ON THE HYPERBOLICITY OF BASE SPACES FOR MAXIMALLY VARIATIONAL FAMILIES OF SMOOTH PROJECTIVE VARIETIES

Ya Deng, Dan Abramovich

To cite this version:

Ya Deng, Dan Abramovich. ON THE HYPERBOLICITY OF BASE SPACES FOR MAXIMALLY VARIATIONAL FAMILIES OF SMOOTH PROJECTIVE VARIETIES: Analytic Shafarevich hyperbolicity conjecture. 2019. hal-01874489v2

HAL Id: hal-01874489

https://hal.science/hal-01874489v2

Preprint submitted on 22 May 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE HYPERBOLICITY OF BASE SPACES FOR MAXIMALLY VARIATIONAL FAMILIES OF SMOOTH PROJECTIVE VARIETIES

YA DENG, WITH AN APPENDIX BY DAN ABRAMOVICH

Abstract. For smooth families with maximal variation, whose general fibers have semi-ample canonical bundle, the generalized Viehweg hyperbolicity conjecture states that the base spaces of such families are of log general type. This deep conjecture was recently proved by Popa-Schnell using the theory of Hodge modules and a theorem by Campana-Păun. In this paper we prove that those base spaces are pseudo Kobayashi hyperbolic, as predicted by the Lang conjecture: any complex quasi-projective manifold is pseudo Kobayashi hyperbolic if it is of log general type. As a consequence, we prove the Brody hyperbolicity of moduli spaces of polarized manifolds with semi-ample canonical bundle. This answers a question by Viehweg-Zuo in 2003. We also prove the Kobayashi hyperbolicity of base spaces of effectively parametrized families of minimal projective manifolds of general type. This generalizes previous work by To-Yeung, in which they further assumed that these families are canonically polarized.

Contents

0. Introduction

0.1. **Main theorems.** A complex space X is said to be *pseudo Kobayashi hyperbolic*, if X is hyperbolic modulo a proper Zariski closed subset $\Delta \subsetneq X$, that is, the Kobayashi pseudo distance $d_X : X \times X \to [0, +\infty[$ of X satisfies that $d_X(p,q) > 0$ for every pair of distinct points $p, q \in X$ not both contained in Δ. In particular, any non-constant holomorphic map $\gamma : \mathbb{C} \to X$ has image $\gamma(\mathbb{C}) \subset \Delta$. When such Δ is an empty set, this definition reduces to the usual definition of *Kobayashi hyperbolicity*, and the Kobayashi pseudo distance d_X is a distance. Proven by Parshin and Arakelov in the early 70's, *Shafarevich's hyperbolicity conjecture* states that a non-isotrivial smooth family of curves of genus $g \geqslant 2$ over a non hyperbolic curve has to be isotrivial, that is, all the fibers are isomorphic. One aim of this paper is to prove a result which can be seen as some sort of *analytic Shafarevich hyperbolicity conjecture* in higher dimensions.

Date: Wednesday 22$^{\text{nd}}$ May, 2019.

2010 *Mathematics Subject Classification.* 32Q45, 14D07, 14E99.

Key words and phrases. pseudo Kobayashi hyperbolicity, Brody hyperbolicity, moduli spaces, Viehweg-Zuo question, polarized variation of Hodge structures, Viehweg-Zuo Higgs bundles, Finsler metric, positivity of direct images, Griffiths curvature formula of Hodge bundles.

Theorem A. *Let $f_U : U \to V$ be a smooth projective morphism between complex quasi-projective manifolds with connected fibers. Assume that the general fiber of f_U has semi-ample canonical bundle, and f_U is of maximal variation, that is, the general fiber can only be birational to countably other fibers. Then the base space V is pseudo-Kobayashi hyperbolic.*

As a consequence of Theorem A, we prove affirmatively a conjecture by Viehweg-Zuo [VZ03, Question 0.2] on the Brody hyperbolicity of moduli spaces for polarized manifolds with semi-ample canonical sheaf.

Theorem B (Brody hyperbolicity of moduli spaces)**.** *Consider the moduli functor $\mathscr{P}_h$ of polarized manifolds with semi-ample canonical sheaf introduced by Viehweg [Vie95, §7.6], where h is the Hilbert polynomial associated to the polarization $\mathscr{H}$. Assume that for some quasi-projective manifold V there exists a smooth family $(f_U : U \to V, \mathscr{H}) \in \mathscr{P}_h(V)$ for which the induced moduli map $\varphi_U : V \to P_h$ is quasi-finite over its image, where P_h denotes to be the quasi-projective*[1] *coarse moduli scheme for $\mathscr{P}_h$. Then the base space V is Brody hyperbolic, that is, there are no non-constant entire holomorphic curves $\gamma : \mathbb{C} \to V$.*

As a byproduct, we reduce the pseudo Kobayashi hyperbolicity of varieties to the existence of certain negatively curved Higgs bundles (which we call *Viehweg-Zuo Higgs bundles* in Definition 2.1). This provides a main building block for our recent work [Den19] on the hyperbolicity of bases of log Calabi-Yau pairs.

Another aim of the paper is to prove affirmatively a folklore conjecture on the *Kobayashi hyperbolicity* for moduli spaces of minimal projective manifolds of general type, which can be thought of as an analytic refinement of Theorem B in case the fibers have big and nef canonical bundle.

Theorem C. *Let $f_U : U \to V$ be a smooth family of minimal projective manifolds of general type over the quasi-projective manifold V. Assume that the family f_U is* effectively parametrized, *that is, the Kodaira-Spencer map*

$$\rho_y : \mathscr{T}_{V,y} \to H^1(U_y, \mathscr{T}_{U_y}) \tag{0.1.1}$$

is injective for each point $y \in V$, where $\mathscr{T}_{U_y}$ denotes the tangent bundle of the fiber $U_y := f_U^{-1}(y)$. Then the base space V is Kobayashi hyperbolic.

0.2. **Previous related results.** Theorem A is closely related to the *Viehweg hyperbolicity conjecture*: let $f_U : U \to V$ be a maximally variational smooth family of polarized manifolds with semi-ample canonical bundle over a quasi-projective manifold V, then the base V must be of log-general type. In the series of works [VZ01, VZ02, VZ03], Viehweg-Zuo construct in a first step a big subsheaf of symmetric log differential forms of the base (so-called *Viehweg-Zuo sheaves*). Built on this result, Viehweg hyperbolicity conjecture was shown by Kebekus-Kovács [KK08a, KK08b, KK10] when V is a surface or threefold, by Patakfalvi [Pat12] when V is compact or admits a non-uniruled compactification, and it was completely solved by Campana-Păun [CP15b], in which they proved a vast generalization of the famous generic semipositivity result of Miyaoka (see also [CP15a,CP16,Sch17a] for other different proofs). More recently, using deep theory of Hodge modules, Popa-Schnell [PS17] constructed Viehweg-Zuo sheaves on the base space V of the smooth family $f_U : U \to V$ of projective manifolds whose geometric generic fiber admits a good minimal model. Combining this with the aforementioned theorem of Campana-Păun, they proved that such base space V is of log general type. Therefore, Theorem A is predicted by a famous conjecture of Lang (cf. [Lan91, Chapter VIII. Conjecture 1.4]), which stipulates that a complex quasi-projective manifold is pseudo Kobayashi hyperbolic if and only if it is of log general type. To our knowledge, Lang's conjecture is by now known for the trivial case of curves, for general

[1] The quasi-projectivity of P_h was proved by Viehweg in [Vie95].

hypersurface X in the complex projective space $\mathbb{C}P^n$ of high degrees [Bro17, Dem18, Siu15] as well as their complements $\mathbb{C}P^n \setminus X$ [BD19], for projective manifolds whose universal cover carries a bounded strictly plurisubharmonic function [BD18], for quotients of bounded (symmetric) domains [Rou16, CRT19, CDG19], and for subvarieties on abelian varieties [Yam19]. Theorem A therefore provides some new evidences for Lang's conjecture.

Theorem B was first proved by Viehweg-Zuo [VZ03, Theorem 0.1] for moduli spaces of *canonically polarized* manifolds. Combining the approaches by Viehweg-Zuo [VZ03] with those by Popa-Schnell [PS17], very recently, Popa-Taji-Wu [PTW18, Theorem 1.1] proved Theorem B for moduli spaces of *polarized* manifolds with big and semi-ample canonical bundles. As we will see below, our work owes a lot to the general strategies and techniques in their work [VZ03, PTW18].

The Kobayashi hyperbolicity of moduli spaces $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geqslant 2$ has long been known to us by the work of Royden and Wolpert [Roy75, Wol86]. The first important breakthrough on higher dimensional generalizations was made by To-Yeung [TY15], in which they proved Kobayashi hyperbolicity of the base V considered in Theorem C when the canonical bundle K_{U_y} of each fiber $U_y := f_U^{-1}(y)$ of $f_U : U \to V$ is further assumed to be ample (see also [BPW17, Sch17b] for alternative proofs). Differently from the approaches in [VZ03, PTW18], their strategy is to study the curvature of the generalized Weil-Petersson metric for families of canonically polarized manifolds, along the approaches initiated by Siu [Siu86] and later developed by Schumacher [Sch08, Sch10, Sch12]. For the smooth family of Calabi-Yau manifolds (resp. orbifolds), Berndtsson-Păun-Wang [BPW17] and Schumacher [Sch17b] (resp. To-Yeung [TY18]) proved the Kobayashi hyperbolicity of the base once this family is assumed to be effectively parametrized.

0.3. **Strategy of the proof.** For the smooth family $f_U : U \to V$ of canonically polarized manifolds with maximal variation, Viehweg-Zuo [VZ03] constructed certain negatively twisted Higgs bundles (which we call *Viehweg-Zuo Higgs bundles* in Definition 2.1) $(\tilde{\mathscr{E}}, \tilde{\theta}) := (\bigoplus_{q=0}^{n} \mathscr{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^{n} \mathbb{1} \otimes \theta_{n-q,q})$, over some smooth projective compactification Y of a certain birational model $\tilde{V}$ of V, where $\mathscr{L}$ is some big and nef line bundle on Y, and $\left(\bigoplus_{q=0}^{n} E^{n-q,q}, \bigoplus_{q=0}^{n} \theta_{n-q,q}\right)$ is a Higgs bundle induced by a polarized variation of Hodge structure defined over a Zariski open set of $\tilde{V}$. In a recent remarkable paper [PTW18], Popa-Taji-Wu introduced several new inputs to develop Viehweg-Zuo's strategy in [VZ03], which enables them to construct those Higgs bundles on base spaces of smooth families whose geometric generic fiber admits a good minimal model (see also Theorem 2.21 for a weaker statement as well as a slightly different proof). As we will see in the main content, the Viehweg-Zuo Higgs bundles (VZ Higgs bundles for short) are the crucial tools in proving our main results.

When each fibers $U_y := f_U^{-1}(y)$ of the smooth family $f_U : U \to V$ considered in Theorem A have ample or big and nef canonical bundles, let us briefly recall the general strategies in proving the *algebraic degeneracy* of V in [VZ03, PTW18]. A certain sub-Higgs bundle $(\mathscr{F}, \eta)$ of $(\tilde{\mathscr{E}}, \tilde{\theta})$ with log poles contained in the divisor $D := Y \setminus \tilde{V}$ gives rise to a morphism

$$\tau_{\gamma,k} : \mathscr{T}_{\mathbb{C}}^{\otimes k} \to \gamma^*(\mathscr{L}^{-1} \otimes E^{n-k,k}) \tag{0.3.1}$$

for any entire curve $\gamma : \mathbb{C} \to \tilde{V}$. If $\gamma : \mathbb{C} \to \tilde{V}$ is Zariski dense, by the Kodaira-Nakano vanishing (when K_{U_y} is ample) and Bogomolov-Sommese vanishing theorems (when K_{U_y} is big and nef), one can verify that $\tau_{\gamma,1}(\mathbb{C}) \not\equiv 0$. Hence there is some $m > 0$ (depending on γ) so that $\tau_{\gamma,m}$ factors through $\gamma^*(\mathscr{L}^{-1} \otimes N^{n-m,m})$, where $N^{n-m,m}$ is the kernel of the Higgs field $\theta_m : E^{n-m,m} \to E^{n-m-1,m+1} \otimes \Omega_Y(\log D)$. Applying Zuo's theorem [Zuo00] on the negativity of $N^{n-m,m}$, a certain positively curved metric for $\mathscr{L}$ can produce a singular hermitian metric on $\mathscr{T}_{\mathbb{C}}$ with the *Gaussian curvature* bounded from above by a negative constant, which contradicts with the (Demailly's) Ahlfors-Schwarz lemma [Dem97, Lemma 3.2]. However, this approach did not provide enough information for the Kobayashi pseudo distance of the base V. Moreover, the use of vanishing theorem cannot show $\tau_{Y,1}(\mathbb{C}) \not\equiv 0$ when fibers of $f_U : U \to V$ is not minimal manifolds of general type.

One of the main results in the present paper is to apply the VZ Higgs bundle to construct a (possibly degenerate) Finsler metric F on some birational model $\tilde{V}$ of the base V, whose holomorphic sectional curvature is bounded above by a negative constant (say *negatively curved Finsler metric* in Definition 2.9.(ii)). A bimeromorphic criteria for pseudo Kobayashi hyperbolicity in Lemma 2.10 states that, the base is pseudo Kobayashi hyperbolic if F is *positively definite* over a Zariski dense open set. Let us now briefly explain our idea of the constructions. By factorizing through some sub-Higgs sheaf $(\mathscr{F}, \eta) \subseteq (\tilde{\mathscr{E}}, \tilde{\theta})$ with logarithmic poles *only* along the boundary divisor $D := Y \setminus \tilde{V}$, one can define a morphism for any $k = 1, \ldots, n$:

$$\tau_k : \mathrm{Sym}^k \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k}, \tag{0.3.2}$$

where $\mathscr{L}$ is some big line bundle over Y equipped with a *positively curved* singular hermitian metric $h_{\mathscr{L}}$. Then for each k, the hermitian metric h_k on $\tilde{\mathscr{E}}_k := \mathscr{L}^{-1} \otimes E^{n-k,k}$ induced by the Hodge metric as well as $h_{\mathscr{L}}$ (see Proposition 2.2 for details) will give rise to a Finsler metric F_k on $\mathscr{T}_Y(-\log D)$ by taking the k-th root of the pull-back $\tau_k^* h_k$. However, the holomorphic sectional curvature of F_k might not be negatively curved. Inspired by the aforementioned work of Schumacher, To-Yeung and Berndtsson-Păun-Wang [Sch12, Sch17b, TY15, BPW17] on the curvature computations of generalized Weil-Petersson metric for families of canonically polarized manifolds, we define a convex sum of Finsler metrics

$$F := \Big(\sum_{k=1}^{n} \alpha_k F_k^2\Big)^{1/2} \quad \text{with } \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+ \tag{0.3.3}$$

on $\mathscr{T}_Y(-\log D)$, to offset the unwanted positive terms in the curvature $\Theta_{\tilde{\mathscr{E}}_k}$ by negative contributions from the $\Theta_{\tilde{\mathscr{E}}_{k+1}}$ (the last order term was $\Theta_{\tilde{\mathscr{E}}_n}$ is always semi-negative by the Griffiths curvature formula). We proved in Proposition 2.20 that for proper $\alpha_1, \ldots, \alpha_n > 0$, the holomorphic sectional curvature of F is negative and bounded away from zero. To summarize, we establish an *algorithm* for the construction of Finsler metrics via VZ Higgs bundles.

