta''}F)\Psi_X,\Psi_Y >< \Psi_Y,g > dXdY.$$

One then applies (3.8) and the following result, deduced from (2.2) in dimension n'':

$$(2\pi)^{-n''}\int_{\mathbb{R}^{2n''}} < f,\Psi_X >< \Psi_{X''},\Psi_{T''} >_{n''} dX'' =< f,\Psi_{X',T''} > .$$

Formula (3.6) follows from that and from an analogous result about $< \Psi_Y, g >$. □

For every $X' = (x',\xi')$ in $\mathbb{R}^{2n'}$, set:

$$K_{n'}(X') = \prod_{j=1}^{n'} \left(1+x_j^2\right)\left(1+\xi_j^2\right). \tag{3.10}$$

Lemma 3.3. *For every function G satisfying hypothesis (H), for every X' and Y' in $\mathbb{R}^{2n'}$ and Z'' in $\mathbb{R}^{2n''}$:*

$$K_{n'}(X'-Y')\left|\pi^{-n'}\int_{\mathbb{R}^{2n'}} F(Z',Z'')\ \Phi_{n'}(X',Y',Z')dZ'\right| \leq 9^{n'}N_{n'}(F), \tag{3.11}$$

$$N_{n'}(F) = \sum_{(\alpha,\beta)\in I_2(n')} \|\partial_x^\alpha\partial_\xi^\beta F\|_{L^\infty(\mathbb{R}^{2n})}$$

where $I_2(n')$ is the set of multi-indices α such that $\alpha_j \leq 2$ and $\beta_j \leq 2$ for all $j \leq n$, and $\alpha_j = \beta_j = 0$ for $j > n'$.

Proof. Let $I(X',Y',Z'')$ be the left side of (3.11). Integrations by parts show that for all X and Y in $\mathbb{R}^{2n'}$:

$$I(X',Y',Z'') \leq \pi^{-n'}\int_{\mathbb{R}^{2n'}} \left|(LF)\left(Z'+\frac{X'+Y'}{2},Z''\right)\right| e^{-|Z'|^2}dZ'$$

where L is the differential operator defined by

$$L = \prod_{j\leq n'} L_{z_j}L_{\zeta_j} \qquad\qquad L_{z_j} = \sum_{k=0}^{2} a_k(z_j)\partial_{z_j}^k$$

$$a_0(z) = 3-4z^2 \qquad\qquad a_1(z) = 4z \qquad\qquad a_2(z) = -1.$$

We get as a consequence that

$$I(X',Y',Z'') \leq \sum_{(\alpha,\beta)\in I_2(n')} \left\|\partial_x^\alpha\partial_\xi^\beta F\right\|_{L^\infty(\mathbb{R}^{2n})} \prod_{j\leq n'} C_{\alpha_j}C_{\beta_j}$$

with

$$C_k = \pi^{-1/2}\int_{\mathbb{R}} |a_k(z)|e^{-z^2}dz \qquad\qquad k = 0,1,2.$$

The formula (3.11) then follows from the fact that $\max(C_0,C_1,C_2) \leq 3$. □

End of the proof of Lemma 2.1. The subset E may be any subset of $\{1,...,n\}$, but we can assume in the proof that $E = \{1,...,n'\}$ with $0 \leq n' \leq n$. In this case, we use the above notations, and we set $x = (x',x'')$ with $x' = (x_1,...x_{n'})$ and $x'' = (x_{n'+1},...x_n)$, etc.

Using (3.6) and Lemma 3.3 to obtain an upper bound on the right side, one gets:

$$\left| < Op^{Weyl}(e^{\frac{1}{4}\Delta''}F)f,g > \right| \leq ...$$

$$... \leq 9^{n'}(2\pi)^{-2n'-n''} N_{n'}(F) \int K_{n'}(X'-Y')^{-1}| < f, \Psi_{X',T''} >< \Psi_{Y',T''}, g > |dX'dY'dT''$$

According to Schur's Lemma, this is smaller than

$$... \leq 9^{n'}(2\pi)^{-2n'-n''} N_{n'}(F) \|K_{n'}^{-1}\|_{L^1(\mathbb{R}^{2n'})} ...$$

$$... \left[\int | < f, \Psi_{X',T''} > |^2 dX'dT'' \right]^{1/2} \left[\int | < \Psi_{Y',T''}, g > |^2 dY'dT'' \right]^{1/2}.$$

