Some properties of a new partial order on Dyck paths

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Introduction

This article defines a new partial order on extremely classical combinatorial objects, namely Dyck paths, and describes some of its basic or subtle properties.

Let us start by explaining in some detail how this partial order was discovered. The starting point was the study of the intervals in the Tamari lattice. This was initiated in [Cha07], where these intervals were enumerated. One interesting point made there was the fact that the number of Tamari intervals of size n is exactly the number of triangulations of size n, found by Tutte in one of his foundational articles on planar maps [Tut63]. It was only understood much later that the set of Tamari intervals admits a natural and interesting partial order. Indeed, the set of intervals in any poset can be ordered by the relation $[a, b] \leq [c, d]$ if and only if $a \leq c$ and $b \leq d$. When drawing carefully the Hasse diagram for these posets of Tamari intervals, one obtains a picture which apparently has been considered previously from another point of view. This other context is the study of homotopy associativity and more precisely the cellular diagonal for the associahedra that is required to define the tensor product of A_∞ algebras [Lod11].

By admitting that this is really the same picture, one can consider that Tamari intervals can be gathered into higher-dimensional cells, and that every such cell has unique top and bottom elements. There is a distinguished top cell, made of all intervals whose maximum is the unique maximum of the Tamari lattice. Elements in this top cell are indexed by planar binary trees. It appears that every Tamari interval in this top cell is the top element of some unique cell. Taking the bottom elements of these cells gives another subset indexed by planar binary trees. The partial order that we study in the present article is nothing else than the order induced on this subset of the poset of all Tamari intervals. Note that this description is just the original motivation, but instead we will start here from a purely combinatorial definition and will not prove that it coincides with this former description.

Let us now present the contents of the article. The first section contains the definition of the posets $\mathscr{D}_n$ for $n \geq 0$, starting from a directed graph that turns out to be their Hasse diagram. Some basic properties are given, including their number of maximal elements and their relationship to the Tamari lattices.

Then come several sections that prepare the ground for the first main result, by a careful combinatorial description of the intervals in $\mathscr{D}_n$.

Sections 2 and 3 present a structure of graded monoid on the disjoint union of all elements of $\mathscr{D}_n$ for $n \geq 1$, and some compatibility of the monoid product with the partial orders. This is then built upon in the section 4 to define a graded monoid structure on the disjoint union of all sets of intervals in $\mathscr{D}_n$ for $n \geq 1$. All these monoids are shown to be free. Section 4 also contains a result to the effect that some principal upper ideals are products of smaller principal upper ideals.

Section 5 is the last piece of the combinatorial puzzle, namely the study of a specific subset of core intervals in $\mathscr{D}_{n+2}$ and the construction of a bijection with intervals in $\mathscr{D}_n$ together with a catalytic data.

All this combinatorial preparation is then used in section 6 to prove the first main result: the number of intervals in $\mathscr{D}_n$ is also given by another formula of Tutte, namely his formula for the number of number of rooted bicubic planar maps. This is achieved by deducing a functional equation from the combinatorial description, and then solving this catalytic equation. This enumerative result is yet another instance of the growing connection between Tamari and related posets and various kinds of maps, see for example [FPR17, Fan18b, Fan18a].

Section 7 proves the second main result: all the posets $\mathscr{D}_n$ are meet-semilattices, which means that any two elements have a greatest lower bound. This requires a somewhat technical study of properties in $\mathscr{D}_n$ of Dyck paths that share a common prefix of some length. The proof also gives an algorithm for computing the meet.

Section 8 briefly evoques another interesting aspect of the posets $\mathscr{D}_n$, namely the probable existence of many derived equivalences between some of their intervals. This certainly deserves further investigation.

Section 9 tells a surprising story, namely an unexpected connection with another familly of objects called the Hochschild polytopes, maybe not so well-known, that appeared in the works of Saneblidze in algebraic topology [San11, San09]. One proves that there is a bijection between elements in a specific interval F_n of $\mathscr{D}_{n+2}$ and the vertices of the cell complexes of Saneblidze that realize the Hochschild polytopes. This bijection is probably also an isomorphism of posets, which would prove that the order of Saneblidze is a lattice. This question is left open here, for lack of a strong enough motivation.

The final section 10 contains one result and various remarks. The result is an unexpected symmetry for some kind of h-polynomial that enumerates vertices according to a coloring of the covering relations in $\mathscr{D}_n$.

Section 11 is an appendix recalling some classical properties of Tamari lattices.

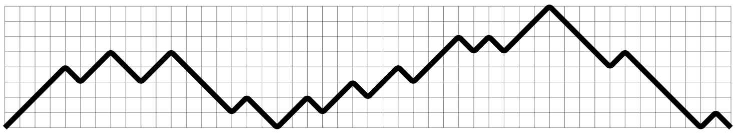
1 Construction and first properties

A *Dyck path* of size $n \geq 0$ is a lattice path from $(0,0)$ to $(2n,0)$ using only north-east and south-east steps and staying weakly above the horizontal line.

We will also consider Dyck paths of size n as words of length $2n$ in the alphabet $\{0,1\}$, where 1 stands for a north-east step and 0 for a south-east step.

The next figure is a typical example.

The *area* under a Dyck path is the surface of the domain between the horizontal line and the Dyck path.



Every Dyck path can be uniquely written as the concatenation of several blocks, where in every block the only vertices on the horizontal line are the first and last ones. If there is only one block, the Dyck path is said to be *block-indecomposable*. The example displayed above has 3 blocks.

Inside a Dyck path, a subsequence of consecutive steps is called a *subpath* if it starts and ends at the same height and keeps strictly above that height in between. Here the height is the vertical coordinate, increased by north-east steps and decreased by south-east steps.

Let us say that a subpath x of a Dyck path w is *movable* if it is preceded in w by the letter 0 and

- either x ends at the last letter of w,
- or x is followed in w by the letter 1.

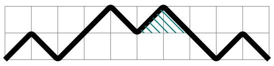


Figure 1: A subpath which is not movable.

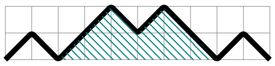


Figure 2: A movable subpath.

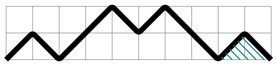


Figure 3: A movable subpath.

For every Dyck path w and every movable subpath x in w, let $N(w, x)$ be the number of consecutive 0 letters that appear just before x. For any integer $1 \leq i \leq N(w, x)$ (corresponding to a choice among the consecutive 0 letters just before x), let us define another Dyck path $M(w, x, i)$ as the following word:

- first the initial part of w until the letter before the chosen 0,
- then x,
- then the 0 letters starting at the chosen 0,
- then the final part of w after x.

Graphically, this transformation corresponds to sliding the subpath x in the north-west direction by one or several steps.

Let $\mathscr{D}_n$ be the set of Dyck paths of size n. Let us introduce a directed graph Γ_n with vertex set $\mathscr{D}_n$. It has edges from every Dyck path w to all Dyck paths

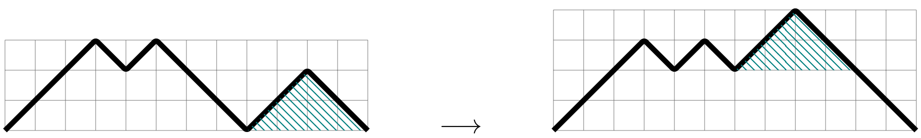


Figure 4: A covering relation $w \to w'$

$M(w, x, i)$ where x is a movable subpath of w and i an integer between 1 and $N(w, x)$.

Let $w_{\min}$ be the unique Dyck path made of alternating 1 and 0.

Proposition 1.1. *The directed graph Γ_n is connected and acyclic with $w_{\min}$ as unique source element.*

Proof. First, every edge in Γ_n increases strictly the area under the Dyck path, so there cannot be any oriented cycles.

Let w be any Dyck path not equal to $w_{\min}$. Then w has a least one block x of maximal height at least 2. This block ends by a sequence of at least two 0 preceded by a 1. Let w' be the Dyck path obtained by exchanging this 1 and that sequence of 0 except the last one. Then w' has a strictly smaller area than w and there is an edge in Γ_n from w' to w.

Therefore, by induction on the area, for every Dyck path w, there exists a path in Γ_n from $w_{\min}$ to w. □

Proposition 1.2. *The directed graph Γ_n is the Hasse diagram of a partial order.*

Proof. This means that Γ_n is acyclic and transitively reduced. It remains only to prove the second property.

So let us consider an edge $w \to M(w, x, i)$ and assume that one can find a sequence (S) of edges $w \to M(w, x', i') \to \cdots \to M(w, x, i)$.

Every edge in Γ_n can be considered as a sequence of several moves where a subpath is slided by just one step in the north-west direction. As recalled in §11, these moves are exactly the cover relations in the Tamari partial order on the same set of Dyck paths.

Therefore, one can refine the sequence (S) of edges in Γ_n into a sequence of cover moves in the Tamari lattice. By lemma 11.1, the interval in the Tamari lattice between w and $M(w, x, i)$ is just a chain obtained by repeated sliding of the subpath x. This implies that all the intermediate vertices of the sequence (S) are obtained by sliding the same subpath x.

But in the graph Γ_n, there cannot be any two consecutive edges where exactly the same subpath is slided. Indeed, after being slided once, the subpath is followed by a letter 0, hence no longer movable. It follows that the sequence (S) is reduced to the single edge $w \to M(w, x, i)$. □

Let us denote by $\leq$ the partial order relation on $\mathscr{D}_n$ thus defined. The edges of Γ_n are now understood as the cover relations in the poset $(\mathscr{D}_n, \leq)$.

We propose to call this partial order the *dexter order* on Dyck paths. This choice of terminology is motivated by the fact that a symmetry is broken when compared to the Tamari lattice. This is also linked to some pattern-exclusion in interval-posets for the Tamari lattices, which reminds of the chirality of seashells.

1.1 First properties

Proposition 1.3. *The maximal elements of $(\mathscr{D}_n, \leq)$ are exactly the block-indecomposable Dyck paths that do not contain any subpath that is both preceded by* 0 *and followed by* 1.

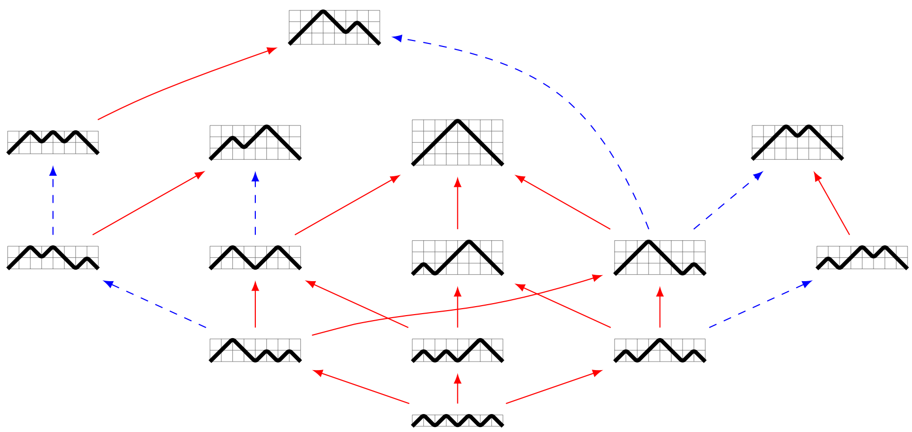


Figure 5: Hasse diagram of the poset $(\mathscr{D}_4, \leq)$. Edge colors will be explained and used in section 10.1.

