The Weyl Symbol of Schrödinger Semigroups

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Abstract

In this paper, we study the Weyl symbol of the Schrödinger semigroup e^{-tH}, $H = -\Delta + V$, $t > 0$, on $L^2(\mathbb{R}^n)$, with nonnegative potentials V in $L^1_{\rm loc}$. Some general estimates like the L^∞ norm concerning the symbol u are derived. In the case of large dimension, typically for nearest neighbor or mean field interaction potentials, we prove estimates with parameters independent of the dimension for the derivatives $\partial_x^\alpha \partial_\xi^\beta u$. In particular, this implies that the symbol of the Schrödinger semigroups belongs to the class of symbols introduced in [2] in a high-dimensional setting. In addition, a commutator estimate concerning the semigroup is proved.

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1 Introduction.

Let V be a nonnegative function in $L^2_{\rm loc}(\mathbb{R}^n)$. It is known that $H = -\Delta + V(x)$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$ and we also denote by H its unique selfadjoint extension. We may also suppose that V is only in $L^1_{\rm loc}(\mathbb{R}^n)$ and use Theorem X.32 in [22] to define H as a selfadjoint operator with a suitable domain. In this paper, we are interested in the Weyl symbol $u(\cdot, t)$ of e^{-tH}, for each $t > 0$. Since this operator is bounded in $L^2(\mathbb{R}^n)$, its Weyl symbol is a priori a tempered distribution $U(t)$ on $\mathbb{R}^{2n}$ which satisfies,

$$< e^{-tH} f, g > = < U(t), H(f, g, \cdot) >, \tag{1}$$

for all f and g in $\mathcal{S}(\mathbb{R}^n)$, where $H(f, g, x, \xi)$ is the Wigner function (c.f. [6] or [20]),

$$H(f, g, x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iv\cdot\xi} f\left(x - \frac{v}{2}\right) \overline{g\left(x + \frac{v}{2}\right)} dv. \tag{2}$$

The aim of this work is to study this Weyl symbol when V is a C^∞ potential describing a large number of particles in interaction, either for a nearest neighbor interaction model in a lattice, or for a mean field approximation model.

Our hypotheses on the interaction potentials will in particular imply that $u(\cdot,t)$, the symbol of e^{-tH}, is a C^∞ function on $\mathbb{R}^{2n}$, belonging for each fixed n, to the usual class of Hörmander with the constant metric [15]. This follows from the general results about the pseudodifferential calculus: see [13] or earlier results of [14]. However, these classical results depend on the dimension n, and cannot be applied when the number of particles in interaction tends to infinity.

Under appropriate hypotheses covering both nearest neighbor interaction and mean field approximation, we shall prove that, for each $t>0$, the Weyl symbol $u(\cdot,t)$ of e^{-tH} belongs to some set of symbols, of a form already studied in [4] for the composition of symbols and in [2] for norm estimates. The sets of symbols defined in [4] and [2] are characterized by estimates in the L^∞ norm, for the derivatives $\partial_x^\alpha \partial_\xi^\beta F$, only involving multi-indices (α,β) such that all the α_j and β_j are bounded by the same integer m, independently of the dimension. The idea that estimates for such multi-indices (with $m=2$) are sufficient to ensure L^2 bounds, goes back to Coifman-Meyer [5] (for a quantization different from the Weyl quantization).

