Global exact controllability of bilinear quantum systems on compact graphs and energetic controllability

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Institut Fourier, Université Grenoble Alpes
100 Rue des Mathématiques, 38610 Gières, France
alessandro.duca@univ-grenoble-alpes.fr

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Abstract

The aim of this work is to study the controllability of the bilinear Schrödinger equation on compact graphs. In particular, we consider the equation (BSE) $i\partial_t\psi = -\Delta\psi + u(t)B\psi$ in the Hilbert space $L^2(\mathscr{G},\mathbb{C})$, with $\mathscr{G}$ being a compact graph. The Laplacian $-\Delta$ is equipped with self-adjoint boundary conditions, B is a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$ with $T > 0$. We provide a new technique leading to the global exact controllability of the (BSE) in $D(|\Delta|^{s/2})$ with $s \geq 3$. Afterwards, we introduce the "energetic controllability", a weaker notion of controllability useful when the global exact controllability fails. In conclusion, we develop applications of the main results involving for instance star graphs.

1 Introduction

In quantum mechanics, any state of a closed system is mathematically represented by a wave function ψ in the unit sphere of a Hilbert space $\mathscr{H}$. We consider the evolution of a particle confined in a network shaped as compact graph $\mathscr{G}$ and subjected to a controlling external field.

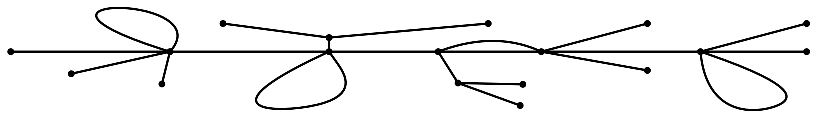

Figure 1: A compact graph is a one-dimensional domain composed by finite vertices (points) connected by edges (segments) of finite lengths.

A standard choice for such setting is to represent the action of the field by an operator B and its intensity by a real function u; we also impose $\mathscr{H} := L^2(\mathscr{G},\mathbb{C})$. The evolution of ψ is modeled by the bilinear Schrödinger equation in $\mathscr{H}$

$$\text{(BSE)} \qquad \begin{cases} i\partial_t\psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0,T), \\ \psi(0) = \psi_0, & T > 0. \end{cases}$$

The Laplacian $A = -\Delta$ is equipped with self-adjoint boundary conditions, B is a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$. When the (BSE) is well-posed, we call Γ_t^u the unitary propagator generated by $A + u(t)B$.

The aim of this work is to study the controllability of the bilinear Schrödinger equation (BSE) according to the structure of the graph $\mathscr{G}$, the definition of the domain of A and the choice of B.

The controllability of finite-dimensional quantum systems modeled by the (BSE), when A and B are $N \times N$ Hermitian matrices, is well-known for being linked to the rank of the Lie algebra spanned by

A and B (see [Alt02, Cor07]); nevertheless the Lie algebra rank condition can not be used for infinite-dimensional quantum systems (see [Cor07]).

The *global approximate controllability* of the (BSE) has been proved with different techniques in literature. We refer to [Mir09, Ner10] for Lyapunov techniques, while we cite [BCMS12, BGRS15] for adiabatic arguments and [BdCC13, BCS14] for Lie-Galerking methods.

The *exact controllability* of infinite-dimensional quantum systems is in general a more delicate matter. When we consider the linear Schrödinger equation, the controllability and observability properties are reciprocally dual. Different results were developed by addressing directly or by duality the control problem with different techniques: multiplier methods [Lio83, Mac94], microlocal analysis [BLR92, Bur91, Leb92] and Carleman estimates [BM08, LT92, MOR08]. In any case, when one considers graph type domains, a complete theory is far from being formulated. Indeed, the interaction between the different components of a graph may generate unexpected phenomena (see [DZ06]).

The bilinear Schrödinger equation is well-know for not being exactly controllable in the Hilbert space where it is defined when B is a bounded operator and $u \in L^2((0,T),\mathbb{R})$ with $T>0$ (even though it is well-posed in such space). We refer to the work [BMS82] by Ball, Mardsen and Slemrod where the well-posedness and the non-controllability of the equation are proved (see also [Tur00]).
As a consequence, the exact controllability of bilinear quantum systems can not proved with the classical techniques valid for the linear Schrödinger equation and weaker notions of controllability are necessary.

The turning point for this kind of studies was the idea of controlling the equation in subspaces of $D(A)$ introduced by Beauchard in [Bea05]. Following this approach, different works were developed for the (BSE) in $\mathscr{G}=(0,1)$ by considering $A=-\Delta_D$ the Dirichlet Laplacian such that

$$D(-\Delta_D)=H^2((0,1),\mathbb{C})\cap H_0^1((0,1),\mathbb{C})), \qquad -\Delta_D\psi:=-\Delta\psi, \qquad \forall\psi\in D(-\Delta_D).$$

For instance, in [BL10], Beauchard and Laurent prove the *well-posedness* and the *local exact controllability* of the bilinear Schrödinger equation in $H^s_{(0)}:=D(|-\Delta_D|^{s/2})$ for $s=3$. For the *global exact controllability* in $H^3_{(0)}$, we refer to [Duc18b], while we mention [Duc18c, Mor14, MN15] for *simultaneous exact controllability* results in $H^3_{(0)}$ and $H^4_{(0)}$.

Studying the controllability of the bilinear Schrödinger equation on compact graphs presents an additional problem, which can be understood by considering $(\lambda_k)_{k\in\mathbb{N}^*}$ the ordered sequence of eigenvalues of A. Nevertheless there exists $\mathcal{M}\in\mathbb{N}^*$ such that

$$\inf_{k\in\mathbb{N}^*}|\lambda_{k+\mathcal{M}}-\lambda_k|>0 \tag{1}$$

(as showed in [Duc18a, *relation* (2)]), the spectral gap $\inf_{k\in\mathbb{N}^*}|\lambda_{k+1}-\lambda_k|>0$ is only valid when $\mathscr{G}=(0,1)$. This hypothesis is crucial for the techniques developed in [BL10, Duc18c, Duc18b, Mor14], which can not be directly applied without imposing further assumptions.

As far as we know, the bilinear Schrödinger equation on compact graphs has only been studied in the seminal work [Duc18a]. There, the author ensures that, if there exist $C>0$ and $\tilde{d}\geq 0$ such that

$$|\lambda_{k+1}-\lambda_k|\geq\frac{C}{k^{\tilde{d}}}, \qquad \forall k\in\mathbb{N}^*,$$

then the *well-posedness* and the *global exact controllability* of the (BSE) can be guaranteed in some spaces $D(|A|^{s/2})$ with $s\geq 3$ depending on $\tilde{d}$.

1.1 Main results

In the current manuscript, we introduce an alternative set of assumptions to the one adopted in [Duc18a]. In particular, we hypothesize the existence of an entire function G such that $G\in L^\infty(\mathbb{R},\mathbb{R})$ and so that there exist $J,I>0$ such that

$$|G(z)|\leq Je^{I|z|}, \qquad \forall z\in\mathbb{C}.$$

We also assume that $(\lambda_k)_{k\in\mathbb{N}^*}$ are pairwise distinct numbers, $\{\pm\sqrt{\lambda_k}\}_{k\in\mathbb{N}^*}$ are simple zeros of G and there exist $\tilde{d}\geq 0$ and $C>0$ such that

$$|G'(\pm\sqrt{\lambda_k})|\geq\frac{C}{k^{1+\tilde{d}}}, \qquad \forall k\in\mathbb{N}^*.$$

When these assumptions are verified for suitable $\tilde{d} \geq 0$, we prove that the *global exact controllability* of the (BSE) can be guaranteed in $H^s_{\mathscr{G}} := D(|A|^{s/2})$ with $s \geq 3$ depending on $\tilde{d}$ (see Theorem 3.1). Before providing an application of the result, we formally define the global exact controllability in such spaces.

Definition 1.1. The (BSE) is said to be globally exactly controllable in $H^s_{\mathscr{G}}$ with $s \geq 3$ when, for every $\psi^1, \psi^2 \in H^s_{\mathscr{G}}$ such that $\|\psi^1\|_{L^2(\mathscr{G},\mathbb{C})} = \|\psi^2\|_{L^2(\mathscr{G},\mathbb{C})}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\Gamma^u_T \psi^1 = \psi^2.$$

We consider a *star graph* $\mathscr{G}$ composed by $N \in \mathbb{N}^*$ edges $\{e_j\}_{j \leq N}$. Each edge e_j is parametrized with a coordinate going from 0 to the length of the edge L_j. We set the coordinate 0 in the external vertex belonging to e_j. We denote V_e the set of the external vertices of the graph $\mathscr{G}$ and v its internal vertex (we refer to the identities (2) for the formal definitions of external and internal vertices).

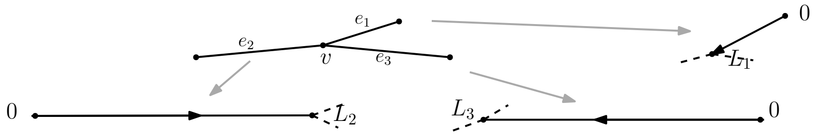

Figure 2: Parametrization of a star graph with $N = 3$ edges.

Definition 1.2. For every $N \in \mathbb{N}^*$, we define $\mathcal{AL}(N)$ such as the set of elements $\{L_j\}_{j \leq N} \in (\mathbb{R}^+)^N$ so that: the numbers $\{1, \{L_j\}_{j \leq N}\}$ are linearly independent over $\mathbb{Q}$ and all the ratios L_k/L_j are algebraic irrational numbers.

Theorem 1.3. *Let $\mathscr{G}$ be a star graph. Let $D(A)$ be the set of functions $f \in H^2(\mathscr{G},\mathbb{C})$ such that:*

- *$f(\tilde{v}) = 0$ for every external vertex $\tilde{v} \in V_e$ (Dirichlet boundary conditions);*
- *f is continuous in the vertex v and $\sum_{e \ni v} \frac{\partial f}{\partial x_e}(v) = 0$ (Neumann-Kirchhoff boundary conditions).*

Let the control field B be such that, for every $\psi \in \mathscr{H}$,

$$\begin{cases} B\psi(x) = (x - L_1)^4 \psi(x), & x \in e_1, \\ B\psi(x) = 0, & x \in \mathscr{G} \setminus e_1. \end{cases}$$

There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable such that, for every $\{L_j\}_{j \leq N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, the (BSE) is globally exactly controllable in

$$H^{4+\epsilon}_{\mathscr{G}}, \qquad \forall \epsilon \in (0, 1/2).$$

When the global exact controllability fails, in the spirit of the results provided in [BC06], we introduce a weaker notion of controllability: the *energetic controllability.* Let $(\varphi_k)_{k \in \mathbb{N}^*}$ be an orthonormal system of $\mathscr{H}$ composed by eigenfunctions of A and $(\mu_k)_{k \in \mathbb{N}^*}$ be the relative eigenvalues.

Definition 1.4. The (BSE) is said to be energetically controllable in $(\mu_k)_{k \in \mathbb{N}^*}$ if, for every $m, n \in \mathbb{N}^*$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ so that

$$\Gamma^u_T \varphi_m = \varphi_n.$$

The energetic controllability guarantees that the energy of the quantum system $i\partial_t \psi = A\psi$ in $L^2(\mathscr{G},\mathbb{C})$ can be controlled in specific energy levels via the external field $u(t)B$. An application of the abstract result, which is stated in Theorem 4.1, is the following theorem.

Theorem 1.5. *Let $\mathscr{G}$ be a star graph with edges of equal length L. Let $D(A)$ be defined such as in Theorem 1.3. Let the control field B be such that, for every $\psi \in \mathscr{H}$,*

$$\begin{cases} B\psi(x) = (x - L)^2 \psi(x), & x \in e_1, \\ B\psi(x) = 0, & x \in \mathscr{G} \setminus e_1. \end{cases}$$

The (BSE) is energetically controllable in $\left(\frac{k^2\pi^2}{4L^2}\right)_{k \in \mathbb{N}^*}$.

Theorem 1.5 is valid although the spectrum of A presents multiple eigenvalues and the global exact controllability from Theorem 3.1 is not satisfied (also [Duc18a, *Theorem* 2.3] is not guaranteed). In addition, the energetic controllability is ensured with respect to all the energy levels of the quantum system $i\partial_t\psi = A\psi$, since the eigenvalues of A non-repeated with their multiplicity are $\left(\frac{k^2\pi^2}{4L^2}\right)_{k\in\mathbb{N}^*}$.

The energetic controllability is useful when it is not possible to fully characterize the spectrum of A because of the complexity of the graph $\mathscr{G}$. By studying the structure of $\mathscr{G}$, it is possible to explicit some eigenvalues $(\mu_k)_{k\in\mathbb{N}^*}$ and verify if the system is energetically controllable in $(\mu_k)_{k\in\mathbb{N}^*}$. In Section 4.1, we discuss some examples where the result is satisfied, *e.g* graphs containing self-closing edges.

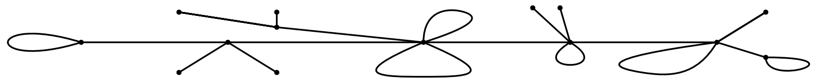

Figure 3: Example of compact graph containing more self-closing edges.

1.2 Scheme of the work

In Section 2, we present the main assumptions adopted in the work and the well-posedness of the (BSE) in $H^s_{\mathscr{G}}$ with suitable $s > 0$ (Proposition 2.1).
In Section 3, we prove the global exact controllability of the (BSE) in such spaces. The result is ensured for generic graphs in Theorem 3.1, while specific star graphs are considered in Theorem 3.2 and Corollary 3.4. Applications of these results are developed in Section 3.2 containing the proof of Theorem 1.3.
In Section 4, we enounce the energetic controllability of the (BSE) in Theorem 4.1. We develop different applications of the outcome in Section 4.1 where the proof of Theorem 1.5 is also provided.
In Appendix A, we prove the global approximate controllability of the (BSE) in $H^s_{\mathscr{G}}$ with suitable $s > 0$.
In Appendix B, we present some spectral results adopted in the work, while we study the solvability of the so-called moments problems in Appendix C.

2 Preliminaries

Let $\mathscr{G}$ be a compact graph composed by $N \in \mathbb{N}^*$ edges $\{e_j\}_{j\leq N}$ of lengths $\{L_j\}_{j\leq N}$ and $M \in \mathbb{N}^*$ vertices $\{v_j\}_{j\leq M}$. We call V_e and V_i the external and the internal vertices of $\mathscr{G}$, *i.e.*

(2) $$V_e := \big\{v \in \{v_j\}_{j\leq M} \mid \exists! e \in \{e_j\}_{j\leq N} : v \in e\big\}, \qquad V_i := \{v_j\}_{j\leq M} \setminus V_e.$$

We study graphs equipped with a metric, which parametrizes each edge e_j with a coordinate going from 0 to its length L_j. A graph is compact when it is composed by a finite number of vertices and edges of finite length. We consider functions $f := (f^1, ..., f^N) : \mathscr{G} \to \mathbb{C}$ with domain a compact metric graph $\mathscr{G}$ so that $f^j : e_j \to \mathbb{C}$ for every $j \leq N$. We denote

$$\mathscr{H} = L^2(\mathscr{G}, \mathbb{C}) = \prod_{j\leq N} L^2(e_j, \mathbb{C}).$$

The Hilbert space $\mathscr{H}$ is equipped with the norm $\|\cdot\|_{L^2}$ and the scalar product

$$\langle\psi, \varphi\rangle_{L^2} := \sum_{j\leq N} \langle\psi^j, \varphi^j\rangle_{L^2(e_j,\mathbb{C})} = \sum_{j\leq N} \int_{e_j} \overline{\psi^j}(x)\varphi^j(x)dx, \qquad \forall\psi, \varphi \in \mathscr{H}.$$

In the bilinear Schrödinger equation (BSE), we consider the Laplacian A being self-adjoint and we denote $\mathscr{G}$ as quantum graph. From now on, when we introduce a quantum graph $\mathscr{G}$, we implicitly define on $\mathscr{G}$ a self-adjoint Laplacian A. Formally, $D(A)$ is characterized via the following boundary conditions.

Boundary conditions. Let $\mathscr{G}$ be a quantum compact graph.

($\mathcal{NK}$) A vertex $v \in V_i$ is equipped with Neumann-Kirchhoff boundary conditions when every $f \in D(A)$ is continuous in v and $\sum_{e\ni v} \frac{\partial f}{\partial x_e}(v) = 0$ (the derivatives have ingoing directions in v).

($\mathcal{D}$) A vertex $v \in V_e$ is equipped with Dirichlet boundary conditions when $f(v) = 0$ for every $f \in D(A)$.

($\mathcal{N}$) A vertex $v \in V_e$ is equipped with Neumann boundary conditions when $\partial_x f(v) = 0$ for every $f \in D(A)$.

Notations. Let $\mathscr{G}$ be a quantum compact graph.

- The graph $\mathscr{G}$ is said to be equipped with ($\mathcal{D}$) (or ($\mathcal{N}$)) when every $v \in V_e$ is equipped with ($\mathcal{D}$) (or ($\mathcal{N}$)) and every $v \in V_i$ with ($\mathcal{NK}$).
- The graph $\mathscr{G}$ is said to be equipped with ($\mathcal{D}/\mathcal{N}$) when every $v \in V_e$ is equipped with ($\mathcal{D}$) or ($\mathcal{N}$), while every $v \in V_i$ with ($\mathcal{NK}$).