Proof. The property of w being maximal is equivalent to the non-existence of movable subpaths in w.

According to their definition, movable subpaths can exist in two distinct ways. The first way is when the movable subpath is the last block of w. This happens if and only if w contains at least two blocks. The second way is the existence of a subpath preceded by 0 and followed by 1.

Therefore being maximal is equivalent to being block-indecomposable and not having this second kind of subpaths. □

Proposition 1.4. *The sets of maximal elements are counted by the Motzkin numbers (A1006).*

Proof. Removing the initial 1 and the final 0 defines a bijection from the set of maximal elements to the set $\mathscr{M}$ of Dyck paths that do not contain any subpath that is both preceded by 0 and followed by 1. Then the decomposition into blocks of such a path has at most two blocks and these blocks satisfy the same condition. One therefore gets

$$M = 1 + tM + t^2M^2 = 1 + t + 2t^2 + 4t^3 + 9t^4 + \dots \tag{1}$$

for the generating series M of the set $\mathscr{M}$. This is the usual equation for the generating series of Motzkin numbers. □

Let us now give examples of intervals in $\mathscr{D}_n$ where some properties do not hold. The interval between $w_{\min}$ and the Dyck path $(1,1,1,1,0,0,1,0,0,1,0,0)$ is not semi-distributive, and therefore not congruence-uniform ; it is also not extremal. For these notions, see general references on lattice theory such as [FJN95, Grä11]. The interval between $w_{\min}$ and the Dyck path

$$(1,1,1,1,1,0,0,1,0,0,0,1,0,0)$$

has a Coxeter polynomial with some roots not on the unit circle (see [Cha12, Len99] for the representation-theoretic context).

Remark 1.5. *There are two simple natural inclusions of posets of $\mathscr{D}_n$ in $\mathscr{D}_{n+1}$, by concatenation of the Dyck path $(1,0)$ at the start or at the end.*

1.2 Relations to other posets

In this subsection, we prove that the partial order $(\mathscr{D}_n, \leq)$ stands somewhere between the Tamari partial order [FT67, MHPS12] and the less well-known comb partial order introduced by Pallo in [Pal03].

The Tamari lattice structure on Dyck paths is recalled in §11.

Proposition 1.6. *Let w and w' be two Dyck paths in $\mathscr{D}_n$ such that $w \leq w'$. Then w is smaller than w' in the Tamari order.*

Proof. By transitivity, it is enough to prove this under the additional assumption that there is an edge $w \to w'$ in Γ_n. It is clear that such an edge can be performed using a sequence of several cover relations in the Tamari lattice, which are just made by sliding a subpath by one step in the north-west direction. □

The comb partial order (or left-arm rotation order) was initially defined in [Pal03] as a partial order on the set of binary trees. Its cover relations are a subset of the cover relations of the Tamari lattice, namely left-to-right rotations $(ab)c \to a(bc)$ of binary trees where the two rotated vertices are on the leftmost branch of the first binary tree. Using the same bijection σ as [BB09] and the proof of their Prop. 2.1, one can see that these restricted rotations correspond precisely to Tamari cover moves in $\mathscr{D}_n$ where the slided subpath is at height 0.

To summarize, the cover moves in the comb partial order on $\mathscr{D}_n$ are the Tamari cover moves where the slided subpath is at height 0.

Proposition 1.7. *Let w and w' be two Dyck paths in $\mathscr{D}_n$ such that w is smaller than w' in the comb partial order. Then $w \leq w'$.*

Proof. This holds because every cover move in the comb poset is also a cover move in $(\mathscr{D}_n, \leq)$, as a subpath at height 0 cannot be followed by the letter 0. □

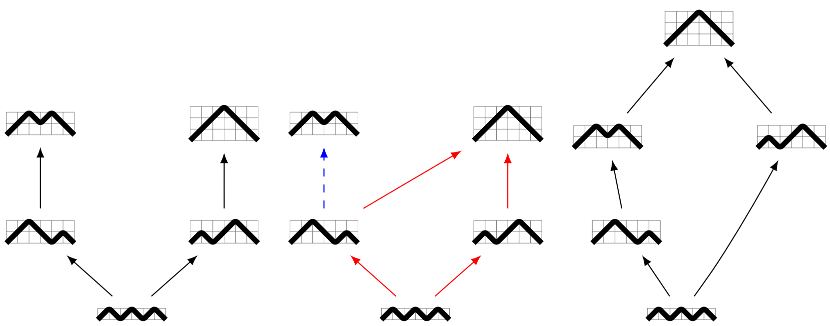


Figure 6: Comparison of three partial orders: comb, dexter and Tamari

2 A monoid on pseudo-Dyck paths

Let us first introduce another kind of decomposition for block-indecomposable Dyck paths.

Proposition 2.1. *Let $k \geq 0$ be an integer. Every block-indecomposable Dyck path ending with exactly $k+1$ letters 0 can be uniquely written as*

$$(1, w_1, 1, w_2, 1, w_3, \dots, 1, w_k, 1, 0^{k+1}) \tag{2}$$

for some Dyck paths $w_1, w_2, \dots, w_k$. This defines a bijection between block-indecomposable Dyck paths ending with exactly $k+1$ letters 0 and sequences of k Dyck paths.

Proof. Clearly, every Dyck path of the given shape (2) is block-indecomposable and ends with exactly $k+1$ letters 0.

Conversely, let us start with a block-indecomposable Dyck path w ending with exactly $k+1$ letters 0. The aim is to recover all the w_i. Consider the word v_k obtained from w by removing the final $(1, 0^{k+1})$ part. The final height of v_k is k. Let w_k be the largest suffix of v_k in which the height is greater than or equal to k. The letter before w_k in w must be 1. Then one can define v_{k-1} by removing $(1, w_k)$ at the end of v_k. Iterating this process of removing the largest suffix of height staying above the final height, one defines $w_1, \dots, w_k$ that are all Dyck paths. This yields the desired decomposition, which is clearly unique. □

Let us call this decomposition the *level-decomposition* of w. We will write

$$w = \mathscr{L}(w_1, w_2, \dots, w_k)$$

to denote this situation.

In fact, both the decomposition into blocks and the level-decomposition are special cases of products inside a monoid structure. Let us describe this monoid.

It turns out to be more convenient to work (for a moment) with Dyck paths minus their initial and final letters. Let us therefore define a *pseudo-Dyck path* to be a path obtained in this way from a non-empty Dyck path. For a Dyck path w, let $\overline{w}$ denote the pseudo-Dyck path obtained by removing the initial and final letters of w.

Note that the height in a pseudo-Dyck path can reach the value -1, taking 0 as initial height.

Let us introduce a product $*$ on the set of pseudo-Dyck paths. Let u and v be two such paths. Write $u = (u', 0^k)$ where u' is either empty (if u itself is empty) or ends with the letter 1. Write $v = v'v''$ where v' is the largest prefix where the height is positive or zero. The product is

$$u * v = (u', v', 0^k, v''). \tag{3}$$

This is again a pseudo-Dyck path, as it lies above the concatenation uv, which is itself clearly a pseudo-Dyck path.

For example, one gets

$$(1, 0) * (0, 1) = (1, 0, 0, 1)$$

and

$$(0, 1, 1, 0) * (1, 0, 1, 0) = (0, 1, 1, 1, 0, 1, 0, 0).$$

Lemma 2.2. *The product $*$ is an associative product on the set of pseudo-Dyck paths.*

Proof. Consider three pseudo-Dyck paths u, v, w. Write $u = (u', 0^k)$, $w = w'w''$ and decompose also v in both ways. One has to distinguish two cases, depending on whether $(1, v, 0)$ is block-indecomposable or not. If it is, then $v = (v', 0^\ell)$ where v' ends by the letter 1. The largest prefix of positive height is v itself. In this case, the two ways to compute the product of u, v, w give $(u', v', w', 0^{\ell+k}, w'')$.

If $(1, v, 0)$ is not block-indecomposable, then $v = (v', v'', 0^\ell)$, where v'' ends with 1 and v' is the largest prefix of positive height. In this case, the two ways to compute the product of u, v, w give $(u', v', 0^k, v'', w', 0^\ell, w'')$. □

This defines a monoid $\overline{\mathbb{M}}_1$ on the set of pseudo-Dyck paths. This monoid is graded by the size, which is the number of letters 1. Note that the empty pseudo-Dyck path is the unit of this monoid.

Lemma 2.3. *Let u and v be two Dyck paths. Then the product $\overline{u} * (0, 1) * \overline{v}$ is the pseudo-Dyck path $\overline{uv}$ associated to the concatenation of u and v.*

Proof. Using the definition of $*$, one can check that the pseudo-Dyck path $\overline{u} * (0, 1) * \overline{v}$ is just the concatenation $(\overline{u}, 0, 1, \overline{v})$. Hence the associated Dyck word is $(1, \overline{u}, 0, 1, \overline{v}, 0)$, which is uv. □

Lemma 2.4. *Let w be a block-indecomposable Dyck path with level-decomposition $w = \mathscr{L}(w_1, \dots, w_k)$. Then the pseudo-Dyck path $\overline{w}$ is*

$$\overline{w} = (w_1, 1, 0) * \dots * (w_k, 1, 0). \tag{4}$$

Proof. By induction on k. This is true if $k = 0$ for the empty level-decomposition of $(1, 0)$ and the empty product. If $k = 1$, the product has only one factor and the statement is also true. Otherwise, let w' be the block-indecomposable Dyck path with level-decomposition $w' = \mathscr{L}(w_1, \dots, w_{k-1})$. Then one knows that $\overline{w'} = (w_1, 1, 0) * \dots * (w_{k-1}, 1, 0)$. Computing the $*$ product of this with $(w_k, 1, 0)$, one gets exactly $\overline{w}$ on the left hand side and the expected product on the right hand side. □

This lemma also implies that the product $\overline{u} * \overline{v}$ for two block-indecomposable Dyck paths u and v is $\overline{w}$, where w is the block-indecomposable Dyck path whose level-decomposition is the concatenation of the level-decompositions of u and v.

Lemma 2.5. *The monoid $\overline{\mathbb{M}}_1$ is generated by the elements $(w, 1, 0)$ where w runs over the set of Dyck paths, plus the element $(0, 1)$.*

Proof. By the previous two lemmas, one can first write any pseudo-Dyck path as a $*$ product of some $\overline{w}$ for block-indecomposable Dyck paths w, with intermediate factors $(0, 1)$. Then using the level-decomposition of each w, one can write $\overline{w}$ as a $*$ product of pseudo-Dyck paths of the shape $(w_i, 1, 0)$. □

Proposition 2.6. *The monoid $\overline{\mathbb{M}}_1$ is free on the generators $(w, 1, 0)$ where w runs over the set of Dyck paths, plus the element $(0, 1)$.*

Proof. Because the monoid is generated by these elements, it is enough to compare its generating series with the generating series of the free monoid on the same generators. This is a simple computation with the usual generating series of Catalan numbers. □

3 The same monoid seen on Dyck paths

Let us now go back to Dyck paths. Using the simple bijection $w \to \overline{w}$ between non-empty Dyck paths and pseudo-Dyck paths, the monoid structure $\overline{\mathbb{M}}_1$ can be transported to a monoid structure $\mathbb{M}_1$ on the set of non-empty Dyck paths, where the product will be denoted $\sharp$.