Let us specify this class of symbols. In [4] and [2], we say that a continuous function F on $\mathbb{R}^{2n}$ is in $S_m(M,\rho,\delta)$ (where m is a nonnegative integer, $M\geq 0$, and ρ and δ are two sequences $(\rho_j)_{(j\leq n)}$ and $(\delta_j)_{(j\leq n)}$ of nonnegative real numbers) if, for all multi-indices α and β in $\mathbb{N}^n$ satisfying $0\leq \alpha_j,\beta_j\leq m$, the derivative $\partial_x^\alpha \partial_\xi^\beta F$ is a continuous and bounded function verifying,

$$\|\partial_x^\alpha \partial_\xi^\beta F\|_{L^\infty(\mathbb{R}^{2n})} \leq M \prod_{j\leq n} \rho_j^{\alpha_j} \delta_j^{\beta_j}. \tag{3}$$

In [2], we proved that, if a function F is in $S_2(M,\rho,\delta)$ and if moreover $0<h\rho_j\delta_j\leq 1$ for all $j\leq n$, then the operator $Op_h^{Weyl}(F)$ of Weyl symbol F is bounded in $L^2(\mathbb{R}^n)$ and

$$\|Op_h^{Weyl}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq M \prod_{j=1}^n (1+81\pi h\rho_j\delta_j). \tag{4}$$

In [4] it is proved that, if F is in $S_m(M,\rho,\delta)$ and G in $S_m(M',\rho,\delta)$, $(m\geq 6)$, and if $0<h\rho_j\delta_j\leq 1$ for all $j\leq n$, then the Weyl symbol $C_h(F,G)$ of the composed operator $Op_h^{Weyl}(F)\circ Op_h^{Weyl}(G)$ is in $S_{m-6}(M'',\rho,\delta)$, with

$$M'' = MM' \prod_{j=1}^n (1+Kh\rho_j\delta_j),$$

where K is a universal constant.

By these results, the sets $S_m(M,\rho,\delta)$ can be applied in situations where the dimension n tends to infinity. Then, an element of $S_m(M,\rho,\delta)$ is rather a family $F=(F_n)_{(n\geq 1)}$, where F_n is a function on $\mathbb{R}^{2n}$ satisfying (3), where M is independent of n, and $\rho=(\rho_j)_{(j\geq 1)}$ and $\delta=(\delta_j)_{(j\geq 1)}$ are now infinite sequences, satisfying $0<h\rho_j\delta_j\leq 1$ for all $j\geq 1$. However, without suitable assumptions on these two

sequences, the constants in (4) will not be bounded. But even in this case, we may think for instance of possible applications to thermodynamic limits (see [12]). Under suitable hypotheses on the sequences $\rho = (\rho_j)_{(j\geq 1)}$ and $\delta = (\delta_j)_{(j\geq 1)}$, the products in (3) and (4) remain bounded as the dimension n tends to infinity, and a Weyl calculus in infinite dimension can be considered, in the spirit of B. Lascar [17][18][19]: see [3] for norms estimates.

In the first section, the symbol $u(\cdot, t)$ of the semigroup is proved to be in $L^\infty(\mathbb{R}^{2n})$ with $|u(\cdot, t)| \leq 1$ almost everywhere and additionally, $\int_{\mathbb{R}^n} u(x, \xi, t)dx$ and $\partial_\xi^\beta u(\cdot, t)$ are estimated, without further regularity hypothesis on the interaction potential V. These results are obtained by applying the Feynman Kac formula to the study of the Weyl symbol of Schrödinger semigroups. The usual applications of this formula (c.f. [1][21][26][27]...) rather concern the distributional kernel of this operator. In the second section, we consider the semigroup in a large dimension setting with regular potentials and obtain estimates on all the derivatives of the Weyl symbol proving in particular that, for each $m \geq 1$, it lies in some set $S_m(1, \rho, \delta)$ defined above, with suitable sequences $\rho = (\rho_j)$ and $\delta = (\delta_j)$, where ρ_j and δ_j are independent of j. Supplementary assumptions on potentials regarding the large dimension are naturally necessary at this step. Then, in the third section, two examples of Schrödinger semigroups in large dimension, satisfying the assumptions of section 2, are considered, namely, the nearest neighbor and the mean field approximation potentials. Moreover, a commutator property is also proved.

