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Henri Guenancia

► To cite this version:

Henri Guenancia. Semistability of the tangent sheaf of singular varieties. Algebraic Geometry, 2016, 3 (5), pp.508 - 542. 10.14231/AG-2016-024 . hal-01878999

HAL Id: hal-01878999

<https://hal.science/hal-01878999>

Submitted on 18 Nov 2020

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SEMI-STABILITY OF THE TANGENT SHEAF OF SINGULAR VARIETIES

by

Henri Guenancia

Abstract. — The main goal of this paper is to prove the polystability of the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ of a log canonical pair (X, D) whose canonical bundle $K_X + D$ is ample, generalizing in a significant way a theorem of Enoki. We apply this result and the techniques involved in its proof to get a Bogomolov type inequality for Chern numbers for lc pairs, a version of the semistability theorem for stable varieties (the higher dimensional analogue of Deligne-Mumford’s stable curves) and finally the generic semi positivity of the sheaf of logarithmic differentials of a log canonical pair with pseudo-effective log canonical class, in the spirit of Miyaoka’s theorem.

Contents

Introduction.....	1
1. A Laplacian estimate.....	6
2. Polystability of the logarithmic tangent sheaf.....	8
3. An extension to stable varieties.....	23
4. Generic semipositivity.....	29
5. A Bogomolov type inequality for Chern numbers.....	36
References.....	37

Introduction

Semi-stability. — This paper deals with the notion of slope semi-stability for coherent sheaves on singular varieties using a differential-geometric approach. This notion was first introduced by Mumford [Mum63] in his attempt to construct bounded families of vector bundles over a curve. Recall that a vector bundle E over a complex projective curve C is said to be semistable (resp. stable) if for every proper subbundle F of E , we have

$$\frac{\deg(F)}{\operatorname{rk} F} \leq \frac{\deg(E)}{\operatorname{rk} E} \quad \left(\text{resp. } \frac{\deg(F)}{\operatorname{rk} F} < \frac{\deg(E)}{\operatorname{rk} E} \right)$$

where the degree of F is $\int_C c_1(F)$; the quantity $\deg(F)/\operatorname{rk} F$ is called the slope of F and denoted $\mu(F)$. When one tries to generalize this definition to higher dimensional varieties one first needs an ample line bundle H (called a polarization) to define a degree and thus

a slope: $\deg_H(F) = c_1(F) \cdot H^{n-1}$ and $\mu_H(F) = \deg_H(F)/\text{rk } F$. And more importantly, in order to keep the good moduli properties shared by stable vector bundles over curves, one needs to ask that not only all the subbundles of E have a smaller slope than the one of E , but also all its proper coherent *subsheaves*, cf Definition 2.1.

The Kobayashi-Hitchin correspondence. — One of the major developments regarding stable vector bundles is the celebrated Kobayashi-Hitchin correspondence, relating the stability of a vector bundle E (an algebro-geometric property) to the existence of a special hermitian metric on E (a differential-geometric property):

Kobayashi-Hitchin correspondence. — *Let E be an holomorphic vector bundle on a compact Kähler manifold (X, ω) . Then E is polystable with respect to ω if and only if E admits a Hermite-Einstein metric with respect to ω .*

Recall that a Hermite-Einstein metric h on E is a Hermite-Einstein metric with respect to a Kähler form ω if its Chern curvature tensor $\Theta_h(E) \in \mathcal{C}^\infty(X, \Omega^{1,1} \otimes \text{End}(E))$ satisfies $\text{tr}_\omega(\Theta_h(E)) = \mu \text{Id}_E$ for some constant $\mu \in \mathbb{R}$. Here, polystability means that E is semistable and is the sum of stable subsheaves (that automatically have same slope).

This correspondence is due to Kobayashi-Lübke [KobS82, Lü83] in the direction "Hermite-Einstein \Rightarrow stable" and to Donaldson for Riemann surfaces [Don83], algebraic surfaces [Don85] and manifolds [Don87] and to Uhlenbeck-Yau [UY86, UY89] for Kähler manifolds.

The case where $E = T_X$ is the tangent bundle of the manifold is of particular importance because due to the symmetry of the Riemann curvature tensor, if ω is a Kähler form and h is the induced hermitian metric on T_X , then (T_X, h) is Hermite-Einstein with respect to ω if and only if ω is a Kähler-Einstein metric, i.e. $\text{Ric } \omega = \mu \omega$ for some constant $\mu \in \mathbb{R}$. For a lot of reasons, the Kähler-Einstein equation is in general easier to study/solve than the Hermite-Einstein equation, and currently we dispose of a lot of existence results and tools in the Kähler-Einstein theory. For example, when K_X is ample (resp. numerically trivial), the celebrated theorems of Aubin-Yau and Yau [Aub78, Yau78] provide a unique negatively curved Kähler-Einstein metric (resp. Ricci-flat metric) on X living in $c_1(K_X)$ (resp. in any Kähler cohomology class). Bringing these theorem together with the Kobayashi-Hitchin correspondence, we see that:

Corollary. — *Let X be a compact Kähler manifold. Then*

- *If K_X is ample, then T_X is polystable with respect to K_X .*
- *If K_X is numerically trivial, then T_X is polystable with respect to any Kähler class.*

Singular varieties and stability. — In the present paper, we will strive to extend the above result in a singular setting pushing further the very elegant approach of I. Enoki [Eno88] who essentially proved the semistability of the tangent sheaf of X in both cases above assuming merely that X has canonical singularities. His strategy is to use approximate

Kähler-Einstein metric on a resolution X' . By considering them as approximate Hermite-Einstein metrics on $T_{X'}$, he obtains an "approximate slope inequality". Thanks to various estimates (due to Yau) about Monge-Ampère equations, he manages to control the error term in the inequality and to prove that it converges to zero at the end of the approximation process.

In a related though somewhat different framework, Tsuji [Tsu88b] proved the semistability of the tangent bundle of smooth minimal projective manifolds (ie with nef canonical bundle) using a perturbation argument involving conic metrics along some suitable divisor, whose angle tends to 2π . He also states (without proof though) that the result holds as well for a log smooth pair (X, D) such that $K_X + D$ is nef.

We will generalize Enoki's Theorem from three different perspectives: we will merely require that X has log canonical singularities, we will allow a boundary divisor and work in the setting of log pairs, and finally we will show polystability in addition to semistability:

Theorem A. — *Let X be a compact Kähler space and D a reduced Weil divisor such that (X, D) has log canonical singularities*

- *If $K_X + D$ is ample, then the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ is polystable with respect to $K_X + D$.*
- *If $K_X + D \equiv 0$, then $\mathcal{T}_X(-\log D)$ is semistable with respect to any Kähler class. Moreover, if $D = 0$ and X has klt singularities, then \mathcal{T}_X is actually polystable with respect to any Kähler class.*

Note that the ampleness of $K_X + D$ can be replaced nefness without any change in the proof (for the semistability, not the polystability though); however it is not that common that consider stability with respect to non-movable classes, which is why we left it out of the statement.

Compared to the proof in the case of canonical singularities, a lot of serious new difficulties pop up in this general setting, mainly because the presence of boundary divisor D requires to work with cuspidal metrics instead of smooth metrics. Also passing from canonical to log canonical singularities depends upon a finer analysis of the regularity for solutions of degenerate Monge-Ampère equations, which we develop in §1. We outlined the new issues we had to overcome in §2.2.

We should mention that the second item of Theorem A is closely linked with the main result of [GKP] which further investigates the tangent sheaf of varieties with canonical singularities and vanishing first Chern class. Also, some of the techniques used in the proof appear in the very interesting paper [CP14] where one of the clever ideas to treat boundary problems is to use conic metrics (with angles going to 0); we will apply these ideas in the last section.

Stable varieties. — The first item of Theorem A above can be used to deduce properties about the so-called stable varieties. For short, stable varieties are the higher dimensional

analogue of stable curves introduced by Deligne and Mumford, in the sense that they allow to compactify the moduli space of canonically polarized manifolds (say with fixed Hilbert polynomial). Their definition involves three requirements: projectivity, positivity of the canonical bundle and specific singularities, cf §3.1 and the references therein for a more detailed account. For the time being, we just need to know that if X is a stable variety, then K_X can be defined as a ample \mathbb{Q} -line bundle, and the normalization map $\nu : X^\nu \rightarrow X$ satisfies that $\nu^* K_X = K_{X^\nu} + D$ where D is a reduced Weil divisor such that the pair (X^ν, D) has log canonical singularities.

Therefore, Theorem A guarantees that $\mathcal{T}_{X^\nu}(-\log D)$ is semistable with respect to $K_{X^\nu} + D$, and one can easily infer from this that $\text{Aut}(X)$ is a finite group, cf Corollary 2.6. One can also ask whether one can say anything about semistability of the tangent sheaf of X itself. Inspired by the case of stable curves, we introduce the sheaf $\nu_* \mathcal{T}_{X^\nu}(-\log D)$ that should be the "tangent sheaf" we want to look at in this setting where the philosophy is to describe the objects upstairs. For non-normal varieties, the notions of slope and stability exist, but are way more delicate to deal with. We recall them in §3, and manage to prove the following result which is a non-trivial consequence of Theorem A:

Theorem B. — *Let X be a stable variety, $\nu : X^\nu \rightarrow X$ its normalization map, and $D \subset X^\nu$ the conductor of ν . Then the sheaf $\nu_* \mathcal{T}_{X^\nu}(-\log D)$ is semistable with respect to K_X .*

The proof of this theorem requires to understand the relationship between the slope of a sheaf and the one of its pull-back under the normalization map, which happens to be trickier than one could imagine at first sight, cf Remark 3.8.

Generic semipositivity. — Miyaoka generic semi positivity theorem [Miy87] asserts that if X is a normal projective variety, then Ω_X^1 is generically semipositive unless X is uniruled. Recall that a reflexive sheaf \mathcal{E} on a normal projective variety X is generically semi-positive if for any ample polarizations H_1, \dots, H_{n-1} and any coherent quotient \mathcal{F} of \mathcal{E} , the degree of \mathcal{F} with respect to (H_1, \dots, H_{n-1}) is non-negative, i.e. $(c_1(\mathcal{F}) \cdot H_1 \cdot \dots \cdot H_{n-1}) \geq 0$. This celebrated result had a lot of applications and was recently generalized to the case of log smooth pairs (X, D) such that $K_X + D$ is pseudo-effective by Campana-Păun [CP13, CP14]. Capitalizing on their method and the techniques of this paper, we are able to extend this to the following singular case:

Theorem C. — *Let X be an n -dimensional compact Kähler space, D a reduced Weil divisor such that (X, D) has log canonical singularities and let ω be a Kähler form. If $K_X + D$ is pseudo-effective (resp. $-(K_X + D)$ is pseudo-effective), then $\Omega_X^1(\log D)$ (resp. $\mathcal{T}_X(-\log D)$) is generically ω -semipositive.*

In particular, if $K_X + D \equiv 0$, the slope of the logarithmic tangent sheaf vanishes, so the generic semi positivity of $\Omega_X^1(\log D)$ is equivalent to its semistability with respect to any Kähler class. This is how we will prove the second item of Theorem A.

A Bogomolov-type inequality. — In the last section of this paper, we investigate applications of the Kähler-Einstein theory for singular spaces to Chern numbers inequalities. In the smooth case, this is a very fruitful domain (cf §5) but for singular varieties, the lack of understanding of the Kähler-Einstein metrics near the singularities makes it really hard to tackle such global questions. However, the point of this article is to show that even a rough control on the singularities of the metric lets us prove semistability properties. Therefore, one can apply Bogomolov-Gieseker inequality (generalized by Miyaoka) to our semistable logarithmic tangent sheaf in the context we are working in.

More precisely, let (X, D) be a projective log canonical pair of dimension n where D is a reduced divisor, and let $\pi : Y \rightarrow X$ be a log resolution of the pair (X, D) . We denote by Δ the sum of the strict transform of D and the exceptional divisor with zero log discrepancy; to put it in a different way, $K_Y + \Delta - \pi^*(K_X + D)$ is an exceptional divisor with coefficients > -1 .

Theorem D. — *Let (X, D) be a projective log canonical as above. We have the following Bogomolov type inequalities:*

- If $K_X + D$ is ample, then:

$$c_1(T_Y(-\log \Delta))^2 \cdot (\pi^*(K_X + D))^{n-2} \leq \frac{2n}{n-1} c_2(T_Y(-\log \Delta)) \cdot (\pi^*(K_X + D))^{n-2}$$

- If $K_X + D \equiv 0$, then for any ample line bundle H on X , we have:

$$c_1(T_Y(-\log \Delta))^2 \cdot (\pi^*H)^{n-2} \leq \frac{2n}{n-1} c_2(T_Y(-\log \Delta)) \cdot (\pi^*H)^{n-2}$$

Of course, if the non-snc locus $(X, D)_{\text{sing}}$ has codimension > 2 then we can define c_1 and c_2 for the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ as the push forward of the usual cycles on $(X, D)_{\text{reg}}$ by the open immersion. The projection formula yields $c_1(\mathcal{T}_X(-\log D))^2 \cdot (K_X + D)^{n-2} \leq \frac{2n}{n-1} c_2(\mathcal{T}_X(-\log D)) \cdot (K_X + D)^{n-2}$ if $K_X + D$ is ample, and $c_1(\mathcal{T}_X(-\log D))^2 \cdot H^{n-2} \leq \frac{2n}{n-1} c_2(\mathcal{T}_X(-\log D)) \cdot H^{n-2}$ if $K_X + D \equiv 0$.

Organization of the paper. —

- §1: We obtain a Laplacian estimate for solutions of Monge-Ampère equations with degenerate right hand side, which will turn out to be crucial for the proof of the semistability.
- §2: We first recall the definitions of stability, log canonical singularities, logarithmic tangent sheaf, then we state the first part of Theorem A, explain the difficulties and prove it.
- §3: We give the definition of stable varieties as well as semistability in a context of non-normal varieties, and then prove Theorem B.
- §4: Extending the techniques from Campana-Păun's papers [CP13, CP14] to the singular case, we prove generic semipositivity for the sheaf of logarithmic differentials assuming that the logarithmic canonical bundle is pseudo-effective (Theorem 4.1). Then we apply this result to prove the second item of Theorem A.

- §5 We prove a Bogomolov type inequality for Chern numbers on log canonical pairs.

Acknowledgements. This work was initiated near the end of my PhD thesis, and I am grateful to my advisors Sébastien Boucksom and Mihai Păun who generously shared their ideas and helped me develop this article. I am particularly indebted to Mihai Păun who shared his unpublished work [CP14] with me and followed the development of this paper with interest. Finally, I would also like to thank Patrick Graf for helpful discussions, and the referee for reading the manuscript extremely carefully and pointing out several inaccuracies.

1. A Laplacian estimate

This section is devoted to the technical Proposition 1.1 which can be seen as a generalization of the Laplacian estimate obtained using Chern-Lu formula. It will be used in the next section to control the error term in the semistability inequality.

So let us first recall Chern-Lu's formula [Che68, Lu68], which is going to be an essential tool to get the Laplacian estimate. Let (X, ω_X) and (Y, ω_Y) be two Kähler manifolds, and $f : X \rightarrow Y$ an holomorphic map satisfying $\partial f \neq 0$. Then

$$\Delta_{\omega_X} \log |\partial f|^2 \geq \frac{\text{Ric } \omega_X \otimes \omega_Y (\partial f, \bar{\partial} f)}{|\partial f|^2} - \frac{\omega_X \otimes R^Y (\partial f, \bar{\partial} f, \partial f, \bar{\partial} f)}{|\partial f|^2}$$

where ∂f is viewed as a section of $T_X^* \otimes f^* T_Y$.

Using this formula when f is the identity map (but the Kähler forms differ), one can derive so-called Laplacian estimates for the Kähler-Einstein equation provided that the reference metric has upper bounded holomorphic bisectional curvature (cf [JMR11, Section 7]). The following proposition, inspired by [Pău08] (see also [BBE⁺11, Theorem 10.1]), enables to derive (weaker) Laplacian estimates in some cases where the Ricci curvature is not bounded from below:

Proposition 1.1. — *Let X be a compact Kähler manifold of dimension n , ω, ω' two cohomologous Kähler metrics on X . We assume that $\omega' = \omega + dd^c \varphi$, that there exists a smooth function ψ and that we have a constant $C > 0$ satisfying:*

- (i) $\sup_X |\varphi| \leq C$,
- (ii) $\sup_X \Psi \leq C$ and $dd^c \Psi \geq -C\omega$,
- (iii) $\Theta_\omega(T_X) \leq C\omega \otimes \text{Id}_{T_X}$.
- (iii) $\text{Ric } \omega' \geq -C\omega' - C\omega - dd^c \Psi$

Then there exists some constant $M > 0$ depending only on n and C such that

$$\omega' \geq M^{-1} e^\Psi \omega.$$

In the above proposition, we used the operator $d^c := \frac{1}{2i\pi}(\partial - \bar{\partial})$, and we denoted by $\Theta_\omega(T_X)$ the Chern curvature of the hermitian vector bundle (T_X, ω) . The inequality in (iii) is to be understood in the sense of Griffiths semipositivity.