To prove Theorem A, we first note that the VZ Higgs bundles over some birational model $\tilde{V}$ of the base space V were constructed by Popa-Taji-Wu in their elaborate work [PTW18]. Let Y be some smooth projective compactification $\tilde{V}$ with simple normal crossing boundary $D := Y \setminus \tilde{V}$. By our construction of negatively curved Finsler metric F defined in (0.3.3) via VZ Higgs bundles, to show that F is *positively definite* over some Zariski open set, it suffices to prove that $\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1}$ defined in (0.3.2) is *generically injective* (which we call *generic local Torelli* for VZ Higgs bundles in § 2.1). This was proved in Theorem F, by using the degeneration of Hodge metric and the curvature properties of Hodge bundles. In particular, we show that the generic injectivity of τ_1 is indeed an intrinsic feature of all VZ Higgs bundles (not related to the Kodaira dimension of fibers of f!). By a standard inductive argument in [VZ03, PTW18], one can easily show that Theorem A implies Theorem B.

Now we will explain the strategy to prove Theorem C. Note that the VZ Higgs bundles are only constructed over some birational model $\tilde{V}$ of V, which is not Kobayashi hyperbolic in general. This motivates us first to establish a *bimeromorphic criteria for Kobayashi hyperbolicity* in Lemma 2.11. Based on this criteria, in order to apply the VZ Higgs bundles to prove the Kobayashi hyperbolicity of the base V in Theorem C, it suffices to show that

(♠) for any given point y on the base V, there exists a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ constructed over some birational model $\nu : \tilde{V} \to V$, such that $\nu^{-1} : V \dashrightarrow \tilde{V}$ is defined at y.

(♣) The negatively curved Finsler metric F on $\tilde{V}$ defined in (0.3.3) induced by the above VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ is positively definite at the point $\nu^{-1}(y)$.

Roughly speaking, the idea is to produce an abundant supply of *fine* VZ Higgs bundles to construct sufficiently many negatively curved Finsler metrics, which are obstructions to the degeneracy of Kobayashi pseudo distance d_V of V. This is much more demanding than the Brody hyperbolicity and Viewheg hyperbolicity of V, which can be shown by the existence of *only one* VZ Higgs bundle on an arbitrary birational model of V, as mentioned in [VZ02, VZ03, PS17, PTW18].

Let us briefly explain how we achieve both (♠) and (♣).

As far as we see in [VZ03, PTW18], in their construction of VZ Higgs bundles, one has to blow-up the base for several times (indeed twice). Recall that the basic setup in [VZ03, PTW18] is the following: after passing to some smooth birational model $f_{\tilde{U}} : \tilde{U} = U \times_V \tilde{V} \to \tilde{V}$ of $f_U : U \to V$, one can find a smooth projective compactification $f : X \to Y$ of $\tilde{U}^r \to \tilde{V}$

$$\begin{tikzcd} U^r \arrow[d] & \tilde{U}^r \arrow[l, "\overset{\text{bir}}{\sim}"'] \arrow[r, "\subseteq"] \arrow[d] & X \arrow[d, "f"] \\ V & \tilde{V} \arrow[l, "\overset{\text{bir}}{\sim}"', "\nu"] \arrow[r, "\subseteq"] & Y \end{tikzcd} \tag{0.3.4}$$

so that there exists (at least) one hypersurface

$$H \in \left|\ell K_{X/Y} - \ell f^*\mathscr{L}\right| \quad \text{for some} \quad \ell \gg 0 \tag{0.3.5}$$

which is *transverse* to the general fibers of f. Here $\mathscr{L}$ is some big and nef line bundle over Y, and $U^r := U \times_V \times \cdots \times_V U$ (resp. $\tilde{U}^r$) is the r-fold fiber product of $f_U : U \to V$ (resp. $f_{\tilde{U}} : \tilde{U} \to \tilde{V}$). The VZ Higgs bundle is indeed the logarithmic Higgs bundles associated to the Hodge filtration of an auxiliary variation of polarized Hodge structures constructed by taking the middle dimensional relative de Rham cohomlogy on the cyclic cover of X ramified along H.

In order to find such H in (0.3.5), a crucial step in [VZ03,PTW18] is the use of *weakly semi-stable reduction* by Abramovich-Karu [AK00] so that, after changing the birational model $U \to V$ by performing certain (uncontrollable) base change $\tilde{U} := U \times_V \tilde{V} \to \tilde{V}$, one can find a "good" compactification $X \to Y$ of $\tilde{U}^r \to \tilde{V}$ and a finite dominant morphism $W \to Y$ from a smooth projective manifold W such that the base change $X \times_Y W \to W$ is birational to a *mild morphism* $Z \to W$, which is in particular flat with reduced fibers (even fonctorial under fiber products). For our goal (♠), we need a more refined control of the *alteration for the base* in the weakly semistable reduction [AK00, Theorem 0.3], which remains unknown at the moment. Fortunately, as was suggested to us and proved in Appendix A by Abramovich, using moduli of Alexeev stable maps one can establish a $\mathbb{Q}$-*mild reduction* for the family $U \to V$ in place of the *mild reduction* in [VZ03], so that we can also find a "good" compactification $X \to Y$ of $U^r \to V$ without passing the birational models $\tilde{V} \to V$ as in (0.3.4). This is the main theme of Appendix A.

Even if we can apply $\mathbb{Q}$-mild reduction to avoid the first blow-up of the base as in [VZ03, PTW18], the second blow-up is in general inevitable. Indeed, the *discriminant* of the new family $Z_H \to Y \supset V$ obtained by taking the cyclic cover along H in (0.3.5) is in general not normal crossing. One thus has to blow-up this discriminant locus of $Z_H \to Y$ to make it normal crossing as in [PTW18]. Therefore, to assure (♠), it then suffices to show that there exists a compactification $f : X \to Y$ of the smooth family $U^r \to V$ so that for some sufficiently ample line bundle $\mathscr{A}$ over Y,

$$f_*(mK_{X/Y}) \otimes \mathscr{A}^{-m} \text{ is globally generated over } V \text{ for some } m \gg 0. \tag{$*$}$$

Indeed, for any given point $y \in V$, by (∗) one can find H transverse to the fiber $X_y := f^{-1}(y)$, and thus the new family $Z_H \to Y$ will be smooth over an open set containing y. To the bests of our knowledge, (∗) was only known to us when the moduli is canonically polarized [VZ02, Proposition 3.4]. § 1.5 is devoted to the proof of (∗) for the family $U \to V$ in Theorem C (see Theorem D.(iii) below). This in turn achieves (♠).

To achieve (♣), our idea is to take *different cyclic coverings* by "moving" H in (0.3.5), to produce different "fine" VZ Higgs bundles. For any given point $y \in V$, by (♠), one can take a birational model $\nu : \tilde{V} \to V$ so that ν is isomorphic at y, and there exists a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ on the normal crossing compactification $Y \supset \tilde{V}$. To prove that the induced negatively curved Finsler metric F is positively definite at $\tilde{y} := \nu^{-1}(y)$, by our definition of F in (0.3.3), it suffices to show that τ_1 defined in (0.3.2) is *injective* at $\tilde{y}$ in the sense of $\mathbb{C}$-linear map between complex vector spaces

$$\tau_{1,\tilde{y}} : \mathscr{T}_{\tilde{V},\tilde{y}} \xrightarrow{\simeq} \mathscr{T}_Y(-\log D)_{\tilde{y}} \xrightarrow{\rho_{\tilde{y}}} H^1(X_{\tilde{y}}, \mathscr{T}_{X_{\tilde{y}}}) \xrightarrow{\varphi_{\tilde{y}}} \tilde{\mathscr{E}}_{1,\tilde{y}}.$$

As we will see in § 3, when H in (0.3.5) is properly chosen (indeed transverse to the fiber X_y) which is ensured by (∗), $\varphi_{\tilde{y}}$ is injective at $\tilde{y}$. Hence $\tau_{1,\tilde{y}}$ is injective by our assumption of *effective parametrization* (hence $\rho_{\tilde{y}}$ is injective) in Theorem C. This is our strategy to prove Theorem C.

0.4. **Results on the positivity of direct images.** As we explained above, one has to prove some results on the positivity of direct images for families with fibers of general type, which fits our needs in achieving the crucial property (∗).

Theorem D (=Theorem 1.21)**.** *Let $f_U : U \to V$ be a smooth projective morphism of quasi-projective manifolds with connected fibers. Assume that each fiber $X_y := f_U^{-1}(y)$ is a projective manifold of general type, and the set of $z \in V$ with X_z birationally equivalent to X_y is finite. Then*

(i) *for any smooth projective compactification $f : X \to Y$ of $f_U : U \to V$ and any sufficiently ample line bundle $\mathscr{A}$ over Y, $f_*(\ell K_{X/Y})^{\star\star} \otimes \mathscr{A}^{-1}$ is globally generated over V for any $\ell \gg 0$. In particular, $f_*(\ell K_{X/Y})$ is ample with respect to V.*
(ii) *In the same setting as (i), $\det f_*(\ell K_{X/Y}) \otimes \mathscr{A}^{-r_\ell}$ is also globally generated over V for any $\ell \gg 0$, where $r_\ell := \operatorname{rank} f_*(\ell K_{X/Y})$. In particular, the augmented base locus*

$$\mathbf{B}_+\big(\det f_*(\ell K_{X/Y})\big) \subset Y \setminus V.$$

(iii) *For some sufficiently divisible $r \gg 0$, there exists an algebraic fiber space $\tilde{f} : \tilde{X} \to \tilde{Y}$ compactifying $U^r \to V$ so that for ℓ large and divisible enough, $\tilde{f}_*(\ell K_{\tilde{X}/\tilde{Y}}) \otimes \mathscr{L}^{-\ell}$ is globally generated over V. Here $\mathscr{L}$ is some sufficiently ample line bundle over $\tilde{Y}$, and U^r denotes to be the r-fold fiber product of $U \to V$.*

As far as we are aware of, the best known result on Theorem D.(i) is due to Viehweg-Zuo [VZ02, Proposition 3.4.iii)], in which they proved the same result but for canonically polarized family. Theorems D.(i) and D.(ii) also refine a theorem by Kollár [Kol87], in which he proved the bigness (in the sense of Viehweg) of $f_*(\ell K_{X/Y})$ and $\det f_*(\ell K_{X/Y})$ under a weaker assumption that the variation of the family is maximal.

Let us emphasize that we have to apply the $\mathbb{Q}$-mild reduction in the proof of Theorem D.(iii) to find a "good compactification" of $f_U : U \to V$. As we have seen in the work [VZ03, PTW18], this is a crucial step in the construction of VZ Higgs bundles.

The proof of Theorem D.(i) mainly follows the strategy of [Vie90, Theorem 5.2] and [VZ02, Proposition 3.4.iii]. The first step is to prove that $\det f_*(\mu m K_{X/Y})^a \otimes \det f_*(m K_{X/Y})^b$ is ample with respect to V for some $\mu \gg m \gg 0$, and $b \gg a \gg 0$. To prove this, we apply Kollár-Viehweg's ampleness criterion and the BCHM theorem [BCHM10] to reduce the problem to the weak positivity of $f_*(mK_{X/Y})$ with respect to V for $m \gg 0$. We then apply the techniques in [CP17] to obtain the positivity of $K_{X/Y}$ modulo some multiplicity divisors and f-exceptional divisors, whereas the properties of m-Bergman metric and the pluricanonical (L^2-)extension theorem enable us to control these multiplicity divisors.

We also give a partial converse of Theorem D.(ii), which can be seen as a criteria for the *birational isotriviality* of families of general type varieties, and refines a result by Kawamata (cf. [Kaw85]).

Theorem E (=Theorem 1.13)**.** *Let $f : X \to Y$ be an algebraic fiber space between smooth projective manifolds with general fibers of general type. For the integer $m \geqslant 2$ with $f_*(mK_{X/Y})$ non-zero, if the* numerical dimension $\nu\big(\det f_*(mK_{X/Y})\big) = 0$*, then f is* birationally isotrivial*, that is, two general fibers X_y and X_z of the fibration f are birationally equivalent.*

We stress here that we have a concrete loci on Y in which any two fibers are birationally equivalent (see Remark 1.14). To prove Theorem E, we apply the deep results in [CP17] and the properties of line bundles whose numerical dimension is zero studied in [Bou04,BDPP13].

0.5. **Structure of the paper.** The paper is organized as follows. In § 1.1, we recall the Viehweg's *weak positivity for torsion free sheaves* in studying the positivity of direct images, and we prove a slightly more general result on the weak positivity of direct images of *logarithmic relative pluri-canonical bundles*. This result was applied in § 1.2 to obtain a *strong positivity* of the determinant of direct image sheaves. § 1.3 is of independent interest: we apply the recent work by Cao-Păun to give a criterion on *birational isotriviality* for families of projective manifolds of general type. § 1.5 is the the first main technical part of our paper. In this subsection, we prove the "almost ampleness" of relative pluri-canonical bundles as well as their direct images for certain families. The aim of § 1.6 is to provide the basic setup for § 3, combining the $\mathbb{Q}$-*mild reduction* in Appendix A and our main results in § 1.5. § 2 is the core of our paper and is of independent interests. In § 2.1 we give an abstract definition of the VZ Higgs bundles following [VZ02, VZ03, PTW18] for the purpose of further applications. In § 2.3 we prove that any VZ Higgs bundle satisfies a "generic local Torelli". In §§ 2.5 and 2.6 we prove that for any VZ Higgs bundle we can associate it to a Finsler metric with the holomorphic sectional curvature bounded above by a negative constant, which is *non-degenerate* over the Zariski dense open set on which the *local Torelli property* holds. This in turn proves Theorems A and B. § 3 is devoted to the refinements of VZ Higgs bundles, following the approaches in [VZ02,PTW18]. Based on the constructions in § 2, these refined Higgs bundles are applied to produce *sufficiently many negatively curved Finsler metrics* on different birational models of base spaces for effectively parametrized families of minimal projective manifolds of general type, which are the obstructions to the *degeneracy of Kobayashi pseudo distance* of these base spaces. This in turn proves Theorem C. Appendix A is written by Abramovich to introduce the $\mathbb{Q}$-*mild reduction*, which is applied in the present paper to find a good compactification of smooth families *without passing to birational models.*

The techniques in § 1 seems rather involved, since our objective is not merely to prove the hyperbolicity of moduli spaces, but also to study the positivity of direct images combining both the analytic methods and algebraic ones, which (we hope) might bring some new perspectives in this independent subject. The readers who are only interested in the proof of Theorem C can skip § 1.1, § 1.2 and § 1.3 since Proposition 1.9 (which is used to prove Theorem D.(i)) has already been proved by Viehweg [Vie90, Theorem 5.2] when the fibers are further assumed to be *minimal.*

Acknowledgments. This paper is the merger of my previous two articles [Den18a,Den18b] with slight improvements. It owes a lot to the celebrated work [VZ02, VZ03, PTW18], to which I express my gratitudes. I would like to sincerely thank Professors Dan Abramovich, Sébastien Boucksom, Håkan Samuelsson Kalm, Kalle Karu, Mihai Păun, Mihnea Popa, Georg Schumacher, Jörg Winkelmann, Chenyang Xu, Xiaokui Yang, Kang Zuo, and Olivier Benoists, Junyan Cao, Chen Jiang, Ruiran Sun, Lei Wu, Jian Xiao for answering my questions and very fruitful discussions. I thank in particular Junyan Cao and Lei Wu for their careful reading of the early draft of the paper and numerous suggestions. I am particularly grateful to Professor Dan Abramovich for suggesting the $\mathbb{Q}$-mild reduction, and writing Appendix A which provides a crucial step for the present paper. Lastly, I thank Professors Damian Brotbek and Jean-Pierre Demailly for their encouragements and supports.