2 First properties of the symbol of the semigroup.

The first step consists in writing the Weyl symbol of e^{-tH} with the Feynman Kac formula, under rather general hypotheses on the potential V. We make the choice to not first express the symbol with the distribution kernel, in order to avoid the use of Brownian bridges. Let $T > 0$ and n be an integer ≥ 1. We denote by B the Banach space of continuous functions ω on $[0, T]$ taking values into $\mathbb{R}^n$ and vanishing at $t = 0$. This space is endowed with the supremum norm, with the Borel $\sigma-$algebra $\mathcal{B}$ and with the Wiener measure μ of variance 1 (c.f. [16][9][10][11]).

Proposition 2.1. *Let $V \geq 0$ be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $U(t)$ be the Weyl symbol of the operator e^{-tH}, first considered as a tempered distribution on $\mathbb{R}^{2n}$. Then, $U(t)$ is identified with a function $u(\cdot, t)$ in $L^\infty(\mathbb{R}^{2n})$. We have, for each t in $(0, T]$ and for almost every (x, ξ) in $\mathbb{R}^{2n}$,*

$$u(x, \xi, t) = \int_B e^{-i\omega(t)\xi} e^{-\int_0^t V(x - \frac{\omega(t)}{2} + \omega(s))ds} d\mu(\omega). \tag{5}$$

Moreover, the following inequality holds,

$$|u(\cdot, t)| \leq 1, \tag{6}$$

almost everywhere on $\mathbb{R}^{2n}$, for each $t \in [0, T]$.

One notices that the above integral involves all of the trajectories of the Brownian motion with a starting and a finishing point that are symmetric with respect to x.

Proof. Let f and g in $\mathcal{S}(\mathbb{R}^n)$. When $V \geq 0$ belongs to $L^1_{\rm loc}(\mathbb{R}^n)$, one may apply Feynman Kac formula (c.f. B. Simon [24] or [25]) written as,

$$< e^{-tH} f, g >= \int_{\mathbb{R}^n \times B} f(x+\omega(t))\overline{g(x)} e^{-\int_0^t V(x+\omega(s))ds} dx d\mu(\omega). \tag{7}$$

Notice that we use here a scalar product antilinear w.r.t. the second variable.

According to the Wigner function definition, we have for all x and y in $\mathbb{R}^n$,

$$f(x)\overline{g(y)} = \int_{\mathbb{R}^n} H\left(f, g, \frac{x+y}{2}, \xi\right) e^{-i(x-y)\cdot\xi} d\xi.$$

Consequently, for all ω in B,

$$f(x+\omega(t))\overline{g(x)} = \int_{\mathbb{R}^n} H\left(f, g, x + \frac{\omega(t)}{2}, \xi\right) e^{-i\omega(t)\cdot\xi} d\xi.$$

The Weyl symbol $U(t)$ of e^{-tH} being a priori defined as a tempered distribution on $\mathbb{R}^{2n}$, thus satisfies, for all F in $\mathcal{S}(\mathbb{R}^{2n})$,

$$< U(t), F >= \int_{\mathbb{R}^{2n} \times B} F\left(x + \frac{\omega(t)}{2}, \xi\right) e^{-i\omega(t)\cdot\xi} e^{-\int_0^t V(x+\omega(s))ds} dx d\xi d\mu(\omega).$$

The above identity shows that, for all F in $\mathcal{S}(\mathbb{R}^{2n})$,

$$| < U(t), F > | \leq \|F\|_{L^1(\mathbb{R}^{2n})}.$$

As a consequence, $U(t)$ is identified with a function $u(\cdot, t)$ in $L^\infty(\mathbb{R}^{2n})$, with a $L^\infty(\mathbb{R}^{2n})$ norm smaller or equal than 1, and satisfying (5) and (6). The proposition is then proved. $\square$

As a first consequence of Proposition 2.1, we give below two corollaries which do not assume that the potential V is differentiable.