This means that

$$u \sharp v = (1, \overline{u} * \overline{v}, 0). \tag{5}$$

The unit of $\mathbb{M}_1$ becomes $(1,0)$, and the free generators of $\mathbb{M}_1$ are the element $(1,0,1,0)$ and all elements of the shape $(1,w,1,0,0)$ for some Dyck path w.

The monoid structure $\mathbb{M}_1$ interacts nicely with the partial order on $\mathscr{D}_n$.

Lemma 3.1. *Let $w \in \mathscr{D}_n$ be a Dyck path. Assume that $w = w_1 \sharp \dots \sharp w_m$ as a product of generators of the monoid $\mathbb{M}_1$. Let $w \to w'$ be a cover move in the partial order on $\mathscr{D}_n$. Then the decomposition of w' as a product of generators has at most m factors. If this decomposition has exactly m factors, then there is one cover move inside one of the factors, and the other factors are unchanged. Otherwise, several consecutive factors are merged into a new factor, all other factors being unchanged.*

Proof. (A) Assume first that the cover move is sliding one full block of w, and therefore merging two consecutive blocks v and v' of w. One has to understand the level-decomposition of this new block in terms of those of $v = \mathscr{L}(v_1, \dots, v_k)$ and v'. This depends on the height $1 \leq i \leq k+1$ of the block v' after it has been slided.

Assume first that $1 \leq i \leq k$. Then the new level decomposition will be made of $v_1, \dots, v_{i-1}$, then a new term coming from the fusion of $v_i, \dots, v_k$, then the level decomposition of v'. There are strictly less factors in w' than in w, because the factor $(1,0,1,0)$ separating the two blocks disappears.

There remains the case when $i = k+1$. Then the new level decomposition is $v_1, \dots, v_k$, followed by the empty Dyck path, then by the level decomposition of v'. Therefore the number of factors stays the same with just one change, the factor $(1,0,1,0)$ being replaced by the factor $(1,1,0,0)$. This corresponds to a cover move inside this factor.

(B) Assume now that the cover move is happening inside one block v of w. Let $v = \mathscr{L}(v_1, \dots, v_k)$ be the level-decomposition of v. Then the cover move can only happen inside some v_i as a cover move $v_i \to v_i'$. Therefore in the monoid $\mathbb{M}_1$, one gets w' from w by replacing the factor $(1, v_i, 1, 0, 0)$ by $(1, v_i', 1, 0, 0)$. ☐

This means that performing a cover move in a $\sharp$ product can either preserve the product or merge terms of the product, but can never split them.

Let us now give two lemmas that will be used in the next section to define a product $\sharp$ on the set of all intervals.

Lemma 3.2. *Let $w \to w'$ be a cover relation of non-empty Dyck paths. Let u be a non-empty Dyck path. Then there is a cover relation from $u \sharp w$ to $u \sharp w'$.*

Proof. One can assume that u is not $(1,0)$, because it is the unit of $\mathbb{M}_1$. Let $\overline{u} = (u', 0^k)$ be the decomposition of $\overline{u}$ according to its final sequence of 0. Let $\overline{w} = w_1 w_2$ and $\overline{w'} = w_1' w_2'$ be the decompositions whose first term is the longest prefix before going below the horizontal axis. Then $\overline{u} * \overline{w}$ and $\overline{u} * \overline{w'}$

are $(u', w_1, 0^k, w_2)$ and $(u', w'_1, 0^k, w'_2)$. If either $w_1 = w'_1$ or $w_2 = w'_2$, then the statement is clear, as the cover move is inside either w_1 or w_2. Otherwise, the cover move must merge the first two blocks of w by sliding the second block. One can then write $w = (1, w_1, 0, x, w_3)$ and $w' = (1, y, 0, w_3)$ where x is the slided subpath, y is the result of sliding x on w_1 and w_3 may be empty. Then the products $\overline{u} * \overline{w}$ and $\overline{u} * \overline{w'}$ are (when lifted back to Dyck paths) equal to $(1, u', w_1, 0^{k+1}, x, w_3)$ and $(1, u', y, 0^{k+1}, w_3)$. One can slide x to get the wanted cover move. □

Lemma 3.3. *Let $w \to w'$ be a cover relation of non-empty Dyck paths. Let u be a non-empty Dyck path. Then there is a cover relation from $w \sharp u$ to $w' \sharp u$.*

Proof. One can assume that u is not $(1, 0)$, because it is the unit of $\mathbb{M}_1$. Let $\overline{u} = u'u''$ be the decomposition whose first term is the longest prefix before going below the horizontal axis. Let $\overline{w} = (v, 0^k)$ and $w' = (v', 0^{k'})$ be the decompositions according to the final sequences of 0. Then $\overline{w} * \overline{u}$ and $\overline{w'} * \overline{u}$ are $(v, u', 0^k, u'')$ and $(v', u', 0^{k'}, u'')$. If $k = k'$, then the statement is clear, as the cover move is inside v. Otherwise, the cover move must merge the last two blocks of w by sliding the last block. One can then write $w = (w_1, w_2, 0^i, w_3, 0^{k+1})$ and $w' = (w_1, w_2, w_3, 0^{i+k+1})$ for some $i \geq 1$, where w_1 may be empty and $(w_3, 0^{k+1})$ is the slided subpath. Then the right products $\overline{w} * \overline{u}$ and $\overline{w'} * \overline{u}$ are (when lifted back to Dyck paths) equal to $(w_1, w_2, 0^i, w_3, u', 0^k, u'', 0)$ and $(w_1, w_2, w_3, u', 0^{i+k}, u'', 0)$. Then the expected cover move is given by sliding $(w_3, u', 0^{k+1})$, using also the starting 0 of u'' or the final 0 if u'' is empty. □

4 A monoid on intervals

In this section, we will define and study another monoid $\mathbb{M}_2$, on the set of all intervals in the posets $\mathscr{D}_n$ for $n \geq 1$.

Let $[w_1, w'_1]$ in $\mathscr{D}_m$ and $[w_2, w'_2]$ in $\mathscr{D}_n$ be two such intervals.

Lemma 4.1. *In $\mathscr{D}_{m+n-1}$, one has $w_1 \sharp w_2 \leq w'_1 \sharp w'_2$.*

Proof. Let us pick any sequence of cover moves from w_1 to w'_1 and any sequence of cover moves from w_2 to w'_2. Using lemma 3.2 and the chosen cover moves from w_2 to w'_2, one gets a sequence of cover moves from $w_1 \sharp w_2$ to $w_1 \sharp w'_2$. Using then lemma 3.3 and the chosen cover moves from w_1 to w'_1, one gets a sequence of cover moves from $w_1 \sharp w'_2$ to $w'_1 \sharp w'_2$. □

Let us therefore define a product $\sharp$ on the set of all intervals (except the unique interval in $\mathscr{D}_0$) by the formula

$$[w_1, w'_1] \sharp [w_2, w'_2] = [w_1 \sharp w_2, w'_1 \sharp w'_2]. \tag{6}$$

This defines a monoid $\mathbb{M}_2$ on this set of intervals. This monoid is graded by the size minus 1. The unit is the unique interval in $\mathscr{D}_1$.

Proposition 4.2. *The monoid $\mathbb{M}_2$ is free with generators all intervals with maximal elements of the shape $(1, w, 1, 0, 0)$ plus the interval $[(1, 0, 1, 0), (1, 0, 1, 0)]$ in $\mathscr{D}_2$.*

Note that the generators of $\mathbb{M}_2$ are exactly the intervals whose maximal element is a generator of the monoid $\mathbb{M}_1$.

Proof. Take any interval $[w, w']$. Let $w' = w'_1 \sharp \dots \sharp w'_k$ be the unique expression of w' as a product of generators in the monoid $\mathbb{M}_1$. By lemma 3.1, the minimum w can be expressed as $w_1 \sharp \dots \sharp w_k$ where $w_i \leq w'_i$ for every i and the w_i need not be generators in $\mathbb{M}_1$. This implies that the interval $[w, w']$ can be written as the product

$$[w, w'] = [w_1, w'_1] \sharp \dots \sharp [w_k, w'_k].$$

Therefore the monoid $\mathbb{M}_2$ is indeed generated by the proposed generators.

Conversely, given a product of some generators, one can recover uniquely the generator factors by the same procedure of decomposition of the maximum in $\mathbb{M}_1$. Therefore the monoid $\mathbb{M}_2$ is free. □

The monoid $\mathbb{M}_2$ interacts nicely with the partial order.

Theorem 4.3. *The interval $I = [w_1, w'_1] \sharp [w_2, w'_2]$ is isomorphic as a poset to the cartesian product of the intervals $[w_1, w'_1]$ and $[w_2, w'_2]$.*

Proof. The isomorphism is just given by the product $\sharp$ of Dyck paths. By lemma 3.2 and lemma 3.3, every cover move in the cartesian product of intervals is a cover move in I.

Conversely, by lemma 3.1, every cover move in the interval I takes place inside one factor of the maximal element of I, hence either inside w'_1 or w'_2. □

The number of generators of the monoid $\mathbb{M}_2$ is a sequence starting with

$$3, 3, 11, 51, 267, 1507, 8955, 55251, 350827, \dots \tag{7}$$

which apparently has not already been studied.

4.1 Specific corollaries

Let us now state some interesting special cases of theorem 4.3.

First, there is a simple factorisation property, for which we give another direct proof.

Let $[u, v]$ be an interval in $\mathscr{D}_n$. Let $v_1, \dots, v_k$ be the unique decomposition of v into blocks. Because the covering moves in $\mathscr{D}_n$ can only increase the heights, at every point where v touches the horizontal line, the Dyck path u must also touch the horizontal line. So one can decompose u by cutting at these points. This defines the Dyck paths u_i for $i = 1, \dots, k$.

Proposition 4.4. *Every interval $[u, v]$ in $\mathscr{D}_n$ is isomorphic to the cartesian product of the intervals $[u_i, v_i]$ for $i = 1, \dots, k$.*

Proof. The same property used above to decompose u is true for all elements of the interval $[u, v]$. Cutting at the touch points of v defines a bijection, with inverse just given by concatenation, between elements of $[u, v]$ and elements of the product of the intervals $[u_i, v_i]$. This is clearly an isomorphism of posets. □

For $w \in \mathscr{D}_n$, let $I(w)$ be the interval $[w_{\min}, w]$. As a special case of proposition 4.4, every interval $I(w)$ is isomorphic to the product of the intervals $I(w_i)$ over the blocks w_i of w.

For $w \in \mathscr{D}_n$, let $J(w)$ be the interval $[w_{\min}, (1, w, 1, 0, 0)]$ in $\mathscr{D}_{n+2}$.

Keeping the same notations, one has another factorisation result, easily deduced from theorem 4.3.

Theorem 4.5. *Let w be a block-indecomposable Dyck path. The interval $I(w)$ is isomorphic to the product of the intervals $J(w_i)$, where the w_i are the Dyck paths in the level-decomposition of w.*

It follows that in this case the isomorphism type of the interval $I(w)$ only depends on the set of Dyck paths in the level-decomposition of w, and not on their order. Combining with Prop. 4.4, one obtains the following clean statement.