Notations and conventions.

Throughout this article we will work over the complex number field $\mathbb{C}$.

- An algebraic fiber space[2] (or fibration for short) $f : X \to Y$ is a surjective projective morphism between projective manifolds with connected geometric fibers. Any $\mathbb{Q}$-divisor E in X is said to be f-exceptional if $f(E)$ is an algebraic variety of codimension at least two in Y.
- We say that a morphism $f_U : U \to V$ is a *smooth family* if f_U is a surjective smooth projective morphism with connected fibers between quasi-projective varieties.
- For any surjective morphism $Y' \to Y$, and the algebraic fiber space $f : X \to Y$, we denote by $(X \times_Y Y')\tilde{}$ the (unique) irreducible component (say the *main component*) of $X \times_Y Y'$ which dominates Y'.
- Assume that $B := Y \setminus Y_0$ is simple normal crossing and
$$f^*B = \sum W_i + \sum_j a_j V_j + \sum_k b_k V'_k,$$
is normal crossing, where $a_j \geqslant 2$, $b_j \geqslant 1$, $f(V_j)$ is a divisor in Y and V'_k is f-exceptional. We denote by $\Delta_f := \sum_j (a_j - 1)V_j$ the *multiplicity divisor of the fibration* f. If $\Delta_f = 0$, the fibration f is called *semi-stable in codimension one.*
- Let $\mu : X' \to X$ be a birational morphism from a projective manifold X' to a singular variety X. μ is called a *strong desingularization* if $\mu^{-1}(X^{\mathrm{reg}}) \to X^{\mathrm{reg}}$ is an isomorphism. Here X^{reg} denotes to be the smooth locus of X.
- For any birational morphism $\mu : X' \to X$, the *exceptional locus* is the inverse image of the smallest closed set of X outside of which μ is an isomorphism, and denoted by $\mathrm{Ex}(\mu)$.
- Denote by $X^r := X \times_Y \cdots \times_Y X$ the r-fold fiber product of the fibration $f : X \to Y$, $(X^r)\tilde{}$ the *main component* of X^r dominating Y, and $X^{(r)}$ a *strong desingularization* of $(X^r)\tilde{}$.
- For any quasi-projective manifold Y, a Zariski open subset $Y_0 \subset Y$ is called a *big open set* of Y if and only if $\mathrm{codim}_{Y \setminus Y_0}(Y) \geqslant 2$.
- A singular hermitian metric h on the line bundle L is said to be *positively curved* if the curvature current $\Theta_h(L) \geqslant 0$.

1. Positivity of direct images

This section is devoted to the proofs of Theorems D and E, which are used to proved Theorem C.

1.1. Weak positivity of relative pluricanonical bundles.

1.1. Weak positivity of relative pluricanonical bundles. In [Vie83], Viehweg introduced the definition of *weak positivity for torsion free sheaves* to study the Iitaka's $C_{n,m}$-conjecture. In [Vie90, Theorem 2.7] he further proved the weak positivity of direct images of relative pluricanonical bundles $f_*(mK_{X/Y})$ when $K_{X/Y}$ is relatively *semi-ample.* In this section, following the recent fundamental work by Păun-Takayama [PT18], we will provide a generalization of Viehweg's theorem for the purpose of Proposition 1.9. Let us first recall the

[2] Here we follow the definition in [Mor87].

definitions of weak positivity by Viehweg in [Vie83], and the weak positivity in the sense of Nakayama in [Nak04]. In [PT18], the author mainly studied the weak positivity in the sense of Nakayama due to their general statements of the theorems.

For a torsion free sheaf $\mathscr{E}$ on a quasi-projective variety Z, we denote by $S^m\mathscr{E}$ the m-th symmetric tensor product of $\mathscr{E}$, and let $\widehat{S}^m\mathscr{E}$ be the double dual of the sheaf $S^m\mathscr{E}$.

Definition 1.1 (Viehweg)**.** Let Y be a quasi-projective normal variety, and let $\mathscr{G}$ be a torsion free coherent sheaf on Y, whose restriction to some dense Zariski open set $Y_0 \subset Y$ is locally free. Let $\mathscr{H}$ be an ample invertible sheaf over Y.

(i) The sheaf $\mathscr{G}$ is *weakly positive over* Y_0 if for a given number $\alpha > 0$, there exists some $\beta > 0$ such that $\widehat{S}^{\alpha\beta}\mathscr{G} \otimes \mathscr{H}^\beta$ is globally generated over Y_0.
(ii) The sheaf $\mathscr{G}$ is *weakly positive at a point* y (in the sense of Nakayama) if for any integer $\alpha > 0$, there exists an integer $\beta > 0$ such that $\widehat{S}^{\alpha\beta}\mathscr{G} \otimes \mathscr{H}^\beta$ is globally generated at y.
(iii) The sheaf $\mathscr{G}$ is *ample with respect to* Y_0 if for some $\mu > 0$ there exists a morphism
$$\bigoplus \mathscr{H} \to \widehat{S}^\mu\mathscr{G}$$
surjective over Y_0.

Observe that Viehweg's weak positivity requires global generation in Definition 1.1.(i) to hold on a Zariski open set, while Nakayama's weak positivity Definition 1.1.(ii) may be verified on a countable intersection of Zariski open sets only. Hence we cannot apply the results on the weak positivity in the sense of Nakayama in [PT18] directly to show the weak positivity of certain torsion free sheaves.

The following theorem by Berndtsson, Păun and Takayama [BP08, PT18] is a crucial tool in the study of weak positivity. The (positively curved) singular hermitian metrics on torsion free sheaves were defined by Raufi in [Rau15], and we do not recall the definitions here.

Theorem 1.2 (Berndtsson-Păun-Takayama)**.** *Let $f : X \to Y$ be an algebraic fiber space which is smooth over a Zariski open set $Y_0 \subset Y$. Let L be a pseudo-effective line bundle over X endowed with a positively curved singular hermitian metric h. For some Zariski open set $Y_1 \subset Y_0$, assume that for any $y \in Y_1$, one has*

$$H^0\big(X_y, (K_{X_y} + L_y) \otimes \mathscr{J}(h_y)\big) = H^0(X_y, K_{X_y} + L_y) \tag{1.1.1}$$

where $L_y := L_{\restriction X_y}$, $h_y := h_{\restriction X_y}$ and $\mathscr{J}(h_y)$ denotes the multiplier ideal sheaf with respect to the singular hermitian metric h_y. Then

(i) *$f_*(K_{X/Y} + L)$ is locally free over Y_1.*
(ii) *There exists a natural singular hermitian metric, say the* Narasimhan-Simha metric *g_{NS}, over the direct image $f_*(K_{X/Y} + L)$, which is positively curved.*
(iii) *The metric g_{NS} is locally bounded from above over Y_1.*

Now we state the main technical result in this subsection, which is indeed a special case of [PT18, Theorem 2.5.3]. In order to prove their much more general theorem, they have to use the subtle result [ELM+09] in the proof. Here our assumption is less general, and thus the proof is a direct applications of L^2-estimates on (not necessarily compact) complete Kähler manifolds in [Dem82, Théorème 5.1], as shown in [PT18, Proof of Theorem 2.5.4]. Since [PT18, Theorem 2.5.3] only states the weak positivity in the sense of Nakayama (although their proof implies Theorem 1.3 implicitly), we provide a detailed proof here for the sake of completeness.

Theorem 1.3 (Păun-Takayama)**.** *Let $\mathcal{F}$ be a torsion free coherent sheaf over a projective manifold Y, equipped with a positively curved singular hermitian metric $h_{\mathcal{F}}$. Let $Y_1 \subset Y$ be a Zariski open set so that $\mathcal{F}_{\restriction Y_1}$ is locally free, and $h_{\mathcal{F}}$ is locally bounded from above over Y_1. Then $\mathcal{F}$ is weakly positive over Y_1.*

Proof. Take $\mathbb{P}(\mathcal{F}) := \mathrm{Proj}\big(\bigoplus_{m\geqslant 0} S^m\mathcal{F}\big)$ to be the projectivization of $\mathcal{F}$. Denote by $\mathscr{O}(1)$ the tautological line bundle over $\mathbb{P}(\mathcal{F})$, and $\pi' : \mathbb{P}(\mathcal{F}) \to Y$ the natural projection map. Since $\mathcal{F}$ might not be locally free, the projective scheme $\mathbb{P}(\mathcal{F})$ is not smooth in general. We define $\mathbb{P}'(\mathcal{F})$ to be the normalization of $\mathbb{P}(\mathcal{F})$, and $\mu : Z \to \mathbb{P}'(\mathcal{F})$ to be a strong desingularization of $\mathbb{P}'(\mathcal{F})$. Let $Y' \supset Y_1$ be the big open set of Y so that $\mathcal{F}_{\upharpoonright Y'}$ is locally free. Hence $Z' := \pi^{-1}(Y') \to Y'$ is *smooth projective morphism* between quasi-projective manifolds with fibers isomorphic to $\mathbb{P}^{r-1}$, where $r := \mathrm{rank}\,\mathcal{F}$, and $\pi : Z \to Y$ can be seen as a smooth projective compactification of $Z' \to Y'$.

$$\begin{tikzcd}
\mathbb{P}(\mathcal{F}) \arrow[d, "\pi'"] & Z \arrow[l, "\mu"'] \arrow[d, "\pi"] & \supset Z' \arrow[d] \\
Y \arrow[r, equal] & Y & \supset Y'
\end{tikzcd}$$

Write $L := \mu^*\mathscr{O}(1)$. The positively curved singular hermitian metric $h_{\mathcal{F}}$ of $\mathcal{F}$ induces a metric h for $L_{\upharpoonright Z'}$ which is positively curved and locally bounded over $\pi^{-1}(Y_1)$.

Denote by $n = \dim Y$. Take a Kähler form ω on Y. Let us fix an ample line bundle A over Y such that $A\otimes K_Y^{-1}\otimes(\det\mathcal{F})^{-1}$ is sufficiently very ample in the following sense: for any point $y \in Y$, there exists a singular hermitian metric h_y of $A\otimes K_Y^{-1}\otimes(\det\mathcal{F})^{-1}$ which is smooth outside y, so that $\sqrt{-1}\Theta_{h_y}\big(A\otimes K_Y^{-1}\otimes(\det\mathcal{F})^{-1}\big) \geqslant \omega$, and h_y has logarithmic poles around y:

$$-\log h_y \simeq (n+1)\log|t|^2, \tag{1.1.2}$$

where $t := (t_1, \ldots, t_n)$ is some coordinate system of an open set $U_y \ni y$ centering at y.

Since Z' is quasi-projective, the manifold Z' can be equipped with a *complete Kähler form* $\hat\omega$ by [Dem82, Théorème 0.2]. The line bundle $L := \mu^*\mathscr{O}(1)$ is *relatively ample* when restricted to $Z' \to Y'$. One can further assume that $L\otimes\pi^*A_{\upharpoonright Z'}$ is endowed with a smooth hermitian metric h_0 so that the curvature form $\sqrt{-1}\Theta_{h_0}$ is *locally strictly positive* over Z', that is, for any relatively compact subset K of Z', there is an $\varepsilon > 0$ so that $\sqrt{-1}\Theta_{h_0}(L\otimes\pi^*A_{\upharpoonright Z'})_{\upharpoonright K} \geqslant \varepsilon\hat\omega_{\upharpoonright K}$.

Note that

$$K_{Z'}\otimes L^{m+r}\otimes\pi^*(A^2\otimes K_Y^{-1}\otimes(\det\mathcal{F})^{-1})_{\upharpoonright Z'} = L^m\otimes\pi^*A^2_{\upharpoonright Z'}$$

for any $m \in \mathbb{N}$. Let us fix any $y \in Y_1$, and any positive integer $m > 1$. Take relative compact open sets $U_y' \Subset U_y \Subset Y_1$ containing y so that $\mathscr{O}(A)_{\upharpoonright U_y} \simeq \mathscr{O}_{U_y}$, and pick a $\mathscr{C}^\infty$ cut-off function λ such that $\lambda \equiv 1$ over U_y', and $\mathrm{Supp}(\lambda) \subset U_y$. For any section $e \in H^0\big(\mathbb{P}(\mathcal{F}_y), \mathscr{O}(m)_{\upharpoonright\mathbb{P}(\mathcal{F}_y)}\big)$, it can extend to a holomorphic section

$$\sigma \in H^0\big(\mathbb{P}(\mathcal{F}_{\upharpoonright U_y}), L^m\otimes\pi^*A^2_{\upharpoonright\mathbb{P}(\mathcal{F}_{\upharpoonright U_y})}\big) = H^0\big(\pi^{-1}(U_y), K_{Z'}\otimes M_{m\upharpoonright\pi^{-1}(U_y)}\big),$$

where we write $M_m := L^{m+r}\otimes\pi^*(A^2\otimes K_Y^{-1}\otimes(\det\mathcal{F})^{-1})_{\upharpoonright Z'}$. Let us endow M_m with a singular hermitian metric $g_m := h_0\cdot h^{m+r-1}\cdot\pi^*h_y$. Recall that h is locally bounded over Y_1, h_0 is smooth whose curvature form is locally strictly positive, and h_y has log poles at y as (1.1.2). Hence the zero scheme of the multiplier ideal sheaf

$$V\big(\mathscr{J}(g_m)\big) = \pi^{-1}(y) = \mathbb{P}(\mathcal{F}_y), \tag{1.1.3}$$

and there exists an $\varepsilon_y > 0$ so that

$$\sqrt{-1}\Theta_{g_m}(M_m) \geqslant \varepsilon_y\hat\omega \quad \text{over } \pi^{-1}(U_y).$$