Corollary 2.2. *For every multi-index β, for each $t \geq 0$, the derivative $\partial_\xi^\beta u(x, \xi, t)$ understood in the sense of distributions, is a function in $L^\infty(\mathbb{R}^{2n})$, which satisfies,*

$$\|\partial_\xi^\beta u(\cdot, t)\|_{L^\infty(\mathbb{R}^{2n})} \leq t^{|\beta|/2} \prod_{j \leq n} A_{\beta_j}, \qquad A_k = \frac{2^{k/2}}{\sqrt{\pi}} \Gamma((k+1)/2). \tag{8}$$

Let m be a nonnegative integer. If the multi-index β verifies $\beta_j \leq m$ for all $j \leq n$, then we have,

$$\|\partial_\xi^\beta u(\cdot, t)\|_{L^\infty(\mathbb{R}^{2n})} \leq B_m^{|\beta|} t^{|\beta|/2}, \qquad B_m = \max_{k \leq m} A_k. \tag{9}$$

When $m \geq 2$, we have $B_m = A_m$.

Proof. We use the notation $\omega(t) = (\omega_1(t), ..., \omega_n(t))$. In view of Proposition 2.1,

$$\|\partial_\xi^\beta u(\cdot, t)\|_{L^\infty(\mathbb{R}^{2n})} \leq \int_B \prod_{j\leq n} |\omega_j(t)|^{\beta_j} d\mu(\omega).$$

According to Kuo [16] (Chap.1, sect.4 and 5), we know that,

$$\int_B \prod_{j\leq n} |\omega_j(t)|^{\beta_j} d\mu(\omega) = t^{|\beta|/2} \prod_{j\leq n} A_{\beta_j}, \tag{10}$$

where the A_k are given in (8). Since the Gamma function is increasing on $[\frac{3}{2}, +\infty)$ (at least) and since by inspection $A_1 \leq A_2 = A_0$, we see that $B_m = A_m$ for $m \geq 2$. This proves the corollary. □

Corollary 2.3. *If $V \geq 0$ and if the right hand side below defines a convergent integral, then for almost every ξ in $\mathbb{R}^n$, the function $u(\cdot, \xi, t)$ belongs to $L^1(\mathbb{R}^n)$, and we have,*

$$\int_{\mathbb{R}^n} |u(x, \xi, t)| dx \leq \int_{\mathbb{R}^n} e^{-tV(x)} dx. \tag{11}$$

Proof. From (5), we see that

$$|u(x, \xi, t)| \leq \int_B e^{-\int_0^t V(x - \frac{\omega(t)}{2} + \omega(s)) ds} d\mu(\omega).$$

Since the function $x \mapsto e^{-tx}$ is convex, using Jensen inequality

$$e^{-\int_0^t V(x - \frac{\omega(t)}{2} + \omega(s)) ds} \leq \frac{1}{t} \int_0^t e^{-tV(x - \frac{\omega(t)}{2} + \omega(s))} ds$$

and integrating over $x \in \mathbb{R}^n$ and $\omega \in B$, for almost every $\xi \in \mathbb{R}^n$, lead to inequality (11). The corollary is thus proved. □

3 The large dimension setting.

We shall here give Hamiltonians H_Λ for systems with a large number of particles indexed by Λ, for which we shall obtain estimates on the derivatives of the Weyl symbol $u_\Lambda(\cdot, t)$ of e^{-tH_Λ}. These estimates prove in particular that $u_\Lambda(\cdot, t)$ belongs to the class of symbols studied in [4] and [2], allowing a Weyl calculus where all the constants in the inequalities are independent of Λ. The assumptions on the interaction potentials V_Λ are stated below. In the next section, we shall give two examples of Hamiltonians satisfying these hypotheses.