Theorem 4.6. *For any w, the isomorphism type of the interval $I(w)$ only depends on the union of the sets of Dyck paths in the level-decomposition of all blocks of w.*

4.2 Factorisation of principal upper ideals

Let us now study the principal upper ideals in $\mathscr{D}_n$. When looking at the posets $\mathscr{D}_n$ for small n, one can see that some of their principal upper ideals are isomorphic to (products of) principal upper ideals for some smaller Dyck paths. The main result of this section is a general description of that phenomenon.

For a Dyck path w, let $\mathrm{Up}(w)$ be the principal upper ideal generated by w, namely the set of all u such that $w \leq u$.

Let us say that a Dyck path w admits a *strip* if it can be written as $(u, 1, v, 1, 0, 0, 0^k)$ where v is any Dyck path. The strip itself is graphically the horizontal region of width 1 ranging from the letter 1 after u to the letter 0 before 0^k.

Lemma 4.7. *A Dyck path w does not admit a strip if and only if w is empty or w ends by $(1, 0)$.*

Proof. If w ends by $(1, 0)$, it clearly admits no strip.

Suppose that w does not end with $(1, 0)$, and consider the second 0 in the final sequence of 0. There is exactly one block indecomposable Dyck path x inside w that ends with this letter 0. Because the final sequence of 0 in x has length 2, x can be written as expected. □

Suppose now that w admits a strip, so that $w = (u, 1, v, 1, 0, 0, 0^k)$. Let $u' = (u, 1, 0, 0^k)$.

Proposition 4.8. *The principal upper ideal $\mathrm{Up}(w)$ is isomorphic as a poset to the product of $\mathrm{Up}(u')$ and $\mathrm{Up}(v)$.*

Proof. By the hypotheses on w, any sliding move from w either happens inside v, or somewhere before v in which case it can be matched with a sliding move in u'. Moreover the resulting w' has the same properties as w with modified u or v, so that the full principal upper ideal factorizes as expected. □

Conversely, one can realize in this way the product of any two principal upper ideals $\mathrm{Up}(u')$ and $\mathrm{Up}(v)$, assuming only that the first one is not empty.

As the simplest non-trivial example, $\mathrm{Up}((1, 0, 1, 1, 0, 1, 0, 1, 0, 0))$ is isomorphic to the product of $\mathrm{Up}((1, 0, 1, 0))$ by itself.

Let us define a *reduced interval* as an interval whose minimum is either empty or of the shape $(w, 1, 0)$.

From proposition 4.8, one can deduce a bijection between non-reduced intervals and pairs (non-empty interval, interval). A non-reduced interval is the same as an element in $\mathrm{Up}(w)$ for some non-empty w not of the shape $(w, 1, 0)$. Therefore one can use the factorization above to map uniquely this element to a pair of elements in two arbitrary upper ideals, which give two arbitrary intervals (the first one being non-empty).

At the level of generating series for intervals, one gets from this decomposition that

$$f_A = f_R + t(f_A - 1)f_A, \tag{8}$$

where f_A is the generating series for all intervals and f_R the generating series for reduced intervals. With an additional variable s accounting for the number of blocks in the minimum of the intervals, one gets the refined equation

$$f_A = f_R + t(f_A - 1)f_A|_{s=1}. \tag{9}$$

5 Properties of the core intervals

Let us study now the *core intervals*, namely intervals whose minimum has the shape $(v, 1, 0)$ (shape A) and whose maximum has the shape $(1, w, 1, 0, 0)$ (shape B), where v and w are Dyck paths. In this section, one assumes that $n \geq 2$.

Let E_n be the subset of elements of $\mathscr{D}_n$ that have either shape A or shape B.

Lemma 5.1. *The set E_n is a lower ideal in $\mathscr{D}_n$.*

Proof. An element of shape B can only cover elements of shape A or B. An element of shape A can only cover elements of shape A. □

It follows that the Hasse diagram of the induced partial order on E_n is just the restriction of the Hasse diagram of $\mathscr{D}_n$.

For every element $w \in \mathscr{D}_{n-2}$, let us define a chain $E(w)$ of cover moves in E_n. Start from $e_0(w) = (1, 0, w, 1, 0)$. Sliding the second block of $e_0(w)$ to height 1 defines $e_1(w)$, which has one block less then $e_0(w)$. Repeat the same operation and define successive $e_i(w)$ until reaching a block-indecomposable Dyck path $e_k(w)$, which has necessarily shape B. The number k is the number of blocks of w plus 1.

Suppose that w is a concatenation of blocks $w_1, \dots, w_{k-1}$. One can describe the chain $E(w)$ completely: the element $e_{i-1}(w)$ for $1 \leq i \leq k$ is

$$(1, w_1, w_2, \dots, w_{i-1}, 0, w_i, \dots, w_{k-1}, 1, 0)$$

and the last element $e_k(w)$ is just $(1, w, 1, 0, 0)$. One can therefore recover the first element of this chain from its last element.

Lemma 5.2. *The set E_n is the disjoint of the chains $E(w)$ for $w \in \mathscr{D}_{n-2}$.*

Proof. Consider the following map θ from E_n to E_n. If $w \in E_n$ has shape A, slide its second block to height 1 to get $\theta(w)$. Otherwise w can be written as $(1, w', 1, 0, 0)$, so one can define $\theta(w)$ to be $(1, 0, w', 1, 0)$. By the previous remarks, the chains $E(w)$ are nothing but the orbits of θ. □

Note also that the final step in every chain $E(w)$ is a cover move from shape A to shape B. Every such cover move is the last step of such a chain.

The poset E_n is not a disjoint union of total orders, but its structure is organised around this set of chains as follows.

Proposition 5.3. *Consider a cover relation $u_1 \to u_2$ in E_n where u_1 and u_2 are in two distinct chains $E(w_1)$ and $E(w_2)$. Then there exists a cover relation $w_1 \to w_2$ in $\mathscr{D}_{n-2}$.*

Proof. The cover relation $u_1 \to u_2$ must be between two elements of shape A or two elements of shape B, because cover moves from shape A to shape B are inside chains, as noted above. If both u_1 and u_2 have shape B, then the statement is clear. Assume therefore that both u_1 and u_2 have shape A.

In this situation, u_1 can be written as $(1, w_1', 0, w_1'', 1, 0)$, where $w_1 = w_1' w_1''$ is a concatenation of Dyck paths. The cover move $u_1 \to u_2$ can not be the sliding along the 0 between w_1' and w_1'', because this move is in the chain $E(w_1)$. It can therefore only happen inside w_1' or inside w_1''. This implies that u_2 is $(1, w_2', 0, w_2'', 1, 0)$, with one part unchanged and the other part changed by a cover move. Therefore $w_2 = w_2' w_2''$ and this implies that there is a cover move from w_1 to w_2. □

Proposition 5.4. *Let v be an non-empty Dyck path. The top element of the unique chain $E(w)$ containing $(v, 1, 0)$ is the unique minimal element among all Dyck paths $(1, w', 1, 0, 0)$ such that $(v, 1, 0) \leq (1, w', 1, 0, 0)$.*

Proof. Consider any sequence of cover moves from $(v, 1, 0)$ to some $(1, w', 1, 0, 0)$. Note that this takes place entirely inside E_n. The chosen sequence of cover moves is made either of cover moves inside one chain, or of cover moves between chains. By proposition 5.3, the first reached element of shape B must be the last element $(1, w'', 1, 0, 0)$ of some chain $E(w'')$ where $w \leq w''$. The remaining cover moves are between elements of shape B, and therefore $w \leq w'' \leq w'$. The statement follows. □

Let us now use all of this to give a precise description of the set of core intervals. By replacing the bottom element of such an interval $[u, (1, w', 1, 0, 0)]$ to the top element of its chain $E(w)$, one gets an interval between elements of shape B, or equivalently an interval $[w, w']$ in $\mathscr{D}_{n-2}$. The position of u in the chain $E(w)$ is described by an integer i between 0 and the number of blocks of w. This defines a map from the set of core intervals sending $[u, (1, w', 1, 0, 0)]$ to the pair $([w, w'], i)$. Conversely, given any interval $[w, w']$ in $\mathscr{D}_{n-2}$ and any integer i between 0 and the number of blocks of w, one can recover the full chain $E(w)$ and pick u by its index in this chain.

To summarize, there is a bijection between core intervals and pairs made of an arbitrary interval $[w, w']$ and an integer between 0 and the number of blocks of w.

6 Counting the intervals

In this section, we use the previous structural results on intervals to count them.

Theorem 6.1. *The number of intervals in the poset $\mathscr{D}_n$ is 1 for $n = 0$ and*

$$3\frac{2^{n-1}(2n)!}{n!(n+2)!} \quad \textit{for} \quad n \geq 1. \tag{10}$$

This formula describes exactly the sequence A000257, whose first few terms are

$$1, 1, 3, 12, 56, 288, 1584, 9152, \ldots$$

This is also the number of rooted bicubic planar maps on $2n$ vertices [Tut63], the number of rooted Eulerian planar maps with n edges, the number of modern intervals in the Tamari lattice on $\mathscr{D}_n$ and the number of new intervals in the Tamari lattice on $\mathscr{D}_{n+1}$ [Cha07]. For a simple bijection between these last two kinds of intervals, see [Rog18].

The proof uses a recursive description of all the intervals, based on the previous structural results. The good catalytic parameter turns out to be the number of blocks of the minimum of the interval.

Let A_n be the set of all intervals in $\mathscr{D}_n$ for $n \geq 0$. Let R_n be the subset of A_n made of intervals whose minimum has the shape $(w, 1, 0)$ (reduced intervals), plus the interval (empty, empty). Let C_n be the subset of A_n made of intervals whose minimum has the shape $(w, 1, 0)$ and whose maximum has the shape $(1, w, 1, 0, 0)$ (core intervals), plus the interval $(1, 0, 1, 0), (1, 0, 1, 0)$.

Let f_A, f_R, f_C be the associated generating series, with a variable t for the size and a variable s for the number of blocks in the minimum of the interval.

From the factorisation property of principal upper ideals in §4.2, one gets (9), that we repeat here:

$$f_A = f_R + t(f_A - 1)f_A|_{s=1}. \tag{11}$$

Applying the freeness of the monoid of intervals $\mathbb{M}_2$ to the subset of intervals whose minimum ends with $(1, 0)$ gives

$$\frac{f_R - 1}{st} = 1 + \frac{f_A - 1}{st}\frac{f_C}{st}, \tag{12}$$

because the last factor must be a core interval.

From the properties of core intervals obtained in §5, one gets

$$f_C = s^2t^2\left(1 + \frac{sf_A - f_A|_{s=1}}{s-1}\right). \tag{13}$$

All together, these three equations give the functional equation

$$f_A = 1 + st + st(f_A - 1)\left(1 + \frac{sf_A - f_A|_{s=1}}{s-1}\right) + t(f_A - 1)f_A|_{s=1}. \tag{14}$$

Using the general method of [BMJ06] to deduce an algebraic equation from this kind of functional equation with one catalytic parameter, one gets (as the unique pertinent factor) the equation

$$16g^2t^2 - g(8t^2 + 12t - 1) + t^2 + 11t - 1 \tag{15}$$

for the generating series $g = f_A|_{s=1}$, in which one recognizes the known algebraic equation for the sequence A000257. This implies theorem 6.1.