Let us denote by $A_m := [\sqrt{-1}\Theta_{g_m}(M_m), \Lambda_{\hat\omega}]$, which is a semi-positive Hermitian operator acting on $\mathscr{C}^\infty(Z', \Omega_{Z'}^{n+r-1,1}\otimes M_m)$. Moreover, $A_m \geqslant \varepsilon_y\mathbb{1}$ over $\pi^{-1}(U_y)$. Define

$$u := \bar\partial\big((\pi^*\lambda)\sigma\big) = \bar\partial(\pi^*\lambda)\wedge\sigma \in \mathscr{C}^\infty(Z', \Omega_{Z'}^{n+r-1,1}\otimes M_m)$$

which vanishes over $\pi^{-1}(U'_y)$, and is supported in $\pi^{-1}(U_y)$. Then

$$\int_{Z'} \langle A_m^{-1}u, u\rangle_{g_m} dV_{\hat{\omega}} \leqslant \int_{\pi^{-1}(U_y)} \langle A_m^{-1}u, u\rangle_{g_m} dV_{\hat{\omega}} \leqslant \frac{1}{\varepsilon_y}\int_{\pi^{-1}(U_y)} |u|^2_{g_m} dV_{\hat{\omega}} < +\infty$$

where the last inequality is due to (1.1.3) and the relative compactness of U_y in Y_1. By [Dem82, Théorème 5.1], one can solve the $\bar{\partial}$-equation over Z', and thus there exists a section $v \in L^2_{\rm loc}(Z', K_{Z'} \otimes M_m)$ so that $\bar{\partial} v = u$ and

$$\int_{Z'} |v|^2_{g_m} dV_{\hat{\omega}} \leqslant \int_{Z'} \langle A_m^{-1}u, u\rangle_{g_m} dV_{\hat{\omega}} \leqslant \frac{1}{\varepsilon_y}\int_{\pi^{-1}(U_y)} |u|^2_{g_m} dV_{\hat{\omega}} < +\infty. \tag{1.1.4}$$

Hence $\bar{\partial}\big((\pi^*\lambda)\sigma - v\big) = 0$. In particular, the section v is holomorphic over $\pi^{-1}(U'_y)$, and vanishes identically over $\pi^{-1}(y)$ by (1.1.3) and (1.1.4). Then

$$(\pi^*\lambda)\sigma - v \in H^0(Z', K_{Z'} \otimes M_m) = H^0(Z', L^m \otimes \pi^* A^2_{\restriction Z'})$$

extends the given section $e \in H^0\big(\mathbb{P}(\mathcal{F}_y), \mathscr{O}(m)_{\restriction \mathbb{P}(\mathcal{F}_y)}\big) \simeq S^m\mathcal{F}_y$. By the isomorphism

$$H^0(Z', L^m \otimes \pi^* A^2_{\restriction Z'}) \simeq H^0(Y', S^m\mathcal{F} \otimes A^2_{\restriction Y'}),$$

we conclude that for any $m > 1$, $S^m\mathcal{F} \otimes A^2_{\restriction Y'}$ is generated by globally sections at each point of $Y_1 \subset Y'$. By the very definition of the reflexive hull and the fact that $\operatorname{codim}_{Y\setminus Y'}(Y) \geqslant 2$, the natural inclusion

$$H^0(Y', S^m\mathcal{F} \otimes A^2_{\restriction Y'}) \xrightarrow{\simeq} H^0(Y, \widehat{S}^m\mathcal{F} \otimes A^2).$$

is an isomorphism. Hence for any $m > 1$, $\widehat{S}^m\mathcal{F} \otimes A^2$ is also globally generated over $Y_1 \subset Y'$. This leads to the weak positivity of $\mathcal{F}$ over Y_1. □

Theorems 1.2 and 1.3 immediately imply the following.

Corollary 1.4. *Suppose the algebraic fiber space $f : X \to Y$ and the pseudo-effective line bundle L on X are in the same setup as Theorem 1.2. Then the direct image $f_*(K_{X/Y} + L)$ is weakly positive over Y_1.*

We are in a position to prove the main result in this subsection.

Proposition 1.5 (Weak positivity of direct images)**.** *Let $f : X \to Y$ be an algebraic fiber space so that the Kodaira dimension of the general fiber is non-negative. Assume that f is smooth over a dense Zariski open set of $Y_0 \subset Y$ so that both $B := Y \setminus Y_0$ and f^*B are normal crossing. Then for any $m \gg 0$, the direct image $f_*\big(mK_{X/Y} - (m-1)\Delta_f\big)$ is weakly positive over Y_0, where Δ_f is the multiplicity divisor of f.*

Proof. It follows from the work of [BP08, CP17] (see [CP17, Theorem 2.3 and Remark 2.5]) that for $m \gg 0$, $mK_{X/Y}$ can be equipped with the *m-th Bergman metric h_m* so that the curvature current

$$\sqrt{-1}\Theta_{h_m}(mK_{X/Y}) \geqslant m[\Delta_f]. \tag{1.1.5}$$

h_m thus induces a singular metric h of $L := (m-1)(K_{X/Y} - \Delta_f)$ defined by

$$h \overset{\rm loc}{:=} h_m^{\frac{m-1}{m}} \cdot |\sigma_{\Delta_f}|^{2(m-1)}$$

where σ_{Δ_f} is the local defining equation of Δ_f. By (1.1.5), h is positively curved. It follows from [PT18, §3.1.1.(4)] that, (1.1.1) holds for any $y \in Y_0$. Hence by Corollary 1.4, we conclude the weak positivity of $f_*(K_{X/Y} + L) = f_*\big(mK_{X/Y} - (m-1)\Delta_f\big)$ over Y_0. □

Remark 1.6. The weak positivity of the direct images of relative pluricanonical bundles $f_*(mK_{X/Y})$ with $K_{X/Y}$ relative semi-ample was proved by Viehweg in [Vie90, Theorem 2.7] using vanishing theorems. In [PS14], Popa-Schnell proved some variants of Viehweg's weak positivity results using the theory of *Castelnuovo-Mumford regularity* and vanishing theorems. In [Fuj16], Fujino proved that, after passing to a certain base change, the direct image of pluricanonical bundles are locally free and *numerically eventually free* (nef for short), which was refined by Takayama in [Tak16]. In [PT18], Păun-Takayama proved the weak positivity at certain points in the sense of Nakayama for twisted pluricanonical bundles $f_*(mK_{X/Y}+L)$ where L is a pseudo-effective line bundle. In a very recent preprint [Iwa18], Iwai gives a criterion for the weak positivity of torsion free sheaves.

1.2. **From weak positivity to ampleness.** Consider locally free sheaves $\mathscr{E}$ and $\mathscr{Q}$ over a complex manifold X of rank n and r respectively. Suppose that for some $\mu \in \mathbb{N}$, there is a quotient of vector bundles

$$\varphi : S^\mu\mathscr{E} \twoheadrightarrow \mathscr{Q}. \tag{1.2.1}$$

Write $K_x \subset S^\mu\mathscr{E}_x$ for the kernel of $\varphi_x : S^\mu\mathscr{E}_x \to \mathscr{Q}_x$. According to the pioneering work by Viehweg [Vie89, Vie90] and Kollár [Kol90], if K_x varies in $S^\mu\mathscr{E}$ with $x \in X$ "as much as possible", and $\mathscr{E}$ possesses some "semi-positivity", then the vector bundle $\mathscr{Q}$ should be "very positive", *afortiori* its determinant line bundle $\det \mathscr{Q}$.

In order to make this precise, we fix a basis $\mathbf{e} := \{e_1, \ldots, e_n\}$ of $\mathscr{E}_x$ for a point $x \in X$. The inclusion

$$K_x \hookrightarrow S^\mu\mathscr{E}_x$$

defines a point $[K_{\mathbf{e},x}]$ in the Grassmann variety $\mathrm{Grass}(S^\mu\mathbb{C}^n, r)$, which parametrizes r-dimensional quotient spaces of $S^\mu\mathbb{C}^n$. The group $G := SL(n,\mathbb{C})$ acts on $\mathrm{Grass}(S^\mu\mathbb{C}^n, r)$ by changing the basis of $\mathscr{E}_x$. Whereas $[K_{\mathbf{e},x}]$ depends on the chosen basis $\mathbf{e}$ for $\mathscr{E}_x$, the G-orbit G_x of $[K_{\mathbf{e},x}]$ in $\mathrm{Grass}(S^\mu\mathbb{C}^n, r)$ is well defined and depends only on the quotient $\varphi_x : S^\mu\mathscr{E}_x \to \mathscr{Q}_x$ defined in (1.2.1). Note that for two different points $x, y \in X$, either $G_x = G_y$, or $G_x \cap G_y = \varnothing$.

Definition 1.7 (Kollár-Viehweg)**.** For a Zariski open set $X_0 \subset X$, $\ker(\varphi)$ has *maximal variation* over X_0 if for any $x \in X_0$, the set $y \in X_0$ with equal orbit $G_y = G_x$ is finite, and $\dim G_x = \dim G$.

We will need the following crucial ampleness criterion in [Vie90, Ampleness Criterion 5.7].

Theorem 1.8 (Viehweg)**.** *Let Y be a projective manifold, and let $\mathscr{E}$ be a torsion free coherent sheaf defined over Y, which is weakly positive over a dense Zariski open set Y_0 of Y. Let $\mathscr{Q}$ be a reflexive sheaf on Y, which is also locally free over Y_0. Assume that we have a map*

$$\varphi : \widehat{S}^\mu\mathscr{E} \to \mathscr{Q}$$

such that its restriction to Y_0 is a quotient of vector bundles. Assume that the kernel of $\varphi_{\restriction Y_0}$ has maximal variation over Y_0. Then for $b \gg a \gg 0$, the rational map

$$Y \dashrightarrow \mathbb{P}\big(H^0(Y, \mathscr{A})\big)$$

induced by the invertible sheaf $\mathscr{A} := \det(\mathscr{Q})^a \otimes \det(\mathscr{E})^b$, is an embedding when restricted to Y_0. In particular, $\mathscr{A}$ is ample with respect to Y_0.

The following result will be used in the proof of Theorem 1.21.(i). Let us mention that for families of projective manifolds with big and nef canonical bundles, Proposition 1.9 has already been proved by Viehweg [Vie90, Theorem 5.2], and the proof we presented here is also in the same spirit.

Proposition 1.9. *Let $f : X \to Y$ be an algebraic fiber space which is smooth over a Zariski open set $Y_0 \subset Y$. Assume that both $B := Y \setminus Y_0$ and f^*B is normal crossing. Let V be a dense Zariski open set of Y_0 so that for each $y \in V$, K_{X_y} is big, and the set of $z \in V$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite, where $\overset{\text{bir}}{\sim}$ stands for the* birational equivalence. *Then* $\det f_*(\mu m K_{X/Y})^a \otimes \det f_*(mK_{X/Y})^b$ *is ample with respect to V for some $b \gg a \gg 0$ and $\mu \gg m \gg 0$.*

Proof. Since $f_0 = f_{\restriction X_0} : X_0 = f^{-1}(Y_0) \to Y_0$ is a smooth fibration, Siu's invariance of plurigenera implies that, for any $\ell \in \mathbb{N}$, the direct image $f_*(\ell K_{X/Y})$ is locally free over Y_0, with $f_*(\ell K_{X/Y})_y \simeq H^0(X_y, \ell K_{X_y})$. By the theorem of Birkar-Cascini-Hacon-McKernan [BCHM10] (see also [Kol13, Theorem 1.26] for a precise statement), the relative canonical sheaf of rings with respect to $f_0 : X_0 \to Y_0$

$$R(X_0/Y_0, K_{X_0}) := \sum_{m\geqslant 0} (f_0)_*\mathscr{O}(mK_{X_0})$$

is a finitely generated sheaf of $\mathscr{O}_{Y_0}$-algebras, and the (unique) *relative canonical model* for $X_0 \to Y_0$ is defined by

$$X_0^{\text{can}} := \operatorname{Proj}_{Y_0} R(X_0/Y_0, K_{X_0}).$$

Moreover, X_0^{can} is normal with *canonical singularities*, projective over Y_0, and there is a natural birational map $\phi : X_0 \to X_0^{\text{can}}$ with

$$\begin{tikzcd} X_0 \arrow[r, dashed, "\phi"] \arrow[dr, "f"'] & X_0^{\text{can}} \arrow[r, hook, "\iota"] \arrow[d, "f^c"] & \mathbb{P}(F_m) \arrow[dl] \\ & Y_0 & \end{tikzcd} \tag{1.2.2}$$

so that the pushforward by ϕ gives an isomorphism

$$\sum_{m\geqslant 0} f_*\mathscr{O}(mK_{X_0}) \simeq \sum_{m\geqslant 0} f^c_*\mathscr{O}(mK_{X_0^{\text{can}}}).$$

Here we write $F_m := (f_0)_*(mK_{X_0/Y_0})$ which is a locally free. Then there exists $m, \mu \gg 0$ and a natural multiplication map

$$\varphi : \widehat{S}^\mu f_*(mK_{X/Y}) \to \big(f_*(\mu m K_{X/Y})\big)^{\star\star}, \tag{1.2.3}$$

such that the restriction of φ to Y_0, denoted by φ_0, is a quotient map between vector bundles. We further assume that $\mathscr{O}_{\mathbb{P}(F_m)}(\mu) \otimes \mathcal{I}_{X_0^{\text{can}}}$ is *relatively globally generated*, where $\mathcal{I}_{X_0^{\text{can}}}$ is the ideal sheaf of $X_0^{\text{can}} \subset \mathbb{P}(F_m)$. We will show that the kernel of φ_0 *has maximal variation over V*.

Fix any $y \in V$, and we take a basis $\mathbf{e} := \{e_0, \ldots, e_N\}$ of $H^0(X_y, mK_{X_y}) \simeq \mathbb{C}^{N+1}$. The map (1.2.3) gives rise to a short exact sequence

$$0 \to H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\mu) \otimes \mathcal{I}_{X_y^{\text{can}}}\big) \xrightarrow{i_{\mathbf{e},y}} H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\mu)\big) \to H^0\big(X_y^{\text{can}}, \mathscr{O}_{\mathbb{P}^N}(\mu)_{\restriction X_y^{\text{can}}}\big) \to 0, \tag{1.2.4}$$

where $X_y^{\text{can}} := (f^c)^{-1}(y)$ and $\mathcal{I}_{X_y^{\text{can}}}$ is the ideal sheaf of $X_y^{\text{can}} \subset \mathbb{P}^N$. Write $K_{\mathbf{e},y} := H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\mu) \otimes \mathcal{I}_{X_y^{\text{can}}}\big)$. Recall that $\mathscr{O}_{\mathbb{P}^N}(\mu) \otimes \mathcal{I}_{X_y^{\text{can}}}$ is globally generated. Then $[K_{\mathbf{e},y}] \in \operatorname{Grass}(S^\mu\mathbb{C}^{N+1}, r)$ determines $X_y^{\text{can}} \subset \mathbb{P}^N$, where $r := \operatorname{rank} f_*(\mu m K_{X/Y})$. If we take another the basis $\mathbf{e}'$ of $H^0(X_y, mK_{X_y})$, then $[K_{\mathbf{e}',y}]$ determines another subvariety $\tilde{X}_y^{\text{can}} \subset \mathbb{P}^N$ which is projectively equivalent (hence isomorphic) to X_y^{can}. Hence the stabilizer of the action of $G := SL(N+1, \mathbb{C})$ on $\operatorname{Grass}(S^\mu\mathbb{C}^{N+1}, r)$ is contained in $\operatorname{Aut}(X_y^{\text{can}})$, which is finite for X_y^{can} has canonical singularities and is of general type. Write G_y for the G-orbit of $[K_{\mathbf{e},y}]$ in $\operatorname{Grass}(S^\mu\mathbb{C}^{N+1}, r)$, which is independent of the basis $\mathbf{e}$. One thus has $\dim G_y = \dim G$. On the other hand, if $G_z = G_y$ for some other $z \in V$, then X_y^{can} is isomorphic to X_z^{can}, and by the assumption, there exists only finite such $z \in V$. This in turn implies that the kernel of φ_0 has maximal variation over V.