Using (2.2), one shows that $| < Op^{Weyl}(e^{\frac{1}{4}\Delta''} F)f, g > |$ is smaller than

$$\leq 9^{n'}(2\pi)^{-n'} N_{n'}(F) \|K_{n'}^{-1}\|_{L^1(\mathbb{R}^{2n'})} \|f\|_{L^2(\mathbb{R}^n)} \ \ \|g\|_{L^2(\mathbb{R}^n)}.$$

Since $\|K_{n'}^{-1}\|_{L^1(\mathbb{R}^{2n'})} = \pi^{2n'}$, the former inequalities imply that

$$\left| < Op^{Weyl}(e^{\frac{1}{4}\Delta''} F)f, g > \right| \leq \left(\frac{9\pi}{2} \right)^{n'} N_{n'}(F) \|f\|_{L^2(\mathbb{R}^n)} \ \ \|g\|_{L^2(\mathbb{R}^n)}.$$

Lemma 2.1 holds if $E = \{1, ..., n'\}$, a case which we are brought back to by a suitable permutation.

□

4. Proof of Lemma 2.2

Let Δ_j be the operator defined by (2.6) and

$$A_j = I - e^{\frac{1}{4}\Delta_j} \tag{4.1}$$

Lemma 4.1. *One can write*

$$A_j = B_j \partial_{x_j} + C_j \partial_{\xi_j} = D_j \Delta_j \tag{4.2}$$

where the operators B_j, C_j and D_j are bounded in the space C_b of continuous bounded functions on $\mathbb{R}^{2n}$. More precisely,

$$\|A_j\|_{\mathcal{L}(C_b)} \leq 2 \qquad \|B_j\|_{\mathcal{L}(C_b)} \leq \pi^{-1/2} \tag{4.3}$$

$$\|C_j\|_{\mathcal{L}(C_b)} \leq \pi^{-1/2} \qquad \|D_j\|_{\mathcal{L}(C_b)} \leq 1/4.$$

Proof. The first inequality in (4.3) is standard. The expression of the heat operator allows us to write the first equality in (4.2) with

$$(B_j\varphi)(x,\xi) = -\pi^{-1} \int_{\mathbb{R}^2 \times [0,1]} e^{-(u^2+v^2)} u\varphi(x + \theta u e_j, \xi + \theta v e_j) du dv d\theta \tag{4.4}$$

$$(C_j\varphi)(x,\xi) = -\pi^{-1} \int_{\mathbb{R}^2 \times [0,1]} e^{-(u^2+v^2)} v\varphi(x + \theta u e_j, \xi + \theta v e_j) du dv d\theta \tag{4.5}$$

We deduce the bounds on the norms of C_j and D_j in (4.3) from these inequalities. The last inequality (4.2) and the bound on D_j in (4.3) follow by integrating by parts in (4.4) and (4.5). □

End of the proof of Lemma 2.2. For every multi-index (α, β) in $I_2(E)$, one can rewrite the operator $\partial_x^\alpha \partial_\xi^\beta T(E)$ as follows

$$\partial_x^\alpha \partial_\xi^\beta T(E) = \prod_{j \in E} U_j \ \partial_x^{\alpha_j} \partial_\xi^{\beta_j}$$

with

$$U_j = \begin{cases} A_j & \text{if} \quad \alpha_j + \beta_j \geq 2 \\ B_j \partial_{x_j} + C_j \partial_{\xi_j} & \text{if} \quad \alpha_j + \beta_j = 1 \\ D_j \Delta_j & \text{if} \quad \alpha_j + \beta_j = 0 \end{cases}$$

According to the bounds on the norms of the operators A_j to D_j given by Lemma 4.1, if F satisfies hypothesis (H) with $\rho_j = \delta_j \leq 1$, one has:

$$\|\partial_x^\alpha \partial_\xi^\beta T(E) F\|_{L^\infty(\mathbb{R}^{2n})} \leq M 2^{|E|} \prod_{j \in E} \rho_j^2$$

Since $I_2(E)$ contains exactly $9^{|E|}$ elements, this achieves the proof of Lemma 2.2. □

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