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Abstract—In this paper, an achievable \( \eta \)-Nash equilibrium (\( \eta \)-NE) region for the two-user Gaussian interference channel with noisy channel-output feedback is presented for all \( \eta \geq 1 \). This result is obtained in the scenario in which each transmitter-receiver pair chooses its own transmit-receive configuration in order to maximize its own individual information transmission rate. At an \( \eta \)-NE, any unilateral deviation by either of the pairs does not increase the corresponding individual rate by more than \( \eta \) bits per channel use.

Index Terms—Gaussian Interference Channel, Noisy channel-output feedback, \( \eta \)-Nash equilibrium region.

I. INTRODUCTION

The interference channel (IC) is one of the simplest yet insightful multi-user channels in network information theory. An important class of ICs is the two-user Gaussian interference channel (GIC) in which there exist two point-to-point links subject to mutual interference and independent Gaussian noise sources. In this model, each output signal is a noisy version of the sum of the two transmitted signals affected by the corresponding channel gains. The analysis of this channel can be made considering two general scenarios: (1) a centralized scenario in which the entire network is controlled by a central entity that configures both transmitter-receiver pairs; and (2) a decentralized scenario in which each transmitter-receiver pair autonomously configures its transmission-reception parameters. In the former, the fundamental limits are characterized by the capacity region, which is approximated to within a fixed number of bits in [1] for the case without feedback; in [2] for the case with perfect channel-output feedback; and in [3] and [4] for the case with noisy channel-output feedback. In the latter, the fundamental limits are characterized by the \( \eta \)-Nash equilibrium (\( \eta \)-NE) region. The \( \eta \)-NE of the GIC is approximated in the cases without feedback and with perfect channel-output feedback in [5] and [6], respectively.

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In this paper the \( \eta \)-NE region of the GIC is studied assuming that there exists a noisy feedback link from each receiver to its corresponding transmitter. The \( \eta \)-NE region is approximated by two regions for all \( \eta \geq 1 \): a region for which an equilibrium transmit-receive configuration is presented for each of the information rate pairs (an achievable region); and a region for which any information rate pair that is outside of this region cannot be an \( \eta \)-NE (impossibility region). The focus of this paper is on the achievable region.

The results presented in this paper are a generalization of the results presented in [5] and [6], and they are obtained thanks to the analysis of linear deterministic approximations in [7] and [8].