Proof. — The main difficulty is that we do not have a control on the lower bound of $\text{Ric } \omega'$. The trick, inspired by [Pău08], is to add ψ in the Laplacian appearing in Chern-Lu formula. Let us now get into the details. We apply Chern-Lu's formula to $f = \text{id} : (X, \omega') \rightarrow (X, \omega)$. Then $|\partial f|^2 = \text{tr}_{\omega'} \omega$. We denote by g (resp. h) the hermitian metrics induced by ω' (resp. ω). The second term of Chern-Lu formula is easily dealt with:

$$\begin{aligned} -\omega' \otimes R^h(\partial f, \bar{\partial} f, \partial f, \bar{\partial} f) &= -g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}^h \\ &\geq -C g^{i\bar{j}} g^{k\bar{l}} (h_{i\bar{j}} h_{k\bar{l}} + h_{i\bar{l}} h_{k\bar{j}}) \\ &\geq -2C (\text{tr}_{\omega'} \omega)^2. \end{aligned}$$

To analyze $\text{Ric } \omega' \otimes \omega(\partial f, \bar{\partial} f)$, it is convenient to introduce a suitable system of coordinates. More precisely, at a given point $p \in X$, one can choose geodesic coordinates such that $h_{k\bar{l}}(0) = \delta_{kl}$, $g_{i\bar{j}}(0) = \delta_{ij}$ and such that the derivatives of h_{kl} and g_{ij} vanish at p . So, at p , we have

$$\begin{aligned} \text{Ric } \omega' \otimes \omega(\partial f, \bar{\partial} f) &= \sum_{i,j,k,l} g^{i\bar{k}} g^{j\bar{l}} R_{i\bar{j}k\bar{l}}^g h_{k\bar{l}} \\ &\geq - \sum_{i,j,k,l} g^{i\bar{k}} g^{j\bar{l}} (C g_{i\bar{j}} + (C h_{i\bar{j}} + \Psi_{i\bar{j}})) h_{k\bar{l}} \\ &= - \sum_i \left(C g^{i\bar{i}} + g^{i\bar{i}} g^{i\bar{i}} (C + \Psi_{i\bar{i}}) \right) \\ &\geq -C \left(\sum_i g^{i\bar{i}} \right) - \left(\sum_i g^{i\bar{i}} \right) \cdot \left(\sum_j g^{j\bar{j}} (C + \Psi_{j\bar{j}}) \right) \\ &= -C \text{tr}_{\omega'} \omega - (\text{tr}_{\omega'} \omega) \cdot \text{tr}_{\omega'} (C\omega + dd^c \Psi). \end{aligned}$$

At that point, Chern-Lu formula gives us:

$$\Delta_{\omega'} \log \text{tr}_{\omega'} \omega \geq -3C \text{tr}_{\omega'} \omega - C - \text{tr}_{\omega'} dd^c \Psi$$

and therefore:

$$\Delta_{\omega'} (\log \text{tr}_{\omega'} \omega + \Psi) \geq -3C \text{tr}_{\omega'} \omega - C.$$

Setting $A = 3C + 1$, we get as usual:

$$\Delta_{\omega'} (\log \text{tr}_{\omega'} \omega + \Psi - A\varphi) \geq \text{tr}_{\omega'} \omega - nA - C.$$

The end is classic: we choose a point p where $\log \text{tr}_{\omega'} \omega + \Psi - A\varphi$ attains its maximum; then we have

$$\begin{aligned} \log \text{tr}_{\omega'} \omega &\leq (\log \text{tr}_{\omega'} \omega + \Psi - A\varphi)(p) - \Psi + A\varphi \\ &\leq (\log(nA + C) + \sup \Psi + 2A \sup |\varphi|) - \Psi \end{aligned}$$

which gives the expected result since we have a uniform bound on $\|\varphi\|_\infty$ by assumption. \square

Remark 1.2. — Note that it is not clear a priori to deduce the previous estimate using [Pău08] by exchanging the role of ω and ω' because we would no longer have control on the bisectional curvature of ω' .

Combining the previous result and Păun's estimate, we obtain the following estimate:

Corollary 1.3. — *Let X be a compact Kähler manifold of dimension n , ω, ω' two cohomologous Kähler metrics on X . We assume that $\omega' = \omega + dd^c\varphi$ with $\omega'^n = e^{\psi^+ - \psi^-} \omega^n$ for some smooth functions ψ^\pm , and that we have a constant $C > 0$ and some $p > 1$ satisfying:*

- (i) $\sup_X \psi^+ \leq C$ and $\|e^{-\psi^-}\|_{L^p(\omega^n)} \leq C$,
- (ii) $dd^c\psi^\pm \geq -C\omega$.

Then there exists some constant $M > 0$ depending only on n, p, C and ω such that

$$M^{-1}e^{\psi^+}\omega \leq \omega' \leq Me^{-\psi^-}\omega.$$

2. Polystability of the logarithmic tangent sheaf

2.1. Generalities. — In this section, we briefly recall the definitions of slope, semi-stability, various types of singularities appearing in the Minimal Model Program as well as the (logarithmic) tangent sheaf before exposing the previously known results. We refer to [Har80], [KobS87, Chap. V] or [HL10] for more details.

2.1.1. Notion of stability. — In the following, X will be a complex projective normal variety of dimension n , and \mathcal{F} will always denote a coherent sheaf. We write $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ for the dual of \mathcal{F} . We say that \mathcal{F} is reflexive if the natural map

$$j : \mathcal{F} \rightarrow \mathcal{F}^{**}$$

is an isomorphism. For instance, the dual of a coherent sheaf is always reflexive. Moreover, as the kernel of j is exactly the torsion of \mathcal{F} , a reflexive sheaf is automatically torsion-free. We define the rank of a coherent sheaf \mathcal{F} to be its rank at the generic point (or equivalently, consider the Zariski open subset where \mathcal{F} is locally free), we denote it by $\text{rk } \mathcal{F}$.

We need now to define the determinant of a coherent sheaf \mathcal{F} . We set $r = \text{rk } \mathcal{F}$, and we let

$$\det \mathcal{F} := (\Lambda^r \mathcal{F})^{**}$$

be the determinant of \mathcal{F} . It is a rank one reflexive sheaf on X , but if X is not smooth, it is in general not a line bundle, ie it is not locally free. As X is normal, there is a 1-1 correspondence between rank one reflexive sheaves (up to isomorphism) and Weil divisors (up to linear equivalence), the correspondence being given in the usual way $D \mapsto \mathcal{O}_X(D) = \{f, \text{div}(f) \geq -D\}$ in one direction, and in the other direction, given \mathcal{F} a rank one reflexive sheaf, we choose a Weil divisor on X_{reg} representing the line bundle $\mathcal{F}|_{X_{\text{reg}}}$ and take its closure (for the Zariski topology).

We will denote by $c_1(\mathcal{F})$ the equivalence class of any Weil divisor attached to $\det \mathcal{F}$. We can now define the slope:

Definition 2.1. — Let H be an ample line bundle on X . We define the slope of \mathcal{F} with respect to H to be the rational number

$$\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\mathrm{rk} \mathcal{F}}.$$

Let us get now to the definition of slope stability, which goes back to Mumford [Mum63] and Takemoto [Tak72]:

Definition 2.2. — Let \mathcal{E} be a torsion-free coherent sheaf on X , and H be an ample line bundle.

We say that \mathcal{E} is *semistable* (resp. *stable*) with respect to H if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ (resp. every non-zero and proper coherent subsheaf \mathcal{F}), we have

$$\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E}) \quad (\text{resp. } \mu_H(\mathcal{F}) < \mu_H(\mathcal{E})).$$

We say that \mathcal{E} is *polystable* (with respect to H) if \mathcal{E} is the direct sum of stable subsheaves with same slope.

2.1.2. Canonical and log canonical singularities. — Let us now very briefly give a definition of the class of singularities we are going to deal with (see [KM98] for a more detailed account):

Definition 2.3. — Let X be a complex normal variety, D an *effective* Weil \mathbb{Q} -divisor D such that $K_X + D$ is \mathbb{Q} -Cartier and $\pi : X' \rightarrow X$ a log resolution of (X, D) . We define the coefficients a_i by the formula

$$K_{X'} = \pi^*(K_X + D) + \sum a_i E_i$$

where E_i is either an exceptional divisor or the strict transform of a component of D .

· If $D = 0$, we say that X has canonical (resp. log terminal) singularities if for all i , one has $a_i \geq 0$ (resp. $a_i > -1$).

· Else, we say that the pair (X, D) has klt (resp. log canonical) singularities if for all i , one has $a_i > -1$ (resp. $a_i \geq -1$).

So all these singularities are normal so that the notions of slope and semistability make sense for these varieties.

2.1.3. The logarithmic tangent sheaf. — Let us first recall that on an arbitrary complex variety X , one can define the sheaf Ω_X^1 of Kähler differentials (cf [Har77, II.8]) and define the tangent sheaf \mathcal{T}_X of X as the dual $(\Omega_X^1)^*$ of the sheaf of Kähler differentials. If X is smooth, then this sheaf corresponds to the sheaf associated with the tangent bundle T_X . If X is merely normal, then \mathcal{T}_X is a reflexive sheaf so that by the observation above, it can equivalently be defined as push-forward of $\mathcal{T}_{X_{\mathrm{reg}}}$ via the open immersion $X_{\mathrm{reg}} \hookrightarrow X$.

Let us now consider log pairs: let X be a normal projective variety, and D be a reduced Weil divisor on X . If the pair (X, D) is log smooth, that is X is smooth and D is reduced

with simple normal crossings, we have a well defined logarithmic tangent bundle $T_X(-\log D)$ which is the dual of the bundle of logarithmic differentials $\Omega_X^1(\log D)$. It simply consists of vector fields that vanish along D , i.e. if D is locally given by $(z_1 \cdots z_k = 0)$, then the sheaf at stake is the locally free \mathcal{O}_X -module generated by

$$z_1 \frac{\partial}{\partial z_1}, \dots, z_k \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \dots, \frac{\partial}{\partial z_n}.$$

To define its analogue in a more general setting, we proceed as follows. We denote by $(X, D)_{\text{reg}}$ the simple normal crossing (snc for short) locus of the pair (X, D) , i.e. the locus of points $x \in X$ where (X, D) is log smooth in a neighborhood of x . As X is normal, it is smooth in codimension 1, and D is generically smooth, therefore $(X, D)_{\text{reg}}$ is a Zariski open set whose complement has codimension at least 2. Let us denote by $j : (X, D)_{\text{reg}} \hookrightarrow X$ the open immersion.

Definition 2.4. — Let (X, D) be a log pair as above, and denote by $U := (X, D)_{\text{reg}}$ its snc locus. The logarithmic tangent sheaf of (X, D) , denoted by $\mathcal{T}_X(-\log D)$, is defined as $j_* T_U(-\log D|_U)$.

This sheaf is automatically coherent, and hence reflexive by e.g. [Har80, Proposition 1.6]. One can equivalently define the logarithm tangent sheaf as the sheafification of the module of derivations that preserve the ideal sheaf corresponding to D . When (X, D) is log smooth, the logarithmic tangent sheaf is locally free, and we will denote $T_X(-\log D)$ the corresponding vector bundle.

2.2. Statement of the result. — Our main result extends Enoki's theorem (say when K_X is ample) to the case of a log canonical pair (X, D) and also prove the polystability:

Theorem 2.5. — *Let X be a normal projective variety and D be a reduced Weil divisor such that (X, D) is log canonical and $K_X + D$ is an ample \mathbb{Q} -line bundle. Then $\mathcal{T}_X(-\log D)$ is polystable with respect to $K_X + D$.*

Compared to Enoki's setting, a large amount of new difficulties arise in this generalized setting that require a lot of fine analysis and some recent works:

- Getting precise estimates for Monge-Ampère with a very degenerate right hand side (this is the content of Lemma 2.7 which is a combination of Proposition 1.1 and [GW14, Theorem A])
- Working with cuspidal metrics instead of smooth ones (which requires the delicate Proposition 2.8)
- Analyzing the limiting behavior of the approximate Kähler-Einstein metrics (we will use [BG14] and [GW14, Theorem B] in a crucial way) to get polystability

However, these complications come from the relatively large degree of generality that we chose to work with, and we could have obtained weaker generalizations of Enoki's theorem at a less expensive cost, cf Remark 2.10.

2.3. An application to automorphism groups. — A first application of Theorem 2.5 concerns the finiteness of some automorphism groups. Indeed, it is well-known that semistable sheaves with negative slopes do not admit any non-trivial section, and relying on this fact and the above theorem, one can prove:

Corollary 2.6. — *The automorphism group $\text{Aut}(X)$ of a stable variety X is finite.*

For the definition of a stable variety, see Definition 3.1. The finiteness of the automorphism group of such varieties was already well-known, cf [Miy83, BHPS12] or [BG14, Theorem 6.3] for a differential-geometric proof in the normal case.

Proof of Corollary 2.6. — Let $\nu : Y \rightarrow X$ be the normalization of X . By the universal property of the normalization, any automorphism of X can be lifted to an automorphism of Y giving an injection $i : \text{Aut}(X) \rightarrow \text{Aut}(Y)$; observe that any element in $i(\text{Aut}(X))$ preserves the conductor Δ of the normalization. Moreover, the subgroup of $\text{Aut}(Y)$ fixing every connected component of Y has finite index in $\text{Aut}(Y)$, so we can assume that Y is connected.

So we are reduced to showing that given a normal variety Y and a reduced divisor Δ such that the pair (Y, Δ) has log canonical singularities and satisfies $K_Y + \Delta$ is ample, then the automorphism group $\text{Aut}(Y, \Delta)$ of the pair (Y, Δ) is finite; this group is defined as the (closed) subgroup of $\text{Aut}(Y)$ fixing Δ . But any such automorphism will preserve $K_Y + \Delta$, and therefore our group is a subgroup of the automorphism of a polarized variety so it is a linear group (as it embeds into $\text{PGL}(N, \mathbb{C})$ for some large integer N). Therefore $\text{Aut}(Y, \Delta)$ is finite if and only if its tangent space is trivial, but its tangent space is precisely $H^0(Y, \mathcal{T}_Y(-\log \Delta))$, the space of holomorphic vector fields tangent to Δ . Indeed, given a 1-parameter family of such automorphisms (ϕ_t) , each of them preserves $(Y, \Delta)_{\text{reg}}$ the snc locus of the pair, so taking the derivative at $t = 0$ yields a vector field on $(Y, \Delta)_{\text{reg}}$ tangent to Δ . As $\mathcal{T}_Y(-\log \Delta)$ is reflexive and the complement of $(Y, \Delta)_{\text{reg}}$ has codimension at least two, we get an element in $H^0(Y, \mathcal{T}_Y(-\log \Delta))$. The correspondence works just as well in the other direction by restricting the vector field to $(Y, \Delta)_{\text{reg}}$ and taking its flow, which extends to an automorphism of the pair (Y, Δ) by [BBE⁺11, Lemma 5.2].

Now, any non-zero element $\xi \in H^0(Y, \mathcal{T}_Y(-\log \Delta))$ induces an injective sheaf morphism $f : \mathcal{O}_Y \rightarrow \mathcal{T}_Y(-\log \Delta)$ (the kernel is a torsion sheaf, but \mathcal{O}_Y is torsion-free), hence $f(\mathcal{O}_Y)$ is subsheaf of $\mathcal{T}_Y(-\log \Delta)$ with zero slope. This contradicts the semistability of $\mathcal{T}_Y(-\log \Delta)$ whose slope is $-(K_Y + \Delta)^n/n < 0$. \square

2.4. Proof of the Theorem. —

Proof of Theorem 2.5. — We proceed in four main steps.