In our framework, the Laplacian A admits purely discrete spectrum (see [Kuc04, *Theorem* 18]). We define $(\lambda_k)_{k\in\mathbb{N}^*}$ the ordered sequence of eigenvalues of A and a Hilbert basis of $\mathscr{H}$

$$\Phi := (\phi_k)_{k\in\mathbb{N}^*} \tag{3}$$

composed by corresponding eigenfunctions. From [Duc18a, *RemarkA*.4], there exist $C_1, C_2 > 0$ so that

$$C_1 k^2 \leq \lambda_k \leq C_2 k^2, \qquad \forall k \geq 2. \tag{4}$$

For $s > 0$, we define the spaces $H^s = H^s(\mathscr{G}, \mathbb{C}) := \prod_{j=1}^N H^s(e_j, \mathbb{C})$ and

$$h^s = \Big\{ (x_j)_{j\in\mathbb{N}^*} \subset \mathbb{C} \;\big|\; \sum_{j=1}^{\infty} |j^s x_j|^2 < \infty \Big\}$$

equipped with the norm $\big\|(x_j)_{j\in\mathbb{N}^*}\big\|_{(s)} = \big(\sum_{j=1}^{\infty} |j^s x_j|^2\big)^{\frac{1}{2}}$ for every $(x_j)_{j\in\mathbb{N}^*} \in h^s$. Let $[r]$ be the entire part of $r \in \mathbb{R}$. For $s > 0$, we denote

$$H^s_{\mathcal{NK}} := \Big\{ \psi \in H^s \mid \partial_x^{2n}\psi \text{ is continuous in } v,\ n < \big[(s+1)/2\big];$$
$$\sum_{e\in N(v)} \partial_{x_e}^{2n+1}\psi(v) = 0,\ \forall n \in \mathbb{N},\ n < \big[s/2\big],\ \forall v \in V_i \Big\},$$
$$H^s_{\mathscr{G}} = H^s_{\mathscr{G}}(\mathscr{G}, \mathbb{C}) := D(A^{s/2}), \qquad \|\cdot\|_{(s)} := \|\cdot\|_{H^s_{\mathscr{G}}} = \Big(\sum_{k\in\mathbb{N}^*} |k^s \langle \cdot, \phi_k \rangle_{L^2}|^2 \Big)^{\frac{1}{2}}.$$

We introduce the main assumptions adopted in the manuscript by considering $(\mu_k)_{k\in\mathbb{N}^*} \subseteq (\lambda_k)_{k\in\mathbb{N}^*}$ an ordered sequence of some eigenvalues of A and

$$\boldsymbol{\varphi} := (\varphi_k)_{k\in\mathbb{N}^*} \subseteq (\phi_k)_{k\in\mathbb{N}^*}$$

the corresponding eigenfunctions. Let $\eta > 0$, $a \geq 0$, $I := \{(j,k) \in (\mathbb{N}^*)^2 : j \neq k\}$ and

$$\widetilde{\mathscr{H}} := \overline{span\{\varphi_k \mid k \in \mathbb{N}^*\}}^{\,L^2}.$$

Assumptions I $(\boldsymbol{\varphi}, \eta)$**.** The bounded symmetric operator B satisfies the following conditions.

1. There exists $C > 0$ such that
$$|\langle \varphi_k, B\varphi_1 \rangle_{L^2}| \geq \frac{C}{k^{2+\eta}}, \qquad \forall k \in \mathbb{N}^*.$$
2. For every $(j,k),(l,m) \in I$ such that $(j,k) \neq (l,m)$ and $\mu_j - \mu_k - \mu_l + \mu_m = 0$, it holds
$$\langle \varphi_j, B\varphi_j \rangle_{L^2} - \langle \varphi_k, B\varphi_k \rangle_{L^2} - \langle \varphi_l, B\varphi_l \rangle_{L^2} + \langle \varphi_m, B\varphi_m \rangle_{L^2} \neq 0.$$

Assumptions I (η)**.** The couple (A, B) satisfies Assumptions I(Φ, η) with Φ defined in (3).

Assumptions II (φ, η, a)**.** Let $Ran(B|_{H^2_{\mathscr{G}}\cap\widetilde{\mathscr{H}}}) \subseteq H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ and one of the following points be satisfied.

1. When $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$ and $a+\eta \in (0, 3/2)$, there exists $d \in [\max\{a+\eta, 1\}, 3/2)$ such that
$$Ran(B|_{H^{2+d}_{\mathscr{G}}\cap\widetilde{\mathscr{H}}}) \subseteq H^{2+d} \cap H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}.$$

2. When $\mathscr{G}$ is equipped with $(\mathcal{N})$ and $a+\eta \in (0, 7/2)$, there exist $d \in [\max\{a+\eta, 2\}, 7/2)$ and $d_1 \in (d, 7/2)$ such that
$$Ran(B|_{H^{d_1}_{\mathcal{NK}}\cap\widetilde{\mathscr{H}}}) \subseteq H^{d_1}_{\mathcal{NK}} \cap \widetilde{\mathscr{H}}, \qquad Ran(B|_{H^{2+d}_{\mathscr{G}}\cap\widetilde{\mathscr{H}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}.$$

3. When $\mathscr{G}$ is equipped with $(\mathcal{D})$ and $a+\eta \in (0, 5/2)$, there exists $d \in [\max\{a+\eta, 1\}, 5/2)$ such that
$$Ran(B|_{H^{2+d}_{\mathscr{G}}\cap\widetilde{\mathscr{H}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}.$$

   If $a+\eta \geq 2$, then there exists $d_1 \in (d, 5/2)$ such that
$$Ran(B|_{H^{d_1}\cap\widetilde{\mathscr{H}}}) \subseteq H^{d_1} \cap \widetilde{\mathscr{H}}.$$

Assumptions II (η, a)**.** The couple (A, B) satisfies Assumptions II(Φ, η, a) with Φ defined in (3).

2.1 Well-posedness of the bilinear Schrödinger equation

Now, we cite [Duc18a, *Proposition* 3.1] where the well-posedness of the bilinear Schrödinger equation (BSE) is ensured in $H^s_{\mathscr{G}}$ with suitable $s \geq 3$.

Proposition 2.1. [*Duc18a, Proposition* 3.1] *Let $\mathscr{G}$ be a compact quantum graph and (A, B) satisfy Assumptions II$(\eta, \tilde{d})$ with $\eta > 0$ and $\tilde{d} \geq 0$. For any $T > 0$ and $u \in L^2((0,T), \mathbb{R})$, the flow of the (BSE) is unitary in $\mathscr{H}$ and, for any initial data $\psi^0 \in H^{2+d}_{\mathscr{G}}$ with d from Assumptions II$(\eta, \tilde{d})$, there exists a unique mild solution of (BSE) in $H^{2+d}_{\mathscr{G}}$, i.e. a function $\psi \in C^0([0,T], H^{2+d}_{\mathscr{G}})$ such that*

$$\psi(t,x) = e^{-iAt}\psi^0(x) - i\int_0^t e^{-iA(t-s)}u(s)B\psi(s,x)ds, \qquad \forall t \in [0,T]. \tag{5}$$

Remark 2.2. *Let $\varphi := (\varphi_k)_{k\in\mathbb{N}^*} \subseteq (\phi_k)_{k\in\mathbb{N}^*}$ be an orthonormal system of $\mathscr{H}$ made by eigenfunctions of A and*
$$\widetilde{\mathscr{H}} := \overline{span\{\varphi_k \mid k \in \mathbb{N}^*\}}^{L^2}.$$
If (A, B) satisfies Assumptions II$(\varphi, \eta, \tilde{d})$ with $\eta > 0$ and $\tilde{d} \geq 0$, then, for every $\psi^0 \in H^{2+d}_{\mathscr{G}} \cup \widetilde{\mathscr{H}}$ with d from Assumptions II$(\varphi, \eta, \tilde{d})$ and $u \in L^2((0,T), \mathbb{R})$, there exists a unique mild solution of (BSE) in
$$H^{2+d}_{\mathscr{G}} \cup \widetilde{\mathscr{H}}.$$
The statement follows equivalently to Proposition 2.1 as the propagator Γ^u_t preserves the space $H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ when $B : H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}} \longrightarrow H^2_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$.

3 Global exact controllability

Theorem 3.1. *Let $\mathscr{G}$ be a compact quantum graph and $(\lambda_k)_{k\in\mathbb{N}^*}$ be the ordered sequence of eigenvalues of A. Let $G \in L^\infty(\mathbb{R}, \mathbb{R})$ be an entire function such that there exist $J, I > 0$ such that*
$$|G(z)| \leq Je^{I|z|}, \qquad \forall \in \mathbb{C}.$$
The eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ are simple, the numbers $\{\pm\sqrt{\lambda_k}\}_{k\in\mathbb{N}^*}$ are simple zeros of G and there exist $\tilde{d} \geq 0$ and $C > 0$ such that
$$|G'(\pm\sqrt{\lambda_k})| \geq \frac{C}{k^{1+\tilde{d}}}, \qquad \forall k \in \mathbb{N}^*.$$
If the couple (A, B) satisfies Assumptions I(η) and Assumptions II$(\eta, \tilde{d})$ for $\eta > 0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s = 2 + d$ and d from Assumptions II$(\eta, \tilde{d})$.

Proof. **1) Local exact controllability.** For $\epsilon, T > 0$, let

$$O^s_{\epsilon,T} := \big\{\psi \in H^s_{\mathscr{G}} \big| \ \|\psi\|_{L^2} = 1, \ \|\psi - \phi_1(T)\|_{(s)} < \epsilon\big\}, \qquad \phi_1(T) = e^{-i\lambda_1 T}\phi_1.$$

We prove the existence of $T, \epsilon > 0$ so that, for every $\psi \in O^s_{\epsilon,T}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that $\psi = \Gamma^u_T \phi_1$. The result corresponds to the surjectivity, for $T > 0$ sufficiently large, of the map

$$\Gamma^{(\cdot)}_T \phi_1 : u \in L^2((0,T),\mathbb{R}) \longmapsto \psi \in O^s_{\epsilon,T} \subset H^s_{\mathscr{G}}.$$

We decompose $\Gamma^{(\cdot)}_T\phi_1 = \sum_{k\in\mathbb{N}^*} \phi_k(T)\langle\phi_k(T), \Gamma^{(\cdot)}_T\phi_1\rangle_{L^2}$ and we consider the map α such that

$$\alpha(\cdot) = \big(\langle\phi_k(T), \Gamma^{(\cdot)}_T\phi_1\rangle_{L^2}\big)_{k\in\mathbb{N}^*} : L^2((0,T),\mathbb{R}) \longrightarrow Q := \{\mathbf{x} := (x_k)_{k\in\mathbb{N}^*} \in h^s(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}.$$

The local exact controllability is equivalent to the local surjectivity of α. To this end, we use the Generalized Inverse Function Theorem ([Lue69, Theorem 1; p. 240]) and we study the surjectivity of $\gamma(v) := (d_u\alpha(0))\cdot\ v$ the Fréchet derivative of α. The map γ is the sequence of elements

$$\gamma_k(v) := -i\int_0^T v(\tau)e^{i(\lambda_k-\lambda_1)s}d\tau\langle\phi_k, B\phi_1\rangle_{L^2}, \qquad \forall k \in \mathbb{N}^*$$

so that $\gamma : L^2((0,T),\mathbb{R}) \longrightarrow T_\delta Q = \{\mathbf{x} := (x_k)_{k\in\mathbb{N}^*} \in h^s(\mathbb{C}) \mid ix_1 \in \mathbb{R}\}$ with $\alpha(0) = \delta = (\delta_{k,1})_{k\in\mathbb{N}^*}$. The surjectivity of γ corresponds to the solvability of the moments problem, for $(x_k)_{k\in\mathbb{N}^*} \in T_\delta Q$,

$$x_k\langle\phi_j, B\phi_k\rangle^{-1}_{L^2} = -i\int_0^T u(\tau)e^{i(\lambda_k-\lambda_1)\tau}d\tau, \qquad \forall k \in \mathbb{N}^*. \tag{6}$$

In other words, we need to ensure that, for every $(x_k)_{k\in\mathbb{N}} \in \{(x_k)_{k\in\mathbb{N}} \in h^s(\mathbb{C}) \mid ix_1 \in \mathbb{R}\} \subset h^s$, there exists $u \in L^2((0,T),\mathbb{R})$ with $T > 0$ such that the relations (6) are satisfied for every $k \in \mathbb{N}^*$. To this purpose, we notice that $\big(x_k\langle\phi_k, B\phi_1\rangle^{-1}_{L^2}\big)_{k\in\mathbb{N}^*} \in h^{s-2-\eta} = h^{d-\eta} \subseteq h^{\tilde{d}}$ thanks to the point **1.** of Assumptions I(η). As B is symmetric, we have

$$\langle\phi_1, B\phi_1\rangle_{L^2} \in \mathbb{R}, \qquad\qquad ix_1\langle\phi_1, B\phi_1\rangle^{-1}_{L^2} \in \mathbb{R}.$$

From Proposition $C.7$, the solvability of (6) is guaranteed thanks to the identity (1) and since

$$\big(x_k\langle\phi_k, B\phi_1\rangle^{-1}_{L^2}\big)_{k\in\mathbb{N}^*} \in \{(c_k)_{k\in\mathbb{N}^*} \in h^{\tilde{d}}(\mathbb{C}) \mid c_1 \in \mathbb{R}\}.$$

The local exact controllability is proved and the result is also valid for the reversed dynamics (see [Duc18c, *Section* 1.3]). Thus, for every $\psi \in O^s_{\epsilon,T}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that $\phi_1 = \Gamma^u_T\psi$.

2) Global exact controllability. Let $T, \epsilon > 0$ be so that **1)** is valid. Thanks to Theorem $A.2$, for any $\psi_1, \psi_2 \in H^s_{\mathscr{G}}$ such that $\|\psi_1\|_{L^2} = \|\psi_2\|_{L^2} = p$, there exist $T_1, T_2 > 0$, $u_1 \in L^2((0,T_1),\mathbb{R})$ and $u_2 \in L^2((0,T_2),\mathbb{R})$ such that

$$\|\Gamma^{u_1}_{T_1}p^{-1}\psi_1 - \phi_1\|_{(s)} < \epsilon, \qquad \|\Gamma^{u_2}_{T_2}p^{-1}\psi_2 - \phi_1\|_{(s)} < \epsilon, \qquad \Longrightarrow \qquad p^{-1}\Gamma^{u_1}_{T_1}\psi_1, p^{-1}\Gamma^{u_2}_{T_2}\psi_2 \in O^s_{\epsilon,T}.$$

From the point **1)**, there exist $u_3, u_4 \in L^2((0,T),\mathbb{R})$ such that $\Gamma^{u_3}_T\Gamma^{u_1}_{T_1}\psi_1 = \Gamma^{u_4}_T\Gamma^{u_2}_{T_2}\psi_2 = p\phi_1$. In conclusion, there exists $\widetilde{T} > 0$ and $\widetilde{u} \in L^2((0,\widetilde{T}),\mathbb{R})$ such that $\Gamma^{\widetilde{u}}_{\widetilde{T}}\psi_1 = \psi_2$. □

3.1 Global exact controllability of bilinear quantum systems on star graphs

In the current section, we ensure the global exact controllability when $\mathscr{G}$ is a suitable star graph. From now on, when we denote $\mathscr{G}$ as a star graph, we also consider it as a quantum graph.

Theorem 3.2. *Let $\mathscr{G}$ be a star graph equipped with $(\mathcal{D}/\mathcal{N})$ made by edges long $\{L_j\}_{j\le N} \in \mathcal{AL}(N)$. If the couple (A, B) satisfies Assumptions I(η) and Assumptions II(η,ϵ) for $\eta, \epsilon > 0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s = 2 + d$ and d from Assumptions II(η,ϵ).*

Proof. **1) Star graph equipped with $(\mathcal{D})$.** The conditions $(\mathcal{D})$ on V_e imply that, for each $k \in \mathbb{N}^*$,

$$\phi_k = (a_k^1 \sin(\sqrt{\lambda_k}x), ..., a_k^n \sin(\sqrt{\lambda_k}x))$$

for suitable $\{a_k^l\}_{l\leq N} \subset \mathbb{C}$ such that $(\phi_k)_{k\in\mathbb{N}}$ is orthormal in $\mathscr{H}$. The conditions $(\mathcal{NK})$ in the internal vertex $v \in V_i$ ensure that

$$(7) \qquad \begin{cases} a_k^1 \sin(\sqrt{\lambda_k}L_1) = ... = a_k^N \sin(\sqrt{\lambda_k}L_N), \\ \sum_{l\leq N} a_k^l \cos(\sqrt{\lambda_k}L_l) = 0, \end{cases} \quad \Rightarrow \quad \sum_{l=1}^N \cot(\sqrt{\lambda_k}L_l) = 0.$$

We use the provided identities in order to construct an entire function satisfying the hypotheses of Theorem 3.1. To this purpose, we define the maps

$$G(x) := \prod_{l\leq N} \sin(xL_l) \sum_{l\leq N} \cot(xL_l) \qquad \tilde{G}(x) := \prod_{l\leq N} \sin(xL_l) \sum_{l\leq N} \frac{L_l}{\sin^2(xL_l)}.$$

As $|\cos(zL_l)| \leq e^{L_l|z|}$ and $|\sin(zL_l)| \leq e^{L_l|z|}$ for every $l \leq N$ and $z \in \mathbb{C}$, we notice that G is an entire function such that

$$|G(z)| \leq Ne^{|z|\sum_{l=1}^N L_l} \qquad \forall z \in \mathbb{C}.$$

In addition, $G(\lambda_k) = 0$ for every $k \in \mathbb{N}^*$ thanks to (7) and $G \in L^\infty(\mathbb{R},\mathbb{R})$, while

$$G'(x) = -\tilde{G}(x) + H(x), \qquad H(x) := \frac{d}{dx}\Big(\prod_{l\leq N} \cos(xL_l)\Big) \sum_{l\leq N} \cot(xL_l).$$

The identities (7) imply that $H(\sqrt{\lambda_k}) = 0$ and then

$$(8) \qquad G'(\sqrt{\lambda_k}) = -\tilde{G}(\sqrt{\lambda_k}), \qquad \forall k \in \mathbb{N}^*.$$

Now, for $L^* := \min_{l\leq N} L_l$ and $x \in \mathbb{R}$, we have

$$(9) \qquad |\tilde{G}(x)| = \frac{\prod_{l\leq N} |\sin(xL_l)| \sum_{l\leq N} L_l \prod_{k\neq l} \sin^2(xL_k)}{\prod_{l\leq N} \sin^2(xL_l)} \geq L^* \sum_{l\leq N} \prod_{k\neq l} |\sin(xL_k)|.$$

We refer to [DZ06, *Corollary A*.10; (2)], which contains a misprint as it is valid for every

$$\lambda > \frac{\pi}{2}\max\{1/L_j \ : \ j \leq N\}.$$

Thanks to the relations (8) and (9), the mentioned corollary ensures that, for every $\epsilon > 0$, there exists $C_1 > 0$ such that

$$|G'(\pm\sqrt{\lambda_k})| \geq L^* \sum_{l=1}^N \prod_{j\neq l} |\sin(\sqrt{\lambda_k}L_j)| \geq \frac{C_1}{(\sqrt{\lambda_k})^{1+\epsilon}}, \qquad \forall k \in \mathbb{N}^* \quad : \quad \lambda_k > \frac{\pi}{2}\max\{1/L_j \ : \ j \leq N\}.$$

Remark 3.3. *For every $k \in \mathbb{N}^*$ and $j \leq N$, we have $|\phi_k^j(L_j)| \neq 0$, otherwise the $(\mathcal{NK})$ conditions would ensure that $\phi_k^l(L_l) = \phi_k^m(L_m) = 0$ with $l, m \leq N$ so that $\phi_k^l, \phi_k^m \not\equiv 0$ and there would be satisfied*

$$a_k^l \sin(L_l\sqrt{\lambda_k}) = a_k^m \sin(L_m\sqrt{\lambda_k}) = 0$$

with $a_k^l, a_k^m \neq 0$, which is absurd as $\{L_j\}_{j\leq N} \in \mathcal{AL}(N)$.