To finish the proof, by Theorem 1.8 it then suffices to show that $f_*(mK_{X/Y})$ is weakly positive over V, which is ensured by our more general result in Proposition 1.5. The proposition follows. □

1.3. **A criterion for birationally isotrivial family.**

In this subsection we will prove Theorem E. The idea of the proof is inspired by recent results of Cao [Cao18, Cao16] and Cao-Păun [CP17]. Let us start with the following result.

Proposition 1.10. *Let $f : X \to Y$ be any algebraic fiber space. Assume that ℓ is any positive integer with $f_*(\ell K_{X/Y})$ non-zero. If the* numerical dimension $\nu\big(\det f_*(\ell K_{X/Y})\big) = 0$ *(see [BDPP13] for the definition), then*

(i) *for any birational morphism $\psi : Y' \to Y$, defining X' to be strong desingularization of the main component $(X \times_Y Y')\tilde{}$ dominating Y'*

(1.3.1)
$$\begin{tikzcd} X' \arrow[r] \arrow[dr, "f'"'] & X \times_Y Y' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

one has

$$\nu\big(\det f'_*(\ell K_{X'/Y'})\big) = 0.$$

(ii) *For any positive integer m so that $f_*(mK_{X/Y})$ is non-zero, one has*

$$\nu\big(\det f_*(mK_{X/Y})\big) = 0,$$

and $f_*(mK_{X/Y})$ is flat *over a Zariski open set of Y.*

Proof. Denote by $F_\ell := f_*(\ell K_{X/Y})$ (resp. $F'_\ell := f'_*(\ell K_{X'/Y'})$), which is torsion free over Y (resp. Y'). By [CP17, §4] (or Proposition 1.5 in the logarithmic setting) there exists a positively curved singular hermitian metric (Narasimhan-Simha metric) h_ℓ (resp. h'_ℓ) over F_ℓ (resp. F'_ℓ). Hence the line bundle $\det F_\ell$ (resp. $\det F'_\ell$) has a positive curvature current denoted by Ξ, (resp. Ξ') induced by h_ℓ (resp. h'_ℓ). Let $V \subset Y$ be the big open set so that $\psi : \psi^{-1}(V) \xrightarrow{\simeq} V$ is an isomorphism. Then

$$(F_\ell, h_\ell)_{\upharpoonright V} \simeq (F'_\ell, h'_\ell)_{\upharpoonright \psi^{-1}(V)},$$

and thus $\Xi_{\upharpoonright V} \simeq \Xi'_{\upharpoonright \psi^{-1}(V)}$. In particular, $\psi_*(\Xi') = \Xi$ in the sense of pushforward of positive currents. Hence there exists an ψ-exceptional divisor E (may not be effective!) so that

$$\det F'_\ell \overset{\text{num}}{\equiv} \psi^* \det F_\ell + E. \tag{1.3.2}$$

Take an effective ψ-exceptional divisor E' so that $E' - E$ is effective as well. It follows from [Leh13, Theorem 1.1.(1)] that

$$0 = \nu(\det F_\ell) = \nu(\psi^* \det F_\ell + E') \geqslant \nu(\psi^* \det F_\ell + E) = \nu(\det F'_\ell) \geqslant 0.$$

This proves Claim (i).

Let us prove Claim (ii). Since the numerical dimension $\nu(\det F_\ell)$ is a birational invariant, we may assume that, after passing to a new birational model of the fibration $f : X \to Y$ as in (1.3.1), f is smooth over $Y_0 \subset Y$, and both $B := Y \setminus Y_0$ and f^*B are normal crossing divisors. Recall that the Narasimhan-Simha metric h_ℓ over F_ℓ induces a singular metric $h_{\ell,\det}$ for the line bundle $\det F_\ell$ whose curvature current $\sqrt{-1}\Theta_{h_{\ell,\det}}(\det F_\ell) = \Xi$ is positive. By [BDPP13] and the assumption that $\nu(\det F_\ell) = 0$, one has

$$\det F_\ell \overset{\text{num}}{\equiv} \sum_{i=1}^{p} \lambda_i D_i \quad \lambda_i \in \mathbb{Q}^+ \text{ for } i = 1, \ldots p,$$

where $\sum_{i=1}^{p} D_i$ is an *exceptional divisor* in the sense of [Bou04, Definition 3.10]. In particular, by [Bou04, Proposition 3.13], $\sum_{i=1}^{p} \lambda_i [D_i]$ is the *unique* positive current in $c_1(\det F_\ell)$, and thus

$$\Xi = \sum_{i=1}^{p} \lambda_i [D_i].$$

In particular, $\sqrt{-1}\Theta_{h_{\ell,\det}}(\det F_\ell) \equiv 0$ over $Y \setminus \cup_{i=1}^{p} D_i$.

By [CP17, Eq. (5.10)], there exists another positively-curved singular hermitian metric h' of $\det F_\ell$ so that

$$\sqrt{-1}\Theta_{h'}(\det F_\ell) - \varepsilon\sqrt{-1}\Theta_{h_{m,\det}}(\det F_m) \geqslant 0 \tag{1.3.3}$$

for some $\varepsilon > 0$. Recall that $c_1(\det F_\ell)$ contains only one positive current $\sum_{i=1}^{p} \lambda_i [D_i]$. Then

$$\sqrt{-1}\Theta_{h'}(\det F_\ell) = \sum_{i=1}^{p} \lambda_i [D_i].$$

It follows from (1.3.3) that

$$\sqrt{-1}\Theta_{h_{m,\det}}(\det F_m) = \sum_{i=1}^{p} \lambda_i' [D_i], \quad \lambda_i' \in \mathbb{R}^{\geqslant 0} \text{ for } i = 1, \ldots p. \tag{1.3.4}$$

By [BDPP13, Theorem 3.7], $\sum_{i=1}^{p} \lambda_i' D_i$ is also an exceptional divisor, which is thus the unique positive current in $c_1(\det F_m)$. This in turn implies that the numerical dimension $\nu\big(\det f_*(mK_{X/Y})\big) = 0$ for any $m \in \mathbb{N}^*$. Moreover, by (1.3.4) together with Lemma 1.11 below, over $Y_0 \setminus \cup_{i=1}^{p} D_i$ the Narasimhan-Simha metric h_m of $f_*(mK_{X/Y})$ is smooth and the curvature tensor

$$\Theta_{h_m}(F_m) \equiv 0 \quad \text{over} \quad Y_0 \setminus \cup_{i=1}^{p} D_i. \tag{1.3.5}$$

This proves Claim (ii). □

Lemma 1.11 ([CP17, Corollary 2.9])**.** *Let E be a vector bundle over a (possibly non-compact) Kähler manifold X, equipped with a positively-curved singular hermitian metric h_E. Assume that $\Theta_{\det h_E}(\det E) \equiv 0$ over an open (Euclidean topology) set $U \subset X$, then over U, h_E is smooth, and $\Theta_{h_E}(E) \equiv 0$.*

Remark 1.12. In [CP17, Remark 5.10], the authors asked the following question: for any algebraic fiber space $f : X \to Y$, assume that $c_1\big(\det f_*(\ell K_{X/Y})\big) = 0$ for some non-zero $f_*(\ell K_{X/Y})$, then for any birational model $f' : X' \to Y'$ as in (1.3.1), does it follow that $f'_*(\ell K_{X'/Y'})$ is flat? Proposition 1.10.(ii) can be seen as an answer to their question.

We are now in a position to prove Theorem E.

Theorem 1.13 (=Theorem E)**.** *Let $f : X \to Y$ be an algebraic fiber space between smooth projective manifolds with general fibers of general type. Let $\ell \geqslant 2$ be any positive integer such that $f_*(\ell K_{X/Y})$ is non-zero, and the numerical dimension $\nu\big(\det f_*(\ell K_{X/Y})\big) = 0$. Then f is* birationally isotrivial*, that is, two general fibers X_y and X_z of the fibration f are birationally equivalent.*

Proof. By Proposition 1.10.(i) one can assume that f is smooth over a non-empty Zariski open set $Y_0 \subset Y$, and both $B := Y \setminus Y_0$ and f^*B are normal crossing divisors. Take $\mu \gg m \gg 0$, so that the natural multiplication map

$$\varphi : \widehat{S}^\mu F_m \longrightarrow (F_{\mu m})^{\star\star} \tag{1.3.6}$$

is surjective over Y_0. We denote by $\mathcal{F}_{\mu m} \subset (F_{\mu m})^{\star\star}$ the image of φ, which is also torsion free, and coincides with $(F_{\mu m})^{\star\star}$ over Y_0 when $\mu \gg 0$. Since the Narasimhan-Simha metric h_m on F_m induces positively-curved metric h_m^μ over $\widehat{S}^\mu F_m$, the quotient metric $h_{\mathcal{F}}$ on $\mathcal{F}_{\mu m}$ induced by h_m^μ is also positively curved by [PT18, Lemma 2.3.4], and thus the induced metric $h_{\mathcal{F},\det}$ on the determinant $\det \mathcal{F}_{\mu m}$ is positively curved as well.

On the other hand, the inclusion

$$\det(\mathcal{F}_{\mu m}) \hookrightarrow \det\big((F_{\mu m})^{\star\star}\big) = \det(F_{\mu m}),$$

induces an effective divisor

$$T \in |\det(F_{\mu m}) - \det(\mathcal{F}_{\mu m})|.$$

Hence

$$\sqrt{-1}\Theta_{h_{\mathcal{F},\det}}\big(\det(\mathcal{F}_{\mu m})\big) + T \in c_1\big(\det(F_{\mu m})\big).$$

By (1.3.4), there exists an effective exceptional divisor (in the sense of [Bou04, Definition 3.10]) $\sum_{i=1}^p \mu_i D_i$ so that $\sum_{i=1}^p \mu_i [D_i]$ is the unique positive current in $c_1(\det F_{\mu m})$. Then

$$\sqrt{-1}\Theta_{h_{\mathcal{F},\det}}\big(\det(\mathcal{F}_{\mu m})\big) + [T] = \sum_{i=1}^p \mu_i [D_i].$$

In particular,

$$\sum_{i=1}^p \mu_i [D_i] - [T] \geqslant 0,$$

and

$$\sqrt{-1}\Theta_{h_{\mathcal{F},\det}}\big(\det(\mathcal{F}_{\mu m})\big) \equiv 0 \quad \text{over} \quad Y \setminus \cup_{i=1}^p D_i. \tag{1.3.7}$$

By Lemma 1.11 again, $\Theta_{h_{\mathcal{F}}}(\mathcal{F}_{\mu m}) \equiv 0$ over $Y_0 \setminus \cup_{i=1}^p D_i$. Recall that the restrictions $F_{m\upharpoonright Y_0}$ and $F_{\mu m\upharpoonright Y_0}$ are locally free, and the restriction of φ defined in (1.3.6) to Y_0

$$\varphi_{\upharpoonright Y_0} : S^\mu F_{m\upharpoonright Y_0} \twoheadrightarrow F_{\mu m\upharpoonright Y_0}$$

is surjective. In particular, over the Zariski open set $V := Y_0 \setminus \cup_{i=1}^p D_i$, $\widehat{S}^\mu F_{m\upharpoonright V} = S^\mu F_{m\upharpoonright V}$, and $\mathcal{F}_{\mu m\upharpoonright V} = F_{\mu m\upharpoonright V}$, and the restriction $\varphi_{\upharpoonright V}$ is a quotient map between vector bundles. Hence both the curvature tensors of $(S^\mu F_m, h_m^\mu)_{\upharpoonright V}$ and $(F_{\mu m}, h_{\mathcal{F}})_{\upharpoonright V}$ vanish identically. Since $h_{\mathcal{F}}$ is the quotient metric induced by h_m^μ, the *second fundamental form* with respect to $\varphi_{\upharpoonright V}$ thus vanishes identically. We denote by $E := \ker \varphi$. Then $E_{\upharpoonright V}$ is a *flat subbundle* of $S^\mu F_{m\upharpoonright V}$.

In other words, for any $y \in V$, we take an open set $U \subset V$ containing y so that there exists a holomorphic frame $e_0, e_1, \ldots, e_N \in H^0(U, F_m)$ which trivializes $F_m \simeq U \times \mathbb{C}^{N+1}$ so that $\nabla(e_i) \equiv 0$ for $i = 0, \ldots, N$, where ∇ is the hermitian connection with respect to the metric h_m^μ. We can also take such a holomorphic frame $f_1, \ldots, f_r \in H^0(U, E)$ which trivialize $E_{\upharpoonright U}$. Then

$$\varphi(f_j) = \sum_{|\alpha|=\mu} a_{j\alpha} e_0^{\alpha_0} e_1^{\alpha_1} \cdots e_N^{\alpha_N}, \tag{1.3.8}$$

where $a_{j\alpha} \in \mathbb{C}$ are all *constant* for any $j = 1, \ldots, r$ and α.

Now we will pursue the similar strategy in the proofs of [CH17, Proposition 4.1] or [Cao16, Proposition 2.8] to show the birational equivalence of general fibers. We denote by X_0^{can} the relative canonical model for $X_0 \to Y_0$ as in the proof of Proposition 1.9. By (1.2.2) and (1.2.4), for $\mu \gg m \gg 0$, (1.3.8) shows that X_0^{can} over U is a subvariety of $U \times \mathbb{P}^N \simeq \mathbb{P}(F_m)_{\restriction U}$ defined by equations

$$\{\sum_{|\alpha|=\mu} a_{j\alpha} z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_N^{\alpha_N}\}_{j=1,\ldots,r}.$$

Recall that $a_{j\alpha}$'s are all constant, then $f^c : X_0^{\text{can}} \to Y_0$ are locally trivial. The theorem follows. □

Remark 1.14. (i) The proof of Theorem 1.13 further indicates the locus of Y in which any two fibers are birationally equivalent. More precisely, in the same setting as Theorem 1.13, let Y_0 be the maximal Zariski open set of Y over which f is smooth, and let D be the only effective divisor which is numerically equivalent to $\det f_*(\ell K_{X/Y})$. Then for any $y, y' \in Y_0 \setminus D$, X_y is birationally equivalent to $X_{y'}$.
(ii) It is worthwhile mentioning that in [Kaw85] Kawamata proved the subadditivity of Kodaira dimensions for algebraic fiber spaces (Iitaka $C_{n,m}$-conjecture) whose geometric generic fiber admits a good minimal model. For such algebraic fiber spaces $f : X \to Y$, in [Kaw85, Theorem 1.1.(i)] he further showed that there exists a *certain* positive integer ℓ such that the Kodaira dimension

$$\kappa\big(\det f_*(\ell K_{X/Y})\big) \geqslant \operatorname{Var}(f). \tag{1.3.9}$$

By [BCHM10] we know the existence of good minimal models for varieties of general type. Hence (1.3.9) holds for algebraic fiber spaces whose general fibers are of general type. In particular, when $\kappa\big(\det f_*(\ell K_{X/Y})\big) \leqslant 0$ for the positive integer ℓ in (1.3.9), f must be birationally isotrivial. Theorem 1.13 can therefore be seen as a further refinement of Kawamata's result.