Step 1. Reduction to the log smooth case

Let (Y, Δ) be a log canonical pair with Δ a reduced Weil divisor such that $K_Y + \Delta$ is ample, and let \mathcal{G} be a coherent subsheaf of $\mathcal{T}_Y(-\log \Delta)$ of rank $r > 0$. We choose $\pi : X \rightarrow Y$

a log resolution of the pair that is an isomorphism over its snc locus; we denote by Δ' the strict transform of Δ . There exists a π -exceptional divisor $E = \sum a_i E_i$ such that $K_X + \Delta' = \pi^*(K_Y + \Delta) + E$, and we have $a_i \geq -1$ for all i by the log canonical assumption. Let us set $F = \sum_{a_i=-1} E_i$ the "purely log canonical part" and $D := \Delta' + F$; for some reasons that will appear later, we want to find a coherent subsheaf \mathcal{F} of $\mathcal{T}_X(-\log D)$ such that $\pi_* c_1(\mathcal{F}) = c_1(\mathcal{G})$. We consider the snc locus $(Y, \Delta)_{\text{reg}}$ of the pair, which open subset of Y whose complement has codimension at least 2, and we set $U := \pi^{-1}((Y, \Delta)_{\text{reg}})$ and let $j : U \rightarrow X$ denote the open immersion. Then the sheaf $\mathcal{F} := (j_*(\pi^*\mathcal{G})|_U) \cap \mathcal{T}_X(-\log D)$ is a coherent subsheaf of $\mathcal{T}_X(-\log D)$ that satisfies $\pi_* c_1(\mathcal{F}) = c_1(\mathcal{G})$ on $(Y, \Delta)_{\text{reg}}$, hence on the whole Y .

Finally, as $c_1(\mathcal{T}_Y(-\log \Delta))$ is represented by $-(K_Y + \Delta)$, the projection formula yields the following equivalences:

$$\begin{aligned}
 (2.1) \quad & \frac{c_1(\mathcal{G}) \cdot (K_Y + \Delta)^{n-1}}{r} \leq \frac{c_1(\mathcal{T}_Y(-\log \Delta)) \cdot (K_Y + \Delta)^{n-1}}{n} \\
 \iff & \frac{c_1(\mathcal{F}) \cdot \pi^*(K_Y + \Delta)^{n-1}}{r} \leq \frac{-\pi^*(K_Y + \Delta)^n}{n} \\
 \iff & \frac{c_1(\mathcal{F}) \cdot \pi^*(K_Y + \Delta)^{n-1}}{r} \leq \frac{-(K_X + \Delta' - E) \cdot \pi^*(K_Y + \Delta)^{n-1}}{n} \\
 \iff & \frac{c_1(\mathcal{F}) \cdot \pi^*(K_Y + \Delta)^{n-1}}{r} \leq \frac{c_1(\mathcal{T}_X(-\log D)) \cdot \pi^*(K_Y + \Delta)^{n-1}}{n}
 \end{aligned}$$

as E is π -exceptional.

Therefore we are reduced to showing the semistability of $\mathcal{T}_X(-\log D)$ with respect to $\pi^*(K_Y + \Delta)$. Actually, we will prove a bit more and show that for every generically injective sheaf morphism $\mathcal{F} \rightarrow \mathcal{T}_X(-\log D)$, we have the expected slope inequality.

Step 2. Construction of appropriate cusp metrics

Let us introduce some notations first. We choose A an ample line bundle on X , ω_A a Kähler form whose cohomology class is $c_1(A)$, and ω_0 a Kähler form on Y representing $c_1(K_Y + \Delta)$. Recall that we denote by F the "purely log canonical" part of E , ie $F = \sum_{a_i=-1} E_i$, we set $X_0 := X \setminus D$, and we fix two parameters $\varepsilon, t > 0$ for our regularization process. Finally, we consider for each i such that $a_i > -1$ a regularizing family $(\theta_{i,\varepsilon})_{\varepsilon>0}$ of $(1,1)$ -forms in $c_1(a_i E_i)$ such that $\theta_{i,\varepsilon}$ converges to the singular current $a_i[E_i]$ when ε goes to zero. An explicit formula for $\theta_{i,\varepsilon}$ can be given in terms of hermitian metrics h_i on $\mathcal{O}_X(E_i)$ as well as sections s_i cutting out the divisors E_i : $\theta_{i,\varepsilon} = a_i \left(\frac{\varepsilon^2 |D'_i s_i|^2}{(|s_i|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \theta_{E_i}}{|s_i|^2 + \varepsilon^2} \right)$ where D'_i is the $(1,0)$ -part of the Chern connection of $(\mathcal{O}_X(E_i), h_i)$, and θ_{E_i} is its curvature. In the following, we will work with the metric ω satisfying:

$$(2.2) \quad \text{Ric } \omega = -\omega + t\omega_A + [D] - \sum_{a_i > -1} \theta_{i,\varepsilon}.$$

This metric ω belongs to $c_1(\pi^*(K_Y + \Delta) + tA)$ and depends on ε and t but we choose not to mention its dependence to keep the notation lighter. It is smooth on X_0 , and has cusp singularities along D , ie around points where D is given by $(z_1 \cdots z_r = 0)$, ω is uniformly equivalent to the model cusp metric (also called Poincaré metric) $\sum_{k=1}^r \frac{dz_k \wedge d\bar{z}_k}{|z|^2 \log^2 |z|^2} + \sum_{k>r} dz_k \wedge d\bar{z}_k$. The existence (and uniqueness) of ω is guaranteed by results of Kobayashi [KobR84] and Tian-Yau [TY87], who obtained it as the solution of the Monge-Ampère equation

$$(\pi^*\omega_0 + t\omega_A + dd^c\varphi)^n = \frac{e^{\varphi+f} \prod_i (|s_i|^2 + \varepsilon^2)^{a_i}}{\prod_j |t_j|^2} \omega_A^n$$

where f is some smooth function determined by ω_A as well as the hermitian metrics chosen on E and D , and $(s_i = 0)$ (resp. $(t_j = 0)$) cuts out E_i for i such that $a_i > -1$ (resp. the j -th component of D). Moreover, one can rewrite this equation in a perhaps more standard form using the potential $\varphi_P = -\sum_j \log \log^2 |t_j|^2$ of the cusp metric (also called Poincaré metric) along D : setting $\omega_P := \pi^*\omega_0 + t\omega_A + dd^c\varphi_P$ and $\psi := \varphi - \varphi_P$, the equation becomes

$$(2.3) \quad (\omega_P + dd^c\psi)^n = \prod_i (|s_i|^2 + \varepsilon^2)^{a_i} e^{\psi+F} \omega_P^n$$

where F is a smooth function when read in the quasi-coordinates, cf [KobR84] or [Gue12, Lemma 4.3]. The metric $\omega = \omega_P + dd^c\psi$ (that depends strongly on ε and t) satisfies the following property, which will be a cornerstone of the proof:

Lemma 2.7. — *For every fixed $t > 0$ and any section $s = s_i$ or $s = t_j$, the integral*

$$\int_{X_0} \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_A \wedge \omega^{n-1}$$

converges to 0 when ε goes to 0.

Proof. — Assume for the moment that we can prove that there is a constant C depending only on $t > 0$ such that

$$(2.4) \quad \omega \geq C^{-1} \prod_{a_i > 0} (|s_i|^2 + \varepsilon^2)^{a_i} \omega_P.$$

In view of the equation (2.3) satisfied by ω and the above assumption, we have

$$\mathrm{tr}_\omega(\omega_A) \omega^n \leq C \frac{\prod_{a_i > -1} (|s_i|^2 + \varepsilon^2)^{a_i} e^{\psi} \omega_P^n}{\prod_{a_i > 0} (|s_i|^2 + \varepsilon^2)^{a_i}}$$

for some $C > 0$ under control (depending on $\sup |F|$ and a constant $M > 0$ such that $\omega_A \leq M\omega_P$). We need here another non-trivial input, as we want to get rid of ψ . This is possible by invoking [GW14, Theorem A] (or more precisely the outcome of sections 2.3 and 2.4 where the Monge-Ampère equation studied there is exactly (2.3) with $t > 0$ fixed) that provides a bound $\sup |\psi| \leq C$ independent of ε (but depending on t).

Combining these observations, we get:

$$\frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_A \wedge \omega^{n-1} \leq \frac{C\varepsilon^2}{|s|^2 + \varepsilon^2} \cdot \prod_{-1 < a_i < 0} (|s_i|^2 + \varepsilon^2)^{a_i} \omega_P^n.$$

Checking the convergence to 0 of our integral is now local on $\text{Supp}(D + E)$. As this divisor has simple normal crossing support, we can use Fubini's theorem to reduce our problem to a one-dimensional one. We will be done once we've prove that both integrals

$$\int_{\mathbb{D}} \frac{\varepsilon^2 idz \wedge d\bar{z}}{(|z|^2 + \varepsilon^2)^{1+\delta}} \quad \text{and} \quad \int_{\mathbb{D}} \frac{\varepsilon^2 idz \wedge d\bar{z}}{(|z|^2 + \varepsilon^2) \cdot |z|^2 \log^2 |z|^2}$$

converge to 0, for any $\delta \in (0, 1)$, \mathbb{D} being a small disc centered at $0 \in \mathbb{C}$. For the first integral, we can perform the change of variable $w = z/\varepsilon$, we get the integral

$$\varepsilon^{2(1-\delta)} \int_{|w|^2 \leq 1/\varepsilon} \frac{|dw|^2}{(1 + |w|^2)^{1+\delta}}$$

which goes to zero as $1 > \delta > 0$. As for the second one, we know that $\varepsilon^2(|z|^2 + \varepsilon^2)^{-1} \leq 1$, and that $(|z|^2 \log^2 |z|^2)^{-1} \in L^1(\mathbb{D})$; we then just have to apply the dominated convergence theorem to conclude.

The only thing left to prove now is (2.4) and we would like to use Proposition 1.1 with $\omega' = \omega$ and $\omega = \omega_P$. So we check the four conditions: for (i), we write $\omega = \omega_P + dd^c \psi$ where the L^∞ estimate on ψ comes from [GW14, §2.3-2.4]. For (iii), it is well-known that the reference metric ω_P has bounded geometry so, in particular, it has a bounded curvature tensor. As for (ii) and (iv), we set $\Psi := \sum_{a_i > 0} a_i \log(|s_i|^2 + \varepsilon^2)$; obviously it satisfies (ii) and we have moreover (outside D):

$$\begin{aligned} \text{Ric } \omega &= -\omega + t\omega_A - dd^c \sum_{a_i > 0} a_i \log(|s_i|^2 + \varepsilon^2) + \left(\sum_{-1 < a_i < 0} \frac{a_i \varepsilon^2 \theta_{E_i}}{|s_i|^2 + \varepsilon^2} + \sum_{a_i > 0} a_i \theta_{E_i} \right) \\ &\geq -\omega - C\omega_P - dd^c \sum_{a_i > 0} a_i \log(|s_i|^2 + \varepsilon^2) \\ &= -\omega - C\omega_P - dd^c \Psi \end{aligned}$$

for some constant C big enough. Therefore, inequality (2.4) would follow from Proposition 1.1 if we could establish it for metrics ω, ω' without the compactness assumption. It turns out that in our situation, the reference metric ω_P has bounded curvature tensor, and the unknown metric ω is complete with (qualitatively) bounded Ricci curvature (even bounded holomorphic bisectional curvature) so that we can apply the generalized maximum principle of Yau and mimic the proof of Proposition 1.1 without any serious change. This ends the proof of (2.4), and hence of the lemma. \square

Step 3. Computing the slopes using the singular metrics

We start with a generically injective morphism $\mathcal{F} \rightarrow \mathcal{T}_X(-\log D)$. If $r = \text{rk}(\mathcal{F})$, then this morphism induces $j : \det \mathcal{F} \rightarrow \Lambda^r \mathcal{T}_X(-\log D)$ where by definition, $\det \mathcal{F} := (\Lambda^r \mathcal{F})^{**}$. The strategy consists of endowing $T_X(-\log D)$ (as well as \mathcal{F}) with the hermitian metric induced by ω and computing the slopes of these sheaves using the corresponding representatives of their first Chern classes. Two different types of difficulty appear though: first, as ω is cuspidal along D , it induces a singular hermitian metric on $T_X(-\log D)$, and it

is not clear that one can use it to compute any slope (whether for $T_X(-\log D)$ or \mathcal{F}). And secondly, as ω is not Kähler-Einstein but only approximately, it is not clear that the error term in the slope inequality will vanish when the various regularizing parameters (essentially ε and t) will converge to 0.

We start by addressing the first difficulty. In order to do so, we introduce the singularity set $W = W(\mathcal{F})$ which is defined as the smallest analytic subset of X outside which the sheaf morphism $\mathcal{F} \rightarrow \mathcal{T}_X(-\log D)$ is an injection of vector bundles. We denote by F the vector bundle on $X \setminus W$ such that $\mathcal{F} = \mathcal{O}_X(F)$ there. Without loss of generality, one can assume that \mathcal{F} is saturated in $\mathcal{T}_X(-\log D)$ (it is enough to consider such subsheaves to test the semi stability), so that W has codimension at least two in X .

We denote by h the smooth hermitian metric induced by ω on $T_X(-\log D)$ over $X \setminus D$. It induces also a smooth hermitian metric on F over $X \setminus (W \cup D)$, that we still denote by h . We claim that h can be used to compute the slope of \mathcal{F} :

Proposition 2.8. — *We have:*

$$\int_{X \setminus (W \cup D)} c_1(F, h) \wedge \omega^{n-1} = c_1(\mathcal{F}) \cdot (\pi^*(K_Y + \Delta) + tA)^{n-1}.$$

Proof. — This is far from being obvious, especially because of the singularities of ω along D . On $X \setminus (W \cup D)$, h induces an hermitian metric $h^{\wedge r}$ on $\Lambda^r F$ that we aim to compare with a smooth hermitian metric h_0 on $\det \mathcal{F}$ – notice that $\Lambda^r F$ and $\det \mathcal{F}$ coincide on $X \setminus (W \cup D)$. We are now going to analyze the behavior of $h^{\wedge r}/h_0$ on $W \cup D$. More precisely, if ξ denotes a local trivialization of the line bundle $\det \mathcal{F}$, then the quantity $h^{\wedge r}(\xi)/h_0(\xi)$ is independent of the choice of ξ so it induces a smooth positive function H on $X \setminus (W \cup D)$, and obviously $dd^c \log H = c_1(\det \mathcal{F}, h_0) - c_1(F, h)$, so everything boils down to showing that $\int_{X \setminus (W \cup D)} dd^c \log H \wedge \omega^{n-1} = 0$. For the sake of clarity, we will distinguish three cases, even if they could be treated in a unified manner:

- Near a point $x \in W \setminus D$, then the morphism $j : \det \mathcal{F} \rightarrow \Lambda^r T_X(-\log D)$ is degenerate. Choose trivialization of all bundles at stake. Then there are holomorphic function (f_1, \dots, f_p) where $p = \binom{n}{r}$ such that $j = (f_1, \dots, f_p)$. Then $H = |j|_h^2$ is the squared norm (for some hermitian product on \mathbb{C}^p) of a vector of holomorphic functions, and we claim that $\log H$, possibly after a modification, is the sum of a smooth function and a pluriharmonic function outside $(j = 0)$ – which is nothing else than W intersected with our trivializing chart. To see that, we choose a log resolution ν of the ideal (f_1, \dots, f_p) which produces (above) a holomorphic function g and a trivial ideal sheaf (g_1, \dots, g_p) such that $f_i = g_i g$ for each i . Therefore, $\pi^* \log H = \log \sum |g_i|^2 + \log |g|^2$, and the first summand extends smoothly across the exceptional divisor $(g = 0) = \nu^{-1}(W)$.