Remark 3.3 implies $|G'(\pm\sqrt{\lambda_k})| \neq 0$ for every $k \in \mathbb{N}^*$ and, from the relation (4), there exist $\epsilon > 0$ and $C_2 > 0$ such that

$$|G'(\pm\sqrt{\lambda_k})| \geq \frac{C_2}{k^{1+\epsilon}}, \quad \forall k \in \mathbb{N}^*.$$

We notice that the spectrum of A is simple. Indeed, if there would exist two orthonormal eigenfuctions f and g of A corresponding to the same eigenvalue λ, then $h(x) = f(v)g(x) - g(v)f(x)$ would be another

eigenfunction of A. Now, h is an eigenfunction corresponding to λ and $h(v)=0$ that is impossible thanks to Remark 3.3.
In conclusion, the claim is achieved as Theorem 3.1 is valid with respect to the function G when $\tilde{d}=\epsilon$.
2) Generic star graph. Let $I_1 \subseteq \{1,...,N\}$ be the set of indices of those edges containing an external vertex equipped with $(\mathcal{N})$ and $I_2 := \{1,..,N\}\setminus I_1$. The proof follows from the techniques adopted in **1)** by considering Proposition $B.2$ (instead of [DZ06, *Corollary* $A.10$; (2)]) and the entire map

$$G(x) := \prod_{l\in I_2}\sin(xL_l)\prod_{l\in I_1}\cos(xL_l)\Big(\sum_{l\in I_2}\cot(xL_l)+\sum_{l\in I_1}\tan(xL_l)\Big). \qquad \square$$

Corollary 3.4. *Let $\mathscr{G}$ be a star graph equipped with ($\mathcal{D}/\mathcal{N}$). Let $\mathscr{G}$ satisfy the following conditions with $\widetilde{N}\in 2\mathbb{N}^*$ such that $\widetilde{N}\leq N$.*

- *For every $j\leq \widetilde{N}/2$, the two external vertices of $\mathscr{G}$ belonging to e_{2j-1} and e_{2j} are both equipped with $(\mathcal{D})$ or $(\mathcal{N})$.*
- *The couples of edges $\{e_{2j-1},e_{2j}\}_{j\leq\widetilde{N}/2}$ are long $\{L_j\}_{j\leq\widetilde{N}/2}$, while the edges $\{e_j\}_{\widetilde{N}<j\leq N}$ measure $\{L_j\}_{\widetilde{N}<j\leq N}$. In addition, $\{L_j\}_{j\leq\frac{\widetilde{N}}{2}}\cup\{L_j\}_{\widetilde{N}<j\leq N}\in\mathcal{AL}\big(\frac{\widetilde{N}}{2}+N-\widetilde{N}\big)$.*

If (A,B) satisfies Assumptions I(η) and Assumptions II(η,ϵ) for $\eta,\epsilon>0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s=2+d$ and d from Assumptions II(η,ϵ).

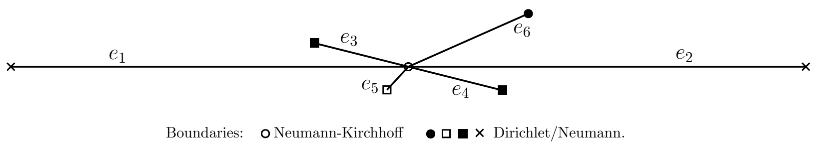

Figure 4: Example of graph described in Corollary 3.4 *with $\widetilde{N}=4$ and $N=6$.*

Proof. Let $I_1\subseteq\{1,...,\widetilde{N}/2\}$ be the set of j such that e_{2j-1} and e_{2j} contain two external vertices of $\mathscr{G}$ equipped with $(\mathcal{N})$ and $I_2 := \{1,..,\widetilde{N}/2\}\setminus I_1$. Let $I_3\subseteq\{\widetilde{N}+1,...,N\}$ be the set of j such that e_j contains an external vertex of $\mathscr{G}$ equipped with $(\mathcal{N})$ and $I_4 := \{\widetilde{N}+1,...,N\}\setminus I_3$. Let

$$(\lambda_k^1)_{k\in\mathbb{N}^*} := \Big(\frac{(2k-1)^2\pi^2}{4L_j^2}\Big)_{\substack{j,k\in\mathbb{N}^*\\ j\in I_1}}, \qquad (\lambda_k^2)_{k\in\mathbb{N}^*} := \Big(\frac{k^2\pi^2}{L_j^2}\Big)_{\substack{j,k\in\mathbb{N}^*\\ j\in I_2}}.$$

We notice that $(\lambda_k^1)_{k\in\mathbb{N}^*}\cup(\lambda_k^2)_{k\in\mathbb{N}^*}\subset(\lambda_k)_{k\in\mathbb{N}^*}$ are the only eigenvalues of A corresponding to eigenfunctions vanishing in the internal vertex v. For every eigenfuction $f\in(\phi_k)_{k\in\mathbb{N}^*}$ of A corresponding to an eigenvalue

$$\lambda = \frac{(2k-1)^2\pi^2}{4L_j^2}\in(\lambda_k^1)_{k\in\mathbb{N}^*}, \qquad k\in\mathbb{N}^*,\ j\in I_1,$$

the eigenfunction f is uniquely defined (up to multiplication for $\alpha\in\mathbb{C}$ such that $|\alpha|=1$) by the identities

$$f^{2j-1}(x) = -f^{2j}(x) = \sqrt{L_j^{-1}}\cos(\sqrt{\lambda}x), \qquad f^l\equiv 0, \qquad \forall l\in\{1,...,N\}\setminus\{2j-1,2j\}.$$

Equivalently, it is valid with $(\lambda_k^2)_{k\in\mathbb{N}^*}$ and then the eigenvalues $(\lambda_k^1)_{k\in\mathbb{N}^*}\cup(\lambda_k^2)_{k\in\mathbb{N}^*}$ are simple. In conclusion, the discrete spectrum of A is simple since, if there would exist a multiple eigenvalue

$$\lambda\in(\lambda_k)_{k\in\mathbb{N}^*}\setminus\Big((\lambda_k^1)_{k\in\mathbb{N}^*}\cup(\lambda_k^2)_{k\in\mathbb{N}^*}\Big),$$

then there would exist two orthonormal eigenfuctions f and g corresponding to the same eigenvalue λ. Now, $h(x)=f(v)g(x)-g(v)f(x)$ would be another eigenfunction corresponding to λ such that $h(v)=0$, which is impossible as it would imply that

$$\lambda\in(\lambda_k^1)_{k\in\mathbb{N}^*}\cup(\lambda_k^2)_{k\in\mathbb{N}^*}.$$

Thus, the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ are simple. The remaining part of proof follows the one of Theorem 3.2 thanks to Proposition $B.2$ by considering the entire function

$$G(x) := \prod_{l\in I_2\cup I_4} \sin(xL_l) \prod_{l\in I_1\cup I_3} \cos(xL_l)\Big(2\sum_{l\in I_2} \cot(xL_l) + 2\sum_{l\in I_1} \tan(xL_l) + \sum_{l\in I_4} \cot(xL_l) + \sum_{l\in I_3} \tan(xL_l)\Big).$$

□

3.2 Applications and proof of Theorem 1.3

In the following theorem, we apply Theorem 3.2 for a specific problem.

Theorem 3.5. *Let $\mathscr{G}$ be a star graph equipped with $(\mathcal{N})$. For every $\psi \in \mathscr{H}$, let B be such that*

$$B(\psi^1, ..., \psi^N) = \big((5x^6 - 24x^5L_1 + 45x^4L_1^2 - 40x^3L_1^3 + 15x^2L_1^4 - L_1^6)\psi^1, 0, ..., 0\big).$$

There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable so that, for every $\{L_j\}_{j\le N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, the problem (BSE) is globally exactly controllable in

$$H^{5+\epsilon}_{\mathscr{G}}, \qquad \forall \epsilon \in (0, 1/2).$$

Proof. The conditions $(\mathcal{N})$ in V_i imply the existence, for every $k \in \mathbb{N}^*$, of $\{a_k^l\}_{l\le N} \subset \mathbb{C}$ such that

$$\phi_k = (a_k^1 \cos(x\sqrt{\lambda_k}), ..., a_k^N \cos(x\sqrt{\lambda_k})).$$

The coefficients $\{a_k^l\}_{l\le N} \subset \mathbb{C}$ are so that $(\phi_k)_{k\in\mathbb{N}^*}$ forms a Hilbert basis of $\mathscr{H}$ and then

$$(10) \qquad 1 = \sum_{l\le N} \int_0^{L_l} |a_k^l|^2 \cos^2(x\sqrt{\lambda_k})dx = \sum_{l\le N} |a_k^l|^2\Big(\frac{L_l}{2} + \frac{\sin(2L_l\sqrt{\lambda_k})}{4\sqrt{\lambda_k}}\Big).$$

For every $k \in \mathbb{N}^*$, the $(\mathcal{NK})$ boundary conditions in V_i ensure

$$(11) \qquad \begin{aligned} &a_k^1 \cos(\sqrt{\lambda_k}L_1) = ... = a_k^N \cos(\sqrt{\lambda_k}L_N), \qquad \sum_{l\le N} a_k^l \sin(\sqrt{\lambda_k}L_l) = 0,\\ &\sum_{l\le N} \tan(\sqrt{\lambda_k}L_l) = 0, \qquad \sum_{l\le N} |a_k^l|^2 \sin(2L_l\sqrt{\lambda_k}) = 0. \end{aligned}$$

The last identities and (10) imply $1 = \sum_{l=1}^N |a_k^l|^2 L_l/2$. Thanks to (11), we have $a_k^l = a_k^1 \frac{\cos(\sqrt{\lambda_k}L_1)}{\cos(\sqrt{\lambda_k}L_l)}$ for $l \ne 1$ and $k \in \mathbb{N}^*$. Thus, $|a_k^1|^2\big(L_1 + \sum_{l=2}^N L_l \frac{\cos^2(\sqrt{\lambda_k}L_1)}{\cos^2(\sqrt{\lambda_k}L_l)}\big) = 2$ for every $k \in \mathbb{N}^*$ and

$$(12) \qquad |a_k^1|^2 = 2\prod_{m=2}^N \cos^2(\sqrt{\lambda_k}L_m)\Big(\sum_{j=1}^N L_j \prod_{m\ne j} \cos^2(\sqrt{\lambda_k}L_m)\Big)^{-1}.$$

Verifying Assumptions $\mathbf{I(3+\epsilon)}$ with $\boldsymbol{\epsilon > 0}$. For every $k \in \mathbb{N}^*$, thanks to the relation (11)

$$\prod_{l\le N} \cos(\sqrt{\lambda_k}L_l) \sum_{l\le N} \tan(\sqrt{\lambda_k}L_l) = 0, \qquad \Longrightarrow \qquad \sum_{l=1}^N \sin(\sqrt{\lambda_k}L_l) \prod_{m\ne l} \cos(\sqrt{\lambda_k}L_m) = 0.$$

Thanks to the relation (4) and Corollary $B.3$, for every $\epsilon > 0$, there exist $C_1, C_2 > 0$ such that,

$$(13) \qquad |a_k^1| \ge \sqrt{\frac{2}{\sum_{l=1}^N L_l \cos^{-2}(\sqrt{\lambda_k}L_l)}} \ge \sqrt{\frac{2}{\sum_{l=1}^N L_l C_1^{-2}\lambda_k^{1+\epsilon}}} \ge \frac{C_2}{k^{1+\epsilon}}, \qquad \forall k \in \mathbb{N}^*.$$

In addition, $\langle \phi_1^l, B\phi_k^l\rangle_{L^2(e_l,\mathbb{C})} = 0$ for $2 \le l \le N$ and, for every $k \in \mathbb{N}^*$,

$$(14) \qquad \langle \phi_1, B\phi_k\rangle_{L^2} = -\frac{120a_k^1a_1^1L_1^6}{(\sqrt{\lambda_k} + \sqrt{\lambda_1})^4} - \frac{120a_k^1a_1^1L_1^6}{(\sqrt{\lambda_k} - \sqrt{\lambda_1})^4} + o(\sqrt{\lambda_k}^{-5}).$$

From the relations (13) and (14), thanks to the relation (4), for every $\epsilon > 0$, there exists $C_3 > 0$, such that for $k \in \mathbb{N}^*$ sufficiently large,

$$|\langle \phi_1, B\phi_k \rangle_{L^2}| \geq \frac{C_3}{k^{5+\epsilon}}. \tag{15}$$

Now, as done in [Duc18a, *Example* 1.2], it is possible to compute $a_k(\cdot)$ and $B_k(\cdot)$ with $k \in \mathbb{N}^*$, analytic functions in $\mathbb{R}^+$, so that

$$a_k(L_1)^2 = (a_k^1)^2, \quad a_1(L_1)a_k(L_1)B_k(L_1) = \langle \phi_1, B\phi_k \rangle_{L^2}$$

and each $a_1(\cdot)a_k(\cdot)B_k(\cdot)$ is non-constant and analytic. Each $a_1(\cdot)a_k(\cdot)B_k(\cdot)$ has discrete zeros $\tilde{V}_k \subset \mathbb{R}^+$ and $\tilde{V} = \bigcup_{k\in\mathbb{N}^*} \tilde{V}_k$ is countable. For every $\{L_l\}_{l\leq N} \in \mathcal{AL}(N)$ so that $L_1 \notin \tilde{V}$,

$$|\langle \phi_1, B\phi_k \rangle_{L^2}| \neq 0, \qquad \forall k \in \mathbb{N}^*. \tag{16}$$

Thus, the point **1.** of Assumptions I$(3+\epsilon)$ is ensured thanks to the relations (15) and (16) since, for every $\epsilon > 0$, there exists $C_4 > 0$ such that

$$|\langle \phi_1, B\phi_k \rangle_{L^2}| \geq \frac{C_4}{k^{5+\epsilon}}, \quad \forall k \in \mathbb{N}^*.$$

Let $(k,j),(m,n) \in I$, $(k,j) \neq (m,n)$ for $I := \{(j,k) \in (\mathbb{N}^*)^2 : j \neq k\}$. We prove the validity of the point **2.** of Assumptions I$(3+\epsilon)$. As above, we compute $F_k(\cdot)$ with $k \in \mathbb{N}^*$, analytic in $\mathbb{R}^+$, such that $\langle \phi_k, B\phi_k \rangle_{L^2} = F_k(L_1)$. Each $F_{j,k,l,m}(\cdot) := F_j(\cdot) - F_k(\cdot) - F_l(\cdot) + F_m(\cdot)$ is non-constant and analytic in $\mathbb{R}^+$, the set of its positive zeros $V_{j,k,l,m}$ is discrete and

$$V := \bigcup_{(j,k),(l,m)\in I \,:\, (j,k)\neq(l,m)} V_{j,k,l,m}$$

is countable. For $\{L_l\}_{l\leq N} \in \mathcal{AL}(N)$ so that $L_1 \notin V \cup \tilde{V}$, the point **2.** of Assumptions I$(3+\epsilon)$ with $\epsilon > 0$ is satisfied.