1.4. ***m*-Bergman metric and pluricanonical extension techniques.** Before we prove Theorem D, we need some technical results. The first one is a *pluricanonical extension theorem* which is a refinement of [Den17, Theorem 2.11] and [Cao16, Theorem 2.10]. Its proof is a combination of the Ohsawa-Takegoshi-Manivel L^2-extension theorem, with the semi-positivity of m-relative Bergman metric studied by Berndtsson-Păun [BP08, BP10] and Păun-Takayama [PT18].

Theorem 1.15 (Pluricanonical L^2-extension). *Let $f : X \to Y$ be an algebraic fiber space so that the Kodaira dimension of the general fiber is non-negative. Assume that f is smooth over a dense Zariski open set of $Y_0 \subset Y$ so that both $B := Y \setminus Y_0$ and f^*B are normal crossing. Let L be any pseudo-effective line bundle L on X equipped with a positively curved singular metric h_L with algebraic singularities satisfying the following property*

(i) *There exists some regular value $z \in Y$ of f, such that for some $m \in \mathbb{N}$, all the sections $H^0\big(X_z, (mK_X + L)_{\restriction X_z}\big)$ extends locally near z.*
(ii) $H^0\big(X_z, (mK_{X_z} + L_{\restriction X_z}) \otimes \mathscr{J}(h_{L\restriction X_z}^{\frac{1}{m}})\big) \neq \varnothing.$

Then for any regular value y of f satisfying that

(i) *all sections $H^0\big(X_y, mK_{X_y} + L_{\restriction X_y}\big)$ extends locally near y,*
(ii) *the metric $h_{L\restriction X_y}$ is not identically equal to $+\infty$,*

both the restriction maps in the diagram

$$\begin{tikzcd}
H^0(X, mK_{X/Y} + L + f^*A_Y) \arrow[r, two heads] & H^0\big(X_y, (mK_{X_y} + L_{\restriction X_y}) \otimes \mathscr{J}(h_{L\restriction X_y}^{\frac{1}{m}})\big) \\
H^0\big(X, mK_{X/Y} - m\Delta_f + L + f^*A_Y\big) \arrow[u, hook] \arrow[ur, two heads] &
\end{tikzcd}$$

are both surjective. Here A_Y is a universal *ample line bundle on Y which does not depend on L, f and m.*

Proof. Thanks to [BP10, A.2.1], the assumptions in the theorem imply that there exists a m-relative Bergman type metric $h_{m,B}$ on $mK_{X/Y}+L$ with respect to h_L such that the curvature current $i\Theta_{h_{m,B}}(mK_{X/Y}+L)\geqslant 0$. Thus $h:=h_{m,B}^{\frac{m-1}{m}}\cdot h_L^{\frac{1}{m}}$ defines a possible singular metric on

$$\widetilde{L}:=\frac{m-1}{m}(mK_{X/Y}+L)+\frac{1}{m}L=(m-1)K_{X/Y}+L,$$

with $i\Theta_h(\widetilde{L})\geqslant 0$.

Take any $s\in H^0\big(X_y,(mK_{X_y}+L_{\restriction X_y})\otimes\mathscr{J}(h_{L\restriction X_y}^{\frac{1}{m}})\big)$. It follows from the construction of the m-relative Bergman kernel metric that $|s|^2_{h_{m,B}}$ is $\mathscr{C}^0$-bounded. Then we see that

$$\begin{aligned}\int_{X_y}|s|^2_{\omega,h}dV_{X_y,\omega} &= \int_{X_y}|s|_{h_{m,B}}^{\frac{2(m-1)}{m}}|s|^{\frac{2}{m}}_{\omega,h_L^{\frac{1}{m}}}dV_{X_y,\omega}\\ &\leqslant C\int_{X_y}|s|^{\frac{2}{m}}_{\omega,h_L^{\frac{1}{m}}}dV_{X_y,\omega}<+\infty,\end{aligned}$$

which implies that $s\in H^0\big(X_y,(K_X+\tilde{L}+f^*(A_Y-K_Y))_{\restriction X_y}\otimes\mathscr{J}(h_{\restriction X_y})\big)$. Take A_Y sufficiently ample such that A_Y-K_Y-B separates $(2n+1)$-jets everywhere, where $n:=\dim Y$. We then can apply the Ohsawa-Takegoshi-Manivel L^2-extension theorem (see [CDM17, Dem16]) for $K_X+\widetilde{L}+f^*(A_Y-K_Y-B)$, to extend s to a section S in $H^0\big(X,(K_{X/Y}+\widetilde{L}+f^*A_Y)\otimes\mathscr{J}(h)\big)$. In conclusion, the restriction

$$H^0\big(X,mK_{X/Y}+L+f^*(A_Y-B)\big)\twoheadrightarrow H^0\big(X_y,(mK_{X_y}+L_{\restriction X_y})\otimes\mathscr{J}(h_{L\restriction X_y}^{\frac{1}{m}})\big)$$

is surjective.

On the other hand, as in (1.1.5), the m-Bergman metric $h_{m,B}$ of $mK_{X/Y}+L$ also has certain singularities along the multiplicity divisor Δ_f of the fibration f, which forces the extended section of s vanishes on Δ_f. More concretely, the curvature of the m-relative Bergman metric

$$i\Theta_{h_{m,B}}(mK_{X/Y}+L)\geqslant m[\Delta_f]$$

where $[\Delta_f]$ is the positive (1,1)-current associated to the effective divisor Δ_f. One thus has

$$i\Theta_h(\tilde{L})\geqslant\frac{m-1}{m}i\Theta_{h_{m,B}}(mK_{X/Y}+L)+\frac{1}{m}i\Theta_{h_L}\geqslant(m-1)[\Delta_f].$$

By the assumption the support $|\Delta_f|$ is simple normal crossing, which in turn implies that the multiplier ideal

$$\mathscr{J}(h)\subseteq\mathscr{O}_X\big(-(m-1)\Delta_f\big).$$

Recall that

$$S\in H^0\big(X,(K_{X/Y}+\widetilde{L}+f^*A_Y)\otimes\mathscr{J}(h)\big),$$

then one can divide S by $(m-1)\Delta_f$ to obtain a holomorphic section

$$S'\in H^0\big(X,mK_{X/Y}+L-(m-1)\Delta_f+f^*(A_Y-B)\big).$$

By definition $f^*B\geqslant\Delta_f$. The theorem immediately follows from that $\Delta_f\cap X_y=\varnothing$. □

We will apply a technical lemma in [CP17, Claim 3.5] to prove Theorem 1.21.(i). Let us first recall some definitions of singularities of divisors in [Vie95, Chapter 5.3] in a slightly different language.

Definition 1.16. Let X be a smooth projective variety, and let $\mathscr{L}$ be a line bundle such that $H^0(X, \mathscr{L}) \neq \varnothing$. One defines

$$e(\mathscr{L}) = \sup\Big\{\frac{1}{c(D)} \mid D \in |\mathscr{L}| \text{ is an effective divisor}\Big\} \tag{1.4.1}$$

where

$$c(D) := \sup\{c > 0 \mid (X, c \cdot D) \text{ is a klt divisor}\}$$

is the *log canonical threshold* of D.

Viehweg showed that one can control the lower bound of $e(\mathscr{L})$.

Lemma 1.17 ([Vie95, Corollary 5.11]). *Let X be a smooth projective variety equipped with a very ample line bundle $\mathscr{H}$, and let $\mathscr{L}$ be a line bundle such that $H^0(X, \mathscr{L}) \neq \varnothing$.*

(i) *Then there is a uniform estimate*

$$e(\mathscr{L}) \leqslant c_1(\mathscr{H})^{\dim X - 1} \cdot c_1(\mathscr{L}) + 1. \tag{1.4.2}$$

(ii) *Let $Z := X \times \cdots \times X$ be the r-fold product. Then for $\mathscr{M} := \bigotimes_{i=1}^r \mathrm{pr}_i^* \mathscr{L}$, one has $e(\mathscr{M}) = e(\mathscr{L})$.*

Lemma 1.18 (Cao-Păun). *Let $f : X \to Y$ be an algebraic fiber space so that the Kodaira dimension of the general fiber is non-negative. Assume that f is smooth over a dense Zariski open set of $Y_0 \subset Y$ so that both $B := Y \setminus Y_0$ and f^*B are normal crossing. Then there exists some positive integer $C \geqslant 2$ so that for any $m \geqslant m_0$ and $a \in \mathbb{N}$, any $y \in Y_0$ and any section*

$$\sigma \in H^0(X_y, amCK_{X_y}),$$

there exists a section

$$\Sigma \in H^0\big(X, f^*A_Y - af^* \det f_*(mK_{X/Y}) + amr_mCK_{X/Y} + a(P_m + F_m)\big) \tag{1.4.3}$$

whose restriction to the fiber X_y is equal to $\sigma^{\otimes r_m}$. Here F_m and P_m are effective divisors on X (independent of a) such that F_m is f-exceptional with $f(F_m) \subset \mathrm{Supp}(B)$, $\mathrm{Supp}(P_m) \subset \mathrm{Supp}(\Delta_f)$, $r_m := \mathrm{rank} f_*(mK_{X/Y})$, and A_Y is the universal ample line bundle on Y defined in Theorem 1.15.

Since [CP17, Claim 3.5] does not provide an effective estimate for the coefficients in (1.4.3), we will give a sketch proof of Lemma 1.18 to show how to apply Lemma 1.17 to achieve that. This proof is exactly the same as [CP17, Claim 3.5].

Sketch proof of Lemma 1.18. To make the proof less technical, we may assume that $X \to Y$ is a smooth fibration. Write $r = \mathrm{rank} f_*(mK_{X/Y})$ for short. Consider the r-fold fiber product $X^r := X \times_Y X \times_Y \cdots \times_Y X$ of f. Let $f^r : X^r \to Y$ be the natural induced fibration, and let $\mathrm{pr}_i : X^r \to X$ be the projection on the i-th factor. Then

$$K_{X^r/Y} = \bigotimes_{i=1}^r \mathrm{pr}_i^*(K_{X/Y}), \quad \text{and} \quad f_*^r(K_{X^r/Y}^{\otimes m}) = \bigotimes_{i=1}^r f_*(mK_{X/Y}).$$

We see that there exists a natural morphism

$$\det f_*(mK_{X/Y}) \to \bigotimes_{i=1}^r f_*(mK_{X/Y}),$$

which induces a zero divisor Γ of the section

$$H^0\big(X^r, mK_{X^r/Y} - f^{r*} \det f_*(mK_{X/Y})\big)$$

such that Γ does not contain any fiber of f^r. Then there exists for $\varepsilon_m \in \mathbb{Q}^+$ small enough, such that for each fiber X_y^r of $f^r : X^r \to Y$, $(X_y^r, \varepsilon_m \Gamma_{\restriction X_y^r})$ is a klt pair.

Indeed, one can apply Lemma 1.17 to control the lower bound of ε_m. Take a very ample line bundle $\mathscr{A}$ over X and fix a point $z \in Y$. Write $d := \dim X_y$. Since $f : X \to Y$ is a flat family,

$$e(mK_{X_y}) \leqslant c_1(\mathscr{A})^{d-1} \cdot c_1(mK_{X_y}) + 1 = m \cdot c_1(\mathscr{A})^{d-1} \cdot c_1(K_{X_z}) + 1,$$

by (1.4.2) for any $y \in Y$. Note that $X_y^r = X_y \times \cdots \times X_y$ is the r-fold product of X_y. Since $\Gamma_{\restriction X_y^r}$ is a zero divisor of a non-zero global section in

$$H^0(X_y^r, mK_{X_y^r}) = H^0\big(X_y^r, \bigotimes_{i=1}^{r} \mathrm{pr}_i^*(K_{X_y}^{\otimes m})\big).$$

By Lemma 1.17 for any $m \gg 0$ and any $y \in Y$, the log canonical threshold

$$c(\Gamma_{\restriction X_y^r}) \geqslant \frac{1}{e\big(\bigotimes_{i=1}^{r} \mathrm{pr}_i^*(K_{X_y}^{\otimes m})\big)} = \frac{1}{e(mK_{X_y})} \geqslant \frac{1}{m \cdot c_1(\mathscr{A})^{d-1} \cdot c_1(K_{X_z}) + 1} \geqslant \frac{2}{(C-1)m} \tag{1.4.4}$$

for some $C \in \mathbb{N}$ which does not depend on m. We thus can take $\varepsilon_m = \frac{1}{(C-1)m}$.

Write $L_r := mK_{X^r/Y} - f^{r*} \det f_*(mK_{X/Y})$, which is equipped with a singular hermitian metric h induced by Γ. Then by our choice of C, for any $y \in Y$

$$\mathscr{J}(h_{\restriction X_y^r}^{\otimes \varepsilon_m}) = \mathscr{O}_{X_y^r}.$$

By Siu's invariance of plurigenera, for any $k \in \mathbb{N}$ with $k\varepsilon_m \in \mathbb{N}$, all the sections $H^0(X_y^r, kK_{X_y^r} + k\varepsilon_m L_{r\restriction X_y^r})$ extends locally near y for any $y \in Y$. Applying Theorem 1.15 to $X^{(r)}$ with $L = L_r$, there exists an ample line bundle A_Y over Y such that, the following surjection holds

$$H^0\big(X^r, kK_{X^r/Y} + k\varepsilon_m L_r + f^{r*}A_Y\big) \twoheadrightarrow H^0(X_y^r, kK_{X_y^r} + k\varepsilon_m L_{r\restriction X_y^r}). \tag{1.4.5}$$

Let $i_y : X_y \hookrightarrow X_y^r$ be the diagonal embedding. For any $\sigma \in H^0\big(X_y, k(1+\varepsilon_m m)K_{X_y}\big)$, there is a natural section $s \in H^0(X_y^r, kK_{X_y^r} + k\varepsilon_m L_{r\restriction X_y^r})$ such that $i_y^* s = \sigma^{\otimes r}$. By (1.4.5), s extends to a section $S \in H^0\big(X^r, kK_{X^r/Y} + k\varepsilon_m L_r + f^{r*}A_Y\big)$. Denote by $\Sigma \in H^0\big(X, f^*A_Y + rk(1+\varepsilon_m m)K_{X/Y} - k\varepsilon_m f^* \det f_*(mK_{X/Y})$ the restriction of S to the diagonal $X \hookrightarrow X^r$. By the following commutative diagram

$$\begin{tikzcd}
S \in H^0\big(X^r, kK_{X^r/Y} + k\varepsilon_m L_r + f^{r*}A_Y\big) \arrow[r, two heads] \arrow[d] & H^0(X_y^r, kK_{X_y^r} + k\varepsilon_m L_{r\restriction X_y^r}) \arrow[d] \\
H^0\big(X, f^*A_Y + rk(1+\varepsilon_m m)K_{X/Y} - k\varepsilon_m f^* \det f_*(mK_{X/Y})\big) \arrow[r] & H^0(X_y, rk(1+\varepsilon_m m)K_{X_y}),
\end{tikzcd}$$

Σ extends σ^r. The lemma is obtained by setting $\varepsilon_m = \frac{1}{(C-1)m}$, $k = \frac{a}{\varepsilon_m}$. □

1.5. **Positivity of the direct images.** This section is devoted to the proof of Theorem D, which refines results by Viehweg-Zuo [VZ02, Proposition 3.4] and [VZ03, Proposition 4.3], and a theorem by Kollár [Kol87]. We first recall the definition of *Kollár family of varieties with semi-log canonical singularities* (*slc family* for short).