We suppose that the functions V_Λ are given, nonnegative, C^∞ on $\mathbb{R}^\Lambda$, the set of mappings from Λ into $\mathbb{R}$, for each finite subset Λ in Γ, for a given infinite countable set Γ. For all integers $m \geq 0$, we denote by $\mathcal{M}_m(\Lambda)$ the set of multi-indices α in $\mathbb{N}^\Lambda$ such that $0 \leq \alpha_j \leq m$, for all $j \in \Lambda$. For each multi-index α, $S(\alpha)$ denotes the set of sites $j \in \Lambda$ such that $\alpha_j \neq 0$.

Set $m \geq 1$. We also assume that there exists $C_m > 0$ such that, for all finite subsets Λ in Γ, for all α in $\mathcal{M}_m(\Lambda)$, we have for all $x \in \mathbb{R}^\Lambda$

$$\sum_{0 \neq \beta \leq \alpha} |\partial^\beta V_\Lambda(x)| \leq C_m |S(\alpha)|. \tag{12}$$

We set,

$$H_\Lambda = -\Delta_\Lambda + V_\Lambda(x) \tag{13}$$

and $U_\Lambda(t)$ denotes the Weyl symbol of e^{-tH_Λ} which, according to Proposition 2.1, is a tempered distribution on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ identified with a function $u_\Lambda(\cdot, t)$ in $L^\infty(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)$.

Theorem 3.1. *With these notations, let the functions $V_\Lambda \geq 0$ in $C^\infty(\mathbb{R}^\Lambda)$ be given for all finite subsets Λ of Γ. Let $m \geq 1$. We suppose that there exists $C_m > 0$ independent of Λ, such that (12) is satisfied. For each $t > 0$, and for every finite subset Λ of Γ, let $u_\Lambda(\cdot, t)$ be the Weyl symbol of e^{-tH_Λ}, which is identified with a function in $L^\infty(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)$ in view of Proposition 2.1. Then, for each α and β in $\mathcal{M}_m(\Lambda)$, the derivative $\partial_x^\alpha \partial_\xi^\beta u(\cdot, t)$, understood in the sense of distributions, is a function in $L^\infty(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)$ which satisfies,*

$$\|\partial_x^\alpha \partial_\xi^\beta u(\cdot, t)\|_{L^\infty(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)} \leq (m!)^{|S(\alpha)|} e^{tC_m|S(\alpha)|} B_m^{|S(\beta)|} t^{|\beta|/2}, \tag{14}$$

where B_m is defined in (8) and (9), and C_m in (12) (these constants are independent of Λ).

Proof. According to Proposition 2.1, we have for all F in $\mathcal{S}(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)$,

$$| < \partial_x^\alpha \partial_\xi^\beta U_\Lambda(t), F > | \leq \|F\|_{L^1(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)} \sup_{(x,\omega) \in \mathbb{R}^\Lambda \times B} \left| \partial_x^\alpha e^{-\int_0^t V_\Lambda(x+\omega(s))ds} \right| \int_B \prod_{j \in \Lambda} |\omega_j(t)|^{\beta_j} d\mu(\omega). \tag{15}$$

We shall use a multi-dimensional variant of Faà di Bruno formula due to Constantine Savits [7]. For each multi-index α, denote by $F(\alpha)$ the set of mappings φ from the set of multi-indices $0 \neq \beta \leq \alpha$ into the set of integers ≥ 0, such that

$$\sum_{0 \neq \beta \leq \alpha} \varphi(\beta)\beta = \alpha.$$

Constantine Savits formula is rewritten as,

$$\partial^\alpha e^{W(x)} = \alpha! e^{W(x)} \sum_{\varphi \in F(\alpha)} \prod_{0 \neq \beta \leq \alpha} \frac{1}{\varphi(\beta)!} \left[\frac{\partial^\beta W(x)}{\beta!} \right]^{\varphi(\beta)}. \tag{16}$$