II. DECENTRALIZED GAUSSIAN INTERFERENCE CHANNELS WITH NOISY CHANNEL-OUTPUT FEEDBACK

Consider the two-user decentralized Gaussian interference channel with noisy channel-output feedback (D-GIC-NOF) depicted in Figure 1. Transmitter \( i \), with \( i \in \{1,2\} \), communicates with receiver \( i \) subject to the interference produced by transmitter \( j \), with \( j \in \{1,2\} \setminus \{i\} \). There are two independent and uniformly distributed messages, \( W_i \in \mathcal{W}_i \), with \( \mathcal{W}_i = \{1,2,\ldots,2^{N_i R_i}\} \), where \( N_i \) denotes the fixed block-length in channel uses and \( R_i \) the information transmission rate in bits per channel use. At each block, transmitter \( i \) sends the codeword \( X_i = (X_{i,1},X_{i,2},\ldots,X_{i,N_i})^T \in C_i \subseteq \mathbb{R}^{N_i} \), where \( C_i \) is the codebook of transmitter \( i \). The channel coefficient from transmitter \( j \) to receiver \( i \) is denoted by \( h_{ij} \); the channel coefficient from transmitter \( i \) to receiver \( i \) is denoted by \( h_{ii} \); and the channel coefficient from channel-output \( i \) to transmitter \( i \) is denoted by \( c_{ii} \). All channel coefficients are assumed to be non-negative real numbers. At a given channel...
use \( n \in \{1, 2, \ldots, N\} \), with
\[
N = \max(N_1, N_2),
\] (1)
the channel output at receiver \( i \) is denoted by \( \bar{Y}_{i,n} \). During channel use \( n \), the input-output relation of the channel model is given by
\[
\bar{Y}_{i,n} = h_{ii} X_{i,n} + h_{ij} X_{j,n} + n_{i,n},
\] (2)
where \( X_{i,n} = 0 \) for all \( n \) such that \( N \geq n > N_i \) and \( n_{i,n} \) is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of receiver \( i \). Let \( d > 0 \) be the finite feedback delay measured in channel uses. At the end of channel use \( n \), transmitter \( i \) observes \( \bar{Y}_{i,n} \), which consists of a scaled and noisy version of \( \bar{Y}_{i,n-d} \). More specifically,
\[
\bar{Y}_{i,n} = \left\{ \begin{array}{ll}
\sum_{i,n} \bar{Z}_{i,n} & \text{for } n \in \{1, 2, \ldots, d\} \\
\sum_{i,n} \bar{Z}_{i,n-d} + \sum_{i,n} \bar{Z}_{i,n} & \text{for } n \in \{d+1, d+2, \ldots, N\},
\end{array} \right.
\] (3)
where \( \bar{Z}_{i,n} \) is a real Gaussian random variable with zero mean and unit variance that represents the noise in the feedback link of transmitter-receiver pair \( i \). The random variables \( \bar{Z}_{i,n} \) and \( \bar{Z}_{i,n} \) are assumed to be independent. In the following, without loss of generality, the feedback delay is assumed to be one channel use, i.e., \( d = 1 \). The encoder of transmitter \( i \) is defined by a set of deterministic functions \( f_{i,1}^{(N)}, f_{i,2}^{(N)}, \ldots, f_{i,N}^{(N)} \), with \( f_{i,1}^{(N)} : \mathcal{W}_i \times \mathbb{R} \rightarrow \mathcal{X}_i \) and for all \( n \in \{2, 3, \ldots, N\} \), \( f_{i,n}^{(N)} : \mathcal{W}_i \times \mathbb{R}^{n-1} \rightarrow \mathcal{X}_i \), such that
\[
X_{i,1} = f_{i,1}^{(N)} (W_i, \Omega_i),
\] (4a)
\[
X_{i,n} = f_{i,n}^{(N)} (W_i, \Omega_i, \bar{Y}_{i,1}, \bar{Y}_{i,2}, \ldots, \bar{Y}_{i,n-1}),
\] (4b)
where \( \Omega_i \) is an additional index randomly generated. The index \( \Omega_i \) is assumed to be known by both transmitter \( i \) and receiver \( i \), while unknown by transmitter \( j \) and receiver \( j \).

The components of the input vector \( X_i \) are real numbers subject to an average power constraint
\[
\frac{1}{N} \sum_{n=1}^{N_i} \mathbb{E}[X_{i,n}^2] \leq 1.
\] (5)

The decoder of receiver \( i \) is defined by a deterministic function \( \psi_i^{(N)} : \mathcal{X}_i \rightarrow \mathcal{W}_i \). At the end of the communication, receiver \( i \) uses the vector \( \left( \bar{Y}_{i,1}, \bar{Y}_{i,2}, \ldots, \bar{Y}_{i,N} \right) \) and the index \( \Omega_i \) to obtain an estimate
\[
\hat{W}_i = \psi_i^{(N)} (\Omega_i, \bar{Y}_{i,1}, \bar{Y}_{i,2}, \ldots, \bar{Y}_{i,N}),
\] (6)
of the message index \( W_i \). A transmit-receive configuration for transmitter-receiver pair \( i \), denoted by \( s_i \), can be described in terms of the block-length \( N_i \), the rate \( R_i \), the codebook \( \mathcal{C}_i \), the encoding functions \( f_{i,1}^{(N)}, f_{i,2}^{(N)}, \ldots, f_{i,N}^{(N)} \), and the decoding function \( \psi_i^{(N)} \) etc. The average error probability at decoder \( i \) given the configurations \( s_1 \) and \( s_2 \), denoted by \( p_i(s_1, s_2) \), is given by
\[
p_i(s_1, s_2) = \Pr [ W_i \neq \hat{W}_i ].
\] (7)

Within this context, a rate pair \( (R_1, R_2) \in \mathbb{R}^2_+ \) is said to be achievable if it complies with the following definition.

**Definition 1 (Achievable Rate Pairs):** A rate pair \( (R_1, R_2) \in \mathbb{R}^2_+ \) is achievable if there exists at least one pair of configurations \( (s_1, s_2) \) such that the decoding bit error probabilities \( p_1(s_1, s_2) \) and \( p_2(s_1, s_2) \) can be made arbitrarily small by letting the block-lengths \( N_1 \) and \( N_2 \) grow to infinity.