Verifying Assumptions II$(3+\epsilon_1, \epsilon_2)$ with $\epsilon_1, \epsilon_2 > 0$ so that $\epsilon_1 + \epsilon_2 \in \left(0, \frac{1}{2}\right)$. Let

$$P(x) := (5x^6 - 24x^5 L_1 + 45x^4 L_1^2 - 40x^3 L_1^3 + 15x^2 L_1^4 - L_1^6).$$

For $m > 0$, we notice $B : H^m \longrightarrow H^m$ and $\partial_x(B\psi)(\tilde{v}) = 0$ for every $\tilde{v} \in V_e$ since $\partial_x P(0) = 0$. Now, $\partial_x(B\psi)(v) = (B\psi)(v) = 0$ with $v \in V_i$ as $\partial_x P(L_1) = P(L_1) = 0$ and then $B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}$. Moreover, $\partial_x^2 P(L_1) = \partial_x^3 P(L_1) = 0$, which imply $B : H^m_{\mathcal{NK}} \longrightarrow H^m_{\mathcal{NK}}$ for every $m \in \left(0, \frac{9}{2}\right)$. For $d \in \left[3+\epsilon_1+\epsilon_2, \frac{7}{2}\right)$ and $d_1 \in \left(d, \frac{7}{2}\right)$, there follow

$$Ran(B|_{H^{d_1}_{\mathcal{NK}}}) \subseteq H^{d_1}_{\mathcal{NK}}, \qquad Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq Ran(B|_{H^{2+d}\cap H^{1+d}_{\mathcal{NK}}\cap H^2_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}.$$

The point **2.** of Assumptions II$(3+\epsilon_1, \epsilon_2)$ with $\epsilon_1, \epsilon_2 > 0$ so that $\epsilon_1 + \epsilon_2 \in \left(0, \frac{1}{2}\right)$ is valid.

Conclusion. The couple (A, B) satisfies Assumptions I$(3+\epsilon)$ and Assumptions II$(3+\epsilon_1, \epsilon_2)$ with $\epsilon_1, \epsilon_2 > 0$ so that $\epsilon_1 + \epsilon_2 \in \left(0, \frac{1}{2}\right)$. Theorem 3.2 guarantees the global exact controllability of the (BSE) in $H^s_{\mathscr{G}}$ with $s = 2 + d$ and $d \in \left[3+\epsilon_1+\epsilon_2, \frac{7}{2}\right)$. □

Proof of Theorem 1.3. Theorem 1.3 is proved as [Duc18a, *Example* 1.2] that is stated for $N = 4$. The only difference between the two results is that Theorem 1.3 is ensured from the validity of Theorem 3.2 instead of [Duc18a, *Theorem* 2.4], which is only valid for $N \leq 4$. □

4 Energetic controllability

Let us recall the notation $(\varphi_k)_{k\in\mathbb{N}^*} \subseteq (\phi_k)_{k\in\mathbb{N}^*}$ indicating an orthonormal system of $\mathscr{H}$ made by some eigenfunctions of A. Let $(\mu_k)_{k\in\mathbb{N}^*}$ be the ordered sequence of corresponding eigenvalues. We refer to Definition 1.4 for the formal definition of energetic controllability.

Theorem 4.1. *Let $\mathscr{G}$ be a compact quantum graph and one of the following points be verified.*

1. *There exists an entire function G such that $G \in L^\infty(\mathbb{R},\mathbb{R})$ and there exist $J, I > 0$ so that*
$$|G(z)| \leq J e^{I|z|}, \qquad \forall z \in \mathbb{C}.$$
The numbers $\{\pm\sqrt{\mu_k}\}_{k\in\mathbb{N}^*}$ are simple zeros of G and there exist $\tilde{d} \geq 0$ and $C > 0$ so that
$$|G'(\pm\sqrt{\mu_k})| \geq \frac{C}{k^{1+\tilde{d}}}, \qquad \forall k \in \mathbb{N}^*.$$

2. *For every $\epsilon > 0$, there exist $C > 0$ and $\tilde{d} \geq 0$ so that $|\mu_{k+1} - \mu_k| \geq \frac{C}{k^{\tilde{d}}}$ for each $k \in \mathbb{N}^*$.*

If (A, B) satisfies Assumptions I(φ, η) and Assumptions II$(\varphi, \eta, \tilde{d})$ for $\eta > 0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ for $s = 2 + d$ with d from Assumptions II$(\varphi, \eta, \tilde{d})$ and energetically controllable in $(\mu_k)_{k\in\mathbb{N}^*}$.

Proof. From Remark 2.2, the (BSE) is well-posed in $H^s_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ with $s = 2 + d$ and d from Assumptions II$(\varphi, \eta, \tilde{d})$. The statement of Theorem 3.1 holds in $\widetilde{\mathscr{H}}$ when the point **1.** is valid, while the validity of [Duc18a, *Theorem* 2.3] in $\widetilde{\mathscr{H}}$ is guaranteed by **2.** . The global exact controllability is provided in $H^s_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ and the energetic controllability follows as $\varphi_k \in H^s_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ for every $k \in \mathbb{N}^*$. □

Let $\mathscr{G}$ be a generic compact quantum graph. By watching the structure of the graph and the boundary conditions of $D(A)$, it is possible to construct some eigenfuctions $(\varphi_k)_{k\in\mathbb{N}^*}$ of A corresponding to some eigenvalues $(\mu_k)_{k\in\mathbb{N}^*}$. For instance, we consider $\mathscr{G}$ containing a self-closing edge e_1 of length 1.

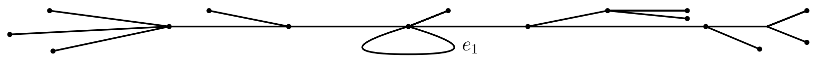

Figure 5: Example of compact graph containing a self-closing edge.

We define $\varphi := (\varphi_k)_{k\in\mathbb{N}^*}$ such that $\varphi_k = \big(\sqrt{2}\sin(2k\pi x), 0, ..., 0\big)$ and the corresponding eigenvalues $(\mu_k)_{k\in\mathbb{N}^*} = (4k^2\pi^2)_{k\in\mathbb{N}^*} \subseteq (\lambda_k)_{k\in\mathbb{N}^*}$, which satisfy the gap condition
$$\inf_{k\in\mathbb{N}^*} |\mu_{k+1} - \mu_k| = 12\pi^2 > 0.$$

If Assumptions I(φ, η) and Assumptions II$(\varphi, \eta, 0)$ are satisfied for $\eta > 0$, then Theorem 4.1 implies the energetic controllability in $(\mu_k)_{k\in\mathbb{N}^*}$. As we do in the proof of Theorem 4.4, this approach is also valid when $\mathscr{G}$ contains more self-closing edges (*e.g.* Figure 3).

Remark. *The idea described above can be adopted when $\mathscr{G}$ contains suitable sub-graphs denoted "uniform chains". A uniform chain is a sequence of edges of equal length L connecting $M \in \mathbb{N}^*$ vertices $\{v_j\}_{j\leq M}$ such that, if $M \geq 3$, then $v_2, ..., v_{M-1} \in V_i$. Moreover, one of the following assumptions is valid.*

- *The vertices $v_1, v_M \in V_e$ are equipped with $(\mathcal{D})$.*
- *The vertices $v_1 = v_M$ belong to V_i.*
- *The number of vertices $M \in \{2, 3\}$ and $v_1, v_M \in V_e$ are equipped with $(\mathcal{N})$.*

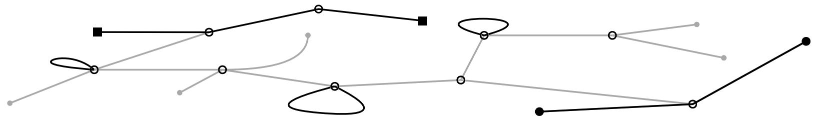

Figure 6: The figure underlines the uniform chains in a generic compact graph.

Let $\mathscr{G}$ contain $\widetilde{N} \in \mathbb{N}^*$ uniform chains $\{\widetilde{\mathscr{G}}_j\}_{j\leq\widetilde{N}}$, composed by edges of lengths $\{L_j\}_{j\leq\widetilde{N}} \in \mathcal{AL}(\widetilde{N})$. Let $I_1 \subseteq \{1,...,\widetilde{N}\}$ and $I_2 \subseteq \{1,...,\widetilde{N}\} \setminus I_1$ be respectively the sets of indices j such that the external vertices of $\widetilde{\mathscr{G}}_j$ are equipped with $(\mathcal{N})$ and $(\mathcal{D})$, while $I_3 := \{1,...,\widetilde{N}\} \setminus (I_1 \cup I_2)$. We consider the eigenvalues $(\mu_k)_{k\in\mathbb{N}^*}$ obtained by reordering

$$\Big(\frac{(2k-1)^2\pi^2}{4L_j^2}\Big)_{\substack{k,j\in\mathbb{N}^*\\ j\in I_1}} \cup \Big(\frac{k^2\pi^2}{L_j^2}\Big)_{\substack{k,j\in\mathbb{N}^*\\ j\in I_2}} \cup \Big(\frac{(2k-1)^2\pi^2}{L_j^2}\Big)_{\substack{k,j\in\mathbb{N}^*\\ j\in I_3}}.$$

As in the proof of [Duc18a, Lemma A.2], the Roth's Theorem [Duc18a, Proposition A.1] ensures that, if $\{L_j\}_{j\leq\widetilde{N}} \in \mathcal{AL}(\widetilde{N})$, then for every $\epsilon > 0$, there exists $C > 0$ so that

$$|\mu_{k+1} - \mu_k| \geq \frac{C_\epsilon}{k^\epsilon}, \qquad \forall k \in \mathbb{N}^*,$$

with $\epsilon > 0$ and $C_\epsilon > 0$ depending on ϵ. In conclusion, if Assumptions I(φ, η) and Assumptions II$(\varphi, \eta, \epsilon)$ are satisfied for $\eta > 0$, then Theorem 4.1 implies the energetic controllability in $(\mu_k)_{k\in\mathbb{N}^*}$

4.1 Applications and proof of Theorem 1.5

Proof of Theorem 1.5. Let us assume $N = 3$. The $(\mathcal{D})$ conditions to the external vertices V_e imply

$$\phi_k = (a_k^1 \sin(\sqrt{\mu_k}x), a_k^2 \sin(\sqrt{\mu_k}x), a_k^3 \sin(\sqrt{\mu_k}x))$$

with suitable $(a_k^1, a_k^2, a_k^3) \in \mathbb{C}^3$. From the $(\mathcal{NK})$ in $v \in V_i$, there follow $\sum_{l\leq 3} a_k^l \cos(\sqrt{\mu_k}L) = 0$ and $a_k^m \sin(\sqrt{\mu_k}L) = c \in \mathbb{R}$ for every $m \leq 3$. When $c \neq 0$, we have the eigenvalues $\big(\frac{(2k-1)^2\pi^2}{4L^2}\big)_{k\in\mathbb{N}^*}$ corresponding to the eigenfunctions $(g_k)_{k\in\mathbb{N}^*}$ so that

$$g_k = \Big(\sqrt{\frac{2}{3L}}\sin\Big(\frac{(2k-1)\pi}{2L}x\Big), \sqrt{\frac{2}{3L}}\sin\Big(\frac{(2k-1)\pi}{2L}x\Big), \sqrt{\frac{2}{3L}}\sin\Big(\frac{(2k-1)\pi}{2L}x\Big)\Big), \qquad \forall k \in \mathbb{N}^*.$$

When $c = 0$, we obtain the eigenvalues $\big(\frac{k^2\pi^2}{L^2}\big)_{k\in\mathbb{N}^*}$ of multiplicity two that we associate to the couple of sequences of eigenfunctions $(f_k^1)_{k\in\mathbb{N}^*}$ and $(f_k^2)_{k\in\mathbb{N}^*}$ such that, for every $k \in \mathbb{N}^*$,

$$f_k^1 := \Big(-\sqrt{\frac{4}{3L}}\sin\Big(\frac{k\pi}{L}x\Big), \sqrt{\frac{1}{3L}}\sin\Big(\frac{k\pi}{L}x\Big), \sqrt{\frac{1}{3L}}\sin\Big(\frac{k\pi}{L}x\Big)\Big),$$

$$f_k^2 := \Big(0, -\sqrt{\frac{1}{L}}\sin\Big(\frac{k\pi}{L}x\Big), \sqrt{\frac{1}{L}}\sin\Big(\frac{k\pi}{L}x\Big)\Big).$$

Moreover, $(f_k^1)_{k\in\mathbb{N}^*} \cup (f_k^2)_{k\in\mathbb{N}^*} \cup (g_k)_{k\in\mathbb{N}^*}$ is an Hilbert basis of $\mathscr{H}$ and the eigenvalues of A (not considering their multiplicity) are $\big(\frac{k^2\pi^2}{L^2}\big)_{k\in\mathbb{N}^*} \cup \big(\frac{(2k-1)^2\pi^2}{4L^2}\big)_{k\in\mathbb{N}^*}$

Verifying Assumptions I$(\varphi, 1)$. We reorder $(f_k^1)_{k\in\mathbb{N}^*} \cup (g_k)_{k\in\mathbb{N}^*}$ in $\varphi = (\varphi_k)_{k\in\mathbb{N}^*}$. The point **1.** of Assumptions I$(\varphi, 1)$ is verified as there exists $C_1, C_2 > 0$ such that

$$|\langle\varphi_1, B\varphi_k\rangle_{L^2}| \geq \frac{C_1\sqrt{\mu_k}\sqrt{\mu_1}}{(\mu_k - \mu_1)^2} \geq \frac{C_2}{k^3}, \qquad \forall k \in \mathbb{N}^*.$$

After, there exist $C_3, C_4 > 0$ so that $B_{k,k} := \langle\varphi_k, B\varphi_k\rangle_{L^2} = C_3 + C_4 k^{-2}$ for every $k \in \mathbb{N}^*$ and $\mu_k = \frac{\pi^2 k^2}{4L^2}$. Now, if $\mu_j - \mu_k - \mu_l + \mu_m = \frac{\pi^2}{4L^2}(j^2 - k^2 - l^2 + m^2) = 0$, then

$$B_{j,j} - B_{k,k} - B_{l,l} + B_{m,m} = C_4(j^{-2} - k^{-2} - l^{-2} + m^{-2}) \neq 0,$$

which implies the point **2.** of Assumptions I$(\varphi, 1)$.

Verifying Assumptions II$(\varphi, 1, 0)$ and conclusion. The operator B stabilizes the spaces H^m with $m > 0$ and $\overline{span\{\varphi_k :\ k \in \mathbb{N}^*\}}^{L^2} \cap H^2_{\mathscr{G}}$, ensuring the point **1.** of Assumptions II$(\varphi, 1, 0)$. Since

$$\inf_{j,k\in\mathbb{N}^*} |\mu_k - \mu_j| = \frac{\pi^2}{4L^2},$$

the point **2.** of Theorem 4.1 holds and the global exact controllability is proved in $H^3_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$. As $\varphi_k \in H^3_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ for every $k \in \mathbb{N}^*$, the energetic controllability follows in $\big(\frac{k^2\pi^2}{4L^2}\big)_{k\in\mathbb{N}^*}$.

When $N > 3$, the spectrum contains simple eigenvalues relative to some eigenfunctions $(g_k)_{k\in\mathbb{N}^*}$ and multiple eigenvalues each one corresponding to $N-1$ eigenfunctions $\{f_{k;j}\}_{l\le N-1}$ with $k \in \mathbb{N}^*$. For each $k \in \mathbb{N}^*$, we construct $\{f_{k;j}\}_{l\le N-1}$ such that only the functions $\{f_{k;j}\}_{l\le N-2}$ vanish in e_1. We reorder $(f_{k;N-1})_{k\in\mathbb{N}^*} \cup (g_k)_{k\in\mathbb{N}^*}$ in $\varphi = (\varphi_k)_{k\in\mathbb{N}^*}$ and the proof is achieved as for $N = 3$. □

Theorem 4.2. *Let $\mathscr{G}$ be a star graph equipped with ($\mathcal{D}/\mathcal{N}$) and containing two edges e_1 and e_2 long 1. Let e_1 and e_2 connect the internal vertex of $\mathscr{G}$, equipped with ($\mathcal{NK}$), with two external vertices both equipped with ($\mathcal{D}$).*

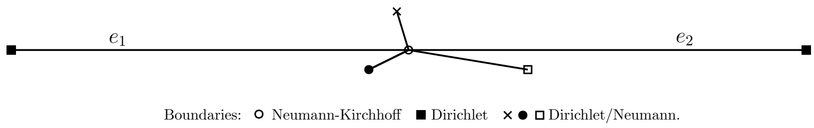

Figure 7: Example of star graph described by Theorem 4.2 with $N = 5$.

Let $B\psi = \big(x^2(\psi^1(x) - \psi^2(x)), x^2(\psi^2(x) - \psi^1(x)), 0, ..., 0\big)$ for every $\psi \in \mathscr{H}$. There exists $(\varphi_k)_{k\in\mathbb{N}^*} \subset (\phi_k)_{k\in\mathbb{N}^*}$ such that the (BSE) is globally exactly controllable in $H^3_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ and energetically controllable in $(k^2\pi^2)_{k\in\mathbb{N}^*}$.

Proof. Let $\mu = (\mu_k)_{k\in\mathbb{N}^*}$ and $\varphi = (\varphi_k)_{k\in\mathbb{N}^*}$ be such that $\mu_k = k^2\pi^2$, $\varphi^1_k = -\varphi^2_k = \sin(k\pi x)$ and $\varphi^l_k = 0$ for every $k \in \mathbb{N}^*$ and $3 \le l \le N$. The claim follows as Theorem 29 from the validity of the point **2.** of Theorem 4.1 with $\tilde{d} = 0$. □

Theorem 4.3. *Let $\mathscr{G}$ be a star graph equipped with ($\mathcal{D}$) and composed by $\frac{N}{2}$ couples of edges $\{e_{2j-1}, e_{2j}\}_{j\le\frac{N}{2}}$ long $\{L_j\}_{j\le\frac{N}{2}} \in \mathcal{AL}(\frac{N}{2})$ with $N \in 2\mathbb{N}^*$.*

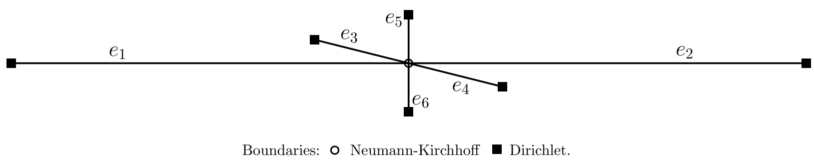

Figure 8: Example of star graph described by Theorem 4.3 with $N = 6$.