For each $t > 0$ and for almost all ω in B, we apply this formula with

$$W(x) = -\int_0^t V_\Lambda(x + \omega(s))ds.$$

Since $V_\Lambda \geq 0$, we obtain

$$\sup_{(x,\omega)\in\mathbb{R}^\Lambda\times B}\left|\partial_x^\alpha e^{-\int_0^t V_\Lambda(x+\omega(s))ds}\right| \leq \alpha! \sum_{\varphi\in F(\alpha)}\prod_{0\neq\beta\leq\alpha}\frac{1}{\varphi(\beta)!}\left[\frac{t\|\partial^\beta V_\Lambda\|_{L^\infty}}{\beta!}\right]^{\varphi(\beta)}.$$

Besides,

$$\sum_{\varphi\in F(\alpha)}\prod_{0\neq\beta\leq\alpha}\frac{1}{\varphi(\beta)!}\left[\frac{t\|\partial^\beta V_\Lambda\|_{L^\infty}}{\beta!}\right]^{\varphi(\beta)} \leq \exp\left[\sum_{0\neq\beta\leq\alpha}\frac{t\|\partial^\beta V_\Lambda\|_{L^\infty}}{\beta!}\right].$$

The last factor in (15) is bounded using (10)(9) and the above right hand side is bounded using the hypothesis (12). We then deduce that,

$$| < \partial_x^\alpha\partial_\xi^\beta U_\Lambda(t), F > | \leq \alpha!\|F\|_{L^1(\mathbb{R}^{2n})}e^{tC_m|S(\alpha)|}B_m^{|S(\beta)|}t^{|\beta|/2}.$$

Since α is in $\mathcal{M}_m(\Lambda)$, we have $\alpha! \leq (m!)^{|S(\alpha)|}$. The proof of Theorem 3.1 is then completed. □

Remark 3.2. *Theorem 3.1 shows that, if the family of functions (V_Λ) verifies (12) with $C_m > 0$, then, for all $t > 0$, and for each $m \geq 0$, the family of functions $u_\Lambda(\cdot, t)$ belongs to the class $S_m(1, \rho, \delta)$ defined in (3), where $\rho_j = m!e^{tC_m}$ and $\delta_j = B_m\sqrt{t}$ for all $j \in \Gamma$. We remark that ρ_j and δ_j depend on m but not on Λ, and thus, not on the dimension.*

4 Examples and application.

4.1 Two examples.

We shall in this section give two examples of families of potentials (V_Λ) satisfying (12) for all integers $m \geq 1$. The first one corresponds to the nearest neighbor interaction in a lattice and the second one corresponds to the mean field approximation model.

Example 4.1. *Set $\Gamma = \mathbb{Z}^d$ $(d \geq 1)$. Let F and G be two nonnegative functions in $C^\infty(\mathbb{R})$, bounded together with all their derivatives. For each finite subset Λ of Γ, we set,*

$$V_\Lambda(x) = \sum_{j\in\Lambda} F(x_j) + \sum_{\substack{(j,k)\in\Lambda^2\\ |j-k|_\infty=1}} G(x_j - x_k). \tag{17}$$

Then, for all integers $m \geq 1$, there exists $C_m > 0$ such that the family of functions (V_Λ) satisfies (12).

Example 4.2. *Let Γ be an infinite countable set. Fix a function $G \geq 0$ in $C^\infty(\mathbb{R})$, bounded together with all its derivatives. Let, for each finite subset Λ of Γ,*

$$V_\Lambda(x) = \frac{1}{|\Lambda|}\sum_{(j,k)\in\Lambda^2} G(x_j - x_k). \tag{18}$$

Then, for any integer $m \geq 1$, there is $C_m > 0$ such that the family of potentials (V_Λ) is verifying (12).