Definition 1.19 (slc family). An *slc family* is a flat proper morphism $f : X \to B$ such that:

(i) each fiber $X_b := f^{-1}(b)$ is a projective variety with slc singularities.
(ii) $\omega_{X/B}^{[m]}$ is flat.
(iii) The family $f : X \to B$ satisfies the *Kollár condition*, which means that, for any $m \in \mathbb{N}$, the reflexive power $\omega_{X/B}^{[m]}$ commutes with arbitrary base change.

To make Definition 1.19.(iii) precise, for every base change $\tau : B' \to B$, given the induced morphism $\rho : X' = X \times_B B' \to X$ we have that the natural homomorphism $\rho^*\omega^{[m]}_{X/B} \to \omega^{[m]}_{X'/B'}$ is an isomorphism. Let us collect the basic properties of slc families, as is well-known to the experts.

Lemma 1.20. *Let $g : Z \to W$ be a surjective morphism between quasi-projective manifolds with connected fibers, which is birational to an slc family $g' : Z' \to W$ whose generic fiber has at most Gorenstein canonical singularities. Then*

(i) *the total space Z' is normal and has only canonical singularities at worst.*
(ii) *If $\nu : W' \to W$ is a dominant morphism with W' smooth quasi-projective, then $Z' \times_W W' \to W'$ is still an slc family whose generic fiber has at most Gorenstein canonical singularities, and is birational to $(Z \times_W W')\tilde{} \to W'$.*
(iii) *Denote by Z'^r the r-fold fiber product $Z' \times_W \cdots \times_W Z'$. Then $g'^r : Z'^r \to W$ is also an slc family whose generic fiber has at most Gorenstein canonical singularities. Moreover, Z'^r is birational to the main component $(Z^r)\tilde{}$ of Z^r dominating W.*
(iv) *Let $Z^{(r)}$ be a desingularization of $(Z^r)\tilde{}$. Then $(g^{(r)})_*(\ell K_{Z^{(r)}/W}) \simeq (g'^r)_*(\ell K_{Z'^r/W})$ is reflexive for every sufficiently divisible $\ell > 0$.*

Now let us state and prove our main result on the positivity of direct images.

Theorem 1.21 (=Theorem D)**.** *Let $f_0 : X_0 \to Y_0$ be a smooth family of projective manifolds of general type. Assume that for any $y \in Y_0$, the set of $z \in Y_0$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite.*

(i) *For any smooth projective compactification $f : X \to Y$ of $f_0 : X_0 \to Y_0$ and any sufficiently ample line bundle A_Y over Y, $f_*(\ell K_{X/Y})^{\star\star} \otimes A_Y^{-1}$ is globally generated over Y_0 for any $\ell \gg 0$. In particular, $f_*(\ell K_{X/Y})$ is ample with respect to Y_0.*
(ii) *In the same setting as (i), $\det f_*(\ell K_{X/Y}) \otimes A_Y^{-r_\ell}$ is also globally generated over Y_0 for any $\ell \gg 0$, where $r_\ell = \operatorname{rank} f_*(\ell K_{X/Y})$. In particular, $\mathbf{B}_+\big(\det f_*(\ell K_{X/Y})\big) \subset Y \setminus Y_0$.*
(iii) *For some $r \gg 0$, there exists an algebraic fiber space $f : X \to Y$ compactifying $X_0^r \to Y_0$, so that $f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ is globally generated over Y_0 for ℓ large and divisible enough. Here X_0^r denotes to be the r-fold fiber product of $X_0 \to Y_0$, and A_Y is some sufficiently ample line bundle over Y.*

Proof. Let us first show that, to prove Claims (i) and (ii), one can assume that both $B := Y \setminus Y_0$ and f^*B are normal crossing.

For the arbitrary smooth projective compactification $f' : X' \to Y'$ of $f_0 : X_0 \to Y_0$, we take a log resolution $\nu : Y \to Y'$ with centers supported on $Y' \setminus Y_0$ so that $B := \nu^{-1}(Y' \setminus Y_0)$ is a simple normal crossing divisor. Define X to be strong desingularization of the main component $(X' \times_{Y'} Y)\tilde{}$ dominant over Y

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & X' \times_{Y'} Y \arrow[r] \arrow[d] & X' \arrow[d, "f'"] \\ & Y \arrow[r, "\nu"] & Y' \end{tikzcd} \tag{1.5.1}$$

so that f^*B is normal crossing. By [Vie90, Lemma 2.5.a], there is the inclusion

$$\nu_* f_*(mK_{X/Y}) \hookrightarrow f'_*(mK_{X'/Y'}) \tag{1.5.2}$$

which is an isomorphism over Y_0 for each $m \in \mathbb{N}$. Hence for any ample line bundle A over Y', once $f_*(mK_{X/Y})^{\star\star} \otimes (\nu^*A)^{-1}$ is globally generated over $\nu^{-1}(Y_0) \simeq Y_0$ for some $m \geqslant 0$, $f'_*(mK_{X'/Y'})^{\star\star} \otimes A^{-1}$ will be also globally generated over Y_0. As we will see, Claim (ii) is a direct consequence of Claim (i). This proves the above statement.

(i) Let us fix a sufficiently ample line bundle A_Y on Y. Assume that both $B := Y \setminus Y_0$ and f^*B are normal crossing. It follows from Proposition 1.9 that one can take some $b \gg a \gg 0$, $\mu \gg m \gg 0$ and $s \gg 0$ such that $\mathscr{L} := \det f_*(\mu m K_{X/Y})^{\otimes a} \otimes \det f_*(mK_{X/Y})^{\otimes b}$ is ample over Y_0. In other words, $\mathbf{B}_+(\mathscr{L}) \subset \mathrm{Supp}(B)$. By the definition of augmented base locus, one can even arrange $a, b \gg 0$ such that there exists a singular hermitian metric h_1 of $\mathscr{L} - 4A_Y$ which is smooth over Y_0, and the curvature current $\sqrt{-1}\Theta_{h_{\mathscr{L}}}(\mathscr{L}) \geqslant \omega$ for some Kähler form ω in Y. Denote by $r_1 := \mathrm{rank} f_*(\mu m K_{X/Y})$ and $r_2 := \mathrm{rank} f_*(mK_{X/Y})$. It follows from Lemma 1.18 that for any sections

$$\sigma_1 \in H^0(X_y, a\mu m C K_{X_y}), \quad \sigma_2 \in H^0(X_y, bmCK_{X_y}),$$

there exists effective divisors Σ_1 and Σ_2 such that

$$\Sigma_1 + af^* \det f_*(m\mu K_{X/Y}) - f^*A_Y \overset{\text{linear}}{\sim} am\mu r_1 C K_{X/Y} + P_1 + F_1$$
$$\Sigma_2 + bf^* \det f_*(mK_{X/Y}) - f^*A_Y \overset{\text{linear}}{\sim} bmr_2 C K_{X/Y} + P_2 + F_2$$

and

$$\Sigma_{1\upharpoonright X_y} = \sigma_1^{\otimes r_1}, \quad \Sigma_{2\upharpoonright X_y} = \sigma_2^{\otimes r_2}.$$

Here F_i is f-exceptional with $f(F_i) \subset \mathrm{Supp}(B)$, $\mathrm{Supp}(P_i) \subset \mathrm{Supp}(\Delta_f)$ for $i = 1, 2$.

Write $N := am\mu r_1 C + bmr_2 C$, $P := P_1 + P_2$ and $F := F_1 + F_2$. Fix any $y \in Y_0$. Then the effective divisor $\Sigma_1 + \Sigma_2$ induces a singular hermitian metric h_2 for the line bundle $L_2 := NK_{X/Y} - f^*\mathscr{L} + 2f^*A_Y + P + F$ such that $h|_{X_y}$ is not identically equal to $+\infty$, and so is the singular hermitian metric $h := f^*h_1 \cdot h_2$ over $L_0 := L_2 + f^*\mathscr{L} - 4f^*A_Y = NK_{X/Y} - 2f^*A_Y + P + F$. In particular, when ℓ sufficiently large, the multiplier ideal sheaf $\mathscr{J}(h^{\frac{1}{\ell}}_{\upharpoonright X_y}) = \mathscr{O}_{X_y}$. By Siu's invariance of plurigenera, all the global sections $H^0\big(X_y, (\ell K_X + L_0)_{\upharpoonright X_y}\big) \simeq H^0\big(X_y, (\ell + N)K_{X_y}\big)$ extends locally, and we thus can apply Theorem 1.15 to obtain the desired surjectivity

$$H^0\big(X, \ell K_{X/Y} + L_0 - \ell\Delta_f + f^*A_Y\big) \twoheadrightarrow H^0\big(X_y, (\ell + N)K_{X_y}\big), \tag{1.5.3}$$

Recall that $\mathrm{Supp}(P) \subset \mathrm{Supp}(\Delta_f)$. Then $\ell\Delta_f \geqslant P$ for $\ell \gg 0$, and one has the inclusion of sheaves

$$\ell K_{X/Y} + L_0 - (\ell - 1)\Delta_f + f^*A_Y \hookrightarrow (N + \ell)K_{X/Y} - f^*A_Y + F.$$

which is an isomorphism over X_0 . By (1.5.3) this implies that the direct image sheaves $f_*(\ell K_{X/Y} - f^*A_Y + F)$ are globally generated over some Zariski open set $U_y \subset Y_0$ containing y for $\ell \gg 0$. Since y is an arbitrary point in Y_0, the direct image $f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1}$ is globally generated over Y_0 for $\ell \gg 0$ by noetherianity. Recall that F is f-exceptional with $f(F) \subset \mathrm{Supp}(B)$. Then there is an injection

$$f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1} \hookrightarrow f_*(\ell K_{X/Y})^{\star\star} \otimes A_Y^{-1}$$

which is an isomorphism over Y_0. Hence $f_*(\ell K_{X/Y})^{\star\star} \otimes A_Y^{-1}$ is also globally generated over Y_0. By Definition 1.1.(iii), $f_*(\ell K_{X/Y})$ is ample with respect to Y_0 for $\ell \gg 0$. The first claim follows.

(ii) The trick to prove the second claim has already appeared in [Den17] in proving a conjecture by Demailly-Peternell-Schneider. We first recall that $f_*(\ell K_{X/Y})$ is locally free outside a codimension 2 analytic subset of Y. By the proof of Theorem 1.21.(i), for ℓ sufficiently large and divisible, $f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1}$ is locally free and generated by global sections over Y_0, where F is some f-exceptional effective divisor. Therefore, its determinant $\det f_*(\ell K_{X/Y} + F) \otimes A_Y^{-r_\ell}$ is also globally generated over Y_0, where $r_\ell := \mathrm{rank} f_*(\ell K_{X/Y})$. Since F is f-exceptional and effective, one has

$$\det f_*(\ell K_{X/Y} + F) \otimes A_Y^{-r_\ell} = \det f_*(\ell K_{X/Y}) \otimes A_Y^{-r_\ell},$$

and therefore, $\det f_*(\ell K_{X/Y}) \otimes A_Y^{-r\ell}$ is also globally generated over Y_0. By the very definition of the augmented base locus $\mathbf{B}_+(\bullet)$ we conclude that

$$\mathbf{B}_+\big(\det f_*(\ell K_{X/Y})\big) \subset \mathrm{Supp}(B).$$

The second claim is proved.

(iii) We combine the ideas in [VZ03, Proposition 4.1] as well as the pluricanonical extension techniques in Theorem 1.15 to prove the result. By Corollary A.2, there exists a smooth projective compactification Y of Y_0 with $B := Y \setminus Y_0$ simple normal crossing, a non-singular finite covering $\psi : W \to Y$, and an slc family $g' : Z' \to W$, which extends the family $X_0 \times_{Y_0} W$. By Lemma 1.20.(iii) for any $r \in \mathbb{Z}_{>0}$, the r-fold fiber product $g''^r : Z''^r \to W$ is still an slc family, which compactifies the smooth family $X_0^r \times_{Y_0} W \to W_0$, where $W_0 := \psi^{-1}(Y_0)$. Note that Z''^r has canonical singularities.

Take a smooth projective compactification $f : X \to Y$ of $X_0^r \to Y_0$ so that f^*B is normal crossing. Let $Z \to Z''^r$ be a strong desingularization of Z'', which also resolves this birational map $Z''^r \dashrightarrow \widetilde{(X \times_Y W)}$. Then $g : Z \to W$ is smooth over $W_0 := \psi^{-1}(Y_0)$.

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "g"] \arrow[rrr, bend left] & Z''^r \arrow[r, dashed] \arrow[d, "g''^r"] & X \arrow[d, "f"] & \widetilde{(X \times_Y W)} \arrow[l] \arrow[d] \\
W \arrow[r, equal] & W \arrow[r, "\psi"] & Y & W \arrow[l, "\psi"']
\end{tikzcd}$$

Let $\tilde{Z}$ be a strong desingularization of Z', which is thus smooth over $W_0 := \psi^{-1}(Y_0)$. For the new family $\tilde{g} : \tilde{Z} \to W$, we denote by $\tilde{Z}_0 := \tilde{g}^{-1}(W_0)$. Then $\tilde{Z}_0 \to W_0$ is also a smooth family, and any fiber of Z_w with $w \in W_0$ is a projective manifold of general type. By our assumption in the theorem, for any $w \in W_0$, the set of $w' \in W_0$ with $\tilde{Z}_{w'} \overset{\mathrm{bir}}{\sim} \tilde{Z}_w$ is finite as $\psi : W \to Y$ is a finite morphism. We thus can apply Theorems 1.21.(i) and 1.21.(ii) to our new family $\tilde{g} : \tilde{Z} \to W$.