Let B be such that $B\psi = ((B\psi)^1, ..., (B\psi)^N)$ for every $\psi \in \mathscr{H}$ and

$$(B\psi)^{2j} = -(B\psi)^{2j-1} = \sum_{l=1}^{N/2} \frac{L_l^{1/2}}{L_j^{1/2}} x^2 \Big(\psi^{2l}\Big(\frac{L_l}{L_j}x\Big) - \psi^{2l-1}\Big(\frac{L_l}{L_j}x\Big)\Big), \qquad \forall j \le \frac{N}{2}.$$

There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable so that, for every $\{L_j\}_{j\le N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, there exists $(\varphi_k)_{k\in\mathbb{N}^*} \subseteq (\phi_k)_{k\in\mathbb{N}^*}$ such that (BSE) is globally exactly controllable in $H^{3+\epsilon}_{\mathscr{G}} \cap \widetilde{\mathscr{H}}$ with $\epsilon \in (0, 1/2)$ and energetically controllable in $\big(\frac{k^2\pi^2}{L_j^2}\big)_{\substack{k,j\in\mathbb{N}^* \\ j\le N/2}}$.

Proof. Let $(\mu_k)_{k\in\mathbb{N}^*} \subset (\lambda_k)_{k\in\mathbb{N}^*}$ be obtained by reordering $\big(\frac{k^2\pi^2}{L_j^2}\big)_{k\in\mathbb{N}^*}$ for every $j \le N/2$ and $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal system of $\mathscr{H}$ made by corresponding eigenfunctions. For $k \in \mathbb{N}^*$, there exist $m(k) \in \mathbb{N}^*$ and $l(k) \le N/2$ so that $\varphi^n_k \equiv 0$ for $n \ne 2l(k)$, $2l(k) - 1$ and

$$\mu_k = \frac{m(k)^2\pi^2}{L^2_{l(k)}}, \qquad \varphi^{2l(k)-1}_k(x) = -\varphi^{2l(k)}_k(x) = \sqrt{\frac{1}{L_{l(k)}}} \sin(\sqrt{\mu_k}x).$$

Let $[r]$ be the entire part of $r\in\mathbb{R}^+$. For $k\in\mathbb{N}^*$ and $C=4\min_{l\leq N}L_l$, we have

$$|\langle\varphi_1,B\varphi_k\rangle_{L^2}|=\Bigg|\sum_{l=1}^{N}\Bigg\langle\varphi_k^l(x),\sum_{n=1}^{N/2}\frac{L_n^{\frac{1}{2}}x^2}{L_{[(l+1)/2]}^{\frac{1}{2}}}\Big(\varphi_1^{2n-1}\Big(\frac{L_n}{L_{[(l+1)/2]}}x\Big)-\varphi_1^{2n}\Big(\frac{L_n}{L_{[(l+1)/2]}}x\Big)\Big)\Bigg\rangle_{L^2(e_l)}\Bigg|$$

$$=\Big|\int_0^{L_{l(k)}}\frac{4x^2}{L_{l(k)}}\sin\Big(\frac{m(1)\pi x}{L_{l(k)}}\Big)\sin\Big(\frac{m(k)\pi x}{L_{l(k)}}\Big)dx\Big|\geq C\Big|\int_0^1 x^2\sin(m(1)\pi x)\sin(m(k)\pi x)dx\Big|.$$

Assumptions I$(\varphi,1)$ and Assumptions II$(\varphi,1,\epsilon)$ with $\epsilon\in(0,\frac{1}{2})$ hold as in Theorem 1.5 and Theorem 3.5. We consider the techniques adopted in the proof of [Duc18a, *Lemma A*.2] which are due to the Roth's Theorem [Duc18a, *Proposition A*.1]. For every $\epsilon>0$, there exists $C_\epsilon>0$ so that

$$|\mu_{k+1}-\mu_k|\geq\frac{C_\epsilon}{k^\epsilon},\qquad\forall k\in\mathbb{N}^*.$$

The claim follows since the hypotheses **2.** of Theorem 4.1 is verified with $\tilde{d}=\epsilon\in(0,\frac{1}{2})$. $\square$

Theorem 4.4. *Let $\mathscr{G}$ be a compact quantum graph. Let the first $\widetilde{N}\leq N$ edges $\{e_j\}_{j\leq\widetilde{N}}$ of the graph be self-closing edges of lengths $\{L_j\}_{j\leq\widetilde{N}}$ (e.g Figure 3). For $\psi=(\psi^1,...\psi^N)$, let B be such that*

$$(B\psi)^l=\sum_{j\leq\widetilde{N}}x^2\Big(\frac{L_jx}{L_l}-L_j\Big)\psi^j\Big(\frac{L_j}{L_l}x\Big),\qquad (B\psi)^m\equiv 0,\qquad\forall l\leq\widetilde{N},\quad\widetilde{N}<m\leq N.$$

There exists $\mathcal{C}\subset(\mathbb{R}^+)^{\widetilde{N}}$ countable so that, if $\{L_j\}_{j\leq\widetilde{N}}\in\mathcal{AL}(\widetilde{N})\setminus\mathcal{C}$, then there exists $(\varphi_k)_{k\in\mathbb{N}^*}\subseteq(\phi_k)_{k\in\mathbb{N}^*}$ such that (BSE) is globally exactly controllable in $H^{3+\epsilon}_{\mathscr{G}}\cup\widetilde{\mathscr{H}}$ with $\epsilon\in(0,1/2)$ and energetically controllable in $\big(\frac{k^2\pi^2}{L_j^2}\big)_{\substack{k,j\in\mathbb{N}^*\\ j\leq\widetilde{N}}}$.

Proof. Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be such that, for each $k\in\mathbb{N}^*$, there exist $m(k)\in\mathbb{N}^*$ and $l(k)\leq\widetilde{N}$ such that $\mu_k=\frac{4m(k)^2\pi^2}{L_{l(k)}^2}$, $\varphi_k^{l(k)}(x)=\sqrt{\frac{2}{L_{l(k)}}}\sin(\sqrt{\mu_k}x)$ and $\varphi_k^n\equiv 0$ for every $n\neq l(k)$ and $n\leq N$. Now, $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal system made by eigenfunctions of A and the claim yields as Theorem 4.3. $\square$

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A Appendix: Global approximate controllability

Definition A.1. The (BSE) is said to be globally approximately controllable in $H^s_{\mathscr{G}}$ with $s>0$ when, for every $\psi\in H^s_{\mathscr{G}}$, $\widehat{\Gamma}\in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi\in H^s_{\mathscr{G}}$ and $\epsilon>0$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\widehat{\Gamma}\psi-\Gamma^u_T\psi\|_{(s)}<\epsilon$.

Theorem A.2. *Let (A,B) satisfy Assumptions I(η) and Assumptions II$(\eta,\tilde{d})$ for $\eta>0$ and $\tilde{d}\geq 0$, then the (BSE) is globally approximately controllable in $H^s_{\mathscr{G}}$ for $s=2+d$ with d from Assumptions II$(\eta,\tilde{d})$.*

Proof. In the point **1)** of the proof, we suppose that (A,B) admits a non-degenerate chain of connectedness (see [BdCC13, Definition 3]). We treat the general case in the point **2)** of the proof.

1) (a) Preliminaries. Let π_m be the orthogonal projector $\pi_m:\mathscr{H}\to\mathscr{H}_m:=\overline{span\{\phi_j\ :\ j\leq m\}}^{L^2}$ for every $m\in\mathbb{N}^*$. Up to reordering of $(\phi_k)_{k\in\mathbb{N}^*}$, the couples $(\pi_m(A+u_0B)\pi_m,\pi_mB\pi_m)$ for $m\in\mathbb{N}^*$ admit non-degenerate chains of connectedness in $\mathscr{H}_m$. Let $\|\cdot\|_{BV(T)}=\|\cdot\|_{BV((0,T),\mathbb{R})}$ and $|\!|\!|\cdot|\!|\!|_{(s)}:=|\!|\!|\cdot|\!|\!|_{L(H^s_{\mathscr{G}},H^s_{\mathscr{G}})}$ for $s>0$. Let $B:H^{s_1}_{\mathscr{G}}\to H^{s_1}_{\mathscr{G}}$ with $s_1>0$ and $s\in[0,s_1+2)$.

Claim. $\forall\epsilon>0,\ \exists N_1\in\mathbb{N}^*,\ \widetilde{\Gamma}_{N_1}\in U(\mathscr{H})\ :\ \pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1}\in SU(\mathscr{H}_{N_1})$,

$$\|\widetilde{\Gamma}_{N_1}\phi_j-\widehat{\Gamma}\phi_j\|_{(s)}<\epsilon,\qquad\forall j\leq N.\tag{17}$$

Let $N, N_1 \in \mathbb{N}^*$ be such that $N_1 \geq N$. We apply the orthonormalizing Gram-Schmidt process to $(\pi_{N_1}\widehat{\Gamma}\phi_j)_{j\leq N}$ and we define the sequence $(\widetilde{\phi}_j)_{j\leq N}$ that we complete in $(\widetilde{\phi}_j)_{j\leq N_1}$, an orthonormal basis of $\mathscr{H}_{N_1}$. The operator $\widetilde{\Gamma}_{N_1}$ is the unitary map such that $\widetilde{\Gamma}_{N_1}\phi_j = \widetilde{\phi}_j$ for every $j \leq N_1$. The provided definition implies $\lim_{N_1\to\infty} \|\widetilde{\Gamma}_{N_1}\phi_j - \widehat{\Gamma}\phi_j\|_{(s)} = 0$ for every $j \leq N$. Thus, for every $\epsilon > 0$, there exists $N_1 \in \mathbb{N}^*$ large enough satisfying the claim.

1) (b) Finite dimensional controllability. Let T_{ad} be the set of $(j,k) \in \{1,...,N_1\}^2$ such that $B_{j,k} := \langle\phi_j, B\phi_k\rangle_{L^2} \neq 0$ and $|\lambda_j - \lambda_k| = |\lambda_m - \lambda_l|$ with $m, l \in \mathbb{N}^*$ implies $\{j,k\} = \{m,l\}$ for $B_{m,l} = 0$. For every $(j,k) \in \{1,...,N_1\}^2$ and $\theta \in [0, 2\pi)$, we define $E^\theta_{j,k}$ the $N_1 \times N_1$ matrix with elements $(E^\theta_{j,k})_{l,m} = 0$, $(E^\theta_{j,k})_{j,k} = e^{i\theta}$ and $(E^\theta_{j,k})_{k,j} = -e^{-i\theta}$ for $(l,m) \in \{1,...,N_1\}^2 \setminus \{(j,k),(k,j)\}$. Let $E_{ad} = \big\{E^\theta_{j,k} \;:\; (j,k) \in T_{ad},\ \theta \in [0,2\pi)\big\}$ and $Lie(E_{ad})$. Fixed v a piecewise constant control taking value in E_{ad} and $\tau > 0$, we introduce the control system on $SU(\mathscr{H}_{N_1})$

$$\begin{cases} \dot{x}(t) = x(t)v(t), & t \in (0,\tau), \\ x(0) = Id_{SU(\mathscr{H}_{N_1})}. \end{cases} \tag{18}$$

> **Claim.** (18) is controllable, *i.e.* for $R \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}^*$, $M_1, ..., M_p \in E_{ad}$, $\alpha_1, ..., \alpha_p \in \mathbb{R}^+$ such that $R = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}$.

For every $(j,k) \in \{1,...,N_1\}^2$, we define the $N_1 \times N_1$ matrices $R_{j,k}$, $C_{j,k}$ and D_j as follow. For $(l,m) \in \{1,...,N_1\}^2 \setminus \{(j,k),(k,j)\}$,we have $(R_{j,k})_{l,m} = 0$ and $(R_{j,k})_{j,k} = -(R_{j,k})_{k,j} = 1$, while $(C_{j,k})_{l,m} = 0$ and $(C_{j,k})_{j,k} = (C_{j,k})_{k,j} = i$. Moreover, for $(l,m) \in \{1,...,N_1\}^2 \setminus \{(1,1),(j,j)\}$, $(D_j)_{l,m} = 0$ and $(D_j)_{1,1} = -(D_j)_{j,j} = i$. We consider the basis of $su(\mathscr{H}_{N_1})$

$$\mathbf{e} := \{R_{j,k}\}_{j,k\leq N_1} \cup \{C_{j,k}\}_{j,k\leq N_1} \cup \{D_j\}_{j\leq N_1}.$$

Thanks to [Sac00, Theorem 6.1], the controllability of (18) is equivalent to prove that $Lie(E_{ad}) \supseteq su(\mathscr{H}_{N_1})$ for $su(\mathscr{H}_{N_1})$ the Lie algebra of $SU(\mathscr{H}_{N_1})$. The claim si valid as it is possible to obtain the matrices $R_{j,k}$, $C_{j,k}$ and D_j for every $j,k \leq N_1$ by iterated Lie brackets of elements in E_{ad}.

1) (c) Finite dimensional estimates. From **2)** and $\pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}^*$, $M_1, ..., M_p \in E_{ad}$, $\alpha_1, ..., \alpha_p \in \mathbb{R}^+$ so that

$$\pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}. \tag{19}$$

> **Claim.** For every $l \leq p$ and $e^{\alpha_l M_l}$ from (19), there exist $(T^l_n)_{l\in\mathbb{N}^*} \subset \mathbb{R}^+$ and $(u^l_n)_{n\in\mathbb{N}^*}$ such that $u^l_n : (0, T^l_n) \to \mathbb{R}$ for every $n \in \mathbb{N}^*$ and
>
> $$\lim_{n\to\infty} \|\Gamma^{u^l_n}_{T^l_n}\phi_k - e^{\alpha_l M_l}\phi_k\|_{(s)} = 0, \qquad \forall k \leq N_1, \tag{20}$$
>
> $$\sup_{n\in\mathbb{N}^*} \big(\|u^l_n\|_{BV(T_n)}, \|u^l_n\|_{L^\infty((0,T_n),\mathbb{R})}, T_n\|u^l_n\|_{L^\infty((0,T_n),\mathbb{R})}\big) < \infty. \tag{21}$$

We consider the results developed in [Cha12, *Section* 3.1 & *Section* 3.2] by Chambrion and leading to [Cha12, *Proposition* 6] (also adopted in [Duc18b]). Each $e^{\alpha_l M_l}$ is a rotation in a two dimensional space for every $l \in \{1,...,p\}$ and the mentioned work allows to explicit $\{T^l_n\}_{l\in\mathbb{N}^*} \subset \mathbb{R}^+$ and $\{u^l_n\}_{n\in\mathbb{N}^*}$ satisfying (21) such that $u^l_n : (0, T^l_n) \to \mathbb{R}$ for every $n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \|\pi_{N_1}\Gamma^{u^l_n}_{T^l_n}\phi_k - e^{\alpha_l M_l}\phi_k\|_{L^2} = 0$ for every $k \leq N_1$. As $e^{\alpha_l M_l} \in SU(\mathscr{H}_{N_1})$, we have

$$\lim_{n\to\infty} \|\Gamma^{u^l_n}_{T^l_n}\phi_k - e^{\alpha_l M_l}\phi_k\|_{L^2} = 0, \qquad \forall k \leq N_1. \tag{22}$$

We consider the theory developed by Kato in [Kat53] and $i(A + u(t)B - ic)$ is maximal dissipative in $H^{s_1}_{\mathscr{G}}$ for suitable $c > |||B|||_{(2)}\|u\|_{L^\infty((0,T),\mathbb{R})}$. Let $\mu > c$ and $\widehat{H}^{s_1+2}_{\mathscr{G}} := D(A^{s_1}(i\mu - A)) \equiv H^{s_1+2}_{\mathscr{G}}$. We know $B : \widehat{H}^{s_1+2}_{\mathscr{G}} \subset H^{s_1}_{\mathscr{G}} \to H^{s_1}_{\mathscr{G}}$ and the arguments of [Duc18c, Remark 1.1] imply that $B \in L(\widehat{H}^{s_1+2}_{\mathscr{G}}, H^{s_1}_{\mathscr{G}})$. For $T > 0$ and $u \in BV((0,T),\mathbb{R})$, we have $|||u(t)B(i\mu - A)^{-1}|||_{(s_1)} < 1$ and we denote

$$M := \sup_{t\in[0,T]} |||(i\mu - A - u(t)B)^{-1}|||_{L(H^{s_1}_{\mathscr{G}}, \widehat{H}^{s_1+2}_{\mathscr{G}})} \leq \sup_{t\in[0,T]} \sum_{l\in\mathbb{N}^*} |||(u(t)B(i\mu - A)^{-1})^l|||_{(s_1)} < +\infty,$$

$$N := \left\vert\!\left\vert\!\left\vert\, i\mu - A - u(\cdot)B \,\right\vert\!\right\vert\!\right\vert_{BV\left([0,T],L(\widehat{H}_{\mathscr{G}}^{s_1+2},H_{\mathscr{G}}^{s_1})\right)} < +\infty, \qquad C_1 := \left\vert\!\left\vert\!\left\vert\, A(A+u(T)B-i\mu)^{-1} \,\right\vert\!\right\vert\!\right\vert_{(s_1)} < \infty.$$

We call U_t^u the propagator generated by $(A+uB-ic)$ such that $U_t^u\psi = e^{-ct}\Gamma_t^u\psi$ for every $\psi \in \mathscr{H}$. Thanks to [Kat53, Section 3.10], for every $\psi \in H_{\mathscr{G}}^{s_1+2}$, $\|(A+u(T)B-i\mu)U_t^u\psi\|_{(s_1)} \leq Me^{MN}\|(A-i\mu)\psi\|_{(s_1)}$ and

$$\begin{aligned}\|\Gamma_T^u\psi\|_{(s_1+2)} &= \|A\Gamma_T^u\psi\|_{(s_1)} \leq e^{cT} \left\vert\!\left\vert\!\left\vert\, A(A+u(T)B-i\mu)^{-1} \,\right\vert\!\right\vert\!\right\vert_{(s_1)} \|(A+u(T)B-i\mu)U_t^u\psi\|_{(s_1)} \\ &\leq C_1Me^{MN+cT} \left\vert\!\left\vert\!\left\vert\, (A-i\mu)A^{-1} \,\right\vert\!\right\vert\!\right\vert_{(s_1)} \|A\psi\|_{(s_1)} \leq C_1Me^{MN+cT}\Big(1+\frac{\mu}{\pi^2}\Big)\|\psi\|_{(s_1+2)}.\end{aligned}$$

For every $T > 0$, $u \in BV((0,T),\mathbb{R})$ and $\psi \in H_{\mathscr{G}}^{s_1+2}$, there exists $C(K) > 0$ depending on $K = \big(\|u\|_{BV(T)}, \|u\|_{L^\infty((0,T),\mathbb{R})}, T\|u\|_{L^\infty((0,T),\mathbb{R})}\big)$ such that $\|\Gamma_T^u\psi\|_{(s_1+2)} \leq C(K)\|\psi\|_{(s_1+2)}$. From classical interpolation techniques, for every $s \in [0, s_1+2]$, there exists $C > 0$ such that

$$\left\vert\!\left\vert\!\left\vert\, \Gamma_{T_n^l}^{u_n^l} \,\right\vert\!\right\vert\!\right\vert_{(s)} \leq C. \tag{23}$$

For every $\psi \in H_{\mathscr{G}}^{s_1+2}$, from the Cauchy-Schwarz inequality, $\|A\psi\|_{L^2}^2 \leq \|A^2\psi\|_{L^2}\|\psi\|_{L^2}$ and $\|A^{\frac{3}{2}}\psi\|_{L^2}^4 \leq \big(\langle A^2\psi, A\psi\rangle_{L^2}\big)^2 \leq \|A^2\psi\|_{L^2}^2\|A\psi\|_{L^2}^2$. By iterating the procedure, there exist $n \in \mathbb{N}^*$ and $C_1 > 0$ such that $\|\psi\|_{(s)}^{n+1} \leq C_1\|\psi\|_{L^2}\|\psi\|_{(s_1+2)}^n$. In conclusion, from (22) and (23), the last relation leads to (20).