The aim of transmitter \( i \) is to autonomously choose its transmit-receive configuration \( s_i \) in order to maximize its achievable rate \( R_i \). Note that the rate achieved by transmitter-receiver \( i \) depends on both configurations \( s_1 \) and \( s_2 \) due to mutual interference. This reveals the competitive interaction between both links in the decentralized interference channel.

The fundamental limits of the two-user D-GIC-OF in Figure 1 can be described by six parameters: \( \text{SNR}_1, \text{SNR}_2, \text{INR}_{ij} \), and \( \text{INR}_{ij} \), with \( i \in \{1, 2\} \) and \( j \in \{1, 2\} \{\{i\} \}, \) which are defined as follows:
\[
\text{SNR}_i \triangleq h_i^2, \quad \text{INR}_{ij} \triangleq h_{ij}^2 \quad \text{and} \quad \text{SNR}_{ij} \triangleq h_i^2 + 2 h_i h_{ij} + h_{jj}^2 + 1.
\] (8)

The analysis presented in this paper focuses exclusively on the case in which \( \text{INR}_{ij} > 1 \) for all \( i \in \{1, 2\} \) and \( j \in \{1, 2\} \{\{i\} \} \). The reason for exclusively considering this case follows from the fact that when \( \text{INR}_{ij} \leq 1 \), the transmitter-receiver pair \( i \) is impaired mainly by noise instead of interference. In this case, feedback does not bring a significant rate improvement. Denote by \( \mathcal{C} \) the capacity region of the two-user GIC-OF with fixed parameters \( \text{SNR}_1, \text{SNR}_2, \text{INR}_{12}, \text{INR}_{21}, \text{SNR}_1, \) and \( \text{SNR}_2 \). The achievable region \( \mathcal{C} \) in [4, Theorem 2] and the converse region \( \mathcal{C} \) in [4, Theorem 3] approximate the capacity region \( \mathcal{C} \) to within 4.4 bits [4].

**III. GAME FORMULATION**

The competitive interaction between the two transmitter-receiver pairs in the interference channel can be modeled by the following game in normal-form:
\[
\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}}).
\] (11)

The set \( \mathcal{K} = \{1, 2\} \) is the set of players, that is, the set of transmitter-receiver pairs. The sets \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are the sets of actions of players 1 and 2, respectively. An action of a player \( i \in \mathcal{K} \), which is denoted by \( s_i \in \mathcal{A}_i \), is basically its transmit-receive configuration as described above. The utility function of player \( i \) is \( u_i : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathbb{R}^+ \), and it is defined as the information rate of transmitter \( i \),
\[
u_i(s_1, s_2) = \left\{ \begin{array}{ll}
R_i & \text{if} \ p_i(s_1, s_2) < \epsilon \\
0 & \text{otherwise},
\end{array} \right.
\] (12)
where \( \epsilon > 0 \) is an arbitrarily small number. This game formulation for the case without feedback was first proposed in [9] and [10].

A class of transmit-receive configurations that are particularly important in the analysis of this game is referred to as
the set of \( \eta \)-Nash equilibria (\( \eta \)-NE), with \( \eta > 0 \). This type of configurations satisfy the following definition.

Definition 2 (\( \eta \)-Nash equilibrium): Given a positive real \( \eta \), an action profile \((s^*_1, s^*_2)\) is an \( \eta \)-Nash equilibrium (NE) in the game \( G = (K, \{A_k\}_{k \in K}, \{u_k\}_{k \in K}) \), if for all \( i \in K \) and for all \( s_i \in A_i \), it follows that

\[
u_i(s_i, s^*_1, s^*_2) \leq u_i(s^*_i, s^*_1, s^*_2) + \eta.\]  

(13)

Let \((s^*_1, s^*_2)\) be an \( \eta \)-Nash equilibrium action profile. Then, none of the transmitters can increase its own transmission rate more than \( \eta \) bits per channel use by changing its own transmit-receive configuration and keeping the average bit error probability arbitrarily close to zero. Note that for \( \eta \) sufficiently large, from Definition 2, any pair of configurations can be an \( \eta \)-NE. Alternatively, for \( \eta = 0 \), the definition of Nash equilibrium is obtained [11]. In this case, if a pair of configurations is a Nash equilibrium \((\eta = 0)\), then each individual configuration is optimal with respect to each other. Hence, the interest is to describe the set of all possible \( \eta \)-NE rate pairs \((R_1, R_2)\) of the game in (11) with the smallest \( \eta \) for which there exists at least one equilibrium configuration pair.