- Near a point $x \in D \setminus W$, the hermitian metric h is degenerate but \mathcal{F} is a genuine subbundle of $T_X(-\log D)$. A local holomorphic frame of $T_X(-\log D)$ is given by

$z_1 \frac{\partial}{\partial z_1}, \dots, z_k \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \dots, \frac{\partial}{\partial z_n}$. Also, ω is (possibly non uniformly in t, ε though) equivalent to the standard Kähler metric with cuspidal singularities along D . Therefore, in these coordinates h is equivalent to the matrix $h_{\text{cusp}} := \text{diag}((-\log |z_1|^2)^{-2}, \dots, (-\log |z_k|^2)^{-2}, 1, \dots, 1)$. Also, near x , $\Lambda^r F$ is locally generated by one element $\xi \in \Lambda^r T_X(-\log D)$,

$$\xi = \sum_{I=(i_1, \dots, i_r)} a_I \left(\prod_{\substack{i \in I \\ i \leq k}} z_i \right) \frac{\partial}{\partial z_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{i_r}}$$

where I runs among all (unordered) r -tuples of $\{1, \dots, n\}$. With respect to $\Lambda^r h_{\text{cusp}}$, the squared norm of ξ is equal to $\sum_I |a_I|^2 \prod_{i \in I \cap \{1, \dots, k\}} (-\log |z_i|^2)^{-2}$. We choose a r -tuple I such that $a_I(x) \neq 0$. One can assume that a_I does not vanish up to shrinking the neighborhood we are working on. As $h^{\wedge r}$ is equivalent to $\Lambda^r h_{\text{cusp}}$, we infer that

$$(2.5) \quad \log |\xi|_{h^{\wedge r}}^2 \geq \sum_{i \in I \cap \{1, \dots, k\}} -\log \log^2 |z_i|^2 - C$$

for some given constant $C > 0$. Remember that our ultimate goal is to show that $\int_{X \setminus (W \cup D)} dd^c \log H \wedge \omega^{n-1} = 0$; a way to prove this is to prove that $\log H$ is regular enough so that this integral is actually cohomological. Of course, $\log H$ is not smooth (or even bounded), but the above inequality tends to suggest that it has finite energy in the sense of [GZ07]. But a zero order bound like this does not quite suffice to support this claim, as we also crucially need to bound its complex Hessian from below. As ξ is a (local) holomorphic section over $X \setminus D$ of the hermitian bundle $(\Lambda^r T_X, h)$, one can write there:

$$dd^c \log |\xi|_h^2 = \frac{1}{|\xi|_h^2} \left(|\nabla \xi|^2 - \frac{|\langle \nabla \xi, \xi \rangle|^2}{|\xi|_h^2} \right) - \frac{\langle \Theta_h(\Lambda^r T_X) \xi, \xi \rangle}{|\xi|_h^2}$$

where $\Theta_h(\Lambda^r T_X)$ is the Chern curvature tensor of $(\Lambda^r T_X, h)$, and ∇ is the $(1, 0)$ -part of its Chern connection. As $|\langle \nabla \xi, \xi \rangle|^2 \leq |\nabla \xi|^2 \cdot |\xi|^2$, the term in the parenthesis of the right hand side is non-negative. Moreover, we know from [KobR84, TY87] that $(X \setminus D, \omega)$ has bounded geometry, so in particular ω has bounded holomorphic bisectional curvature. Therefore the curvature tensor of $(\Lambda^r T_X, h)$ is bounded too, so that there is a constant C (depending on ε and t) such that $\Theta_h(\Lambda^r T_X) \leq C\omega \otimes \text{Id}_{\Lambda^r T_X}$ in the sense of Griffiths semipositivity, so in particular we get

$$(2.6) \quad dd^c \log |\xi|_h^2 \geq -C\omega.$$

Still one cannot conclude yet that $\log H$ is in a finite energy class. Indeed, all we wrote requires ξ not to vanish (so it applies as long as we are far away from W) and also the above inequality (2.6) does not say that $\log H$ is quasi-psh as ω is certainly not dominated by a smooth Kähler form on X . The first issue will be addressed by using the methods of the first item above; as for the second, if we introduce $\varphi_P := \sum -\log \log^2 |t_j|^2$ where the t_j 's are sections cutting out the components of D (and the norms being taken with respect to arbitrary smooth hermitian metrics), then $dd^c \varphi_P$ is quasi-psh and there exists ω_0 a fixed smooth Kähler metric on X such that $\omega_0 + dd^c \varphi_P$ has cusp singularities along D . In particular there exists $C > 0$ (depending on t, ε once again) such that $\omega_0 + dd^c \varphi_P \geq C^{-1}\omega$.

Also, it follows from [Gue12, Proposition 2.3] that $\varphi_P \in \mathcal{E}(X, \omega_0)$, i.e. it has finite energy in the sense of [GZ07].

If W were empty, then we would have that for some big constant $C > 0$, $\log H + C\varphi_P$ is a quasi-psh function dominating a multiple of φ_P up to a constant, so it has finite energy too. Also, ω is a finite energy current (its potential is equal to $\varphi_P + O(1)$), therefore $\int_{X \setminus D} dd^c(\log H + C\varphi_P) \wedge \omega^{n-1} = 0$. By linearity and as φ_P has finite energy, we deduce $\int_{X \setminus D} dd^c \log H \wedge \omega^{n-1} = 0$ which had to be showed.

• We deal with the general case now. The morphism $j : \det \mathcal{F} \rightarrow \Lambda^r T_X(-\log D)$ induces a non-zero section t of $\Lambda^r T_X(-\log D) \otimes (\det \mathcal{F})^{-1}$. We choose a log resolution $\nu : \tilde{X} \rightarrow X$ of the ideal sheaf defined by t and D , which is co-supported on $W \cup D$. Let us call D' the strict transform of D , and analyze the behavior of $\nu^*(\log H)$ along $\text{Exc}(\nu) \cup D'$. By the previous two items, $\nu^*(\log H)$ dominates a (global) finite energy function locally outside $\text{Exc}(\nu) \cap D'$, and its complex Hessian is controlled below by a (negative constant) times a cuspidal metric along D' . So let us now see what happens at a point $x \in \text{Exc}(\nu) \cap D'$. Pick a small neighborhood U of x and let ξ be a holomorphic trivialization of $\det \mathcal{F}$ on U . Over $U \setminus (W \cup D)$, ξ is a non-vanishing section of $\Lambda^r T_X(-\log D)$ that can be written $\xi = \sum_{I=(i_1, \dots, i_r)} a_I \left(\prod_{i \in I} z_i \right) \frac{\partial}{\partial z_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{i_r}}$ as above. Of course the ideal sheaf defined by t is given by (a_I) on U . Moreover, as ω is equivalent to a cusp metric along D ,

$$(2.7) \quad \log |\xi|_h^2 = \log \left(\sum_I |a_I|^2 \prod_{i \in I \cap \{1, \dots, k\}} (-\log |z_i|^2)^{-2} \right) + O(1).$$

We write $\nu^* a_I = g_I g$ where g cuts out the exceptional divisor on $\nu^{-1}(U)$ and (g_I) is a trivial ideal sheaf on this open set. Pick now a point \tilde{x} above x , i.e. $\nu(\tilde{x}) = x$. There exists I such that $g_I(\tilde{x}) \neq 0$; moreover, as $\nu^* D$ is an snc divisor, there are holomorphic coordinates $(\tilde{z}_1, \dots, \tilde{z}_n)$ around \tilde{x} such that each of the $\nu^* z_i$ ($1 \leq i \leq k$) is a monomial in the \tilde{z}_j 's. Therefore, around \tilde{x} , we have:

$$\nu^* \log |\xi|_h^2 \geq \log |g|^2 + \sum_{i \in I \cap \{1, \dots, k\}} -\log \log^2 |\nu^* z_i|^2 - C$$

Take for example $\nu^* z_1$; by the observations above, we have $\nu^* z_1 = \tilde{z}_1^{a_1} \dots \tilde{z}_n^{a_n}$ for some nonnegative integers a_1, \dots, a_n . Then $-\log \log^2 |\nu^* z_i|^2 = -2 \log(-\sum_{i=1}^n a_i \log |\tilde{z}_i|^2)$ which by concavity of the logarithm is bigger than $\frac{1}{\sum a_i} \sum_{i=1}^n -\log \log^2 |\tilde{z}_i|^2$. In the end, we obtain:

$$(2.8) \quad \nu^* \log |\xi|_h^2 \geq \log |g|^2 + k \sum_{i=1}^n -\log \log^2 |\tilde{z}_i|^2 - C.$$

Let's gather all what has been said so far. Take σ a section cutting out the exceptional divisor E of ν , that we are going to measure with respect to an arbitrary smooth hermitian norm h_E . Let us set as before $\varphi_P := \sum -\log \log^2 |s_i|^2$ and we consider the function

$$\psi := \nu^* \left(\log \frac{|\xi|_h^2}{|\xi|_{h_0}^2} + C\varphi_P \right) - \log |\sigma|^2$$

on $\nu^{-1}(X \setminus (W \cup D))$. We claim that for C big enough, ψ extends to a quasi-psh function on \tilde{X} having finite energy. Indeed by the bound $dd^c \log |\xi|_h^2 \geq -C\omega$, we get that there exists a Kähler metric ω_0 on X such that $dd^c(\log |\xi|_h^2 + C\varphi_P) \geq -\omega_0$ outside $W \cup D$, for C big enough. Therefore $dd^c\psi \geq -\tilde{\omega}_0$ on $\nu^{-1}(X \setminus (W \cup D))$ for some Kähler form $\tilde{\omega}_0$ on \tilde{X} . As ψ is bounded above thanks to (2.7), it therefore extends as a quasi-psh function on \tilde{X} . Moreover, we retrieve from (2.8) that near each point of the exceptional divisor (or D'), ψ dominates a function satisfying the criterion [BBGZ13, Lemma 2.9] thanks to [Gue12, Proposition 2.3], so ψ satisfies this criterion globally, hence has finite energy. Therefore,

$$\int_{\nu^{-1}(X \setminus (W \cup D))} dd^c\psi \wedge (\nu^*\omega)^{n-1} = 0.$$

Moreover, φ_P has finite energy, and $dd^c \log |\sigma|^2 = -\Theta_{h_E}(E)$ outside E extends smoothly across across the exceptional divisor (and D). As a result, we have

$$\begin{aligned} \int_{X \setminus (W \cup D)} dd^c \log H \wedge \omega^{n-1} &= \int_{\nu^{-1}(X \setminus (W \cup D))} \nu^* dd^c \log H \wedge (\nu^*\omega)^{n-1} \\ &= \int_{\nu^{-1}(X \setminus (W \cup D))} dd^c[\psi + \log |\sigma|^2 - C\nu^*\varphi_P] \wedge (\nu^*\omega)^{n-1} \\ &= - \int_{\tilde{X}} \Theta_{h_E}(E) \wedge (\nu^*\omega)^{n-1} \end{aligned}$$

which vanishes as E is ν -exceptional and $\nu^*\omega$ is a finite energy current. \square

So we know now that we can use h to compute the slope of \mathcal{F} . But outside $W \cup D$, $\mathcal{F} = \mathcal{O}_X(F)$ is a subbundle of T_X , so the Chern curvature tensors of each of the hermitian bundles can be related using the second fundamental form $\beta \in \Omega^{1,0} \otimes \text{Hom}(F, F^\perp)$:

$$\Theta_h(F) = \text{pr}_F(\Theta_h(T_X)|_F) + \beta^* \wedge \beta$$

where pr_F is the orthogonal projection onto F of the endomorphism part. Taking the trace (as endomorphism), and then wedging with ω^{n-1} , we get:

$$(2.9) \quad c_1(F, h) \wedge \omega^{n-1} = \text{tr}_{\text{End}}(\text{pr}_F(\text{tr}_\omega \Theta_h(T_X)|_F)) \omega^n / n + \text{tr}_{\text{End}}(\beta^* \wedge \beta \wedge \omega^{n-1})$$

and integrating this identity over $X \setminus (W \cup D)$, we get thanks to Proposition 2.8:

$$(2.10) \quad n c_1(\mathcal{F}) \cdot (\pi^*(K_Y + \Delta) + tA)^{n-1} \leq \int_{X \setminus (W \cup D)} \text{tr}_{\text{End}}(\text{pr}_F(\text{tr}_\omega \Theta_h(T_X)|_F)) \omega^n.$$

The next goal is then to compute the integral in the right hand side and express it in terms of cohomological quantities:

Proposition 2.9. — *The integral*

$$\int_{X \setminus (W \cup D)} \text{tr}_{\text{End}}(\text{pr}_F(\text{tr}_\omega \Theta_h(T_X)|_F)) \omega^n$$

converges to $-r(K_Y + \Delta)^n$ when ε and then t converge to 0.

Before starting the proof of the proposition, let us try understand what is $\mathrm{tr}_\omega \Theta_h(T_X)$. We choose geodesic coordinates (z_i) for ω around some point x_0 , so that

$$\Theta_h(T_X)_{x_0} = \sum_{j,k,l,m} R_{j\bar{k}l\bar{m}} dz_j \wedge d\bar{z}_k \otimes \left(\frac{\partial}{\partial z_l} \right)^* \otimes \frac{\partial}{\partial \bar{z}_m}.$$

In particular,

$$\mathrm{tr}_\omega \Theta_h(T_X)_{x_0} = \sum_{j,l,m} R_{j\bar{j}l\bar{m}} \left(\frac{\partial}{\partial z_l} \right)^* \otimes \frac{\partial}{\partial \bar{z}_m}$$

and using the Kähler symmetry $R_{j\bar{j}l\bar{m}} = R_{l\bar{m}j\bar{j}}$, we find

$$\mathrm{tr}_\omega \Theta_h(T_X)_{x_0} = \sum_{j,l,m} R_{l\bar{m}j\bar{j}} \left(\frac{\partial}{\partial z_l} \right)^* \otimes \frac{\partial}{\partial \bar{z}_m}.$$

It will be useful to introduce the operator \sharp (relatively to ω) which associates any $(1,1)$ -form α with an endomorphism $\sharp\alpha$ of T_X . More explicitly, \sharp is defined by $\sharp d\bar{z}_j := \sum_k g^{k\bar{j}} \frac{\partial}{\partial z_k}$ is $\omega = i \sum g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$ and $(g^{\alpha\bar{\beta}})$ if the inverse of $(g_{\alpha\bar{\beta}})$. Therefore if $\alpha = \sum_{i,j} \alpha_{i\bar{j}} dz_i \wedge d\bar{z}_j$, then $\sharp\alpha = \sum_{i,j,k} \alpha_{i\bar{j}} g^{k\bar{j}} dz_i \otimes \frac{\partial}{\partial z_k}$. In particular, we have

$$\mathrm{tr}_{\mathrm{End}}(\sharp\alpha) = \mathrm{tr}_\omega \alpha.$$

Recalling that $\mathrm{Ric} \omega = \sum_{j,l,m} R_{l\bar{m}j\bar{j}} dz_l \wedge d\bar{z}_m$, the computation above reads:

$$(2.11) \quad \mathrm{tr}_\omega \Theta_h(T_X) = \sharp(\mathrm{Ric} \omega).$$

We can now resume to the proof:

Proof of Proposition 2.9. — Over $X \setminus D$, our metric ω satisfies the equation (2.2):

$$\mathrm{Ric} \omega = -\omega + t\omega_A - \sum_{a_i > -1} \theta_{i,\varepsilon}$$

therefore $\mathrm{tr}_\omega \Theta_h(T_X) = -\sharp\omega + t\sharp\omega_A - \sum_{a_i > -1} \sharp\theta_{i,\varepsilon}$, and we have three terms to deal with:

- Of course, $\sharp\omega = \mathrm{Id}_{T_X}$, so that $\mathrm{tr}_{\mathrm{End}}(\mathrm{pr}_F(\sharp\omega)|_F) \omega^n = r\omega^n$, which when integrated over $X \setminus (W \cup D)$ yields $r(\pi^*(K_Y + \Delta) + tA)^n$ as ω has finite energy. So our task is to show that as ε and t go to zero, the other terms vanish once properly contracted.

- As for the second term, ω_A is a positive form, so $\sharp\omega_A$ is a positive endomorphism of T_X . Hence:

$$\begin{aligned} \mathrm{tr}_{\mathrm{End}}(\mathrm{pr}_F(\sharp\omega_A)|_F) \omega^n &\leq \mathrm{tr}(\sharp\omega_A) \omega^n \\ &= \mathrm{tr}_\omega \omega_A \omega^n \\ &= n\omega_A \wedge \omega^{n-1} \end{aligned}$$

and $t \int_{X \setminus (W \cup D)} n\omega_A \wedge \omega^{n-1} = tnA \cdot (\pi^*(K_Y + \Delta) + tA)^{n-1}$ is independent of ε and converges to 0 when $t \rightarrow 0$. As a result, so does the non-negative integral $\int_{X \setminus (W \cup D)} \mathrm{tr}(\mathrm{pr}_F(\sharp\omega_A)|_F) \omega^n$.

• The last term is the more subtle to deal with. We are going to show that for each i such that $a_i > -1$, the integral $\int_{X \setminus (W \cup D)} \text{tr}_{\text{End}}(\text{pr}_F(\sharp \theta_{i,\varepsilon}|_F)) \omega^n$ converges to zero. Remember that $\theta_{i,\varepsilon} = a_i \left(\frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \theta_{E_i}}{|s_i|^2 + \varepsilon^2} \right)$; for the sake of clarity, let us drop the index i and decompose θ as $\theta = a(\beta + \gamma)$ where $\beta := \frac{\varepsilon^2 |D' s|^2}{(|s|^2 + \varepsilon^2)^2}$, and $\gamma = \frac{\varepsilon^2 \Theta}{|s|^2 + \varepsilon^2}$.