1) (d) Infinite dimensional estimates.

> **Claim.** There exists $K > 0$ such that for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u\phi_k - \widehat{\Gamma}\phi_k\|_{(s)} \leq \epsilon$ for every $k \leq N$ and $\sup\big(\|u\|_{BV(T)}, \|u\|_{L^\infty((0,T),\mathbb{R})}, T\|u\|_{L^\infty((0,T),\mathbb{R})}\big) < K$.

Let $p = 2$ (the following result is valid for any $p \in \mathbb{N}^*$). Thanks to (20), for every $\epsilon > 0$ and $N_1 \in \mathbb{N}^*$, there exists $n \in \mathbb{N}^*$ large enough such that, for every $k \leq N$,

$$\|\Gamma_{T_n^2}^{u_n^2}\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_2M_2}e^{\alpha_1M_1}\phi_k\|_{(s)} \leq \left\vert\!\left\vert\!\left\vert\, \Gamma_{T_n^2}^{u_n^2} \,\right\vert\!\right\vert\!\right\vert_{(s)} \|\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_1M_1}\phi_k\|_{(s)} + \sum_{l=1}^{N_1} \|\big(\Gamma_{T_n^2}^{u_n^2}\phi_l - e^{\alpha_2M_2}\phi_l\big)\langle\phi_l, e^{\alpha_1M_1}\phi_k\rangle_{L^2}\|_{(s)}$$

$$\leq \left\vert\!\left\vert\!\left\vert\, \Gamma_{T_n^2}^{u_n^2} \,\right\vert\!\right\vert\!\right\vert_{(s)} \|\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_1M_1}\phi_k\|_{(s)} + \|e^{\alpha_1M_1}\phi_k\|_{L^2}\Big(\sum_{l=1}^{N_1}\|\big(\Gamma_{T_n^2}^{u_n^2}\phi_l - e^{\alpha_2M_2}\phi_l\big)\|_{(s)}^2\Big)^{\frac{1}{2}} \leq \epsilon.$$

In the previous inequality, we considered that $e^{\alpha_1M_1}\phi_k \in \mathscr{H}_{N_1}$ and that $\left\vert\!\left\vert\!\left\vert\, \Gamma_{T_n^2}^{u_n^2} \,\right\vert\!\right\vert\!\right\vert_{(s)}$ is uniformly bounded. Thanks to the identities (17) and (19), the triangular inequality achieves the claim.

> **Claim.** When $B : H_{\mathscr{G}}^{s_1} \to H_{\mathscr{G}}^{s_1}$ for $s_1 > 0$, the global approximate controllability is verified in $H_{\mathscr{G}}^s$ with $s \in [s_1, s_1+2)$

For every $\psi \in H_{\mathscr{G}}^s$ and $\widehat{\Gamma} \in U(\mathscr{H})$ so that $\widehat{\Gamma}\psi \in H_{\mathscr{G}}^s$, the quantity $\|\Gamma_T^u\psi - \widehat{\Gamma}\psi\|_{(s)}$ is uniformly bounded in $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ when

$$\sup\big(\|u\|_{BV(T)}, \|u\|_{L^\infty((0,T),\mathbb{R})}, T\|u\|_{L^\infty((0,T),\mathbb{R})}\big) < K \tag{24}$$

thanks to (23). Then, for any $\epsilon > 0$, there exists $N \in \mathbb{N}^*$ so that $\Big(\sum_{k>N}\big|k^s\langle\phi_k, \Gamma_T^u\psi - \widehat{\Gamma}\psi\rangle_{L^2}\big|^2\Big)^{1/2} \leq \epsilon$ for every $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ satisfying (24). Now,

$$\|\Gamma_T^u\psi - \widehat{\Gamma}\psi\|_{(s)} \leq \Big(\sum_{k\leq N}\big|k^s\langle\phi_k, \Gamma_T^u\psi - \widehat{\Gamma}\psi\rangle_{L^2}\big|^2\Big)^{1/2} + \epsilon \leq N^s\|\psi\|_{L^2}\sum_{k\leq N}\|(\Gamma_T^u)^{-1}\phi_k - \widehat{\Gamma}^{-1}\phi_k\|_{(s)} + \epsilon.$$

The point **4)** is also valid for the reversed dynamics (see [Duc18c, *Section* 1.3]) and there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ satisfying (24) so that $\|(\Gamma_T^u)^{-1}\phi_k - \widehat{\Gamma}^{-1}\phi_k\|_{(s)} \leq \epsilon\|\psi\|_{L^2}^{-1}N^{-s-1}$ for every $k \leq N$, which implies $\|\Gamma_T^u\psi - \widehat{\Gamma}\psi\|_{(s)} \leq 2\epsilon$.

1) (e) Conclusion. Let d be defined in Assumptions II$(\eta, \tilde{d})$. If $d < 2$, then $B : H_{\mathscr{G}}^2 \to H_{\mathscr{G}}^2$ and the global approximate controllability is verified in $H_{\mathscr{G}}^{d+2}$ since $d+2 < 4$. If $d \in [2, 5/2)$, then $B : H^{d_1} \to H^{d_1}$ with

$d_1 \in (d, 5/2)$ from Assumptions II$(\eta, \tilde{d})$. Now, $H^{d_1}_{\mathscr{G}} = H^{d_1} \cap H^2_{\mathscr{G}}$, thanks to [Duc18a, *Proposition* 3.2], and $B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}$ implies $B : H^{d_1}_{\mathscr{G}} \to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2 < d_1+2$. If $d \in [5/2, 7/2)$, then $B : H^{d_1}_{\mathcal{NK}} \to H^{d_1}_{\mathcal{NK}}$ for $d_1 \in (d, 7/2)$ and $H^{d_1}_{\mathscr{G}} = H^{d_1}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ from [Duc18a, *Proposition* 3.2]. Now, $B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}$ that implies $B : H^{d_1}_{\mathscr{G}} \to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2 < d_1+2$.

2) Generalization. Let (A, B) do not admit a non-degenerate chain of connectedness. We decompose

$$A + u(\cdot)B = (A + u_0 B) + u_1(\cdot)B, \qquad\qquad u_0 \in \mathbb{R}, \quad u_1 \in L^2((0,T), \mathbb{R}).$$

We notice that, if (A, B) satisfies Assumptions I(η) and Assumptions II$(\eta, \tilde{d})$ for $\eta > 0$ and $\tilde{d} \geq 0$, then [Duc18a, *Lemma C*.2 & *Remark C*.4] are valid. We consider u_0 belonging to the neighborhoods provided by [Duc18a, *Lemma C*.2 & *Remark C*.4] and we denote $(\phi^{u_0}_k)_{k \in \mathbb{N}}$ a Hilbert basis of $\mathscr{H}$ made by eigenfunctions of $A + u_0 B$. The steps of the point **1)** can be repeated by considering the sequence $(\phi^{u_0}_k)_{k \in \mathbb{N}}$ instead of $(\phi_k)_{k \in \mathbb{N}}$ and the spaces $D(|A + u_0 B|^{\frac{s_1}{2}})$ in substitution of $H^{s_1}_{\mathscr{G}}$ with $s_1 > 0$. The claim is equivalently proved since $\big\| |A + u_0 B|^{\frac{s_1}{2}} \cdot \big\| \asymp \| \cdot \|_{(s_1)}$ with $s_1 \in [s, s+2)$, $s = 2 + d$ and d from Assumptions II$(\eta, \tilde{d})$ thanks to [Duc18a, *Remark C*.4]. $\square$

B Appendix: Spectral properties

For $x \in \mathbb{R}$, we denote $E(x)$ the closest integer number to x, $|||\, x \,||| = \min_{z \in \mathbb{Z}} |x - z|$ and $F(x) = x - E(x)$. We notice $|F(x)| = |||\, x \,|||$ and $-\frac{1}{2} \leq F(z) \leq \frac{1}{2}$. Let $\{L_j\}_{j \leq N} \in (\mathbb{R}^+)^N$ and $i \leq N$, we also define

$$n(x) := E\Big(x - \frac{1}{2}\Big), \quad r(x) := F\Big(x - \frac{1}{2}\Big), \quad d(x) := |||\, x - \frac{1}{2} \,|||\,, \quad \widetilde{m}^i(x) := n\Big(\frac{L_i}{\pi} x\Big).$$

In this appendix, we pursue [Duc18a, *Appendix A*], which is based on the techniques from [DZ06, *Appendix A*].

Lemma B.1. *Let $\{L_k\}_{k \leq N} \subset \mathbb{R}^+$, $I_1 \subseteq \{1, ..., N\}$, $I_2 := \{1, ..., N\} \setminus I_1$ and*

$$a(\cdot) := \prod_{i \in I_2} |\sin((\cdot)L_i)| \sum_{i \in I_1} \prod_{\substack{j \in I_1 \\ j \neq i}} |\cos((\cdot)L_j)| + \prod_{i \in I_1} |\cos((\cdot)L_i)| \sum_{i \in I_2} \prod_{\substack{j \in I_2 \\ j \neq i}} |\sin((\cdot)L_j)|.$$

Let $\{\widetilde{L}_j\}_{j \leq N} \subset \mathbb{R}^+$ be such that $\widetilde{L}_j = 2L_j$ when $j \in I_1$ and $\widetilde{L}_j = L_j$ when $j \in I_2$. There exists $C > 0$ such that, for every $x \in \mathbb{R}$, there holds

$$a(x) \geq C \min\Big(\min_{i \leq N} \prod_{j \neq i} |||\, \Big(\widetilde{m}^i(x) + \frac{1}{2}\Big) \frac{\widetilde{L}_j}{L_i} \,|||\,, \ \min_{i \leq N} \prod_{j \neq i} |||\, m^i(x) \frac{\widetilde{L}_j}{L_i} \,||| \Big).$$

Proof. From [DZ06, *relation* (*A*.3)], for every $x \in \mathbb{R}$, there follows

$$2d(x) \leq |\cos(\pi x)| \leq \pi d(x). \tag{25}$$

As $2d\Big(\big(\widetilde{m}^i(x) + \frac{1}{2}\big)\frac{L_j}{L_i}\Big) \leq \Big|\cos\Big(\big(\widetilde{m}^i(x) + \frac{1}{2}\big)\frac{L_j}{L_i}\pi\Big)\Big|$ and $\widetilde{m}^i(x) + \frac{1}{2} = \frac{L_i}{\pi}x - r\big(\frac{L_i}{\pi}x\big)$ for $x \in \mathbb{R}$ and $i, j \leq N$,

$$2d\Big(\Big(\widetilde{m}^i(x) + \frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \leq |\cos(L_j x)| + \left|\sin\left(\pi \frac{L_j}{L_i}\Big|r\Big(\frac{L_i}{\pi}x\Big)\Big|\right)\right|. \tag{26}$$

Now, $|\sin(\pi |r(\cdot)|)| \leq \pi \,|||\, |r(\cdot)| \,|||\, \leq \pi |r(\cdot)| = \pi d(\cdot) \leq \frac{\pi}{2} |\cos(\pi(\cdot))|$ thanks to [DZ06, *relation* (*A*.3)] and (25). For every $x \in \mathbb{R}$, it holds

$$\left|\sin\left(\pi \frac{L_j}{L_i}\Big|r\Big(\frac{L_i}{\pi}x\Big)\Big|\right)\right| \leq \pi \frac{L_j}{L_i}\Big|r\Big(\frac{L_i}{\pi}x\Big)\Big| \leq \frac{\pi L_j}{2L_i} |\cos(L_i x)|. \tag{27}$$

From (26) and (27), there exists $C_1 > 0$ such that, for every $i, j \leq N$,

$$2d\Big(\Big(\widetilde{m}^i(x) + \frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \leq |\cos(L_j x)| + \frac{\pi L_j}{2L_i} |\cos(L_i x)|, \qquad \forall x \in \mathbb{R}^+, \tag{28}$$

$$\Longrightarrow \quad C_1 \prod_{\substack{j\in I_1\\ j\neq i}} d\Big(\Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \leq \prod_{\substack{j\in I_1\\ j\neq i}} |\cos(L_j x)|+|\cos(L_i x)|.$$

From [DZ06, *relation* $(A.3)$], as done in (26) and (27), there exists $C_2>0$ such that

$$2 \,|||\, \Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i} \,|||\, \leq |\sin(L_j x)| + \frac{\pi L_j}{2L_i}|\cos(L_i x)|, \qquad \forall x\in\mathbb{R}, \tag{29}$$

$$\Longrightarrow \quad C_2 \prod_{\substack{j\in I_1\\ j\neq i}} d\Big(\Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \prod_{\substack{j\in I_2\\ j\neq i}} |||\, \Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i} \,||| \leq \prod_{\substack{j\in I_2\\ j\neq i}} |\sin(L_j x)| \prod_{\substack{j\in I_1\\ j\neq i}} |\cos(L_j x)|+|\cos(L_i x)|.$$

Now, $d(x) = |||\, \frac{1}{2}(2x-1) \,||| \geq \frac{1}{2} |||\, 2x-1 \,||| = \frac{1}{2} |||\, 2x \,|||$ for every $x\in\mathbb{R}$ and $d\big(\big(\widetilde{m}^i(x)+\frac{1}{2}\big)\frac{L_j}{L_i}\big) \geq \frac{1}{2} |||\, \Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{2L_j}{L_i} \,|||$, which imply

$$C_2 \prod_{\substack{j\leq N\\ j\neq i}} \frac{1}{2} \,|||\, \Big(\widetilde{m}^i(\cdot)+\frac{1}{2}\Big)\frac{\widetilde{L}_j}{L_i} \,||| \leq a(\cdot)+|\cos(L_i(\cdot))|. \tag{30}$$

Equivalently, from the proof of [DZ06, *Proposition* $A.1$], for every $x\in\mathbb{R}$,

(31)
$$2 \,|||\, m^i(x)\frac{L_j}{L_i} \,||| \leq |\sin(L_j x)| + \frac{\pi L_j}{2L_i}|\sin(L_i x)|, \qquad 2d\Big(m^i(x)\frac{L_j}{L_i}\Big) \leq |\cos(L_j x)| + \frac{\pi L_j}{2L_i}|\sin(L_i x)|,$$

$$\Longrightarrow \quad C_2 \prod_{\substack{j\leq N\\ j\neq i}} \frac{1}{2} \,|||\, m^i(\cdot)\frac{\widetilde{L}_j}{L_i} \,||| \leq a(\cdot)+|\sin(L_i(\cdot))|. \tag{32}$$

The claim follows as [DZ06, *Proposition* $A.1$]. Indeed, if $(\lambda_k)_{k\in\mathbb{N}^*}\subset\mathbb{R}^+$ is so that $a(\lambda_k) \xrightarrow{k\to\infty} 0$, then there exist some $i_0\leq N$ such that $|\sin(\lambda_k L_{i_0})| \xrightarrow{k\to\infty} 0$ or $|\cos(\lambda_k L_{i_0})| \xrightarrow{k\to\infty} 0$. By considering (30) and (32) with $i=i_0$, we have

$$z(\lambda_k) := \min\Big(\min_{i\leq N} \prod_{j\neq i} |||\, \Big(\widetilde{m}^i(\lambda_k)+\frac{1}{2}\Big)\frac{\widetilde{L}_j}{L_i} \,|||, \min_{i\leq N} \prod_{j\neq i} |||\, m^i(\lambda_k)\frac{\widetilde{L}_j}{L_i} \,||| \Big) \xrightarrow{k\longrightarrow\infty} 0.$$