The set of rate pairs that can be achieved at an \( \eta \)-NE is known as the \( \eta \)-Nash equilibrium (\( \eta \)-NE) region.

Definition 3 (\( \eta \)-NE Region): Let \( \eta > 0 \) be fixed. An achievable rate pair \((R_1, R_2)\) is said to be in the \( \eta \)-NE region of the game \( G = (K, \{A_k\}_{k \in K}, \{u_k\}_{k \in K}) \), if there exists a pair \((s^*_1, s^*_2)\) \( \in A_1 \times A_2 \) that is an \( \eta \)-NE and the following holds:

\[
u_1(s^*_1, s^*_2) = R_1 \quad \text{and} \quad u_2(s^*_1, s^*_2) = R_2.\]  

(14)

The \( \eta \)-NE regions of the two-user GIC with and without perfect channel-output feedback have been approximated to within a constant number of bits in [5] and [6], respectively. The next section introduces a generalization of these results.

IV. MAIN RESULTS

A. Achievable \( \eta \)-Nash Equilibrium Region

Let the \( \eta \)-NE region (Definition 3) of the D-GIC-NOF be denoted by \( \mathcal{N}_\eta \). This section introduces a region \( \mathcal{N}'_\eta \) that is achievable using a coding scheme that combines rate splitting [12], common randomness [5], [6], block Markov superposition coding [13] and backward decoding [14]. In the following, this coding scheme is referred to as randomized Han-Kobayashi scheme with noisy channel-output feedback (RHK-NOF). This coding scheme is presented in [8] and uses the same techniques of the schemes in [5] and [6]. Therefore, the focus of this section is on the results rather than the description of the scheme. A motivated reader is referred to [15]. The RHK-NOF is proved to be an \( \eta \)-NE action profile with \( \eta \geq 1 \). That is, any unilateral deviation from the RHK-NOF by any of the transmitter-receiver pairs might lead to an individual rate improvement which is at most one bit per channel use.

The description of the achievable \( \eta \)-Nash region \( \mathcal{N}'_\eta \) is presented using the constants \( a_{1,i} \): the functions \( a_{2,i} : [0, 1] \rightarrow \mathbb{R}_+ \), \( a_{3,i} : [0, 1]^3 \rightarrow \mathbb{R}_+ \), \( \eta \in \{3, 4, 5, 6\} \), and \( a_{7,i} : [0, 1]^3 \rightarrow \mathbb{R}_+ \), which are defined as follows, for all \( i \in \{1, 2\} \) with \( j \in \{1, 2\} \setminus \{i\} \):

\[
a_{1,i} = \frac{1}{2} \log \left( 2 + \frac{\text{SNR}_i}{\text{INR}_j} \right) - \frac{1}{2}, \quad \text{(15a)}
\]

\[
a_{2,i}(\rho) = \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right) - \frac{1}{2}, \quad \text{(15b)}
\]

\[
a_{3,i}(\rho, \mu) = \frac{1}{2} \log \left( \frac{\text{SNR}_i \left( b_{2,i}(\rho) + 1 \right) + b_{1,i}(1) + 1}{\text{SNR}_i \left( (1-\mu) b_{2,i}(\rho) + 1 \right) + b_{1,i}(1) + 1} \right). \quad \text{(15c)}
\]
Denote by $\mathcal{N}_{\text{PF}}$ the achievable $\eta$-NE region of the two-user D-GIC-POF presented in [6]. The region $\mathcal{N}_{\text{POF}}$ can be obtained as a special case of Theorem 1 as shown by the following corollary.