The first few terms of these series are

$$f_A = 1+st+\left(2s^2+s\right)t^2+\left(5s^3+5s^2+2s\right)t^3+\left(14s^4+21s^3+15s^2+6s\right)t^4+\cdots$$

$$f_R = 1+st+2s^2t^2+\left(5s^3+3s^2\right)t^3+\left(14s^4+16s^3+8s^2\right)t^4+\cdots$$

$$f_C = 2s^2t^2+\left(s^3+s^2\right)t^3+\left(2s^4+3s^3+3s^2\right)t^4+\cdots$$

6.1 Refinement of enumerative correspondence

It follows from theorem 6.1 that the number of intervals in $(\mathscr{D}_n, \leq)$ is the same as the number of modern intervals in the Tamari partial order on the same set. In this section, a conjectural refinement of this equality is proposed.

Let us define a map κ from Dyck paths of size n to planar rooted binary trees with n inner vertices, by induction on n. When $n = 0$, the empty Dyck path is sent to the binary tree that is just one leaf. If the Dyck path w is block-indecomposable, it can be written $w = (1, w', 0)$ for some smaller Dyck path w'. Then $\kappa(w)$ is obtained from $\kappa(w')$ by adding one inner vertex (and two leaves) on the rightmost leaf. Otherwise, cutting w before the last block defines two smaller Dyck paths w_1 and w_2 such that $w = w_1w_2$. Then $\kappa(w)$ is defined by grafting the root of $\kappa(w_1)$ on the second leaf from the right of $\kappa(w_2)$.

Proposition 6.2. *The map κ is a bijection.*

Proof. It is enough to be able to build the inverse by induction. Let us simply sketch the construction. Consider a planar binary tree t. Let v be the parent vertex of its rightmost leaf.

If v is directly connected to two leaves, then one can remove v to get another binary tree t', apply the inverse of κ by induction to get a Dyck path w' and define $w = (1, w', 0)$. Clearly $\kappa(w) = t$.

Otherwise, cut the tree t along the left branch of the vertex v. This gives two trees t_1 (above the cut) and t_2 (below the cut). Applying the inverse of κ by induction gives two Dyck paths w_1 and w_2. Define w to be their concatenation w_1w_2. Clearly again $\kappa(w) = t$. □

Proposition 6.3. *The bijection κ from Dyck paths to planar rooted binary trees is such that, when $t = \kappa(w)$, the number of vertices on the rightmost branch of t is the number of final zeros of w.*

Proof. By induction on n. This is obvious for $n = 0$. The two possible steps (for block-indecomposable w or otherwise) in the inductive definition of the bijection κ do preserve this property, as can be readily checked. □

Conjecture 6.4. *The bijection κ from Dyck paths to planar rooted binary trees is such that, when $t = \kappa(w)$, the number of modern intervals with minimum t in the Tamari lattice is the size of the dexter upper ideal with minimum w.*

7 Semilattice property

The aim of this section is to prove that $\mathscr{D}_n$ is a lower semilattice, namely any two elements have a meet.

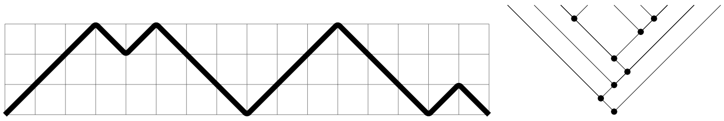


Figure 7: Illustration of the bijection κ

7.1 Some technical lemmas

This subsection proves several useful lemmas.

Lemma 7.1. *Let $u \in \mathscr{D}_n$, with a letter 0 at position $i+1$. Consider the set $R_i(u)$ of elements $v \in \mathscr{D}_n$ such that $u \leq v$, the first i letters of v are the first i letters of u and v has a letter 1 at position $i+1$.*

Either $R_i(u)$ is empty or $R_i(u)$ has a unique minimal element.

This will follow from lemma 7.2.

Keeping the same notations, let us denote by p the prefix of u before position $i+1$. Then u has a unique expression of the form

$$u = p\,0^{\ell}\,X_0 0^{k_0}\,X_1 0^{k_1}\,\ldots X_r 0^{k_r}\,X_{r+1}, \tag{16}$$

where for $i \leq r$ each X_i is a non-empty Dyck path, $\ell > 0$, all $k_i > 0$ and the final X_{r+1} is any Dyck path (possibly empty). This expression is obtained by starting after the prefix and repeatedly do the following : go down along the 0 until reaching a point before the letter 1 (or final), then move on the path until reaching the first point at the same height that is followed by a letter 0 (or final).

Each of the X_i is made of one or several blocks. Such a block is *movable* if it can be slided without changing the prefix, and *frozen* otherwise.

In each X_i for $i \leq r$, all the blocks but the last one are movable. In the last Dyck path X_{r+1}, all the blocks are movable.

If there is no movable block in u, the set $R_i(u)$ in the statement of lemma 7.1 will be empty. Indeed any cover move not changing the prefix will keep the shape of the expression (16) fixed, so that no block becomes movable and X_0 can never be slided.

Let us therefore from now on assume the contrary. In this case, let us denote by $\mathrm{Rise}_i(u)$ the element obtained from u by sliding the first subpath in the first X_j that contains a movable block, up to the height of the last block of the previous X_{j-1} or up to the end of the prefix of u if $j = 0$. In the resulting Dyck path, there is a movable block inside X_{j-1} if $j > 0$.

Lemma 7.2. *Let $u \in \mathscr{D}_n$, with a letter 0 at position $i+1$. Let $v \in \mathscr{D}_n$ such that $u \leq v$, the first i letters of v are the first i letters of u and v has a letter 1 at position $i+1$. Then $\mathrm{Rise}_i(u) \leq v$.*

Proof. The proof will be by decreasing induction on u.

Because $u \leq v$, v must be greater than at least one element that covers u and does not change the prefix. If $\mathrm{Rise}_i(u) \leq v$, the statement holds directly.

The other possible such cover moves can be classified into several types:

(1) strictly inside one of the blocks of X_k with $k \leq j$,

(2) one block of X_j is slided,

(3) somewhere on the right of X_j.

Let first assume that there is a cover move $u \leq u'$ of type (1) with $u' \leq v$. In u', one finds the same first movable block as in u. By induction, one also has $\mathrm{Rise}_i(u') \leq v$. There is a sequence of cover moves $u \leq u' \leq \mathrm{Rise}_i(u')$. But these cover moves commute, so that $\mathrm{Rise}_i(u) \leq \mathrm{Rise}_i(u')$.

Let us now assume that there is a cover move $u \leq u'$ of type (3) with $u' \leq v$. In u', one finds the same first movable block as in u. By induction, one also has $\mathrm{Rise}_i(u') \leq v$. There is a sequence of cover moves $u \leq u' \leq \mathrm{Rise}_i(u')$. But these cover moves also commute, so that again $\mathrm{Rise}_i(u) \leq \mathrm{Rise}_i(u')$.

There remains to assume that there is a cover move $u \leq u'$ of type (2) with $u' \leq v$. If the slided block is not the first block or the second block of X_j, then one can argue as in the preceding case, because the first movable block is not affected.

If the slided block is the second block of X_j, then X_j must contain at least 3 blocks, for otherwise the second block is not movable. Then one can check directly that there is a cover move $\mathrm{Rise}_i(u) \to \mathrm{Rise}_i(u')$.

If the slided block is the first block of X_j, it can either be slided too high or too low, compared to the exact height of the end of X_{j-1} or of the end of the prefix of u. In both cases, it becomes (part of) a frozen block. The existence of v then forces that there is another movable block in X_k for some $k \geq j$.

If the first block B of X_j is slided too low in u', one can apply repeatedly Rise_i to u' until this operator is sliding again the slided block B itself. Let us call u'' the result. Then by induction, one has $u'' \leq v$. One can also see that $\mathrm{Rise}_i(u) \leq u''$ by repeating on $\mathrm{Rise}_i(u)$ all the same slidings performed from u' to u''.

The first block B of X_j can only be slided too high in u' if there exists X_{j-1}, because otherwise the prefix would change. In this case, one can apply repeatedly Rise_i to u' until this operator is making movable the block containing the slided block B itself. Let us call u'' the result. Then by induction, one has $u'' \leq v$. On the other hand, one can apply an operator similar to Rise_i but using the second available moving block on $\mathrm{Rise}_i(u)$ until making the slided block B movable again. In other words, one replays on $\mathrm{Rise}_i(u)$ all the sliding moves performed on u'. Let us call the result $\bar{u}$. Then there is a cover move $\bar{u} \to u''$.

In all cases, one obtains that $\mathrm{Rise}_i(u) \leq v$. $\square$

Let us now give the proof of lemma 7.1. This is just an iteration of lemma 7.2, where the pair (u, v) is replaced by the pair $(\mathrm{Rise}_i(u), v)$ that satisfies the same hypotheses as long as $j > 1$. At each step the index j containing the first movable block is decreasing by one. In the last step, when $j = 1$, the element $\mathrm{Rise}_i(u)$ becomes an element of $R_i(u)$ which is smaller than any element of $R_i(u)$.

For v, w in $\mathscr{D}_n$, let $M(v, w)$ be the set of Dyck paths that are smaller than both v and w. This set is never empty.

Lemma 7.3. *Let v and w in $\mathscr{D}_n$ with a common prefix of length i. Then for every element u in $M(v, w)$, there exists u' in $M(v, w)$ such that $u \leq u'$ and u' has the same prefix of length i as v and w.*

Proof. By induction on i. This is true for $i = 1$, for $u' = u$. Assume now that v and w have a common prefix of length $i + 1$ and let u'_i be defined by induction hypothesis for the shorter common prefix of length i. If the last letter in the common prefix of length $i + 1$ is the same as the letter of u'_i at position $i + 1$, one can take u' to be u'_i.

Because $u'_i \in M(v, w)$, there remains only the case where the letter of u'_i at position $i+1$ is 0 and the last letter in the common prefix of length $i+1$ of v and w is 1. Let us apply lemma 7.1 to u'_i. The set $R_i(u'_i)$ is not empty as it contains v and w. One can therefore take u' to be its unique minimal element. □

7.2 Proof of semilattice property

Let us start with more lemmas.

Let w be a Dyck path such that the i-th step in w is 1 and this step does not start at height 0. On the right from the start i_0 of i-th step, move on the path w until meeting a point i_1 at the same height and followed by a 0 step. This must happen, as the height of i_0 is not zero. Between i_0 and i_1, there is in w a non-empty sequence of subpaths $x_1, \dots, x_N$.

Let us define another Dyck path $\mathrm{Desc}_i(w)$ by sliding down in w the subpath x_N as much as possible, namely by exchanging x_N with all the consecutive 0 steps on its right. Then there is a covering move $\mathrm{Desc}_i(w) \to w$ that is sliding up the subpath x_N.

Lemma 7.4. *Let w be a Dyck path such that the i-th step in w is 1 and this step does not start at height 0. Let $u \leq w$ such that the i-th letter of u is 0 and v and w share the same prefix before the i-th letter. Then u is smaller than $\mathrm{Desc}_i(w)$.*

Proof. The proof will be by induction on increasing w.