Let us verify that these two examples satisfy (12). For the first term $\sum_{j\in\Lambda} F(x_j)$ of the Example 4.1, the only derivation multi-indices yielding a non zero result are the indices with exactly one non zero component. Let $\beta = (\beta_j)_{j\in\Lambda}$ be such an index, i.e., $\beta_j = 0$ for $j \neq s$, with $s \in S(\alpha)$. Then

$$\left|\partial^\beta \sum_{j\in\Lambda} F(x_j)\right| = \left|F^{(\beta_s)}(x_s)\right| \leq \max_{k\leq m} ||F^{(k)}||_{L^\infty(\mathbb{R}^\Lambda)},$$

and there are at most $m|S(\alpha)|$ such indices.

For the second term, the β's yielding a non zero result have, either exactly one non zero component, or exactly two, which moreover, correspond to neighbor points of Λ. The first case, $\beta = (\beta_j)_{j\in\Lambda}$ with $\beta_j = 0$ for $j \neq s$ with $s \in S(\alpha)$ is rather similar to the case of F, in that the maximal order of derivation of G is m. But there are at most $m|S(\alpha)|$ such indices, which give $2 \times 3^d m|S(\alpha)|$ terms, since we must take into account the maximal number of neighbors of s in Λ (3^d) and the fact that we derive the first or the second x. The second case is for $\beta = (\beta_j)_{j\in\Lambda}$ with $\beta_j = 0$ for $j \neq s, s'$ where $s, s' \in S(\alpha)$ with $|s - s'| = 1$ and concerns at most $2|S(\alpha)|3^d m^2$ terms, the maximal order of derivation being $2m$.

For the second example, the computations are similar but there are more terms, since the indices s, s' are not supposed to be neighbors. One roughly needs to replace 3^d by $|S(\alpha)|$ in the preceding computations, but dividing by $|\Lambda|$ allows to keep the same estimates.

4.2 Application.

For all finite subsets Λ in Γ, choose a function $p_\Lambda \geq 0$ in the Schwarz space $\mathcal{S}(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)$. It is known that the Weyl operator $\mathrm{Op}^{\mathrm{Weyl}}(p_\Lambda)$ is trace class. We suppose that its trace equals 1.

For all finite subsets Λ in Γ, suppose that the functions $V_\Lambda \geq 0$ in $\mathbb{R}^\Lambda$ are given satisfying the conditions of Example 4.1 or Example 4.2, and denote by H_Λ the Hamiltonian defined in (13). Let A be a function on $\mathbb{R}$ and denote by A_j the multiplication operator by the function $A(x_j)$ ($j \in \Lambda$). The function A is chosen to be polynomial to avoid a long development on pseudodifferential operators.

Proposition 4.3. *With these notations, there exists a constant $C > 0$ such that, for all finite subsets Λ of Γ, for each t in $(0, 1]$, for every j in Λ, we have,*

$$\left|\mathrm{Tr}([A_j, e^{-tH_\Lambda}] \circ \mathrm{Op}^{\mathrm{Weyl}}(p_\Lambda))\right| \leq C\sqrt{t}.$$

Proof. Let $F_{\Lambda,t}$ be the Weyl symbol of the commutator $[A_j, e^{-tH_\Lambda}]$. According to Theorem 3.1 and to the Weyl calculus in one dimension, there exists $C > 0$ such that, for all finite Λ in Γ, and for any t in $(0, 1]$,

$$\|F_{\Lambda,t}\|_{L^\infty(\mathbb{R}^\Lambda\times\mathbb{R}^\Lambda)} \leq C\sqrt{t}.$$

It is known that,

$$\mathrm{Tr}([A_j, e^{-tH_\Lambda}] \circ \mathrm{Op}^{\mathrm{Weyl}}(p_\Lambda)) = (2\pi)^{-|\Lambda|} \int_{\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda} F_{\Lambda,t}(x,\xi) p_\Lambda(x,\xi) dx d\xi,$$

$$\mathrm{Tr}(\mathrm{Op}^{\mathrm{Weyl}}(p_\Lambda)) = (2\pi)^{-|\Lambda|} \int_{\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda} p_\Lambda(x,\xi) dx d\xi = 1.$$

The proposition then follows. □

References

[1] M. Aizenman, B. Simon, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), no. 2, 209-273.