- Start with γ . One can choose $C > 0$ big enough such that $\pm \Theta \leq C \omega_A$. As the \sharp operator respects positivity (and hence inequalities), we have

$$\begin{aligned} |\text{tr}_{\text{End}}(\text{pr}_F(\sharp \gamma)|_F))| \omega^n &\leq \frac{C \varepsilon^2}{|s|^2 + \varepsilon^2} \text{tr}(\text{pr}_F((\sharp \omega_A)|_F)) \omega^n \\ &\leq \frac{n C \varepsilon^2}{|s|^2 + \varepsilon^2} \omega_A \wedge \omega^{n-1} \end{aligned}$$

hence

$$\left| \int_{X \setminus (W \cup D)} \text{tr}_{\text{End}}(\text{pr}_F(\sharp \gamma)|_F)) \omega^n \right| \leq C' \int_{X \setminus (W \cup D)} \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_A \wedge \omega^{n-1}$$

which converges to zero as ε goes to zero thanks to Lemma 2.7.

- The term involving β cannot be treated in the same way, as it explodes too fast. However, we are going to take advantage of the facts that β has a sign and that $\beta + \gamma$ belongs to a fix *contractible* cohomological class. So we have similarly as before:

$$\text{tr}_{\text{End}}(\text{pr}_F(\sharp \beta)|_F)) \omega^n \leq n \beta \wedge \omega^{n-1}.$$

Writting β as $(\beta + \gamma) - \gamma$, we get:

$$\int_{X \setminus (W \cup D)} \text{tr}(\text{pr}_F((\sharp \beta)|_F)) \omega^n \leq C \int_{X \setminus (W \cup D)} \left(\theta \wedge \omega^{n-1} + \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_A \wedge \omega^{n-1} \right).$$

Using Lemma 2.7 and the fact that θ is smooth and ω has finite energy, we obtain that the right hand side converges to

$$C E \cdot (\pi^*(K_Y + \Delta) + tA)^{n-1}$$

when ε goes to zero, E being the $(\pi$ -exceptional) divisor representing the cohomology class of θ . Therefore, our (non-negative) integral converges to 0 as t goes to 0, which ends the proof of Proposition 2.9.

□

Combining the inequality (2.10) with Proposition 2.9, we have shown that the slope of \mathcal{F} is less than or equal to the slope of $T_X(-\log D)$, which concludes the proof of the semistability of $\mathcal{T}_Y(-\log \Delta)$ with respect to $K_Y + \Delta$.

Step 4. Polystability

Let us prove that the sheaf $\mathcal{T}_Y(-\log \Delta)$ is polystable with respect to $K_Y + \Delta$. Using an induction argument, it would be enough to prove that whenever there exists a sheaf $\mathcal{G} \subset \mathcal{T}_Y(-\log \Delta)$ with same slope as $\mathcal{T}_Y(-\log \Delta)$, then there also exists another sheaf \mathcal{G}'

such that $\mathcal{T}_Y(-\log \Delta) \simeq \mathcal{G} \oplus \mathcal{G}'$. Indeed, if \mathcal{G} is chosen of minimal rank amongst the subsheaves of maximal slope, then \mathcal{G} has to be stable, and we can run the whole argument with \mathcal{G}' instead of $\mathcal{T}_Y(-\log \Delta)$.

So let us assume the existence of such a subsheaf \mathcal{G} . By the arguments of Step 1, we can find a subsheaf \mathcal{F} of $\mathcal{T}_X(-\log D)$ with same slope as $\mathcal{T}_X(-\log D)$, and that we may assume saturated (hence reflexive). Let us set $V = \pi^{-1}((Y, \Delta)_{\text{reg}}) = X \setminus \text{Supp}(E)$, $V_0 := V \setminus D = X \setminus \text{Supp}(D + E)$, and finally $U_0 = V_0 \setminus W$.

Going back to inequality (2.9); combining it with Proposition 2.9 and the fact that \mathcal{F} has same slope as $\mathcal{T}_X(-\log D)$, we obtain that the integral

$$(2.12) \quad \int_{U_0} \text{tr}_{\text{End}}(\beta_{t,\varepsilon}^* \wedge \beta_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1}$$

converges to zero when ε and then t go to zero. Here $\beta_{t,\varepsilon}$ is the second fundamental form of $F \subset T_X$ with respect to the hermitian metric induced by $\omega_{t,\varepsilon}$.

The crucial input we need now is in [BG14] (more precisely Theorem 4.5 combined with the estimates from section §5.5 and 5.6) where it is proved that there exists a Kähler metric ω_∞ on V_0 such that $\omega_{t,\varepsilon}$ converges to ω_∞ in the topology of \mathcal{C}^∞ convergence on the compact subsets of V_0 . Let β_∞ the second fundamental form of $F \subset T_X$ induced by the Kähler metric ω_∞ on U_0 . By the smooth convergence of $\omega_{t,\varepsilon}$ to ω_∞ on the compacts of V_0 , Fatou's lemma applied to (2.12) shows that:

$$\int_{U_0} \text{tr}(\beta_\infty^* \wedge \beta_\infty) \wedge \omega_\infty^{n-1} = 0$$

so that $\text{tr}(\beta_\infty^* \wedge \beta_\infty)$ and hence β_∞ vanishes on U_0 . In particular, we get an holomorphic splitting

$$(2.13) \quad T_X(-\log D)|_{U_0} \simeq \mathcal{F}|_{U_0} \oplus \mathcal{F}|_{U_0}^\perp.$$

We would like to extend this decomposition to $V = \pi^{-1}((Y, \Delta)_{\text{reg}})$ and push it forwards to $(Y, \Delta)_{\text{reg}}$ where it would extend to Y by reflexivity. The difficulty here is that $U_0 \subset V$ has codimension 1 because of D . So we have to show first how extend the splitting across D and then use reflexivity arguments to get it on V .

Another way to reformulate (2.13) is to say that there is an holomorphic surjection of vector bundles $p : T_X(-\log D) \rightarrow \mathcal{F}$ over U_0 . Or better, we can view it as morphism $p : T_X(-\log D) \rightarrow T_X(-\log D)$ over U_0 satisfying $p^2 = p$, and whose norm is less than 1, ie for every $v \in T_X(-\log D)$, we have $|p(v)|_{\omega_\infty} \leq |v|_{\omega_\infty}$. We want to extend p to $U \cap V$. The additional information we need lies in [GW14, Theorem B] where it is shown that ω_∞ is equivalent to a cusp metric along D on V . Now we choose a point $x \in D \cap V$, such that D is locally given by $D = (z_1 \cdots z_k = 0)$ for some coordinates (z_i) around x . Then

$$e_i := \begin{cases} z_i \frac{\partial}{\partial z_i} & \text{if } 1 \leq i \leq k \\ \frac{\partial}{\partial z_i} & \text{if } i > k \end{cases}$$

defines an holomorphic frame of $T_X(-\log D)$ around x . By the previously cited result, the ω_∞ -norm of e_i is equivalent to $(\log \frac{1}{|z_i|^2})^{-1}$ if $1 \leq i \leq k$, and 1 if $i > k$. Let us consider $(p_{ij}(z))$ the matrix of p with respect to the basis (e_i) . We now that there exists a constant C such that $|p(e_i)|_{\omega_\infty} \leq C$ for all i . On the other hand, as ω_∞ is equivalent to the cusp metric, we have

$$\begin{aligned} |p(e_i)|_{\omega_\infty}^2 &= \left| \sum_{j=1}^n p_{ij}(z) e_j \right|_{\omega_\infty}^2 \\ &\geq C^{-1} \left(\sum_{j=1}^r \frac{|p_{ij}(z)|^2}{\log^2 \frac{1}{|z_i|^2}} + \sum_{j=r+1}^n |p_{ij}(z)|^2 \right). \end{aligned}$$

Therefore, $|p_{ij}(z)|$ is uniformly bounded near x if $j > k$, and $|p_{ij}(z)| \leq \log^2 \frac{1}{|z_i|^2}$ if $1 \leq j \leq k$. In any case, p_{ij} is locally L^2 near x (with respect to a smooth volume form), and therefore it extends analytically across D . As a consequence, we get a holomorphic splitting

$$\mathcal{T}_X(-\log D)|_{U \cap V} \simeq \mathcal{F}|_{U \cap V} \oplus \mathcal{F}|_{U \cap V}^\perp$$

As the complement of $U \cap V$ in V has codimension at least 2 and all the sheaves at play are reflexive, the above decomposition extends as a sheaf isomorphism

$$\mathcal{T}_X(-\log D)|_V \simeq \mathcal{F}|_V \oplus (j_{U \cap V})_*(\mathcal{F}|_{U \cap V}^\perp)$$

where $j_{U \cap V} : U \cap V \rightarrow V$ is the open immersion. Pushing forward by π , we get on $(Y, \Delta)_{\text{reg}}$:

$$\mathcal{T}_Y(-\log \Delta)|_{(Y, \Delta)_{\text{reg}}} \simeq \mathcal{G}|_{(Y, \Delta)_{\text{reg}}} \oplus \mathcal{G}'|_{(Y, \Delta)_{\text{reg}}}$$

where \mathcal{G}' is the (reflexive) sheaf on Y defined as $(j_{(Y, \Delta)_{\text{reg}}} \circ \pi \circ j_{U \cap V})_*(\mathcal{F}|_{U \cap V}^\perp)$ where $j|_{\cdot} : \cdot \rightarrow Y$ generically denotes any open immersion from \cdot to Y . And here again, the reflexivity of these sheaves leads to the expected splitting:

$$\mathcal{T}_Y(-\log \Delta) \simeq \mathcal{G} \oplus \mathcal{G}'$$

on Y , ending the proof of the polystability of $\mathcal{T}_Y(-\log \Delta)$. □

Remark 2.10. — We already outlined in the introduction and in §2.2 that Theorem 2.5 is a significant generalization of what was known before, i.e. the semistability of the tangent sheaf in the case of canonical singularities, cf [Eno88]. However, two consequential simplifications occur if one is interested in a weakened form of Theorem 2.5:

- First, if one only focuses on semistability and not polystability, then instead of using the inequality relating the Chern curvature form of a subbundle F to the one of the ambient bundle T_X (cf (2.9)), one can use a weaker form of it as in [CP14, equation (1.2)] which enables to endow $\det F$ with a smooth metric instead of the singular metric induced by ω (the singularities occurring because F is a subbundle only on an open set, and because ω may have cuspidal singularities along a divisor), thus avoiding the use of the technical Lemma 2.8.

· Second, if $\Delta = 0$, then the approximate Kähler-Einstein metrics ω are smooth and don't have cuspidal singularities anymore. In that case, the crucial Lemma 2.8 becomes much easier and was already treated in the item $(**)$ of the proof of [KobS87, Theorem 8.3].

3. An extension to stable varieties

3.1. Stable varieties. — Once we have studied log canonical pairs with ample log canonical bundle, it is very tempting to try to extend these results to varieties (or pairs) with possible non-normal singularities. More precisely, we will consider *stable varieties* in the sense of Kollár-Shepherd-Barron [KSB88] and Alexeev [Ale96]. These varieties are the higher dimensional analogue of stable curves as defined by Deligne-Mumford [DM69] and they arise in the compactification of the moduli space of smooth canonically polarized projective varieties; their precise definition stated below, though we refer to the very nice paper [Kov12] and the references therein for more details and insight on all the objects involved.

Definition 3.1. — A stable variety is an equidimensional and reduced complex projective variety X with semi-log canonical singularities such that K_X is ample.

Let us recall that X has semi-log canonical singularities if the only singularities of X in codimension 1 are ordinary double points (i.e. locally analytically isomorphic to $(xy = 0) \in \mathbb{C}^{n+1}$), X satisfies Serre's S_2 property, is \mathbb{Q} -Gorenstein, and the pair (X^ν, D^ν) formed by the normalization of X and its conductor divisor has log canonical singularities. Here the conductor of the normalization is essentially the Weil divisor determined by the preimage of the ordinary double points locus by the normalization map.

It is important to notice that stable varieties may be reducible, which is a source of difficulties in view of a theory of semistability for coherent sheaves. Indeed, if we want to mimic the definition of the slope of a coherent sheaf \mathcal{F} , we need to define the $(n-1)$ -cycle $c_1(\mathcal{F})$. We can still do it on X_{reg} , but there is an issue when we want to extend it to the whole X as the complement of X_{reg} has codimension one in general. Actually, there is a very general notion of slope, defined in a high degree of generality using Hilbert polynomials (see e.g. [HL10]), and of course agreeing with the usual one on smooth varieties. We are now going to review these notions briefly.

3.2. Slope and stability on non-normal varieties. — On non-normal varieties, defining the slope of a coherent sheaf is more complicated. We recall the definition, cf [HL10]. Let X be a n -equidimensional reduced projective scheme over \mathbb{C} , L an ample \mathbb{Q} -line bundle, and \mathcal{F} a coherent sheaf on X of dimension n (which means that the support of \mathcal{F} has dimension n). We know that the Hilbert polynomial $P(\mathcal{F})$ given by $m \mapsto \chi(X, \mathcal{F} \otimes L^{\otimes m})$ can be uniquely written in the form:

$$P(\mathcal{F}, m) := \sum_{k=0}^n a_k(\mathcal{F}) \frac{m^k}{k!}$$

where $a_k(\mathcal{F})$ are integers. Note that for m large enough, the vanishing of the higher cohomology implies that $P(\mathcal{F}, m) = h^0(X, \mathcal{F} \otimes L^{\otimes m})$. Of course, this polynomial depends on \mathcal{F} and on the polarization L . If X is integral, then the rank of \mathcal{F} is defined as the rank of \mathcal{F} at the generic point (as \mathcal{F} is locally free on some dense open set). In this more general framework, we define the rank of \mathcal{F} as a convex combination of the rank of \mathcal{F} restricted to each irreducible components. More precisely, if $X = X_1 \cup \dots \cup X_r$ are the irreducible components of X and if L_i (resp. \mathcal{F}_i) denotes the restriction of L (resp. \mathcal{F}) to X_i , then we define

$$\mathrm{rk}(\mathcal{F}) := \frac{\sum_{i=1}^r (L_i^n) \mathrm{rk}(\mathcal{F}_i)}{(L^n)}.$$

One can also check that $\mathrm{rk}(\mathcal{F})$ coincides with $a_n(\mathcal{F})/a_n(\mathcal{O}_X)$, cf [Laz04, Remark 1.1.26]. If \mathcal{F} has constant rank (ie $\mathrm{rk}(\mathcal{F}_i)$ is independent of i), then the rank of \mathcal{F} is the same as $\mathrm{rk}(\mathcal{F}_i)$ for any i .

Now we need to define the degree of \mathcal{F} with respect to L . To do it, we introduce the suitable combination of the coefficients a_k that gives in the smooth case the usual degree. More precisely:

$$\deg_L(\mathcal{F}) := a_{n-1}(\mathcal{F}) - \mathrm{rk}(\mathcal{F}) a_{n-1}(\mathcal{O}_X)$$

and finally, one can define the slope as the quotient of the degree by the rank:

$$\mu_L(\mathcal{F}) := \frac{\deg_L(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}.$$

Of course this definition only make sense for sheaves with positive rank.

Remark 3.2. — If $X = X_1 \amalg \dots \amalg X_s$ is the disjoint sum of its irreducible components, and if \mathcal{F} has constant rank, then it follows from the definition that the slope of \mathcal{F} with respect to a polarization L on X is the sum of the slopes of $\mathcal{F}|_{X_i}$ with respect to $L|_{X_i}$.

The first thing to check is whether this definition generalizes the definition of slope that we gave for normal varieties. Recall that if \mathcal{F} is a coherent sheaf on a normal variety we defined $c_1(\mathcal{F})$ to be the $(n-1)$ -cycle represented by the closure of the Weil divisor attached to $\det(\mathcal{F}|_{X_{\mathrm{reg}}})$.

Proposition 3.3. — *Let (X, L) be a polarized normal variety of dimension n , and \mathcal{F} a torsion-free coherent sheaf on X . Then we have*

$$\deg_L(\mathcal{F}) = (c_1(\mathcal{F}) \cdot L^{n-1})$$

and in particular, $\mu_L(\mathcal{F})$ agrees with $(c_1(\mathcal{F}) \cdot L^{n-1})/\mathrm{rk}(\mathcal{F})$.

Proof. — Let $\pi : Y \rightarrow X$ a resolution of X such that π is an isomorphism over X_{reg} . We will show that these two quantities above coincide with $(c_1(\pi^* \mathcal{F}) \cdot \pi^* L^{n-1})$.