Necessarily, u is smaller than at least one of the elements covered by w. If $u \leq \mathrm{Desc}_i(w)$, the statement holds.

Because of the shared prefix, the other possible down-sliding moves from w are of three types:

(1) strictly inside one of the x_i,

(2) splitting some x_i,

(3) somewhere on the right of x_N.

Assume first that $u \leq w'$ where $w' \to w$ is of type (3). By induction, $u \leq \mathrm{Desc}_i(w')$. There is a chain of cover moves $\mathrm{Desc}_i(w') \to w' \to w$. These two cover moves commute if the slided subpath in the move $w' \to w$ is not the first subpath after x_N. Otherwise, one can find a chain of two cover moves from $\mathrm{Desc}_i(w')$ to $\mathrm{Desc}_i(w)$. In all cases, $\mathrm{Desc}_i(w') \leq \mathrm{Desc}_i(w)$.

Assume now that $u \leq w'$ where $w' \to w$ is of type (1). By induction, $u \leq \mathrm{Desc}_i(w')$. There is a chain of cover moves $\mathrm{Desc}_i(w') \to w' \to w$. These cover moves commute, and therefore $\mathrm{Desc}_i(w') \leq \mathrm{Desc}_i(w)$.

Assume then that $u \leq w'$ where $w' \to w$ is of type (2). By induction, $u \leq \mathrm{Desc}_i(w')$. There is a chain of cover moves $\mathrm{Desc}_i(w') \to w' \to w$. If $i < N - 1$, these cover moves commute, and therefore $\mathrm{Desc}_i(w') \leq \mathrm{Desc}_i(w)$.

If $i = N$, then one can apply twice the induction step to obtain that $u \leq \mathrm{Desc}_i(\mathrm{Desc}_i(w'))$. But $\mathrm{Desc}_i(\mathrm{Desc}_i(w')) \leq \mathrm{Desc}_i(w)$ because one can split x_N after it has been slided down.

If $i = N - 1$, one can also apply twice the induction step, to get that $u \leq \mathrm{Desc}_i(\mathrm{Desc}_i(w'))$. One then checks that $\mathrm{Desc}_i(\mathrm{Desc}_i(w')) \leq \mathrm{Desc}_i(w)$ holds also in this case, by just one cover move.

One therefore deduces the statement in all cases. □

Keeping the same notations, let $s_i(w)$ denote $\mathrm{Desc}_i^N(w)$, the image of w under the N-times iteration of the application Desc_i.

Lemma 7.5. *Let w be a Dyck path such that the i-th step in w is 1 and this step does not start at height 0. Let $S_i(w)$ be the set of all Dyck paths u such that $u \leq w$, the i-th letter of u is 0 and u and w share the same prefix before the i-th letter. Then $s_i(w)$ is the unique maximal element of $S_i(w)$.*

Proof. First note that $s_i(w)$ is indeed in $S_i(w)$.

Let u be an element of $S_i(w)$. One can apply lemma 7.4 to the pair (u, w) to get another pair (u, w') where $w' = \mathrm{Desc}_i(w)$. Either $w' = s_i(w)$, or this new pair satisfies again the hypotheses of lemma 7.4. One can therefore repeat this exactly N times, until reaching the pair $(u, s_i(w))$. In particular, $u \leq s_i(w)$. □

Theorem 7.6. *The poset $(\mathscr{D}_n, \leq)$ is a meet-semilattice.*

Proof. Let v and w be two Dyck paths, and let us look for their meet. One can assume that v and w are not equal.

Start from the left, until meeting a difference between v and w. Let i be the last common point. One can assume that w is above v just after the point i, by exchanging v and w if necessary. Note that the height of i cannot be zero.

One can therefore apply lemma 7.5 to w for the step after position i, and obtain an element w' which is maximal among all elements smaller than w that share the same prefix followed by the letter 0.

This gives a new pair of elements (v, w') with $w' \leq w$. Let us prove that $M(v, w) = M(v, w')$. The inclusion $M(v, w') \subseteq M(v, w)$ is clear because $w' \leq w$.

Conversely, let u be an element of $M(v, w)$. Using lemma 7.3, one can find u' in $M(v, w)$ with $u \leq u'$ and u' share the common prefix of v and w. Note that the first letter after the common prefix in u' is 0, because $u' \leq v$. It therefore follows from the definition of w' that $u' \leq w'$.

Hence $M(v, w) = M(v, w')$, and the common prefix of v and w' is strictly longer. Therefore iterating this whole procedure on pairs of Dyck paths ends at a common Dyck path z, that is smaller than v and w and such that $M(z, z) = M(v, w)$. This z is therefore the meet of v and w. □

8 Derived equivalences of intervals

We now turn briefly to a more subtle equivalence between intervals, namely *derived equivalence*, which is defined as follows. One can consider any finite poset P as a small category, with a unique morphism $x \to y$ if and only if $x \leq y$. Then the category of modules over P with coefficients in some base field K can be defined as the category of functors from P to finite-dimensional vector spaces over K. This is an abelian category, with enough projectives and injectives, and finite global dimension. One can therefore associate to P the (bounded) derived category $\mathsf{D}_K(P)$ of this category of modules.

Two posets P and Q are said to be *derived equivalent* (over K) if there is a triangle-equivalence between $\mathsf{D}_K(P)$ and $\mathsf{D}_K(Q)$.

In this section, we conjecture the existence of derived equivalences between some particular kinds of intervals in $\mathscr{D}_n$.

Recall from section 4.1 that the intervals $J(w) = I((1, w, 1, 0, 0))$ are the factors in the cartesian factorisation of the intervals $I(w)$.

Conjecture 8.1. *For any w, the derived isomorphism type of the interval $J(w)$ only depends on the union of the sets of Dyck paths in the level-decomposition of all blocks of w.*

This conjecture is based on experimental evidence, namely the coincidence of some invariants of posets (Coxeter polynomials) which only depend on the derived categories.

Note the striking similarity of this conjecture with theorem 4.6. This conjecture may even be a characterisation of derived equivalence classes.

As a special case, if two words w and w' are related by a permutation of their blocks, then $J(w)$ and $J(w')$ should be derived-equivalent. As the simplest possible non-trivial example, let us consider the posets $J(1, 0, 1, 1, 0, 0)$ and $J(1, 1, 0, 0, 1, 0)$. Both have 9 elements and share the same Coxeter polynomial $\Phi_1^2\Phi_2\Phi_3\Phi_5$, where the Φ_d are the cyclotomic polynomials. These posets are in fact related by a flip-flop in the sense of Ladkani [Lad07] (mapping two elements near the top of $J(1, 1, 0, 0, 1, 0)$ to two elements near the bottom of $J(1, 0, 1, 1, 0, 0)$), and therefore derived-equivalent.

As another special case, if two block-indecomposable words w and w' are related by a permutation of their level-decomposition, then $J(w)$ and $J(w')$ should be derived-equivalent. As the simplest possible non-trivial example, let us consider the posets $J(1, 1, 0, 1, 1, 0, 0, 0)$ and $J(1, 1, 1, 0, 1, 0, 0, 0)$. Both have 27 elements and share the same Coxeter polynomial $\Phi_2\Phi_4\Phi_{18}\Phi_{54}$. It is not clear if these intervals are derived equivalent.

To illustrate the general case in the simplest possible way, consider the posets $J(1, 0, 1, 1, 1, 0, 0, 0)$ and $J(1, 1, 0, 0, 1, 1, 0, 0)$. Both have 20 elements and share the same Coxeter polynomial $\Phi_1^2\Phi_2^2\Phi_3\Phi_5\Phi_6^2\Phi_7$. In this case, there is no obvious flip-flop to prove the expected derived equivalence.

8.1 About the notion of f-vector

Beware that this section is very speculative.

The classical notion of f-vector is attached to cellular or simplicial complexes, where it records the number of cells of every dimension.

In some famillies of posets, including the one studied here, but also the Tamari lattices, the cambrian lattices and many of their relatives, the pictures of the Hasse diagram of intervals, when visually inspected by the human eye, strongly suggest the existence of a cellular complex whose skeleton would be the Hasse diagram.

Although we will not try to give and justify a precise definition here, there is one way to find the f-vector of this putative cell-complex, using only the partial order. Namely, every cell can be identified with its minimal and maximal elements. These two elements must form something like a minimal spherical interval in the poset.

There is a general phenomenon, observed in several families of posets, that derived equivalence of posets often come together with an equality between f-vectors. One important instance is the conjectured derived equivalence between cambrian lattices and lattices of order ideals in root posets.

This phenomenon seems also to be present in the intervals of the dexter lattices, at least in the intervals $J(w)$. One expects that derived equivalent $J(w)$ will share the same f-vector, but not conversely.

9 The Hochschild polytope as an interval

In this section, we explain an unexpected connection between a specific interval in $\mathscr{D}_{n+2}$ and a cell complex called the Hochschild polytope, introduced in algebraic topology by Saneblidze [San09, San11]. Our initial reason for looking at this particular interval was its appearance inside the interval of largest cardinality among all $I(w)$.

For $n \geq 1$, let F_n be the interval in $\mathscr{D}_{n+2}$ between $(1, 1, 0, 0, (1, 0)^n)$ and $(1, 1^n, 0^n, 1, 0, 0)$. This is indeed an interval, as one can move from the former to the latter by sliding (in their order from left to right) all the initial blocks $(1, 0)$ (all of them are slided to the maximal possible height, except the last one that is slided to height 1).

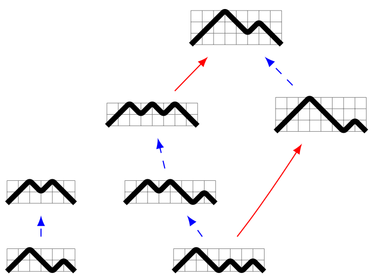


Figure 8: The Hochschild intervals F_1 and F_2.

Let us define a *valley* in a Dyck path to be a subword $(0, 1)$, which means a local minimum for the height function. Similarly, a *peak* is a local maximum of height.

Note that all elements of F_n start with $(1, 1)$. Moreover, all the valleys in all elements of F_n have height 0 or 1. This follows from the next lemma, as this is true for the maximum of F_n.

Lemma 9.1. *Let w be a Dyck path having only valleys at height 0 or 1. If $y \leq w$, then y has the same property.*

Proof. It is enough to prove this when y is covered by w.

If w has a valley at height 0, then y has the same valley. One can therefore assume that w is block-indecomposable, and has only valleys at height 1. Then one can check that all possible down cover moves from w can only create a valley at level 0 or 1. ☐

Lemma 9.2. *Let $w \in F_n$. Then the height of the valleys in w is weakly decreasing from left to right.*

Proof. Otherwise, there is a valley of height 0 followed by a valley of height 1. The subpath after the valley of height 0 must be slided at some point in any chain of cover moves from w to the maximum of F_n, but this would create a valley of height at least 2. This is absurd. $\square$

Lemma 9.3. *For $n \geq 1$, every element of F_n ends either with $(0, 1, 0)$ or with $(0, 1, 0, 0)$.*

Proof. The only other possibility is to end with $(1, 1, 0, 0)$, because of the shape of the maximum of F_n. But then this final subpath can not be slided with the result being still below the maximum of F_n. This is absurd. $\square$

Lemma 9.4. *The set F_n can be described as the set of Dyck paths starting with $(1, 1)$, having only valleys of height 0 or 1, where these heights are decreasing from left to right, and ending either by $(0, 1, 0)$ or by $(0, 1, 0, 0)$.*

Proof. Let us call Q_n this set of elements. By the preceding lemmas and remarks, Q_n contains F_n. For the converse inclusion, one only needs to check the two following statements: (1) an element of Q_n which is not the maximum of F_n can be covered by another element of Q_n and (2) an element of Q_n which is not the minimum of F_n covers another element of Q_n.