As [DZ06, *Proposition* $A.1$], the lemma is proved since $z(\lambda_k)$ converges to 0 at least as fast as $a(\lambda_k)$ thanks to the identities (28), (29) and (31). $\square$

Proposition B.2. *Let* $\{L_j\}_{j\leq N}\subset\mathbb{R}$, $I_1\subseteq\{1,...,N\}$ *and* $I_2:=\{1,...,N\}\setminus I_1$. *If* $\{L_j\}_{j\leq N}\in\mathcal{AL}(N)$, *then, for every* $\epsilon>0$, *there exists* $C_\epsilon>0$ *such that, for every* $x>\max\{\pi/2L_j : j\leq N\}$, *we have*

$$\prod_{j\in I_2}|\sin(xL_j)|\sum_{j\in I_1}\prod_{\substack{k\in I_1\\ k\neq j}}|\cos(xL_k)| + \prod_{j\in I_1}|\cos(xL_j)|\sum_{j\in I_2}\prod_{\substack{k\in I_2\\ k\neq j}}|\sin(xL_k)| \geq \frac{C_\epsilon}{x^{1+\epsilon}}.$$

Proof. The claim is due to Lemma $B.1$ and to the Schmidt's Theorem [DZ06, *Theorem* $A.8$], which implies that, for every $\epsilon>0$ and $i\leq N$, there exist $C_1(i), C_2(i), C_3(i)>0$ such that, for every $x\in\mathbb{R}$,

$$\prod_{\substack{j\leq N\\ j\neq i}} |||\, \Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{\widetilde{L}_j}{L_i} \,||| \geq \frac{C_1(i)}{(2\widetilde{m}^i(x)+1)^{1+\epsilon}} \geq \frac{C_1(i)}{\big(\frac{2L_i}{\pi}x+1\big)^{1+\epsilon}} \geq \frac{C_2(i)}{x^{1+\epsilon}} \tag{33}$$

and $\prod_{\substack{j\leq N\\ j\neq i}} |||\, m^i(x)\frac{\widetilde{L}_j}{L_i} \,||| \geq C_3(i)x^{-1-\epsilon}$ for every $x>\frac{\pi}{2}\max\{1/L_j : j\leq N\}$. The statement follows with $C_\epsilon:=\min\big(\min_{i\leq N} C_2(i), \min_{i\leq N} C_3(i)\big)$. $\square$

Corollary B.3. *Let $\{L_k\}_{k\leq N}\in\mathcal{AL}(N)$ with $N\in\mathbb{N}$. Let $\{\omega_n\}_{n\in\mathbb{N}}$ be the unbounded sequence of positive solutions of the equation*

$$\sum_{l\leq N}\sin(xL_l)\prod_{m\neq l}\cos(xL_m)=0, \qquad x\in\mathbb{R}. \tag{34}$$

For every $\epsilon>0$, there exists $C_\epsilon>0$ so that $|\cos(\omega_n L_l)|\geq\frac{C_\epsilon}{\omega_n^{1+\epsilon}}$ for every $l\leq N$ and $n\in\mathbb{N}$.

Proof. If there exists $\{\omega_{n_k}\}_{k\in\mathbb{N}}$, subsequence of $\{\omega_n\}_{n\in\mathbb{N}}$, such that $|\cos(L_j\omega_{n_k})|\xrightarrow{k\to\infty}0$ for some $j\leq N$, then there exists $i\leq N$ such that $i\neq j$ and $|\cos(L_i\omega_{n_k})|\xrightarrow{k\to\infty}0$ thanks to (34). From (28), we have $\prod_{j\neq i}d\Big(\Big(\widetilde{m}^i(\omega_{n_k})+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big)\xrightarrow{k\to\infty}0$ and (as in the proof of Proposition $B.1$) there exists $C_2>0$ so that

$$C_2|\cos(L_i\omega_n)|\geq\prod_{j\neq i}d\Big(\Big(\widetilde{m}^i(\omega_n)+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big)=\prod_{j\neq i}|||\,\frac{1}{2}\Big(\Big(\widetilde{m}^i(\omega_n)+\frac{1}{2}\Big)\frac{2L_j}{L_i}-1\Big)\,|||\,.$$

The last identity and the techniques leading to the equation (33) achieve the claim. ☐

C Appendix: Moments problems

Let $\mathbb{Z}^*=\mathbb{Z}\setminus\{0\}$ and $\mathbf{\Lambda}=(\lambda_k)_{k\in\mathbb{Z}^*}\subset\mathbb{R}^+$ be an ordered sequence of pairwise distinct numbers such that there exist $\mathcal{M}\in\mathbb{N}^*\setminus\{1\}$ and $\delta>0$ such that

$$\inf_{\{k\in\mathbb{Z}^*\ :\ k+\mathcal{M}\neq0\}}|\lambda_{k+\mathcal{M}}-\lambda_k|\geq\delta\mathcal{M}. \tag{35}$$

From (35), there does not exist $\mathcal{M}$ consecutive $k,k+1\in\mathbb{Z}^*$ such that $|\lambda_{k+1}-\lambda_k|<\delta$. This leads to a partition of $\mathbb{Z}^*$ in subsets that we call E_m with $m\in\mathbb{Z}^*$. This partition also defines an equivalence relation in $\mathbb{Z}^*$ such that $k\sim n$ if and only if there exists $m\in\mathbb{Z}^*$ such that $k,n\in E_m$. Now, $\{E_m\}_{m\in\mathbb{Z}^*}$ are the corresponding equivalence classes and $i(m):=|E_m|\leq\mathcal{M}-1$. For every $\mathbf{x}:=(x_k)_{k\in\mathbb{Z}^*}$, we define $\mathbf{x}^m:=(x_l)_{l\in E_m}$ for $m\in\mathbb{Z}^*$.

Let $\widehat{\mathbf{h}}=(h_j)_{j\leq i(m)}\in\mathbb{C}^{i(m)}$ with $m\in\mathbb{Z}^*$. For every $m\in\mathbb{Z}^*$, we denote $F_m(\widehat{\mathbf{h}}):\mathbb{C}^{i(m)}\to\mathbb{C}^{i(m)}$ the matrix with elements, for every $j,k\leq i(m)$,

$$F_{m;j,k}(\widehat{\mathbf{h}}):=\begin{cases}\prod_{\substack{l\neq j\\ l\leq k}}(h_j-h_l)^{-1}, & j\leq k,\\ 1, & j=k=1,\\ 0, & j>k.\end{cases}$$

For each $k\in\mathbb{Z}^*$, there exists $m(k)\in\mathbb{Z}^*$ such that $k\in E_{m(k)}$. Let $F(\mathbf{\Lambda})$ be the linear operator on $\ell^2(\mathbb{Z}^*,\mathbb{C})$ such that $F(\mathbf{\Lambda}):D(F(\mathbf{\Lambda}))\to\ell^2(\mathbb{Z}^*,\mathbb{C})$ and

$$(F(\mathbf{\Lambda})\mathbf{x})_k=\Big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})\mathbf{x}^{m(k)}\Big)_k, \qquad \forall\mathbf{x}=(x_k)_{k\in\mathbb{Z}^*}\in D(F(\mathbf{\Lambda})),$$

$$H(\mathbf{\Lambda}):=D(F(\mathbf{\Lambda}))=\big\{\mathbf{x}:=(x_k)_{k\in\mathbb{Z}^*}\in\ell^2(\mathbb{Z}^*,\mathbb{C})\ :\ F(\mathbf{\Lambda})\mathbf{x}\in\ell^2(\mathbb{Z}^*,\mathbb{C})\big\}\,.$$

Remark C.1. *We call $F_m(\mathbf{\Lambda}^m)^{-1}$ the inverse matrix of $F_m(\mathbf{\Lambda}^m)$ for $m\in\mathbb{Z}^*$. Now, $F(\mathbf{\Lambda}):H(\mathbf{\Lambda})\to Ran(F(\mathbf{\Lambda}))$ is invertible and $F(\mathbf{\Lambda})^{-1}$ is so that*

$$(F(\mathbf{\Lambda})^{-1}\mathbf{x})_k=\big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^{-1}\mathbf{x}^{m(k)}\big)_k, \qquad \forall\mathbf{x}\in Ran(F(\mathbf{\Lambda})), \quad k\in\mathbb{Z}^*.$$

Let $F(\mathbf{\Lambda})^*$ be the infinite matrix so that $(F(\mathbf{\Lambda})^*\mathbf{x})_k=\big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*\mathbf{x}^{m(k)}\big)_k$ for any $\mathbf{x}=(x_k)_{k\in\mathbb{Z}^*}$ and $k\in\mathbb{Z}^*$, where $F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*$ is the transposed matrix of $F_{m(k)}(\mathbf{\Lambda}^{m(k)})$. For $T>0$, let $\mathbf{e}$ and $\mathbf{\Xi}$ be sequences of functions in $L^2((0,T),\mathbb{C})$ so that

$$\mathbf{e}:=(e^{i\lambda_k(\cdot)})_{k\in\mathbb{Z}^*}, \qquad \mathbf{\Xi}:=(\xi_k(\cdot))_{k\in\mathbb{Z}^*}=F(\mathbf{\Lambda})^*\mathbf{e}.$$

Remark C.2. *When $H(\mathbf{\Lambda})$ is dense in $\ell^2(\mathbb{Z}^*,\mathbb{C})$, we consider $F(\mathbf{\Lambda})^*$ as the unique adjoint operator of $F(\mathbf{\Lambda})$ in $\ell^2(\mathbb{Z}^*,\mathbb{C})$ with domain $H(\mathbf{\Lambda})^* := D(F(\mathbf{\Lambda})^*)$. As in Remark C.1, we define $(F(\mathbf{\Lambda})^*)^{-1}$ the inverse operator of $F(\mathbf{\Lambda})^* : H(\mathbf{\Lambda})^* \to Ran(F(\mathbf{\Lambda})^*)$ and $(F(\mathbf{\Lambda})^*)^{-1} = (F(\mathbf{\Lambda})^{-1})^*$.*

Theorem C.3 (Theorem 3.29; [DZ06])**.** *Let $(\lambda_k)_{k\in\mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying* (35)*. If $T > 2\pi/\delta$, then $(\xi_k)_{k\in\mathbb{Z}^*}$ forms a Riesz Basis in the space $X := \overline{span\{\xi_k|\ k\in\mathbb{Z}^*\}}^{L^2}$.*

Lemma C.4. *Let $\mathbf{\nu} := (\nu_k)_{k\in\mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying* (35)*. Let G be an entire function such that $G \in L^\infty(\mathbb{R},\mathbb{R})$ and there exist $J, I > 0$ such that $|G(z)| \leq Je^{I|z|}$ for every $z \in \mathbb{C}$. If $(\nu_k)_{k\in\mathbb{Z}^*}$ are simple zeros of G such that there exist $\tilde{d} \geq 0$, $C > 0$ such that*

$$|G'(\nu_k)| \geq \frac{C}{|k|^{1+\tilde{d}}}, \qquad \forall k \in \mathbb{Z}^*,\ \nu_k \neq 0, \tag{36}$$

then there exists $C > 0$ so that $Tr\Big(F_m(\mathbf{\nu}^m)^* F_m(\mathbf{\nu}^m)\Big) \leq C \min\{|l| \in E_m\}^{2(1+\tilde{d})}$ for every $m \in \mathbb{Z}^*$.

Proof. The proof is composed as follows.

1. First, we construct $(v_k)_{k\in\mathbb{Z}^*}$ a biorthogonal sequence to $(e^{i\nu_k(\cdot)})_{k\in\mathbb{Z}^*}$ in $L^2((0,T),\mathbb{C})$ with $T > 0$ sufficiently large and we estimate the L^2-norm of v_k for every $k \in \mathbb{Z}^*$.
2. Second, we characterize $(\xi_k)_{k\in\mathbb{Z}^*} = F(\mathbf{\Lambda})^*(e^{i\nu_k(\cdot)})_{k\in\mathbb{Z}^*}$, a Riesz basis of a suitable subspace of $L^2((0,T),\mathbb{C})$, and its biorthogonal sequence.
3. Third, we use the obtained estimates in order to provide an upper bound for $|(F(\mathbf{\nu})\mathbf{x})_k|$ with $k \in \mathbb{N}^*$ and $\mathbf{x} \in \ell^2(\mathbb{Z}^*,\mathbb{C})$. The result leads to the statement.

Construction of a biorthogonal sequence. Let $T > \max(2\pi/\delta, 2I)$. For every $k \in \mathbb{Z}^*$, we define $G_k(z) := G(z)(z-\nu_k)^{-1}$. Thanks to the Paley-Wiener's Theorem [DZ06, *Theorem* 3.19], for every $k \in \mathbb{Z}^*$, there exists $w_k \in L^2(\mathbb{R},\mathbb{R})$ with support in $[-I, I]$ such that

$$G_k(z) = \int_{-I}^{I} e^{izs} w_k(s)ds = \int_{-T/2}^{T/2} e^{izs} w_k(s)ds = \int_0^T e^{izt} e^{-iz\frac{T}{2}} w_k(t - T/2)dt.$$

For $j, k \in \mathbb{Z}^*$ and $c_k := G'(\nu_k)$, we call $v_k(t) := e^{i\nu_k\frac{T}{2}} w_k(t-T/2)$ and $\langle v_k, e^{i\nu_j(\cdot)}\rangle_{L^2((0,T),\mathbb{C})} = \delta_{k,j} G_k(\nu_k) = \delta_{k,j} G'(\nu_k) = \delta_{k,j} c_k$. The sequence $(v_k)_{k\in\mathbb{Z}^*}$ is biorthogonal to $(e^{i\nu_k(\cdot)}/c_k)_{k\in\mathbb{Z}^*}$ and $(v_k/c_k)_{k\in\mathbb{Z}^*}$ is biorthogonal to $(e^{i\nu_k(\cdot)})_{k\in\mathbb{Z}^*}$. Thanks to the Plancherel's identity, $\|v_k\|_{L^2((0,T),\mathbb{C})} = \|G_k\|_{L^2(\mathbb{R},\mathbb{R})}$. We show that, from the Phragmén-Lindelöf Theorem (*e.g.* [You80, *p.* 82; *Theorem* 11]), there exists $C_1 > 0$ such that

$$\|v_k\|_{L^2((0,T),\mathbb{C})} = \|G_k\|_{L^2(\mathbb{R},\mathbb{R})} \leq C_1, \qquad \forall k \in \mathbb{Z}^*. \tag{37}$$

First, G is entire, while there exist I and J such that $|G(z)| \leq Je^{I|z|}$ for every $z \in \mathbb{C}$. Second, there exists $M > 0$ so that $|G(x)| \leq M$ for every $x \in \mathbb{R}$. From [You80, *p.* 82; *Theorem* 11], we have $|G(x+iy)| \leq Me^{I|y|}$ for $x, y \in \mathbb{R}$. For every $k \in \mathbb{Z}^*$, there exists $c_1 > 0$, not depending on k, so that

$$\|G_k\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \overline{G_k(x)} G_k(x)\, dx = \int_{\mathbb{R}} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2}\, dx \leq \int_{|x-\nu_k|\leq 1} \overline{G(x)} G(x)(x-\nu_k)^{-2}\, dx + M^2 c_1.$$

The Cauchy Integral Theorem leads to (37) as there exists $c_2 > 0$, not depending on k, so that $\int_{|x-\nu_k|\leq 1} \frac{\overline{G(x)}G(x)}{(x-\nu_k)^2} dx \leq \int_0^\pi \big|\overline{G(\nu_k + e^{i\theta})} G(\nu_k + e^{i\theta})\big| d\theta \leq M^2 \int_0^\pi e^{2I\sin(\theta)}\, d\theta \leq M^2 c_2$.