Using the projection formula and the fact that on X_{reg} , the cycles $\pi_* c_1(\pi^*(\mathcal{F}))$ and $c_1(\mathcal{F})$ coincide, we see that $(c_1(\pi^* \mathcal{F}) \cdot \pi^* L^{n-1}) = (c_1(\mathcal{F}) \cdot L^{n-1})$, so we did half of the job. The other equality requires a bit more work.

First, let us prove that $\deg_L(\mathcal{F}) = \deg_{\pi^*L}(\pi^*\mathcal{F})$. If we define \mathcal{G} to be the sheaf $\pi^*\mathcal{F} \otimes \pi^*L^{\otimes m}$, then the degeneracy of Leray spectral sequence provides the equality

$$(3.1) \quad \chi(Y, \mathcal{G}) = \sum_{q \geq 0} (-1)^q \chi(X, R^q \pi_* \mathcal{G}).$$

Moreover, the projection formula gives us a isomorphism

$$R^q \pi_*(\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \pi^*L^{\otimes m}) \longrightarrow R^q \pi_*(\pi^*\mathcal{F}) \otimes_{\mathcal{O}_X} L^{\otimes m}$$

As \mathcal{F} is torsion-free, it is locally free in codimension one (cf e.g. [KobS87, 5.15]), so if we apply the projection formula once again, we obtain that the natural morphism

$$R^q \pi_*(\pi^*\mathcal{F}) \longrightarrow R^q \pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an isomorphism in codimension two. Combining these two observations, we get a morphism

$$R^q \pi_*(\pi^*\mathcal{F} \otimes_{\mathcal{O}_Y} \pi^*L^{\otimes m}) \longrightarrow R^q \pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} L^{\otimes m}$$

which is an isomorphism in codimension two. Therefore, as $\pi_* \mathcal{O}_Y = \mathcal{O}_X$, we get from (3.1):

$$\chi(Y, \pi^*\mathcal{F} \otimes \pi^*L^{\otimes m}) = \chi(X, \mathcal{F} \otimes L^{\otimes m}) + \sum_{q > 0} (-1)^q \chi(X, R^q \pi_* \mathcal{O}_Y \otimes \mathcal{F} \otimes L^{\otimes m}) + O(m^{n-2})$$

Finally, as X is normal and π is an isomorphism over X_{reg} , the sheaves $R^q \pi_* \mathcal{O}_Y$ are supported in codimension ≤ 2 for $q > 0$. (cf [Har77, 8.2 & 11.2] for example). Therefore, we obtain:

$$\chi(Y, \pi^*\mathcal{F} \otimes \pi^*L^{\otimes m}) = \chi(X, \mathcal{F} \otimes L^{\otimes m}) + O(m^{n-2}).$$

These two polynomials have same leading term coefficient, namely the top intersection of $c_1(\mathcal{F}) \otimes L$ divided by $n!$ (as π is generically of degree one), so they have the same coefficient in front of the second highest power of m . As this is true for any torsion-free sheaf \mathcal{F} , we have the same result for \mathcal{O}_X too, so in the end we get $\deg_L(\mathcal{F}) = \deg_{\pi^*L}(\pi^*\mathcal{F})$.

To conclude the proof of the proposition, we need to see that $\deg_{\pi^*L}(\pi^*\mathcal{F}) = (c_1(\pi^*\mathcal{F}) \cdot \pi^*L^{n-1})$. By Riemann-Roch-Hirzebruch theorem, we know that

$$\begin{aligned} \chi(Y, \pi^*\mathcal{F} \otimes \pi^*L^{\otimes m}) &= \text{rk}(\mathcal{F}) \cdot (\pi^*L)^n \frac{m^n}{n!} + \left[(c_1(\pi^*\mathcal{F}) \cdot \pi^*L^{n-1}) + \right. \\ &\quad \left. + \frac{1}{2} (c_1(Y) \cdot \pi^*L^{n-1}) \right] \frac{m^{n-1}}{(n-1)!} + O(m^{n-2}). \end{aligned}$$

Applying this to \mathcal{F} and \mathcal{O}_Y , we get the expected equality. \square

Now we would like to see, for non-normal variety, how to relate the slope of a sheaf with the slope of its pull-back by the normalization map. This is the content of the following proposition:

Proposition 3.4. — *Let X be a reduced equidimensional projective scheme over \mathbb{C} , L a polarization, and \mathcal{F} a coherent sheaf which is locally free in codimension one. If $\pi : Y \rightarrow X$ is the normalization of X , then we have:*

$$\mu_{\pi^*L}(\pi^*\mathcal{F}) = \mu_L(\mathcal{F}).$$

Proof. — First, let us define $\mathcal{T} := \pi_*\mathcal{O}_Y/\mathcal{O}_X$, and let set $r = \text{rk}(\mathcal{F})$ (one may notice that \mathcal{F} being locally free in codimension one, it has constant rank). As L and π^*L are ample, we will assume in the following that the power m to which we raise them is large enough so that all the higher cohomology groups involved are zero. As a consequence, we will identify the Euler-Poincaré characteristic with h^0 . Using the projection formula combined with a similar argument as in the previous proof (we use here that \mathcal{F} is locally free in codimension one) we see that

$$h^0(Y, \pi^*\mathcal{F} \otimes \pi^*L^{\otimes m}) = h^0(X, \mathcal{F} \otimes L^{\otimes m} \otimes \pi_*\mathcal{O}_Y) + O(m^{n-2})$$

Moreover, as \mathcal{F} is locally free in codimension one, the short sequence

$$0 \rightarrow \mathcal{F} \otimes L^{\otimes m} \rightarrow \mathcal{F} \otimes L^{\otimes m} \otimes \pi_*\mathcal{O}_Y \rightarrow \mathcal{F} \otimes L^{\otimes m} \otimes \mathcal{T} \rightarrow 0$$

is exact in codimension one. Combining these two identities, we get:

$$a_{n-1}(\pi^*\mathcal{F}) = a_{n-1}(\mathcal{F}) + a_{n-1}(\mathcal{F} \otimes \mathcal{T})$$

where the polarizations which respect to which the Hilbert polynomials are computed are respectively L on X and π^*L on Y .

Therefore, we have:

$$\begin{aligned} \mu_{\pi^*L}(\pi^*\mathcal{F}) &= \frac{1}{r}a_{n-1}(\pi^*\mathcal{F}) - a_{n-1}(\mathcal{O}_Y) \\ &= \frac{1}{r}a_{n-1}(\mathcal{F}) + \frac{1}{r}a_{n-1}(\mathcal{F} \otimes \mathcal{T}) - a_{n-1}(\mathcal{O}_X) + (a_{n-1}(\mathcal{O}_Y) - a_{n-1}(\mathcal{O}_X)) \\ &= \mu_L(\mathcal{F}) + \frac{1}{r}a_{n-1}(\mathcal{F} \otimes \mathcal{T}) - a_{n-1}(\mathcal{T}). \end{aligned}$$

If \mathcal{T} is supported in codimension ≤ 2 , then $\frac{1}{r}a_{n-1}(\mathcal{F} \otimes \mathcal{T}) = a_{n-1}(\mathcal{T}) = 0$ and we are done. Else, it is supported in a codimension one subvariety Z , so it follows from Riemann-Roch formula (in the reducible case, cf [Kol96, Proposition VI.2.7]) that $a_{n-1}(\mathcal{F} \otimes \mathcal{T}) = \text{rk}((\mathcal{F} \otimes \mathcal{T})|_Z) = r \cdot \text{rk}(\mathcal{T}) = r a_{n-1}(\mathcal{T})$, where we used again that \mathcal{F} is locally free in codimension one. In the end, we have proved that the slopes of \mathcal{F} and $\pi^*\mathcal{F}$ agree. \square

Example 3.5. — One should pay attention to the fact the above result is false without the assumption that the sheaf is locally free in codimension one. Indeed, if we choose $X = (xy = 0) \subset \mathbb{P}^2$ to be the union of two lines, then its normalization is \mathbb{P}^1 with two points p', p'' sitting above the node p . Then, if we choose $L = \mathcal{O}_{\mathbb{P}^2}(1)|_X$ and $\mathcal{F} = \mathcal{O}_X \oplus (\mathcal{O}_X/\mathcal{I}_p)$ where \mathcal{I} is the ideal sheaf of the node, we see that $h^0(\mathcal{F} \otimes L^{\otimes m}) = h^0(L^{\otimes m}) + 1$, ie $\mu_L(\mathcal{F}) = 1$. But $\pi^*\mathcal{F} = \mathcal{O}_Y \oplus \mathcal{O}_Y/\mathcal{I}_{p', p''}$ so that $h^0(\pi^*\mathcal{F} \otimes \pi^*L^{\otimes m}) = h^0(\pi^*L^{\otimes m}) + 2$ and therefore $\mu_{\pi^*L}(\pi^*\mathcal{F}) = 2$.

In view of all the observations we made so far, one can introduce a notion of semi-stability which is weaker than the one in [HL10, 1.2.12], but particularly adapted to our context:

Definition 3.6. — Let X be a n -equidimensional reduced projective scheme over \mathbb{C} , and L be an ample divisor. A coherent sheaf \mathcal{E} is said to be semistable with respect to L if for every subsheaf $\mathcal{F} \subset \mathcal{E}$ being locally free in codimension one and satisfying $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$, we have $\mu_L(\mathcal{F}) \leq \mu_L(\mathcal{E})$.

If X is normal and \mathcal{E} is torsion-free, then any subsheaf \mathcal{F} of \mathcal{E} is also torsion-free hence locally free in codimension 1 as X is normal. So we just recover the usual notion of semistability.

3.3. The result. — The general definition above is actually motivated by Proposition 3.4 and tailored for the following Theorem:

Theorem 3.7. — Let X be a stable variety, let $\nu : X^\nu \rightarrow X$ be the normalization of X and Δ be the conductor of ν . Then the sheaf $\nu_* \mathcal{T}_{X^\nu}(-\log \Delta)$ is semistable with respect to K_X .

Proof. — Let $\mathcal{F} \subset \mathcal{T}_X$ be a coherent subsheaf of \mathcal{T}_X which is locally free in codimension one. If $\nu : X^\nu \rightarrow X$ is the normalization of X and Δ be the conductor divisor of ν on X^ν , then we get a generically injective map $\nu^* \mathcal{F} \rightarrow \nu^* \nu_* \mathcal{T}_{X^\nu}(-\log \Delta)$. If we compose it with the natural map $\nu^* \mathcal{T}_X \rightarrow (\nu^* \nu_* \mathcal{T}_{X^\nu}(-\log \Delta))^{**}$, then in view of the lemma 3.9 below, we get a generically injective morphism

$$(3.2) \quad \nu^* \mathcal{F} \rightarrow \mathcal{T}_{X^\nu}(-\log \Delta).$$

Let us now consider $\pi : Y \rightarrow X^\nu$ a log resolution of (X^ν, Δ) that leaves the snc locus untouched, and let $D = \pi^{-1}(\Delta)$. Then there is a natural morphism $\pi^* \mathcal{T}_{X^\nu}(-\log \Delta) \rightarrow \mathcal{T}_Y(-\log D)$ which is an isomorphism over $(X^\nu, \Delta)_{\text{reg}}$. Taking to the double dual and using the reflexivity of $\mathcal{T}_Y(-\log D)$, we end up with a natural isomorphism $(\pi^* \mathcal{T}_{X^\nu}(-\log \Delta))^{**} \simeq \mathcal{T}_Y(-\log D)$. Pulling back (3.2) by π , we get a generically injective morphism $\pi^* \nu^* \mathcal{F} \rightarrow \mathcal{T}_Y(-\log D)$. We proved that in this situation, the slope of $\pi^* \nu^* \mathcal{F}$ with respect to $\pi^* \nu^* K_X$ is less or equal to the slope of $\mathcal{T}_Y(-\log D)$. At this point, we use the fact that \mathcal{F} has constant rank to pass from the slope inequality on each component to the global slope inequality.

By the projection formula, it yields

$$\mu_{\nu^* K_X}(\nu^* \mathcal{F}) \leq \mu_{\nu^* K_X}(\mathcal{T}_{X^\nu}(-\log \Delta)).$$

We would like to push forward to X this inequality. For the left hand side, this can be done thanks to Proposition 3.4 which shows that $\mu_{\nu^* K_X}(\nu^* \mathcal{F}) = \mu_{K_X}(\mathcal{F})$ as \mathcal{F} is locally free in codimension one. Actually, the right hand side is not equal to $\mu_{K_X}(\mathcal{T}_X)$, but there is a correction factor which has the right sign fortunately. More precisely, writing $\mathcal{T} := \nu_* \mathcal{O}_{X^\nu} / \mathcal{O}_X$, we have by the projection formula (m is large and divisible enough):

$$h^0(X^\nu, \nu^* K_X^{\otimes m}) = h^0(X, K_X^{\otimes m} \otimes \nu_* \mathcal{O}_{X^\nu})$$

and also

$$h^0(X^\nu, \mathcal{T}_{X^\nu}(-\log \Delta) \otimes \nu^* K_X^{\otimes m}) = h^0(X, \nu_* \mathcal{T}_{X^\nu}(-\log \Delta) \otimes K_X^{\otimes m})$$

Therefore, we get

$$\begin{aligned} \mu_{\nu^* K_X}(T_{X^\nu}(-\log \Delta)) &= \frac{1}{n} a_{n-1}(T_{X^\nu}(-\log \Delta)) - a_{n-1}(\mathcal{O}_{X^\nu}) \\ &= \left(\frac{1}{n} a_{n-1}(\nu_* T_{X^\nu}(-\log \Delta)) - a_{n-1}(\mathcal{O}_X) \right) + a_{n-1}(\mathcal{O}_X) - a_{n-1}(\mathcal{O}_{X^\nu}) \\ &= \mu_{K_X}(\nu_* T_{X^\nu}(-\log \Delta)) - a_{n-1}(\mathcal{T}). \end{aligned}$$

As \mathcal{T} is supported in codimension ≥ 1 , we have $a_{n-1}(\mathcal{T}) \geq 0$, so that in the end, we have proved

$$\mu_{K_X}(\mathcal{F}) \leq \mu_{K_X}(\nu_* \mathcal{T}_{X^\nu}(-\log \Delta)).$$

□

Remark 3.8. — In general, the sheaf $\nu_* \mathcal{T}_{X^\nu}(-\log \Delta)$ is distinct from the tangent sheaf of X (defined as the sheaf of derivations of \mathcal{O}_X) as already in the case of a (local) node $X = \text{Spec}(\mathbb{C}[x, y, z]/(xy))$ the former sheaf strictly contains the latter one. But given the last step of the proof where we neglected $a_{n-1}(\mathcal{T})$, it could be possible that the same proof yields the semistability of \mathcal{T}_X .

To conclude this section, let us state and prove a result we have used in the course of the proof of Theorem 3.7:

Lemma 3.9. — *Let X be a projective n -equidimensional reduced projective scheme over \mathbb{C} and let $\pi : Y \rightarrow X$ the normalization of X . If \mathcal{E} is a reflexive coherent sheaf on Y , then the reflexive hull $(\pi^* \pi_* \mathcal{E})^{**}$ is naturally isomorphic to \mathcal{E} .*

Proof. — Let us begin by reducing the problem to one of commutative algebra. First, there is a natural map $\pi^* \pi_* \mathcal{E} \rightarrow \mathcal{E}$ inducing $(\pi^* \pi_* \mathcal{E})^{**} \rightarrow \mathcal{E}^{**} \simeq \mathcal{E}$. This map is a morphism between two reflexive sheaves, so if we can show that it is an isomorphism over an open set in Y whose complement has codimension at least two, then our map will actually be a sheaf isomorphism over the whole Y . As a consequence, one can assume that \mathcal{E} is locally free (as \mathcal{E} is reflexive, it will be locally free on such a "big" open set). Now, the problem is local, and as the normalization is an affine morphism, one can assume without loss of generality that $Y = \text{Spec}(B)$, $X = \text{Spec}(A)$, and that $\mathcal{E} = \tilde{B}$. We claim that the sheaf morphism $\mathcal{E}^* \rightarrow (\pi^* \pi_* \mathcal{E})^*$ is an isomorphism, or equivalently that the following morphism of B -modules:

$$\begin{aligned} F : \text{Hom}_B(B, B) &\longrightarrow \text{Hom}_B(B \otimes_A B_A, B) \\ \varphi &\longmapsto \varphi \circ f \end{aligned}$$

is an isomorphism. Here, B_A is B viewed as an A -module and $f : B \otimes_A B_A \rightarrow B$ is defined by $f(\sum b_i \otimes m_i) = \sum b_i m_i$. As f is surjective, F is injective, so we need to prove that F is surjective. Given $\psi \in \text{Hom}_B(B \otimes_A B_A, B)$, we define $\varphi \in \text{Hom}_B(B, B)$ by setting

$\varphi(1) = \psi(1 \otimes 1)$. We want to show that $F(\varphi) = \psi$, and this amounts to proving that for any $b \in B$, we have $\psi(1 \otimes b) = b\psi(1 \otimes 1)$. The crucial input is that B is a subring of the total quotient ring of A , so there exists $s \in A$ which is not zero divisor such that $sb \in A$. Therefore, if we set $x := \psi(1 \otimes b) - b\psi(1 \otimes 1)$, we get $sx = s\psi(1 \otimes b) - sb\psi(1 \otimes 1) = \psi(s \otimes b) - \psi(sb \otimes 1) = \psi(1 \otimes sb) - \psi(sb \otimes 1) = 0$ as $sb \in A$. So $sx = 0$; but s is not a zero divisor in A and hence in B too as B is a subring of the total quotient ring of A . This proves that $x = 0$, and thus F is an isomorphism as claimed. \square

4. Generic semipositivity

What happens if we loosen the positivity assumptions on $K_X + D$ (replacing ample by pseudo-effective)? In view of Miyaoka's semipositivity theorem (cf discussion after Theorem 4.1), it is natural to expect that the sheaf of logarithmic differentials $\Omega_X^1(\log D)$ is generically semipositive with respect to any Kähler class. Recent results of Campana-Păun [CP13, CP14] confirm this in the case where X is smooth and D has simple normal crossings – even if D has merely rational coefficients between 0 and 1.