(1) Let z be an element of Q_n. If z has exactly two peaks, then z is either the maximum of F_n or is covered by this maximum. One can therefore assume that there are at least three peaks in z, hence at least two valleys. If there is a valley of height 0, one can slide up by one step the subpath after the first valley of height 0, which becomes a valley of height 1. The result is still in Q_n. If all valleys have height 1, one can merge the first two peaks.

(2) Let z be an element of Q_n. If it ends with $(0, 1, 0, 0)$, one can slide down this last peak and get another element of Q_n. Otherwise, z ends with $(0, 1, 0)$. If it has a valley at height 1, one can slide down the subpath after the rightmost valley of height 1. Otherwise one can take any peak of height at least 2 and cut it into 2 smaller peaks, except when the only available peak of height at least 2 is at the beginning of z. But then z is the minimum of F_n. $\square$

Remark 9.5. *Note that being block-indecomposable is equivalent inside F_n to having no valley at height 0. Note also that for $n \geq 1$ every element of F_n has at least two peaks, because the unique path with just one peak is not in F_n.*

Let us define $F_{n,b}$ as the subset of block-indecomposable elements of F_n.

Lemma 9.6. *For $n \geq 1$, the subset $F_{n,b}$ form a boolean lattice of cardinality 2^{n-1} with minimum $w_{n,b} = (1, 1, (0, 1)^n, 0, 0)$ and the same maximum as F_n.*

Proof. Because all valleys must have height 1, the shape of possible Dyck paths is strongly constrained by lemma 9.4. The rightmost peak must have height 2, and this forces at least one valley just before the final $(1, 0, 0)$. Every subset of the valleys of $w_{n,b}$ that contains this rightmost one defines a unique element in the interval. The induced partial order is given by inclusion of subsets of valleys. $\square$

Remark 9.7. *This boolean lattice on $F_{n,b}$ can be written as the disjoint union of two boolean lattices of half cardinality: elements ending with $(1,0,1,0,0)$ (bottom part) and the others (top part).*

Lemma 9.8. *The subset of F_n of elements having only valleys of height 0 is a boolean lattice of cardinality 2^{n-1}.*

Proof. The proof is very similar to the proof of lemma 9.6. □

Let us now define a map ρ from F_n to some set of words of length n in the alphabet $\{0,1,2\}$. Let w be a Dyck path in F_n. One reads the word $w \in F_n$ from left to right, while keeping track of an integer N_2 (initially set to 0) and some prefix of the image $\rho(w)$ (initially the empty word). When two consecutive 1 are read in w (except the first two letters of w), the integer N_2 is increased by 1. When a valley of w is read (with height h being either 0 or 1 by lemma 9.1), the word $[h, 2^{N_2}]$ (where the power means that the letter 2 is repeated) is appended to the current prefix, and N_2 is then set back to 0. The result $\rho(w)$ is the prefix obtained after reading all of w. The length of $\rho(w)$ is n because every letter 1 in w (except the two initial ones) contributes a letter in $\rho(w)$.

For example, the image by ρ of $(1,1,1,0,0,1,0,0,1,0) \in F_3$ is $[1,2,0]$. The minimal element of F_n is mapped by ρ to the word $[0,\dots,0]$, and the maximal element to the word $[1,2,\dots,2]$. The image of the Dyck path $w_{n,b} = (1,1,(0,1)^n,0,0)$ is $[1,\dots,1]$.

By the map $w \mapsto \rho(w)$, the number of valleys at height 0 (resp 1) of w becomes the number of letters 0 (resp. 1) in $\rho(w)$.

The construction of ρ can be reversed as follows, proving that it is injective. Starting from any element z in the image $\rho(F_n)$:

(1) split z as a sequence of bricks $[1,2,\dots,2]$ and $[0,2,\dots,2]$, formed by a letter 0 or 1 and the maximal sequence of following 2.

(2) start a new Dyck path at height 0. For each brick, use the number N_2 of 2 in this brick to move up to the appropriate height (one more for the initial brick), then move down to height 0 or 1 according to the first letter of the brick.

(3) when the list of bricks is exhausted, move up by one step and go down to height 0.

For example, one can find in this way that the pre-image of $[1,0,2]$ by ρ, which has two bricks $[1]$ and $[0,2]$, is $(1,1,0,1,1,0,0,0,1,0)$.

Lemma 9.9. *For a cover move $w \to w'$ in F_n, the words $\rho(w)$ and $\rho(w')$ differ by exactly one letter, which increases.*

Proof. One has to distinguish two kinds of cover moves. If the number of peaks is unchanged, then one valley of height 0 becomes a valley of height 1. On the image by ρ, only the letter encoding this height is modified. Otherwise, the number of peaks is decreased by one and the two associated bricks in the image by ρ become just one brick. The valleys corresponding to these two bricks have the same height. In the image by ρ, this means that one subword $[x, 2^M, x, 2^N]$ is replaced by $[x, 2^{M+1+N}]$, where x is either 0 or 1. □

Let us define $F_{n,0}$ and $F_{n,1}$ as the partition of F_n according to the height of the first valley. This corresponds to the decomposition of $\rho(F_n)$ according to the first letter.

Lemma 9.10. *For $n \geq 1$, the set $F_{n,1}$ is the interval between the element $w_{n,1} = (1,1,0,1,0,0,(1,0)^{n-1})$ and the maximum of F_n.*

Proof. The property (♣) of having the first valley at height 1 is preserved by cover moves inside F_n, that can only delete a valley of height 1 if it is followed by another valley of height 1.

Therefore all the elements that are greater than $w_{n,1}$ have property (♣).

Conversely, for any element $w \neq w_{n,1}$ with property (♣), one can find a smaller element w' with property (♣). Either one can easily find such an element w' having the shape $(1,(1,0)^k,0,(1,0)^\ell)$ (by lowering the highest peaks) or w itself has this shape. In the latter case, one can take $w' = w_{n,1}$.

It follows that $w_{n,1}$ is the unique minimal element with property (♣). □

Let us also denote $F_{n,1,0}$, $F_{n,1,1}$ and $F_{n,1,2}$ for the partition of $F_{n,1}$ according to the last letter of the image by ρ.

From lemma 9.9 and lemma 9.10, one deduces:

Lemma 9.11. *The set $F_{n,1,2}$ is an upper ideal in F_n.*

Lemma 9.12. *For $n \geq 1$, the map that inserts $(1,0)$ at the top of the next-to-rightmost peak defines a bijection from $F_{n,1}$ to $F_{n+1,1,2}$. This corresponds to adding 2 at the end of $\rho(w)$.*

Proof. This is easily checked directly for $n = 1$. Let us assume that $n \geq 2$.

First, one deduces from lemma 9.4 that adding $(1,0)$ at the top of the next-to-rightmost peak defines a map from F_n to F_{n+1}. This clearly preserves the height of the first valley, hence defines an injective map from $F_{n,1}$ to $F_{n+1,1}$.

Using the definition of ρ, this application does add 2 at the end of the image by ρ, because it increases the height of the next-to-last peak by one. Therefore its image is contained in $F_{n+1,1,2}$.

Conversely for any element z of $F_{n+1,1,2}$, one can remove $(1,0)$ on the top of the next-to-rightmost peak to define a Dyck path x. This does not change the height of the valleys, in particular the first valley of x has height 1. One just needs to prove that x is in F_n. Using lemma 9.4, one needs only to check the conditions at the beginning and at the end. The condition that x starts by $(1,1)$ can fail if and only if the next-to-rightmost peak of z is its first peak and has height 2. This only happen when z is the maximal element of F_1, but z belongs to F_{n+1} for some $n \geq 1$. The condition at the end is ensured by the final letter 2 in $\rho(z)$, which implies that removing $(1,0)$ on its top does not delete the next-to-rightmost peak. □

Let us define a map μ on F_n by adding $(1,0)$ at the end.

Lemma 9.13. *The image of μ is contained in F_{n+1}.*

Proof. This follows easily from lemma 9.4. □

Using the definition of ρ, one can check that adding $(1,0)$ at the end of any w in F_n corresponds to adding 0 at the end of $\rho(w)$.

Lemma 9.14. *For $n \geq 1$, the map μ is a bijection from $F_{n,1}$ to $F_{n+1,1,0}$.*

Proof. Applying μ on an element of $F_{n,1}$ gives an element of $F_{n+1,1,0}$. This is clearly an injective map.

Conversely, any element w of $F_{n,1,0}$ must end with $(1, 0)$, because its last valley has height 0.

Using lemma 9.4, one can show that cutting this final $(1, 0)$ gives an element of $F_{n,1}$ whose image by μ is w. The only required check is the condition at the end, which follows from the hypothesis that the last letter of $\rho(w)$ is 0. This proves the surjectivity. □

Lemma 9.15. *The map that inserts* $(1, 0)$ *just before the final letter* 0 *defines a bijection from* $F_{n,b}$ *to* $F_{n+1,1,1}$. *This corresponds to adding* 1 *at the end of* $\rho(w)$.

Proof. By lemma 9.6 and the remark following it, there are two inclusions of the set $F_{n,b}$ in $F_{n+1,b}$. One can check that the bottom inclusion is given by inserting $(1, 0)$ just before the final letter 0. Through the application of ρ, this amounts to adding a final 1 to $\rho(w)$. The image is therefore in $F_{n+1,1,1}$. This is clearly an injective map.

Conversely, let w in $F_{n+1,1,1}$. Because the last letter in $\rho(w)$ is 1, and using lemma 9.2, all valleys of w have height 1. Moreover the Dyck path w must end with $(1, 0, 0)$, because it is smaller than the maximum of F_n. But it must in fact end with $(0, 1, 0, 1, 0, 0)$, for otherwise the final letter of $\rho(w)$ would be 2. Hence one can remove $(1, 0)$ just before the final 0, and get an element of $F_{n,b}$, whose image is w. □

Lemma 9.16. *The map that slides down the subpath after the first valley defines a bijection from the subset of* $F_{n,1}$ *where only the first valley has height* 1 *to* $F_{n,0}$. *It amounts to replacing the first letter of* $\rho(w)$ *by* 0.

Proof. Let us consider an element w of $F_{n,1}$ with one valley of height 1 and all other valleys have height 0. There is a unique element w' in $F_{n+1,0}$ that is covered by w, which is obtained by sliding down the subpath after the first valley in w. One can check that $\rho(w')$ is obtained by replacing the first letter of $\rho(w)$ by 0.

Conversely, let w' in $F_{n,0}$. Then w' has only valleys of height zero. Let w be obtained by sliding up the subpath after the first valley of w', by just one step in order to create a valley of height 1. Then one can check that w' is still in F_n using lemma 9.4, because the height of the rightmost peak is always at most 2.