Construction of a Riesz basis. Let $\mathbf{\nu} := (\nu_k)_{k\in\mathbb{Z}^*}$ and $\mathbf{e} := (e^{i\nu_k(\cdot)})_{k\in\mathbb{Z}^*} \subset L^2((0,T),\mathbb{C})$. Thanks to Proposition $C.3$, the sequence of functions $\mathbf{\Xi} = (\xi_k)_{k\in\mathbb{Z}^*} := ((F(\mathbf{\nu})^*\mathbf{e})_k)_{k\in\mathbb{Z}^*}$ forms a Riesz basis in $X := \overline{span\{\xi_k\ :\ k\in\mathbb{Z}^*\}}^{L^2}$. We call $\widetilde{\mathbf{v}} := (\widetilde{v}_k)_{k\in\mathbb{Z}^*}$ the corresponding biorthogonal sequence which is also a Riesz basis of X. From Remark $C.2$, the map $F(\mathbf{\nu})$ is invertible from $H(\mathbf{\nu})^*$ to $Ran(F(\mathbf{\nu})^*)$ and $(F(\mathbf{\nu})^*)^{-1} = (F(\mathbf{\nu})^{-1})^*$. As $\mathbf{v}/\mathbf{c} = (v_k/c_k)_{k\in\mathbb{Z}^*}$ is biorthogonal to $(e^{i\nu_k(\cdot)})_{k\in\mathbb{Z}^*}$, we have $(v_k/c_k)_{k\in\mathbb{Z}^*} = F(\mathbf{\nu})\widetilde{\mathbf{v}}$. Indeed, $\delta_{k,j} = \langle v_k/c_k, ((F(\mathbf{\Lambda})^*)^{-1}\mathbf{\Xi})_j\rangle_{L^2((0,T),\mathbb{C})} = \langle (F(\mathbf{\Lambda})^{-1}\mathbf{v}/\mathbf{c})_k, \xi_j\rangle_{L^2((0,T),\mathbb{C})}$ for every $j, k \in \mathbb{Z}^*$, which implies $(F(\mathbf{\Lambda})^{-1}\mathbf{v}/\mathbf{c})_k = \widetilde{v}_k$. The uniqueness of the biorthogonal family to $\mathbf{\Xi}$ implies the uniqueness of the biorthogonal family to $\mathbf{e}$. From [BL10, *Appendix B*; *Proposition* 19.(2)], there exist $C_2, C_3 > 0$ such that

$$C_2\|\mathbf{x}\|_{\ell^2}^2 \leq \int_0^T |u(s)|^2 ds \leq C_3\|\mathbf{x}\|_{\ell^2}^2, \qquad \forall u(t) = \sum_{k\in\mathbb{Z}^*} \xi_k x_k, \quad \mathbf{x} \in \ell^2(\mathbb{Z}^*, \mathbb{C}). \tag{38}$$

Conclusion. When $u(t) = \sum_{k\in\mathbb{Z}^*} \xi_k x_k$ with $\mathbf{x} \in \ell^2(\mathbb{Z}^*, \mathbb{C})$, the biorthogonality yields to $x_k = \langle \tilde{v}_k, u\rangle_{L^2((0,T),\mathbb{C})}$ for every $k \in \mathbb{Z}^*$. We call $m(k) \in \mathbb{Z}^*$ the number such that $k \in E_{m(k)}$. Thanks to (36), (37), and (38), there exist $C_4, C_5 > 0$ such that, for every $k \in \mathbb{Z}^*$, we have

$$\begin{aligned}|(F(\mathbf{\nu})\mathbf{x})_k| &= |\langle (F(\mathbf{\nu})(\langle \tilde{v}_l, u\rangle_{L^2((0,T),\mathbb{C})})_{l\in\mathbb{Z}})_k| = |\langle v_k/c_k, u\rangle_{L^2((0,T),\mathbb{C})}| \leq \|v_k\|_{L^2((0,T),\mathbb{C})}\|u\|_{L^2((0,T),\mathbb{C})}|c_k|^{-1}\\ &\leq C_3^{\frac{1}{2}}\|G_k\|_{L^2(\mathbb{R},\mathbb{R})}\|\mathbf{x}\|_{\ell^2}|G'(\nu_k)|^{-1} \leq C_4|k|^{1+\tilde{d}}\|\mathbf{x}\|_{\ell^2} \leq C_5 \min_{l\in E_{m(k)}} |l|^{1+\tilde{d}}\|\mathbf{x}\|_{\ell^2}.\end{aligned}$$

Thus, there exists $C_6 > 0$ so that $|(F_{m;j,k}(\mathbf{\nu}^m))| \leq C_6 \min_{l\in E_m} |l|^{1+\tilde{d}}$ for every $j, k \leq i(m)$, which leads to the statement. ☐

Proposition C.5. *Let $(\lambda_k)_{k\in\mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers such that $(\nu_k)_{k\in\mathbb{Z}^*} = \big(sgn(\lambda_k)\sqrt{|\lambda_k|}\big)_{k\in\mathbb{Z}^*}$ satisfies* (35)*. Let exist $C_1, C_2 > 0$ such that*

$$C_1|k| \leq |\nu_k| \leq C_2|k|, \qquad \forall k \in \mathbb{Z}^*, \quad \nu_k \neq 0. \tag{39}$$

Let G be an entire function so that $(\nu_k)_{k\in\mathbb{Z}^*}$ are its simple zeros, $G \in L^\infty(\mathbb{R}, \mathbb{R})$ and there exist $J, I > 0$ such that $|G(z)| \leq Je^{I|z|}$ for every $z \in \mathbb{C}$. If there exist $\tilde{d} \geq 0$ and $C > 0$ such that $|G'(\nu_k)| \geq \frac{C}{|k|^{1+\tilde{d}}}$ for every $k \in \mathbb{Z}^*$ such that $\nu_k \neq 0$, then

$$H(\mathbf{\Lambda}) \subseteq h^{\tilde{d}}(\mathbb{Z}^*, \mathbb{C}).$$

Proof. We show how the upper bound of $Tr\Big(F_m(\mathbf{\nu}^m)^*F_m(\mathbf{\nu}^m)\Big)$ for every $m \in \mathbb{Z}^*$ provided by Lemma $C.4$ leads to an upper bound of $Tr\Big(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)\Big)$. We conclude by discussing how the estimate achieves the claim.

1) Preliminaries. As $\inf_{\substack{k\in\mathbb{Z}^*\\ k+\mathcal{M}\neq 0}} |\nu_{k+\mathcal{M}} - \nu_k| \geq \delta\mathcal{M} \min_{\substack{k\in\mathbb{Z}^*\\ \nu_k\neq 0}}(|\nu_k|, 1)$ with $\delta > 0$ and $\mathcal{M} \in \mathbb{N}^* \setminus \{1\}$,

$$\inf_{\substack{k\in\mathbb{Z}^*\\ k+\mathcal{M}\neq 0}} |\lambda_{k+\mathcal{M}} - \lambda_k| = \inf_{\substack{k\in\mathbb{Z}^*\\ k+\mathcal{M}\neq 0}} \big||\nu_{k+\mathcal{M}}| - |\nu_k|\big|\big||\nu_{k+\mathcal{M}}| + |\nu_k|\big| \geq \min_{\substack{k\in\mathbb{Z}^*\\ \nu_k\neq 0}}(|\nu_k|, 1)\delta\mathcal{M}$$

since $(\lambda_k)_{k\in\mathbb{Z}^*} = \big(sgn(\nu_k)\nu_k^2\big)_{k\in\mathbb{Z}^*}$. Now, $\mathbf{\Lambda} := (\lambda_k)_{k\in\mathbb{Z}^*}$ and $\mathbf{\nu} := (\nu_k)_{k\in\mathbb{Z}^*}$ satisfy (35) with respect to $\delta' := \min_{\substack{k\in\mathbb{Z}^*\\ \nu_k\neq 0}}\{|\nu_k|, 1\}\delta$ and $\mathcal{M}$. This implies that the theory exposed in this appendix and the definitions of the equivalence classes E_m in $\mathbb{Z}^*$ are valid for both the sequences $\mathbf{\Lambda}$ and $\mathbf{\nu}$. We notice $|\lambda_l - \lambda_k| \geq \min\{|\nu_l|, |\nu_k|\}|\nu_l - \nu_k|$ for $l, k \in \mathbb{Z}^*$. Let $m \in \mathbb{Z}^*$ and $I \subseteq E_m$ so that $I \neq \emptyset$. Now, $|I| \leq |E_m| \leq \mathcal{M} - 1$ and

$$\prod_{j,k\in I} |\lambda_k - \lambda_j| \geq \min_{\substack{l\in I\\ \nu_l\neq 0}}|\nu_l|^{|I|} \prod_{j,k\in I} |\nu_k - \nu_j| \geq C_1 \min_{\substack{l\in I\\ \nu_l\neq 0}} |\nu_l| \prod_{j,k\in I} |\nu_k - \nu_j|$$

for $C_1 = \min_{\substack{l\in\mathbb{Z}^*\\ \nu_l\neq 0}}(|\nu_l|^{\mathcal{M}-2}, 1)$. Thus, there exists $C_2 > 0$ so that, for every m and $j, k \in E_m$, we have $|F_{m;j,k}(\mathbf{\Lambda}^m)| \leq C_2|F_{m;j,k}(\mathbf{\nu}^m)| \min\{|\nu_l|^{-1} : \ l \in E_m, \ \nu_l \neq 0\}$. Thanks to (39) and Lemma $C.4$, there exists $C_3 > 0$ such that

$$Tr\Big(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)\Big) \leq C_2^2 \min_{\substack{l\in E_m\\ \nu_l\neq 0}} |\nu_l|^{-2} \ Tr\Big(F_m(\mathbf{\nu}^m)^*F_m(\mathbf{\nu}^m)\Big) \leq C_3 \min_{l\in E_m} |l|^{2\tilde{d}}.$$

2) Conclusion. Let $\rho(M)$ be the spectral radius of a matrix M and let $|||\, M\, ||| = \sqrt{\rho(M^*M)}$ be its euclidean norm. As $\big(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)\big)$ is positive-definite,

$$|||\, F_m(\mathbf{\Lambda}^m)\, |||^2 = \rho\big(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)\big) \leq Tr\Big(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)\Big) \leq C_3 \min_{l\in E_m} |l|^{2\tilde{d}}, \qquad m \in \mathbb{Z}^*.$$

In conclusion, $h^{\tilde{d}}(\mathbb{Z}^*,\mathbb{C}) \subset H(\mathbf{\Lambda})$ as, for every $\mathbf{x} = (x_k)_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{Z}^*,\mathbb{C})$,

$$\|F(\mathbf{\Lambda})\mathbf{x}\|_{\ell^2}^2 \leq \sum_{m\in\mathbb{Z}^*} |||\, F_m(\mathbf{\Lambda}^m)\, |||^2 \sum_{l\in E_m} |x_l|^2 \leq C_3 \sum_{m\in\mathbb{Z}^*} \min_{l\in E_m} |l|^{2\tilde{d}} \sum_{l\in E_m} |x_l|^2 \leq C_3 \|\mathbf{x}\|_{h^{\tilde{d}}}^2 < +\infty. \qquad \square$$

Remark C.6. *If Proposition C.5 is satisfied with $\mathbf{\Lambda} = (\lambda_k)_{k\in\mathbb{Z}^*}$ and $\tilde{d} \geq 0$, then $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}(\mathbb{Z}^*,\mathbb{C})$, which is dense in $\ell^2(\mathbb{Z}^*,\mathbb{C})$. Thanks to Remark C.2, we consider $F(\mathbf{\Lambda})^*$ as the unique adjoint operator of $F(\mathbf{\Lambda})$. As $Tr(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)) = Tr(F_m(\mathbf{\Lambda}^m) F_m(\mathbf{\Lambda}^m)^*)$ for every $m \in \mathbb{Z}^*$, the techniques developed in the proof of Proposition C.5 lead to $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}(\mathbb{Z}^*,\mathbb{C})$.*

Proposition C.7. *Let $(\omega_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}^+ \cup \{0\}$ be an ordered sequence of pairwise distinct numbers so that there exist $\delta, C_1, C_2 > 0$ and $\mathcal{M} \in \mathbb{N}^* \setminus \{1\}$ such that*

$$\inf_{k\in\mathbb{N}^*} |\omega_{k+\mathcal{M}} - \omega_k| \geq \delta\mathcal{M}, \qquad C_1 k^2 \leq |\omega_k| \leq C_2 k^2, \qquad \forall k \in \mathbb{N}^* \setminus \{1\}.$$

Let G be an entire function so that $\{\pm\sqrt{\omega_k}\}_{k\in\mathbb{N}^*}$ are its simple zeros, $G \in L^\infty(\mathbb{R},\mathbb{R})$ and there exist $J, I > 0$ such that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. If there exist $\tilde{d} \geq 0$ and $C > 0$ such that

$$|G'(\pm\sqrt{\omega_k})| \geq \frac{C}{k^{1+\tilde{d}}}, \qquad \forall j \in \mathbb{N}^*,$$

then, for $T > 2\pi/\delta$ and for every $(x_k)_{k\in\mathbb{N}^*} \in h^{\tilde{d}}(\mathbb{N}^*,\mathbb{C})$ with $x_1 \in \mathbb{R}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$x_k = \int_0^T u(\tau) e^{i(\omega_k - \omega_1)\tau} d\tau, \qquad \forall k \in \mathbb{N}^*. \tag{40}$$

Proof. Let $\boldsymbol{\nu} := (\nu_k)_{k\in\mathbb{Z}^*}$ be such that $\nu_k = -\sqrt{\omega_k}$ for $k > 0$ and $\nu_k = \sqrt{\omega_{-k}}$ for $k < 0$. Let

$$\mathbf{\Lambda} := (\lambda_k)_{k\in\mathbb{Z}^*} \quad : \quad \lambda_k = -\omega_k, \quad \forall k > 0; \quad \lambda_k = \omega_{-k}, \quad \forall k < 0,$$

$$\mathbf{\Theta} := (\theta_k)_{k\in\mathbb{Z}^*\setminus\{-1\}} \quad : \quad \theta_k = -\omega_k + \omega_1, \quad \forall k > 0; \quad \theta_k = \omega_{-k} - \omega_1, \quad \forall k < -1.$$

We consider $\mathcal{M}' \in \mathbb{N}^* \setminus \{1\}$ and $\delta' > 0$ so that $\boldsymbol{\nu}$ and $\mathbf{\Lambda}$ satisfy (35) with respect to $\mathcal{M}'$ and δ', while

$$\inf_{\{k\in\mathbb{Z}^*\setminus\{-1\}\ :\ k+\mathcal{M}'\in\mathbb{Z}^*\setminus\{-1\}\}} |\theta_{k+\mathcal{M}'} - \theta_k| \geq \delta'\mathcal{M}'. \tag{41}$$

Let $\{E_m\}_{m\in\mathbb{Z}^*}$ be the equivalence classes in $\mathbb{Z}^*$ defined by $\boldsymbol{\nu}$ and $\mathbf{\Lambda}$ (as in the proof of Proposition $C.5$). Let $-1 \in E_{-1}$. Now, $\{E_m\}_{m\in\mathbb{Z}^*\setminus\{-1\}} \cup \{E_{-1} \setminus \{-1\}\}$ are the equivalence classes in $\mathbb{Z}^* \setminus \{-1\}$ defined by (41). Proposition $C.5$ and Remark $C.6$ imply $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}(\mathbb{Z}^*,\mathbb{C})$. Let $F(\mathbf{\Theta})$ be the operator defined in $\ell^2(\mathbb{Z}^* \setminus \{-1\},\mathbb{C})$. For $m \neq -1$, $F_m(\mathbf{\Theta}^m) = F_m(\mathbf{\Lambda}^m)$ and $F_m(\mathbf{\Theta}^m)^* = F_m(\mathbf{\Lambda}^m)^*$. As in Remark $C.6$,

$$H(\mathbf{\Theta}) \supseteq h^{\tilde{d}}(\mathbb{Z}^* \setminus \{-1\},\mathbb{C}), \qquad H(\mathbf{\Theta})^* \supseteq h^{\tilde{d}}(\mathbb{Z}^* \setminus \{-1\},\mathbb{C}).$$

For $T > 0$, we define in $L^2 := L^2((0,T),\mathbb{C})$ the sequences of functions

$$\mathbf{e} := (e^{i\theta_k(\cdot)})_{k\in\mathbb{Z}^*\setminus\{-1\}}, \qquad \mathbf{\Xi} := (\xi_k(\cdot))_{k\in\mathbb{Z}^*\setminus\{-1\}} = F(\mathbf{\Theta})^*\mathbf{e}.$$

When $T > 2\pi/\delta$, Theorem $C.3$ ensures that $(\xi_k)_{k\in\mathbb{Z}^*\setminus\{-1\}}$ is a Riesz Basis in $X := \overline{span_{k\in\mathbb{Z}^*\setminus\{-1\}}(\xi_k)}^{L^2}$. Thanks to [BL10, *Appendix B*; *Proposition* 19.(2)], the map $M : g \in X \mapsto (\langle \xi_k, g\rangle_{L^2(0,T)})_{k\in\mathbb{Z}^*\setminus\{-1\}} \in \ell^2(\mathbb{Z}^* \setminus \{-1\},\mathbb{C})$ is invertible and

$$\langle \xi_k, g\rangle_{L^2(0,T)} = (F(\mathbf{\Theta})^* \langle \mathbf{e}, g\rangle_{L^2(0,T)})_k, \qquad \forall k \in \mathbb{Z}^* \setminus \{-1\}.$$

Let $\tilde{X} := M^{-1} \circ F(\mathbf{\Theta})^*\big(h^{\tilde{d}}(\mathbb{Z}^*\setminus\{-1\},\mathbb{C})\big)$. The map $(F(\mathbf{\Theta})^*)^{-1} \circ M : g \in \tilde{X} \mapsto (\langle \mathbf{e}, g\rangle_{L^2(0,T)})_{k\in\mathbb{Z}^*\setminus\{-1\}} \in h^{\tilde{d}}(\mathbb{Z}^* \setminus \{-1\},\mathbb{C})$ is invertible. For every $(x_k)_{k\in\mathbb{Z}^*\setminus\{-1\}} \in h^{\tilde{d}}(\mathbb{Z}^* \setminus \{-1\},\mathbb{C})$, there exists $u \in L^2((0,T),\mathbb{C})$ such that

$$x_k = \int_0^T u(\tau) e^{-i\theta_k\tau} d\tau, \qquad \forall k \in \mathbb{Z}^* \setminus \{-1\}. \tag{42}$$

Given $(x_k)_{k\in\mathbb{N}^*} \in h^{\tilde{d}}(\mathbb{N}^*,\mathbb{C})$, we introduce $(\tilde{x}_k)_{k\in\mathbb{Z}^*\setminus\{-1\}} \in h^{\tilde{d}}(\mathbb{Z}^*\setminus\{-1\},\mathbb{C})$ such that $\tilde{x}_k = x_k$ for $k > 0$, while $\tilde{x}_k = \overline{x}_{-k}$ for $k < -1$. Thanks to (42) and to the definition of $\boldsymbol{\Theta}$, there exists $u \in L^2((0,T),\mathbb{C})$ such that

$$\int_0^T u(s)e^{i(\omega_k-\omega_1)s}ds = x_k = \int_0^T \overline{u}(s)e^{i(\omega_k-\omega_1)s}ds, \qquad k \in \mathbb{N}^*\setminus\{1\}.$$

If $x_1 \in \mathbb{R}$, then u is real and (40) is solvable with $u \in L^2((0,T),\mathbb{R})$. □

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