Let us recall that on a normal complex space X endowed with a reduced Weil divisor, the sheaf of logarithmic differentials, that we will denote by $\Omega_X^1(\log D)$ is defined to be the pushforward of the standard logarithmic differentials sheaf $j_*\Omega_{(X,D)_{\text{reg}}}^1(\log D)$ by the open immersion $j : (X, D)_{\text{reg}} \hookrightarrow X$ of the snc locus of the pair (X, D) into X . This sheaf is reflexive, and is isomorphic to the dual of the logarithmic tangent sheaf from Def. 2.4.

Recall also that a reflexive sheaf \mathcal{E} on a normal compact Kähler space X endowed with a Kähler form ω of is said to be generically ω -semipositive if for all coherent quotient \mathcal{F} of \mathcal{E} , the degree of \mathcal{F} with respect to ω is non-negative, i.e. $\int_X c_1(\mathcal{F}) \wedge \omega^{n-1} \geq 0$. By a theorem of Hartshorne [Har71], a vector bundle over a curve is nef if and only if all its quotient bundles have non-negative degree. As a result, the restriction of a generically semipositive sheaf/bundle to a sufficiently general curve is nef (say on a projective variety).

The main theorem of this section is the following:

Theorem 4.1. — *Let X be an n -dimensional compact Kähler space, D a reduced Weil divisor such that (X, D) has log canonical singularities and let ω be a Kähler form. If $K_X + D$ is pseudo-effective (resp. $-(K_X + D)$ is psef), then $\Omega_X^1(\log D)$ (resp. $\mathcal{T}_X(-\log D)$) is generically ω -semipositive.*

In particular, if $K_X + D \equiv 0$, the slope of the logarithmic tangent sheaf vanishes, so the generic semi positivity of $\Omega_X^1(\log D)$ yields its semistability with respect to any Kähler class. If we further assumes that $D = 0$ and X is klt, we can prove polystability on top of the semistability:

Corollary 4.2. — *Let X be a singular Calabi-Yau variety, i.e. X is a complex projective variety with at most klt singularities and such that K_X is numerically trivial. Then \mathcal{T}_X is polystable with respect to any polarization.*

Theorem 4.1 is a variation around Miyaoka's semipositivity theorem [Miy87]. In the case where X is smooth projective, it has been proved by Boucksom-Demailly-Păun-Peternell [BDPP13] that X is uniruled if and only if K_X is not pseudo-effective, so that we essentially recover the same statement. However, as opposed to Miyaoka's theorem, our result works in the context of singular log pairs and holds for (singular) Kähler spaces and not only for algebraic varieties – it was also proved in the smooth Kähler case by Junyan Cao [Cao13]. Also, as there exist rational (hence uniruled) surfaces with only klt singularities and for which K_X is ample [Kol08, Example 43], our assumption that K_X is psef encompasses some new cases even when $D = 0$.

Our result is also closely related with the work of D.Greb, S.Kebekus and T. Peternell who obtained in [GKP, Theorem 1.3] a structure theorem for the tangent sheaf of klt varieties with trivial canonical bundle (and canonical singularities): it decomposes into a sum of stable sheaves after a finite cover, étale in codimension one. Let us finally mention that it is conjectured [Pet12, Conj 1.3] that a projective manifold with nef anticanonical bundle has a generically nef tangent bundle (ie semipositive with respect to every polarization (H_1, \dots, H_{n-1})).

Proof of Theorem 4.1. — Let us begin with the case $K_X + D$ pseudo-effective. Here again, we denote by Y the singular original Kähler space, and choose $\pi : X \rightarrow Y$ a log resolution of (Y, Δ) . We will show that for each coherent subsheaf $\mathcal{F} \subset \mathcal{T}_X(-\log D)$ and every Kähler form ω_0 on Y , we have $\int_X c_1(\mathcal{F}) \wedge (\pi^*\omega_0)^{n-1} \leq 0$. This will as before show that each coherent subsheaf of $\mathcal{T}_Y(-\log \Delta)$ has non-positive slope with respect to ω , and this will conclude by duality (consider a quotient \mathcal{G} of $\Omega_Y^1(\log \Delta)$, it induces a subsheaf $\mathcal{G}^* \subset \mathcal{T}_Y(-\log \Delta)$ with non-positive slope, so that \mathcal{G} has a non-negative slope).

We choose a log resolution $\pi : X \rightarrow Y$ of the pair (Y, Δ) , so we have

$$K_X + D + E = \pi^*(K_Y + \Delta)$$

where $D = \sum D_k$ is a reduced snc divisor (containing the strict transform of Δ as well as exceptional divisor with zero log discrepancy) and $E = \sum a_i E_i$ is an exceptional divisor with positive log discrepancies (i.e. whose coefficients satisfy $a_i < 1$, but are possibly negative).

We would like to endow $T_X(-\log D)$ with a singular metric having semi negative Ricci curvature outside $D \cup E$ in a weak sense, i.e. one would like to solve

$$(4.1) \quad \text{Ric } \omega = -T + [D] + [E]$$

for $\omega \in \{\pi^*\omega_0\}$ and $T \in \pi^*(K_Y + \Delta)$ a positive current. Unfortunately there are obstructions to construct such a metric (as this would amount to solving a Monge-Ampère equation

with non-integrable right-hand side) and use it for our purposes (we would need to know its behavior near the singularities in order to perform integrations by parts for example). Therefore, we will instead approximate a potential such object by regularizing the equation in the following fashion:

- Replace the cohomology class of ω by the Kähler class $\{\pi^*\omega_0 + \delta\omega_X\}$ for some Kähler metric ω_X ,
- Replace D by $(1 - \delta)D$,
- Choose a positive current $T_\delta \in \pi^*(K_Y + \Delta) + \delta\{\omega_X\}$ with analytic singularities,
- Replace $[E]$ by a smooth $(1, 1)$ -form Θ_ε .

For the third item, we use Demailly's regularization theorem [Dem92] inside the big cohomology class $\pi^*(K_Y + \Delta) + \delta\{\omega_X\}$, while for the last item, we choose $\Theta_\varepsilon = \sum_i a_i [\theta_{E_i} + dd^c \log(|s_i|^2 + \varepsilon^2)] = \sum_i a_i \left(\frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \theta_{E_i}}{|s_i|^2 + \varepsilon^2} \right)$ where s_i is a defining section for E_i and θ_{E_i} is the Chern curvature of the smooth hermitian metric $|\cdot|_i$ chosen on $\mathcal{O}_X(E_i)$; set $\theta_E := \sum a_i \theta_{E_i}$. We also choose sections σ_k of the components D_k of D , and pick arbitrary smooth hermitian metrics $|\cdot|_k$ on $\mathcal{O}_X(D_k)$ whose curvature form will be denoted θ_{D_k} ; finally, set $\theta_D = \sum_k \theta_{D_k}$.

Let us now choose an arbitrary volume form dV on X ; one can find a (sup-normalized) potential ψ_δ for T_δ such that $T_\delta = -\text{Ric}(dV) + \theta_D + \theta_E + \delta\omega_X + dd^c\psi_\delta$. In particular, there exists a uniform C (independent of δ) such that $dd^c\psi_\delta \geq -C\omega_X$. Moreover, as T_δ has analytic singularities, ψ_δ is smooth outside a proper analytic set $Z \subset X$ (one should observe that Z depends on δ though); set $X_0 := X \setminus (D \cup Z)$. Thanks to Kołodziej's theorem [Kolo98], one can find a bounded solution $\varphi = \varphi_{\delta, \varepsilon}$ of the following Monge-Ampère equation:

$$(4.2) \quad (\pi^*\omega_0 + \delta\omega_X + dd^c\varphi)^n = \frac{e^{\psi_\delta + C_{\delta, \varepsilon} dV}}{\prod_i (|s_i|^2 + \varepsilon^2)^{a_i} \prod_k |\sigma_k|^{2(1-\delta)}}$$

where $C_{\delta, \varepsilon}$ is a normalizing constant which is bounded uniformly with respect to ε (but not δ !). Although the solution $\omega_{\delta, \varepsilon} := \pi^*\omega_0 + \delta\omega_X + dd^c\varphi$ strongly depends on the parameters δ and ε , we choose not to mention this dependence anymore to lighten notation. So in the following, $\omega = \pi^*\omega_0 + \delta\omega_X + dd^c\varphi$ will denote the solution of the above equation (4.2). It is straightforward to check that

$$(4.3) \quad \text{Ric } \omega = -T_\delta + (1 - \delta)[D] + \delta\theta_D + \Theta_\varepsilon + \delta\omega_X$$

in the sense of currents, which is the regularized version of (4.1).

It is crucial to understand the singularities of ω . Let us first fix δ and ε . We would like to use the laplacian estimate [GP13, Proposition 2.1], so we need to properly regularize $e^{\psi_\delta} / \prod |\sigma_k|^{2(1-\delta)}$. On one hand, Demailly regularization theorem enables us to regularize ψ_δ by smooth sup-normalized functions $\psi_{\delta, \eta}$ such that $dd^c\psi_{\delta, \eta} \geq -C\omega_X$ (up to increasing C) and on the other hand, one can replace $1 / \prod |\sigma_k|^{2(1-\delta)}$ by $1 / \prod (|\sigma_k|^2 + \eta^2)^{1-\delta}$. Everything is in order to apply [GP13, Proposition 2.1], so we obtain that ω is dominated by a metric

with conic singularities with cone angle $2\pi\delta$ along D , i.e. there exists $C_{\delta,\varepsilon}$ such that

$$(4.4) \quad \omega \leq C \left(\omega_X + \sum_k \frac{\langle D'\sigma_k, D'\sigma_k \rangle}{|\sigma_k|^{2(1-\delta)}} \right)$$

so, in particular, ω is smooth outside $D \cup Z$ (by standard bootstrapping arguments). As a result, we can find a family of cut-off functions $(\chi_\eta)_{\eta>0}$ vanishing near $D \cup Z$ and converging to the indicatrix of the complement of this set such that $\|\mathrm{tr}_\omega(dd^c \chi_\eta)\|_{L^1(\omega^n)} \rightarrow 0$. This family can be made explicit if Z is empty cf [CGP13, §9]; else, one considers a log resolution of the ideal corresponding to $D \cup Z$ (with its reduced structure) and construct the family upstairs similarly thanks to (4.4).

Finally, applying Proposition 1.1 to the Monge-Ampère equation (4.2) yields the convergence

$$(4.5) \quad \int_{X_0} \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_X \wedge \omega^{n-1} \rightarrow 0$$

when ε tends to 0 ($\delta > 0$ being fixed) thanks to the same arguments as in Lemma 2.7, whether $(s = 0)$ cuts out a component of E or one of D . To be rigorous, one should only apply Proposition 1.1 once (4.2) is regularized. Basically, one needs to replace $|\sigma_k|^{-2(1-\delta)}$ by $(|\sigma_k|^2 + \eta^2)^{\delta-1}$ for $\eta > 0$ and ψ_δ by a uniformly quasi-psh smooth approximant, which is possible thanks to [Dem92].

Back to our problem, we start with a generically injective morphism $\mathcal{F} \rightarrow \mathcal{T}_X(-D)$. If $r = \mathrm{rk}(\mathcal{F})$, then this morphism induces $L = (\Lambda^r \mathcal{F})^{**} \rightarrow \Lambda^r \mathcal{T}_X(-\log D)$. As X is smooth and L is reflexive of rank one, it is actually locally free, so the above morphism yields a non-zero section u of the bundle $\Lambda^r T_X(-\log D) \otimes L^{-1}$. We want to endow this bundle with a metric. For L , we just choose an arbitrary smooth hermitian metric h_L , and for $\Lambda^r T_X(-\log D)$, we would like to use ω . As ω is singular along $D \cup Z$, we need to work on $X_0 = X \setminus (D \cup Z)$ where ω is a smooth Kähler metric and induces a genuine hermitian metric on $T_X(-\log D)|_{X_0} \simeq T_{X|X_0}$, and hence on $\Lambda^r T_{X|X_0}$. Let us call h this metric; we can then measure u with respect to $h \otimes h_L^{-1}$. For any number $\lambda > 0$, we have on X_0 :

$$(4.6) \quad dd^c \log(|u|^2 + \lambda^2) = \frac{1}{|u|^2 + \lambda^2} \left(|D'u|^2 - \frac{|\langle D'u, u \rangle|^2}{|u|^2 + \lambda^2} - \langle \Theta(\Lambda^r T_X \otimes L^{-1}, h \otimes h_L^{-1})u, u \rangle \right).$$

Here, for a (vector-valued) $(1,0)$ -form α , we used the notation $|\alpha|^2 := \alpha \wedge \bar{\alpha}$. As $|\langle D'u, u \rangle|^2 \leq |D'u|^2 \cdot |u|^2$, we deduce from the equality above:

$$(4.7) \quad dd^c \log(|u|^2 + \lambda^2) \geq \frac{|u|^2}{|u|^2 + \lambda^2} \left(\Theta(L, h_L) - \frac{\langle \Theta(\Lambda^r T_X, h)u, u \rangle}{|u|^2} \right)$$

where the action of $\Theta(\Lambda^r T_X, h)$ is extended to $\Lambda^r T_X \otimes L^{-1}$ by tensorizing it with $\mathrm{Id}_{L^{-1}}$. Let us observe that u is a section of $\Lambda^r T_X(-D) \otimes L^{-1}$ and ω is dominated by a conic metric along D , cf (4.4). Therefore $|u|$ globally bounded on X_0 (of course, the bound depends on both δ and ε).

Now we wedge inequality (4.7) with ω^{n-1} and try to integrate it on X_0 . In order to legitimate this, we need to multiply first by the cut-off function χ_η mentioned a few lines above. Using Stokes theorem, we get:

$$(4.8) \quad - \int_X \log(|u|^2 + \lambda^2) dd^c \chi_\eta \wedge \omega^{n-1} \geq \int_X \frac{\chi_\eta |u|^2}{|u|^2 + \lambda^2} \left(\Theta(L, h_L) - \frac{\langle \Theta(\Lambda^r T_X, h) u, u \rangle}{|u|^2} \right) \wedge \omega^{n-1}$$

Our goal is to study the convergence of each of the terms appearing in the previous inequality when the parameters $\eta, \lambda, \varepsilon, \delta$ (in this order) converge to zero.

- Let us begin with the term on the left hand side. As $\log(|u|^2 + \lambda^2)$ is globally bounded on X_0 (since $\lambda > 0$) and $\text{tr}_\omega(dd^c \chi_\eta) \rightarrow 0$ in $L^1(X_0, \omega^n)$, this term converges to 0 when $\eta \rightarrow 0$, the other parameters $(\lambda, \varepsilon, \delta)$ being fixed.