**Corollary 1** ($\eta$-NE Region with Perfect Output Feedback): Let $\mathcal{N}_{\text{POF}}$ denote the achievable $\eta$-NE region of the two-user D-GIC-POF with fixed parameters $\text{SNR}_{i}$ and $\text{INR}_{ij}$, with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$. Then, the following holds:

$$
\mathcal{N}_{\text{POF}} = \lim_{\text{SNR}_{i} \to \infty} \lim_{\text{SNR}_{j} \to \infty} \mathcal{N}_{\eta}.
$$

Denote by $\mathcal{N}_{\text{POF}}$, the achievable $\eta$-NE region of the two-user D-GIC presented in [5]. The region $\mathcal{N}_{\text{POF}}$ can be obtained as a special case of Theorem 1 as shown by the following corollary.

**Corollary 2** ($\eta$-NE Region without Output Feedback): Let $\mathcal{N}_{\text{POF}}$ denote the achievable $\eta$-NE region of the two-user D-GIC, with fixed parameters $\text{SNR}_{i}$ and $\text{INR}_{ij}$, with $i \in \{1, 2\}$.
Theorem 1: Let $\eta \geq 1$ be fixed. The achievable $\eta$-NE region $\mathcal{N}_\eta$ is given by the closure of all possible achievable rate pairs $(R_1, R_2) \in \mathcal{C}$ in [4, Theorem 2] that satisfy, for all $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, the following conditions:

$$R_i \geq \left( a_{2,i}(\rho) - a_{3,i}(\rho, \mu_j) - a_{4,i}(\rho, \mu_j) - \eta \right)^+,$$

$$R_i \leq \min \left( a_{2,i}(\rho) + a_{3,j}(\rho, \mu_i) + a_{5,j}(\rho, \mu_i) - a_{2,j}(\rho) + \eta, a_{3,i}(\rho, \mu_j) + a_{7,i}(\rho, \mu_1, \mu_2) + 2a_{3,j}(\rho, \mu_i) + a_{5,j}(\rho, \mu_i) - a_{2,j}(\rho) + \eta, a_{2,i}(\rho) + a_{4,i}(\rho, \mu_j) + 2a_{3,j}(\rho, \mu_i) + a_{5,j}(\rho, \mu_i) + a_{7,j}(\rho, \mu_1, \mu_2) - 2a_{2,j}(\rho) + 2\eta \right),$$

for all $(\rho, \mu_1, \mu_2) \in \left[ 0, 1 - \max \left( \frac{1}{\text{INR}_{12}}, \frac{1}{\text{INR}_{21}} \right) \right] \times [0, 1] \times [0, 1].$

Theorem 2: Let $\eta \geq 1$ be fixed. The impossibility region $\mathcal{N}_\eta$ of the two-user D-GIC-NOF is given by the closure of all possible non-negative rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ that satisfy for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

$$\mathcal{B}_{\text{G-IC-NOF}} = \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : R_i \geq L_i \right\},$$

where,

$$L_i = \left( \frac{1}{2} \log \left( 1 + \frac{\text{SNR}_i}{1 + \text{INR}_{ij}} \right) - \eta \right)^+.$$

B. Impossibility Region

This section introduces an impossibility region, denoted by $\mathcal{N}_\eta$. That is, $\mathcal{N}_\eta \supseteq \mathcal{N}_\eta$. More specifically, any rate pair $(R_1, R_2) \in \mathcal{N}_\eta$ is not an $\eta$-NE. This region is described in terms of the convex region $\mathcal{B}_{\text{G-IC-NOF}}$. Here, for the case of the two-user D-GIC-NOF, the region $\mathcal{B}_{\text{G-IC-NOF}}$ is given by the closure of the rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ that satisfy for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

$$\mathcal{B}_{\text{G-IC-NOF}} = \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : R_i \geq L_i \right\},$$

where,

Note that $L_i$ is the rate achieved by the transmitter-receiver pair $i$ when it saturates the power constraint in (5) and treats interference as noise. Following this notation, the impossibility region of the two-user GIC-NOF, i.e., $\mathcal{N}_\eta$, can be described as follows.

V. CONCLUSIONS

In this paper, an achievable $\eta$-Nash equilibrium ($\eta$-NE) region for the two-user Gaussian interference channel with noisy channel-output feedback has been presented for all $\eta \geq 1$. This result generalizes the existing achievable regions of the $\eta$-NE for the cases without feedback and with perfect channel-output feedback.

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