[2] L. Amour, L. Jager, J. Nourrigat, *On bounded pseudodifferential operators in a high-dimensional setting*, to appear in Proc. of the A.M.S.

[3] L. Amour, L. Jager, J. Nourrigat, *Bounded Weyl pseudodifferential operators in Fock space,* preprint, arXiv:1209.2852, sept. 2012.

[4] L. Amour, J. Nourrigat, *Weyl composition of symbols in large dimension*, to appear in J. Anal. Math.

[5] R. L. Coifman, Y. Meyer, *Au delà des opérateurs pseudo-différentiels,* Astérisque, **57**, 1978.

[6] M. Combescure, D. Robert, *Coherent states and applications in mathematical physics,* Theoretical and Mathematical Physics. Springer, Dordrecht, 2012.

[7] G. M. Constantine, T. H. Savits, *A multivariate Faà di Bruno formula with applications,* Trans. of the A.M.S, **348**, (2), (1996), 503-520.

[8] G. B. Folland, *Harmonic analysis in phase space.* Annals of Mathematics Studies, **122**. Princeton University Press, Princeton, NJ, 1989.

[9] L. Gross, *Measurable functions on Hilbert space,* Trans. Amer. Math. Soc. **105** (1962) 372-390.

[10] L. Gross, *Abstract Wiener spaces*, Proc. 5th Berkeley Sym. Math. Stat. Prob, **2**, (1965), 31-42.

[11] L. Gross, *Abstract Wiener measure and infinite dimensional potential theory*, in *Lectures in modern Analysis and applications, II,* Lecture Notes in Math **140**, 84-116, Springer (1970).

[12] B. Helffer, *Semiclassical analysis, Witten Laplacians, and statistical mechanics*, Series in Partial Differential Equations and Applications, 1. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[13] B. Helffer, F. Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Mathematics, 1862. Springer-Verlag, Berlin, 2005.

[14] B. Helffer, D. Robert, *Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles*, J. Funct. Anal. **53** (1983), no. 3, 246-268.

[15] L. Hörmander, *The analysis of linear partial differential operators,* Volume III, Springer, 1985.

[16] H. H. Kuo, *Gaussian measures in Banach spaces.* Lecture Notes in Mathematics, Vol. 463. Springer, Berlin-New York, 1975.

[17] B. Lascar, *Noyaux d'une classe d'opérateurs pseudo-différentiels sur l'espace de Fock, et applications.* Séminaire Paul Krée, 3e année (1976-77), Equations aux dérivées partielles en dimension infinie, Exp. No. 6.

[18] B. Lascar, *Une classe d'opérateurs elliptiques du second ordre sur un espace de Hilbert,* J. Funct. Anal. **35** (1980), no. 3, 316-343.

[19] B. Lascar, *Opérateurs pseudo-différentiels en dimension infinie. Applications.* C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 13, A767-A769.

[20] N. Lerner, *Metrics on the phase space and non self-adjoint pseudo-differential operators,* Birkhäuser Springer, 2010.

[21] B. Øksendal, *Stochastic differential equations. An introduction with applications*, Sixth edition. Universitext. Springer-Verlag, Berlin, 2003.

[22] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic Press, New York-London, 1975.

[23] D. Robert, *Autour de l'approximation semi-classique*, Progress in Mathematics 68, Birkhäuser Boston, Inc., Boston, MA, 1987.

[24] B. Simon, *Functional Integration and quantum Physics,* Second Edition, AMS Chelsea Publishing, Providence, R.I, 2005.

[25] B. Simon, *A Feynman-Kac formula for unbounded semigroups,* Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999), 317-321, CMS Conf. Proc., 28, Amer. Math. Soc., Providence, RI, 2000.

[26] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 447-526.

[27] A-S. Sznitman, *Brownian motion, obstacles and random media*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

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