- As ω has bounded potentials, the dominated convergence theorem guarantees that the integral

$$\int_X \frac{\chi_\eta |u|^2}{|u|^2 + \lambda^2} \Theta(L, h_L) \wedge \omega^{n-1}$$

converges to $c_1(L) \cdot \{\omega\}^{n-1} = c_1(L) \cdot (\pi^*(K_Y + \Delta) + \delta\{\omega_X\})^{n-1}$ when η and λ go to zero. This is a standard argument that we will use repeatedly so here is a justification. If T is a positive current and φ a bounded quasi-psh function, then $dd^c \varphi \wedge T := dd^c(\varphi T)$ is a (signed) measure with zero integral on X (just evaluate this current against the constant function 1). Next, we apply this to $T = \Theta(L, h_L) \wedge \omega^{n-2}$ to get: $\int_X \Theta(L, h_L) \wedge \omega^{n-1} = \int_X \Theta(L, h_L) \wedge \omega^{n-2} \wedge (\pi^* \omega_0 + \delta \omega_X)$, and inductively, we find $\int_X \Theta(L, h_L) \wedge (\pi^* \omega_0 + \delta \omega_X)^{n-2} = c_1(L) \cdot (\pi^*(K_Y + \Delta) + \delta\{\omega_X\})^{n-1}$.

- The remaining term is the more subtle to deal with. First let us introduce a notation: if V is a complex vector space of dimension n , $1 \leq r \leq n$ is an integer, and $f \in \text{End}(V)$, then we denote by $f^{\wedge r}$ the endomorphism of $\Lambda^r V$ defined by

$$f^{\wedge r}(v_1 \wedge \cdots \wedge v_r) := \sum_{i=1}^r v_1 \wedge \cdots \wedge v_{i-1} \wedge f(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_r.$$

Let us add that if V has an hermitian structure and if f is hermitian semipositive, then so is $f^{\wedge r}$ with the induced metric, and we have $\text{tr}(f^{\wedge r}) \leq \binom{n}{r} \text{tr}(f)$. Now we can easily check the following identity:

$$n\Theta(\Lambda^r T_X, h) \wedge \omega^{n-1} = (\sharp \text{Ric } \omega)^{\wedge r} \omega^n.$$

But on X_0 , we have from (4.3) that $\text{Ric } \omega = -T_\delta + \delta\theta_D + \Theta_\varepsilon + \delta\omega_X$, hence $\text{Ric } \omega \leq \delta\theta_D + \Theta_\varepsilon + \delta\omega_X$ so there are now two terms to be controlled:

(*) The first one involves $\delta\omega_X$ (we can treat $\delta\theta_D$ in the same way):

$$\begin{aligned} 0 \leq \langle \sharp(\omega_X)^{\wedge r} u, u \rangle \omega^n &\leq \binom{n}{r} \text{tr}(\sharp\omega_X) |u|^2 \omega^n \\ &= \binom{n}{r} \text{tr}_\omega(\omega_X) |u|^2 \omega^n \\ &= n \binom{n}{r} |u|^2 \omega_X \wedge \omega^{n-1}. \end{aligned}$$

Hence, we get

$$0 \leq \int_X \frac{\chi_\eta |u|^2}{|u|^2 + \lambda^2} \cdot \frac{\langle (\sharp(\delta\omega_X))^{\wedge r} u, u \rangle \omega^n}{|u|^2} \leq n \binom{n}{r} \delta \int_{X_0} \omega_X \wedge \omega^{n-1}$$

and the right hand side equals $n \binom{n}{r} \delta \{\omega_X\} \cdot \{\omega\}^{n-1}$ (because ω has bounded potentials) which is a cohomological quantity depending only on δ , and converging to 0 when this parameter does so.

(**) So now we need to control $\langle (\sharp\Theta_\varepsilon)^{\wedge r} u, u \rangle \omega^n$. We would like to view this term as a cohomological one, but the \sharp operation makes it difficult. So we split this term into a main positive (or negative according to the signe of a_i) part and a smaller correction factor; one can apply to the term with a sign basic inequalities to recover in the end cohomological terms up to a small error factor.

Remember that $\Theta_\varepsilon = \sum a_i \left(\frac{\varepsilon^2 |D' s_i|^2}{(|s_i|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \theta_{E_i}}{|s_i|^2 + \varepsilon^2} \right)$ where θ_{E_i} is the curvature form of the hermitian metric chosen on $\mathcal{O}_X(E_i)$. By linearity, it is enough to consider each of these summands separately. So we pick one and drop the index i to simplify the notations. So we have $\Theta_\varepsilon = a(\beta + \gamma)$ where $\beta = \frac{\varepsilon^2 |D' s|^2}{(|s|^2 + \varepsilon^2)^2}$ and $\gamma = \frac{\varepsilon^2 \theta}{|s|^2 + \varepsilon^2}$; remember that these forms are smooth as long as $\varepsilon > 0$. Let us start with γ : there exists a constant $C > 0$ such that $\pm\gamma \leq C\varepsilon^2/(|s|^2 + \varepsilon^2) \omega_X$. As the \sharp and $^{\wedge r}$ operations preserve positivity, we get that $\pm(\sharp\gamma)^{\wedge r} \omega^n \leq C\varepsilon^2/(|s|^2 + \varepsilon^2) (\sharp\omega_X)^{\wedge r} \omega^n$. But $\sharp\omega_X$ is an positive endomorphism whose trace is $\text{tr}_\omega \omega_X$, and therefore $(\sharp\omega_X)^{\wedge r} \leq \binom{n}{r} \text{tr}_\omega(\omega_X) \text{Id}$. Consequently,

$$\frac{\pm \langle (\sharp\gamma)^{\wedge r} u, u \rangle \omega^n}{|u|^2} \leq \frac{C\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_X \wedge \omega^{n-1}$$

for some $C > 0$ uniform. From (4.5) and the dominated convergence theorem, one deduces that the integral

$$\int_X \frac{\chi_\eta |u|^2}{|u|^2 + \lambda^2} \cdot \frac{\langle (\sharp\gamma)^{\wedge r} u, u \rangle \omega^n}{|u|^2}$$

converges to 0 when η, λ and ε go to zero.

The last term to control is the one involving β . We know that β is non-negative, so $(\# \beta)^{\wedge r} \omega^n \leq n \binom{n}{r} \beta \wedge \omega^{n-1} \text{Id}$ so that

$$\begin{aligned} 0 &\leq \int_X \frac{\chi_\eta |u|^2}{|u|^2 + \lambda^2} \cdot \frac{\langle (\# \beta)^{\wedge r} u, u \rangle \omega^n}{n |u|^2} \leq C \int_X \frac{\chi_\eta |u|^2}{|u|^2 + \lambda^2} \cdot \beta \wedge \omega^{n-1} \\ &\leq C \int_{X_0} \beta \wedge \omega^{n-1} \\ &= C \left(\int_{X_0} (\beta + \gamma) \wedge \omega^{n-1} - \int_{X_0} \gamma \wedge \omega^{n-1} \right) \\ &= C \left(c_1(E_i) \cdot \{\omega\}^{n-1} - \int_{X_0} \gamma \wedge \omega^{n-1} \right) \end{aligned}$$

where we used the fact that ω has bounded potentials (so that none of its powers charge a pluripolar set). We already observed that the second integral converges to 0 when $\varepsilon \rightarrow 0$. As for the first term, it is cohomological (independent of ε) and it equals $\delta^{n-1} c_1(E_i) \cdot \{\omega_X\}^{n-1}$ as E_i is π -exceptional, and thus it converges to 0 when δ goes to 0.

Finally, we showed that if we first make η go to 0, and then λ, ε and at last δ , inequality (4.8) becomes: $L \cdot \{\pi^* \omega_0\}^{n-1} \leq 0$ which is equivalent to $c_1(\mathcal{F}) \cdot \{\pi^* \omega_0\}^{n-1} \leq 0$ or, using (2.1), to $c_1(\mathcal{G}) \cdot \{\pi^* \omega_0\}^{n-1} \leq 0$ which is the expected generic semi positivity inequality for the sheaf of logarithmic forms $\Omega_Y^1(\log \Delta)$.

The case where $-(K_Y + \Delta)$ is pseudo-effective is very similar. Again, it is enough to show that every coherent subsheaf $\mathcal{F} \subset \Omega_X^1(\log D)$ has non-positive slope with respect to any "polarization" $\pi^* \omega_0^{n-1}$ pulled-back from Y . We solve the same Monge-Ampère equation but now the metric satisfies $\text{Ric } \omega_{\varepsilon, \delta} = T_\delta + (1 - \delta)[D] + \delta \theta_D + \Theta_\varepsilon - \delta \omega_X$, where $\omega_{\varepsilon, \delta} \in \pi^* \omega_0 + \delta \omega_X$ and T_δ is a positive current in $c_1(-\pi^*(K_Y + \Delta)) + \delta \{\omega_X\}$ so that $\text{Ric } \omega_{\varepsilon, \delta} \geq \delta \theta_D + \Theta_\varepsilon - \delta \omega_X$. Now, $\Theta(\Omega_X^1) = -\Theta(T_X)^*$, so that $\text{tr}_\omega \Theta(\Omega_X^1) = -\# \text{Ric}$. We can run the same computations as above now, and get the non-positivity of the slope of \mathcal{F} . \square

We can now move on to the proof of the polystability of the tangent sheaf of singular Calabi-Yau varieties:

Proof of Corollary 4.2. — Let us start with a "polarization" ω_0 on Y , and take the notations of the proof above where we work on a resolution $\pi : X \rightarrow Y$ of the Calabi-Yau variety Y , and call E the exceptional divisor of π . As the slope of \mathcal{T}_Y vanishes, Theorem 4.1 gives the ω -semistability of \mathcal{T}_Y already. Now, suppose that there exists a reflexive subsheaf $\mathcal{F} \subset \mathcal{T}_Y$ whose slope with respect to ω_Y vanishes too. As in the case K_Y ample, Step 4, one chooses \mathcal{G} of minimal rank, and it suffices to find an holomorphic complement of \mathcal{G} inside \mathcal{T}_Y . Therefore we are reduced to showing that one can complement over $X \setminus E$ any reflexive subsheaf $\mathcal{F} \subset \mathcal{T}_X$ whose slope with respect to $\pi^* \omega$ vanishes. We denote by W be the minimal analytic set (of codimension at least 2 as \mathcal{F} is reflexive) outside which \mathcal{F} is a subbundle of T_X .

We run the proof of Theorem 4.1 above and as K_Y is numerically trivial one can choose $T_\delta = 0$ from the very beginning (and $D = 0$ of course, as well as $\lambda = 1$). So we end up with a family of Kähler metrics $\omega := \omega_{\varepsilon, \delta}$ cohomologous to $\pi^*\omega_0 + \delta\omega_X$ and solving $\text{Ric } \omega = \Theta_\varepsilon$ where $\Theta_\varepsilon = \sum_{i \in I} a_i \left(\frac{\varepsilon^2 |D's_i|^2}{(|s_i|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \theta_{E_i}}{|s|^2 + \varepsilon^2} \right)$ is a smooth approximant of the current of integration along E , E being defined as the exceptional divisor representing $-K_X$.

We know (from [EGZ09] e.g.) that $\omega_{\varepsilon, \delta}$ converges in $\mathcal{C}_{\text{loc}}^\infty(X \setminus E)$ to a smooth Ricci flat metric ω_∞ inducing a smooth hermitian metric h_∞ on $T_{X|X \setminus E}$.

Inequality (4.7) must become an equality at the limit, so the term we neglected to go from (4.6) to (4.7) necessarily converges to zero. This term is (remember that $\lambda = 1$):

$$\frac{1}{|u|^2 + 1} \left(|D'u|^2 - \frac{|\langle D'u, u \rangle|^2}{|u|^2 + 1} \right) \geq \frac{|D'u|^2}{(|u|^2 + 1)^2}$$

thanks to Cauchy-Schwarz inequality.

Fatou lemma shows that $|D'u|_{\omega_\infty} = 0$ on $X \setminus (W \cup E)$, and therefore $\det \mathcal{F}$ is a parallel subbundle of T_X on this locus. This implies by simple algebraic manipulations that \mathcal{F} itself is a parallel subbundle of T_X . Therefore we get an holomorphic splitting $\mathcal{T}_X = \mathcal{F} \oplus \mathcal{F}^\perp$ over $X \setminus (W \cup E)$. We can push forward this identity to Y , which by reflexivity gives us the expected complement of \mathcal{G} inside \mathcal{T}_Y as $\pi(E \cup W)$ has codimension at least 2. \square

5. A Bogomolov type inequality for Chern numbers

If X is a singular variety (endowed with a Kähler-Einstein metric), then there is in general not possible to reproduce the arguments of Yau and work directly with the metric on the smooth locus to get an appropriate inequality between Chern numbers – this would require to somehow control the curvature of the metric near the singularities, but this seems out of reach for now. Instead, we will rely on the so-called Bogomolov-Gieseker inequality, which asserts that any reflexive sheaf E of rank r on a smooth projective surface which is semistable with respect to some given polarization H satisfies

$$(B) \quad c_1(E)^2 \leq \frac{2r}{r-1} c_2(E)$$

The proof of Bogomolov inequality, generalized by Miyaoka in higher dimension [Miy87, Corollary 4.7], consists of analyzing the high powers of the semistable bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \frac{\det E}{r}$. This bundle has degree zero, hence its twist by an anti-effective divisor cannot admit any section. Identifying the leading order term in Riemann-Roch's formula lead to inequality (B); we refer to [MP97, §3] for more details.

The argument uses Mehta-Ramanathan to reduce the general case to the case of surfaces, so it is important to work with an ample polarization. However, [Sug90, Theorem 6.1] showed that we can get a slightly weaker statement for singular varieties, by proving that if X is \mathbb{Q} -Gorenstein and $\pi : Y \rightarrow X$ is a resolution of singularities, then any rank r bundle E on Y which is semistable with respect to π^*H for some ample divisor H on Y satisfies the

following Bogomolov inequality:

$$c_1(E)^2 \cdot (\pi^* H)^{n-2} \leq \frac{2r}{r-1} c_2(E) \cdot (\pi^* H)^{n-2}.$$

Combining this with Enoki's result about the semistability of the tangent sheaf for canonical varieties with ample canonical class (resp. trivial), he obtained the inequality above with $H = K_X$ (resp. H arbitrary). Actually, using the base point free theorem, he could also generalize it to the case where K_X is nef and big, and also assuming essentially only that K_X is nef (relying on Miyaoka's generic semi positivity theorem for c_2).

Sugiyama's arguments can be transposed to the case of a log pair (X, D) such that $K_X + D$ is \mathbb{Q} -Cartier without any significant changes. So we can use these ideas in our setting ; combined with Theorem A, they yield the theorem below. The context is the following: let (X, D) be a projective log canonical pair of dimension n , and $\pi : Y \rightarrow X$ a log resolution. We denote by Δ the sum of the strict transform of D and the exceptional divisor with zero log discrepancy; to put it in a different way, $K_Y + \Delta - \pi^*(K_X + D)$ is an exceptional divisor with coefficients > -1 . If $K_X + D$ is ample (resp. numerically trivial), then the proof of Theorem A shows that $\mathcal{T}_Y(-\log \Delta)$ is semistable with respect to $\pi^*(K_X + D)$ (resp. π^*H for any ample line bundle H on X). Therefore, we get:

Theorem 5.1. — *Let (X, D) be a projective log canonical as above. We have the following Bogomolov type inequalities:*

- If $K_X + D$ is ample, then:

$$c_1(T_Y(-\log \Delta))^2 \cdot (\pi^*(K_X + D))^{n-2} \leq \frac{2n}{n-1} c_2(T_Y(-\log \Delta)) \cdot (\pi^*(K_X + D))^{n-2}.$$

- If $K_X + D \equiv 0$, then for any ample line bundle H on X , we have:

$$c_1(T_Y(-\log \Delta))^2 \cdot (\pi^*H)^{n-2} \leq \frac{2n}{n-1} c_2(T_Y(-\log \Delta)) \cdot (\pi^*H)^{n-2}.$$

Of course, if the non-snc locus $(X, D)_{\text{sing}}$ has codimension > 2 then we can define c_1 and c_2 for the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ as the push forward of the usual cycles on $(X, D)_{\text{reg}}$ by the open immersion. The projection formula yields:

$$c_1(\mathcal{T}_X(-\log D))^2 \cdot (K_X + D)^{n-2} \leq \frac{2n}{n-1} c_2(\mathcal{T}_X(-\log D)) \cdot (K_X + D)^{n-2}$$

if $K_X + D$ is ample, and

$$c_1(\mathcal{T}_X(-\log D))^2 \cdot H^{n-2} \leq \frac{2n}{n-1} c_2(\mathcal{T}_X(-\log D)) \cdot H^{n-2}$$

if $K_X + D \equiv 0$.

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June 29, 2020

HENRI GUENANCIA, Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651
 USA • E-mail : guenancia@math.sunysb.edu • Url : <http://www.math.sunysb.edu/~guenancia>