Moreover w has exactly one valley of height 1. These two constructions are clearly inverses of each other. □

The Hasse diagram of the interval F_n looks like the graph of vertices and edges of some polytope. It turns out to be related to a familly of cell complexes due to Saneblidze. Namely, one can identify its image by ρ as the set of vertices defined by Saneblidze in [San09].

Theorem 9.17. *The interval* F_n *is mapped by* ρ *to the set of coordinates of the Hochschild polytope of Saneblidze.*

Before entering the proof, let us start by giving a recursive description of these sets Z_n inside $\{0,1,2\}^n$, extracted carefully from this reference and reformulated in simpler terms, as the disjoint union of subsets $Z_n = Z_{n,0} \sqcup Z_{n,1}$. The recursive description also involves a subset $Z_{n,b} \subseteq Z_{n,1}$.

For $n = 1$, this is given by $Z_{n,0} = \{[0]\}$ and $Z_{n,1} = Z_{n,b} = \{[1]\}$.

For $n \geq 1$, the description is given by:

(i) $Z_{n+1,1}$ is made of all elements $[z, 0]$ and $[z, 2]$ for $z \in Z_{n,1}$ and all elements $[z, 1]$ for $z \in Z_{n,b}$,

(ii) $Z_{n+1,b}$ is made of all elements $[z, 1]$ and $[z, 2]$ for $z \in Z_{n,b}$.

(iii) The subset $Z_{n+1,0}$ is made by replacing the initial letter by 0 in all elements of $Z_{n+1,1}$ in which the letter 1 only appears as the first letter.

By induction, all elements of $Z_{n,0}$ (resp. $Z_{n,1}$ and $Z_{n,b}$) start with 0 (resp. 1). Note also that the elements of $Z_{n,b}$ only contains the letters 1 and 2.

Proposition 9.18. *The image of F_n by ρ is equal to Z_n. Moreover $F_{n,b}$ is mapped to $Z_{n,b}$, $F_{n,0}$ to $Z_{n,0}$ and $F_{n,1}$ to $Z_{n,1}$.*

Proof. By induction on $n \geq 1$. The statement holds by inspection for $n = 1$.

Let us assume that the statement holds up to $n-1$. We then need to perform a decomposition of F_n that is parallel to the recursive definition of Z_n.

First, F_n is the disjoint union of $F_{n,0}$ and $F_{n,1}$. Similarly, Z_n is the disjoint union of $Z_{n,0}$ and $Z_{n,1}$, according to the first letter. It is therefore enough to work separately on each part, starting with $F_{n,1}$.

Using the decomposition of $F_{n,1}$ into three parts according to the last letter of the image by ρ, and the bijections stated in lemma 9.14, lemma 9.12 and lemma 9.15, one obtains that $\rho(F_{n,1})$ has the defining property (i) of $Z_{n,1}$.

In order to check that $\rho(F_{n,b})$ has the defining property (ii) of $Z_{n,b}$, one can use lemma 9.6 and the remark following it to check explicitly that elements of $\rho(F_{n+1,b})$ are obtained by adding either 1 or 2 at the end of elements of $\rho(F_{n,b})$.

There remains to check that $\rho(F_{n,0})$ has the defining property (iii) of $Z_{n,0}$. This is exactly provided by lemma 9.16. □

Words on the alphabet $\{0,1,2\}$ in Z_n are considered as vectors in $\mathbb{Z}^n$. The partial order on Z_n introduced by Saneblidze is termwise-comparison: two words z and z' are comparable if and only if $z_i \leq z'_i$ for all $1 \leq i \leq n$.

One certainly expects that the map ρ should give an isomorphism from the partial order on F_n to this partial order on Z_n. One direction of the proof is just lemma 9.9. Completing the proof would require a precise study of the covering relations in both partial orders. Because this does not seem to be central enough in the present article, this is left for another work.

As a corollary of what precedes, one can count the elements of F_n as follows. From lemma 9.8, lemma 9.6, lemma 9.14, lemma 9.12 and lemma 9.15, one deduces the following relations

$$\begin{aligned}
\#F_{n,0} &= 2^{n-1}, \\
\#F_{n,b} &= 2^{n-1}, \\
\#F_{n,1} &= 2\#F_{n-1,1} + \#F_{n-1,b}.
\end{aligned}$$

which imply the following statement.

Proposition 9.19. *The number of elements in F_n is the sequence A045623:*

$$2^{n-2}(n+3) = 2, 5, 12, 28, 64, 144, \dots \tag{17}$$

for $n \geq 1$.

Remark 9.20. *Computing the Coxeter polynomials of the first few lattices in this familly, one observes that they have all their roots on the unit circle. This have been checked by computer up to the lattice with 3328 elements. For example, for the poset F_5 of size 64, the result is $\Phi_1^2\Phi_2^4\Phi_6^4\Phi_7\Phi_{23}^2$. One can see some very regular patterns in these Coxeter polynomials, when expressed as products of $[d]_x = (x^d - 1)/(x - 1)$ factors. This is a little further evidence that this roots-on-the-circle phenomenon could go on for larger cases.*

10 Miscellany

10.1 A symmetry of colored h-polynomials

One can color the edges of the Hasse diagram of $\mathscr{D}_n$ with two colors as follows. When an edge corresponds to sliding a subpath to its highest possible position, this edge is colored red. Other edges are colored blue. For example, see fig. 5 where blue edges are also dashed.

As we will see, this coloring is interesting because the generating polynomial of incoming edges according to their colors has an unexpected symmetry.

As a warm-up, let us start with the simpler generating series of incoming edges, not taking colors into account:

$$A = \sum_{n \geq 0} \sum_{w \in \mathscr{D}_n} x^{C(w)} t^n, \tag{18}$$

where $C(w)$ is the number of elements covered by w.

Using the unique decomposition of Dyck paths into a list of blocks and proposition 4.4, one gets

$$A = \frac{1}{1 - B}, \tag{19}$$

where B is the similar sum restricted to block-indecomposable Dyck paths.

Then using the level-decomposition of block-indecomposables, one gets

$$B = \frac{t}{1 - xtA}. \tag{20}$$

Indeed, the elements covered by $w = \mathscr{L}(w_1, \dots, w_k)$ either come from replacing w_i by some element that it covers, or from sliding down a subpath of w that ends somewhere in the final sequence of letters 0 in w. There are exactly k such additional covered elements.

Together (19) and (20) imply an equation for A which is exactly the well-known equation for the generating series of Narayana polynomials. Note that the Narayana polynomials are the h-vectors of the associahedra, namely the results of the same counting of incoming edges for the Hasse diagram of the Tamari lattices. This hints at a possible cellular structure of the Hasse diagram of $\mathscr{D}_n$, with the same f-vector as the associahedron. This is left for a future study.

Let us now introduce a refined colored version of A:

$$A = \sum_{n\geq 0} \sum_{w \in \mathscr{D}_n} r^{C_r(w)} b^{C_b(w)} t^n, \tag{21}$$

and the associated series B restricted to block-indecomposables. Here $C_r(w)$ and $C_b(w)$ count the red and blue incoming edges at w.

The first equation (19) holds unchanged, whereas (20) must be slightly modified into

$$B = \frac{t}{1 - t(r + b(A-1))}. \tag{22}$$

Indeed, the k elements covered by $w = \mathscr{L}(w_1, \ldots, w_k)$ that do not come from an element covered by some w_i can be either red or blue. They are red exactly when w_i is the empty Dyck path.

By elimination of B, one finds that A satisfies the quadratic equation

$$A^2tb + Atr - 2Atb + At - tr + tb - A + 1 = 0, \tag{23}$$

from which one can deduce the global symmetry

$$A - 1 = (A(1/r, b/r^2, rt) - 1)/r. \tag{24}$$

This symmetry property can be stated as a simple symmetry of the coefficient A_n of t^n in A:

$$\forall n \geq 1 \quad A_n(r, rb) = r^{n-1} A_n(1/r, b/r). \tag{25}$$

The real meaning of this last symmetry is not clear for the moment. It extends the usual symmetry of Narayana numbers.

10.2 m-analogues

One can easily define the same kind of variation for the m-Tamari lattices [BPR12, BMFPR11] instead of the Tamari lattices, using their similar description by sliding subpaths in Dyck paths of slope m. These posets do not seem to be very interesting, at least because their numbers of intervals involve large prime numbers. For example, the first few numbers of intervals for $m = 2$ are given by

$$1, 1, 5, 36, 311, 3001, 31203. \tag{26}$$

10.3 Zeta polynomials and chains

Let us now give a few simple experimental observations related to chains and zeta polynomials in $\mathscr{D}_n$. We have not tried to prove them.

The length of the longest chain in $\mathscr{D}_n$ seems to be A33638, realized between $w_{\min}$ and Dyck paths that ends with as many final repetitions of $(1, 0, 0)$ as possible.

The first few values at -1 of the zeta polynomials of $\mathscr{D}_n$ for $n \geq 1$ are

$$1, -1, 2, -5, 14, -42, 132, -429. \tag{27}$$

One could guess that these should be (up to sign) the Catalan numbers.

The first few values at -2 of the zeta polynomials of $\mathscr{D}_n$ for $n \geq 1$ are

$$1, -2, 7, -29, 131, -625, 3099, -15818. \tag{28}$$

This coincides (up to sign) with the beginning of sequence A007852 that is counting antichains in rooted plane trees on n nodes.

11 About the Tamari lattices

Let us recall that the Tamari lattice [FT67] of size n can be defined on the set of Dyck paths of size n as the partial order induced by transitive closure of some cover relations, namely the exchange in any Dyck path of a letter 0 (assumed to be followed by 1) with the subpath following it. This cover move is equivalent to sliding this subpath by one step in the north-west direction. This description is related in [BB09, §2] to the more classical description using rotation on binary trees.

For a Dyck path $w \in \mathscr{D}_n$, let the *height sequence* of w be $(h_1, h_2, \ldots, h_n)$ where h_i is the height in w just after the i^{th} letter 1. Note that repeated sliding to the north-west of a given subpath x always increase the same subsequence of the height sequence.

Lemma 11.1. *Let w be a Dyck path. Let w' be obtained from w by sliding once or several times the same subpath x in the north-west direction. Then the interval $[w, w']$ in the Tamari lattice is a chain, in other words a total order, and all elements of $[w, w']$ are obtained from w by sliding the subpath x.*

Proof. Only two kinds of Tamari cover moves can happen: either the subpath x itself is slided, or another subpath y is slided.

Assume first that the latter happens, where y is not contained in x. In that case, at least one element h_i of the height sequence get increased, that is not in the subsequence modified when sliding x. Because the height sequence can never decrease, this element h_i is still larger in w' than it was in w, which is absurd.

One can therefore assume that y is contained in x. But sliding x commutes with sliding such y.

Let us pick an arbitrary chain of Tamari cover moves from w to w'. By the commuting relation just explained, one can assume that all slidings of x happen first in this chain.

But then after performing these initial slidings of x, the first step of x must have attained the position it will have in w'. So in fact at this point, the top element w' has been reached already. Therefore our original chain only contains slidings of x.

So the full interval $[w, w']$ is just made of a sequence of slidings of x. □

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