From now on, we will always assume that $\ell \gg 0$ is sufficiently divisible so that $\ell K_{Z'}$ is Cartier. Let A_Y be a sufficiently ample line bundle over Y, so that $A_W := \psi^* A_Y$ is also *sufficiently ample*. Since Z' has canonical singularity, $\tilde{g}_*(\ell K_{\tilde{Z}/W}) = g'_*(\ell K_{Z'/W})$. It follows from Theorem 1.21.(ii) that, for any $\ell \gg 0$, the line bundle

$$\det \tilde{g}_*(\ell K_{\tilde{Z}/W}) \otimes A_W^{-r} = \det g'_*(\ell K_{Z'/W}) \otimes A_W^{-r} \tag{1.5.4}$$

is globally generated over W_0, where $r := \mathrm{rank}\, g'_*(\ell K_{Z'/W})$ depending on ℓ. Then there exists a positively-curved singular hermitian metric $h_{\det}$ on the line bundle $\det g'_*(\ell K_{Z'/W}) \otimes A_W^{-r}$ such that $h_{\det}$ is smooth over W_0.

By the base change properties of slc families (see [BHPS13, Proposition 2.12] and [KP17, Lemma 2.6]), one has

$$\omega^{[\ell]}_{Z''^r/W} \simeq \bigotimes_{i=1}^{r} \mathrm{pr}_i^* \omega^{[\ell]}_{Z'/W}, \quad g''^r_*(\ell K_{Z''^r/W}) \simeq \bigotimes^{r} g'_*(\ell K_{Z'/W}),$$

where $\mathrm{pr}_i : Z''^r \to Z'$ is the i-th directional projection map. Hence $\ell K_{Z''^r}$ is Cartier as well, and we have

$$\bigotimes^{r} g'_*(\ell K_{Z'/W}) \simeq g''^r_*(\ell K_{Z''^r/W}) = g_*(\ell K_{Z/W}).$$

By Lemma 1.20.(iv), $g_*(\ell K_{Z/W})$ is reflexive, and we thus have

$$\det g'_*(\ell K_{Z'/W}) \to \bigotimes^{r} g'_*(\ell K_{Z'/W}) \simeq g_*(\ell K_{Z/W}),$$

which induces a natural effective divisor

$$\Gamma \in |\ell K_{Z/W} - g^* \det g'_*(\ell K_{Z'/W})|$$

such that $\Gamma_{\upharpoonright Z_w} \neq 0$ for any (smooth) fiber Z_w with $w \in W_0$. By (1.4.4), there exists a positive integer C which does not depend on ℓ, so that the log canonical threshold

$$c(\Gamma_{\upharpoonright Z_w}) \geqslant \frac{2}{(C-1)\ell} \tag{1.5.5}$$

for any $\ell \gg 0$. Denote by h the singular hermitian metric on

$$\ell K_{Z/W} - g^* \det g'_*(\ell K_{Z'/W})$$

induced by Γ. By (1.5.5) the multiplier ideal sheaf $\mathscr{J}\left(h_{\upharpoonright Z_w}^{\frac{1}{(C-1)\ell}}\right) = \mathscr{O}_{Z_w}$ for any fiber Z_w with $w \in W_0$. Let us define a positively-curved singular metric $h_{\mathscr{F}}$ for the line bundle $\mathscr{F} := \ell K_{Z/W} - rg^* A_W$ by setting $h_{\mathscr{F}} := h \cdot g^* h_{\rm det}$. Then $\mathscr{J}\left(h_{\mathscr{F}\upharpoonright Z_w}^{\frac{1}{(C-1)\ell}}\right) = \mathscr{O}_{Z_w}$ for any $w \in W_0$.

For any $n \in \mathbb{N}^*$, applying Theorem 1.15 to $n\mathscr{F}$ we obtain the surjectivity

$$H^0\big(Z, (C-1)n\ell K_{Z/W} + n\mathscr{F} + g^* A_W\big) \twoheadrightarrow H^0\big(Z_w, Cn\ell K_{Z_w}\big) \tag{1.5.6}$$

for all $w \in W_0$. In other words,

$$g_*\big(C\ell n K_{Z/W}\big) \otimes A_W^{-(nr-1)}$$

is globally generated over W_0 for any $\ell \gg 0$ and any $n \geqslant 1$.

Since K_{X_y} is big, one thus has

$$r = r_\ell \sim \ell^d \ \text{ as } \ \ell \to +\infty$$

where $d := \dim Z_w \geqslant 2$ (if the fibers of f are curves, one can take a fiber product to replace the original family). Recall that C is a constant which does not depend on ℓ. One thus can take an *a priori* $\ell \gg 0$ so that $r \gg C\ell$. In conclusion, for sufficiently large and divisible m,

$$g_*\big(mK_{Z/W}\big) \otimes A_W^{-2m} = g_*\big(mK_{Z/W}\big) \otimes \psi^* A_Y^{-2m}$$

is globally generated over W_0. Therefore, we have a morphism

$$\bigoplus_{i=1}^{N} \psi^* A_Y^m \to g_*\big(mK_{Z/W}\big) \otimes \psi^* A_Y^{-m}, \tag{1.5.7}$$

which is surjective over W_0. On the other hand, by [Vie90, Lemma 2.5.b], one has the inclusion

$$g_*\big(mK_{Z/W}\big) \hookrightarrow \psi^* f_*(mK_{X/Y}),$$

which is an isomorphism over W_0. (1.5.7) thus induces a morphism

$$\bigoplus_{i=1}^{N} \psi_* \mathscr{O}_W \otimes A_Y^m \to \psi_* g_*\big(mK_{Z/W}\big) \otimes A_Y^{-m} \to \psi_* \psi^*\big(f_*(mK_{X/Y})\big) \otimes A_Y^{-m}, \tag{1.5.8}$$

which is surjective over Y_0. Note that that even if $f_*(mK_{X/Y})$ is merely a coherent sheaf, the projection formula $\psi_*\psi^*\big(f_*(mK_{X/Y})\big) = f_*(mK_{X/Y})) \otimes \psi_* \mathscr{O}_W$ still holds for ψ is finite (see [Ara04, Lemma 5.7]). The trace map

$$\psi_* \mathscr{O}_W \to \mathscr{O}_Y$$

splits the natural inclusion $\mathscr{O}_Y \to \psi_* \mathscr{O}_W$, and is thus surjective. Hence (1.5.8) gives rise to a morphism

$$\bigoplus_{i=1}^{N} \psi_* \mathscr{O}_W \otimes A_Y^m \to \psi_* g_*\big(mK_{Z/W}\big) \otimes A_Y^{-m} \xrightarrow{\Phi} f_*(mK_{X/Y}) \otimes A_Y^{-m}, \tag{1.5.9}$$

which is surjective over Y_0. By taking m sufficiently large, we may assume that $\psi_*\mathscr{O}_W \otimes A_Y^m$ is generated by its global sections. Then $f_*(mK_{X/Y}) \otimes A_Y^{-m}$ is globally generated over Y_0. We complete the proof. □

Remark 1.22. In a recent paper [PX17], Xu-Patakfalvi proved that for an n-dimensional KSBA-stable family $f : (Z, \Delta) \to T$ with finite fiber isomorphism equivalence classes over a normal variety T, $f_*\big((K_{Z/T} + \Delta)^{n+1}\big)$ is ample on T. Their proof relies on some kind of Nakai-Moishezon criterion by Kollár in [Kol90]. In the case of Theorem 1.21, we cannot apply their result to show Theorems 1.21.(i) or 1.21.(ii) directly, as Y_0 might be non-compact.

Since the $\mathbb{Q}$-*mild reduction* in Corollary A.2 holds for any smooth surjective projective morphism with connected fibers and smooth base, it follows from our proof in Theorem 1.21.(iii) and Kawamata's theorem (1.3.9), one still has the *generic* global generation as follows.

Theorem 1.23. *Let $f_U : U \to V$ be a smooth projective morphism between quasi-projective varieties with connected fibers. Assume that the general fiber F of f_U has semi-ample canonical bundle, and f_U is of maximal variation. Then there exists a positive integer $r \gg 0$ and a smooth projective compactification $f : X \to Y$ of $U^r \to V$ so that $f_*(mK_{X/Y}) \otimes \mathscr{A}^{-m}$ is globally generated over some Zariski open subset of V. Here $U^r \to V$ is the r-fold fiber product of $U \to V$, and $\mathscr{A}$ is some ample line bundle on Y.* □

1.6. **Sufficiently many "moving" hypersurfaces.** As we will see in the construction of VZ Higgs bundles in Theorem 3.1, one has to require the following: for the algebraic fiber space $f : X \to Y$ defined in Theorem 1.21.(iii), the positivity of $K_{X/Y}$ must be *almost fonctorial under base changes* (see Theorem 1.24 for a precise statement). Since $f : X \to Y$ is not flat, we are forced to perform the base changes on its $\mathbb{Q}$-mild reductions to study the positivity of relative canonical bundles. Let us state and prove our main result in this subsection, which will be our basic setup in constructing refined VZ Higgs bundles in § 3. The proof we present here follows from [PTW18, Proposition 4.4].

Theorem 1.24. *Let $X_0 \to Y_0$ be a smooth family of minimal projective manifolds of general type over a quasi-projective manifold Y_0. Suppose that for any $y \in Y_0$, the set of $z \in Y_0$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite. Let $Y \supset Y_0$ be the smooth compactification in Corollary A.2. Fix any $y_0 \in Y_0$ and some sufficiently ample line bundle A_Y on Y. Then there exist a birational morphism $\nu : Y' \to Y$ and a new algebraic fiber space $f' : X' \to Y'$ which is smooth over $\nu^{-1}(Y_0)$, so that for any sufficiently large and divisible ℓ, one can find a hypersurface*

$$H \in |\ell K_{X'/Y'} - \ell(\nu \circ f')^* A_Y + \ell E| \tag{1.6.1}$$

satisfying that

- *the divisor $D := \nu^{-1}(Y \setminus Y_0)$ is simple normal crossing.*
- *There exists a reduced divisor S in Y', so that $D + S$ is simple normal crossing, and $H \to Y'$ is smooth over $Y' \setminus D \cup S$.*
- *The exceptional locus* $\mathrm{Ex}(\nu) \subset \mathrm{Supp}(D + S)$, *and $y_0 \notin \nu(D \cup S)$.*
- *The divisor E is effective and f'-exceptional with $f'(E) \subset \mathrm{Supp}(D + S)$.*

Moreover, when $X_0 \to Y_0$ is effectively parametrized *over some open set containing y_0, so is the new family $X' \to Y'$.*

Proof. The proof is a continuation of that of Theorem 1.21.(iii), and we adopt the same notations therein. By (1.5.9) and the isomorphism

$$H^0(Z, \ell K_{Z/W} - \ell g^* A_W) \simeq H^0\big(Z^{\prime r}, \ell K_{Z^{\prime r}/W} - \ell (g^{\prime r})^* A_W\big),$$

the morphism $\Phi : \psi_* g_*\big(\ell K_{Z/W}\big) \otimes A_Y^{-\ell} \to f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ in (1.5.9) gives rise to a natural map

$$\Upsilon : H^0(Z^r, \ell K_{Z^r/W} - \ell (g^r)^* A_W) \to H^0\big(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}\big) \tag{1.6.2}$$

whose image I generates $f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ over Y_0. Note that Υ is fonctorial in the sense that it does not depend on the choice of the birational model $Z \to Z^r$. By the base point free theorem, for any $y \in Y_0$, K_{X_y} is semi-ample, and we can assume that $\ell \gg 0$ is sufficiently large and divisible so that $\ell K_{X/Y}$ is relatively semi-ample over Y_0. Hence we can take a section

$$\sigma \in H^0\big(Z^r, \ell K_{Z^r/W} - \ell (g^r)^* A_W\big) \tag{1.6.3}$$

so that the zero divisor of

$$\Upsilon(\sigma) \in H^0\big(X, \ell K_{X/Y} - \ell f^* A_Y\big) = H^0\big(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}\big),$$

denoted by $H_1 \in |\ell K_{X^{(r)}/Y} - \ell (f^{(r)})^* A_Y|$, is *transverse* to the fiber X_{y_0}. Denote by T the *discriminant locus* of $H_1 \to Y$, and $B := Y \setminus Y_0$. Then $y_0 \notin T \cup B$. Take a log-resolution $\nu : Y' \to Y$ with centers in $T \cup B$ so that both $D := \nu^{-1}(B)$ and $D + S := \nu^{-1}(T \cup B)$ are simple normal crossing. Let X' be a strong desingularization of $(X \times_Y Y')\tilde{}$, and write $f' : X' \to Y'$, which is smooth over $Y_0' := \nu^{-1}(Y_0)$. Set $X_0' := f'^{-1}(Y_0')$. It suffices to show that, there exists a hypersurface H in (1.6.1) with $H_{\upharpoonright (\nu \circ f')^{-1}(V)} = H_{1\upharpoonright (f^{(r)})^{-1}(V)}$, where $V := Y \setminus S' \cup B \subset Y_0$. Since the birational morphism ν is isomorphic at y_0, we can write y_0 as $\nu^{-1}(y_0)$ abusively.

Now we follow the similar arguments in [PTW18, Proposition 4.4] to prove the existence of H (in which they apply their methods for *mild morphisms*). Let W' be a strong desingularization of $W \times_Y Y'$ which is finite at $y_0 \in Y'$. Write $W_0' := \nu'^{-1}(W_0)$. By Lemma 1.20.(ii), the new family $Z'' := Z^r \times_W W' \to W'$ is still an slc family, which compactifies the smooth family $X_0' \times_{Y_0'} W' \to W_0'$. Let M' be a desingularization of Z'' so that it resolves the rational maps to X' as well as Z.

$$\begin{tikzcd}
 & X \arrow[dd, "f"] & & Z \arrow[ll] \arrow[rr] \arrow[dd, "g"] & & Z^r \arrow[dd, "g^r"] \\
X' \arrow[ur, "\mu"] \arrow[dd, "f'"] & & M' \arrow[ll] \arrow[rr] \arrow[ur] \arrow[dd, "h'"] & & Z'' \arrow[ur, "\mu'"] \arrow[dd, "g''"] & \\
 & Y & & W \arrow[ll, "\psi"] \arrow[rr, equal] & & W \\
Y' \arrow[ur, "\nu"] & & W' \arrow[ll, "\psi'"] \arrow[rr, equal] \arrow[ur, "\nu'"] & & W' \arrow[ur, "\nu'"] &
\end{tikzcd}$$

By the properties of slc families, $\mu'^* \omega_{Z^r/W}^{[\ell]} = \omega_{Z''/W'}^{[\ell]}$, which induces a natural map

$$\mu^* : H^0\big(Z^r, \ell K_{Z^r/W} - \ell (g^r)^* A_W\big) \to H^0\big(Z'', \ell K_{Z''/W'} - \ell (\nu' \circ g'')^* A_W\big). \tag{1.6.4}$$

Since both Z^r and Z'' have canonical singularities, one has the following natural morphisms

$$g_*(\ell K_{Z/W}) \simeq (g^r)_*(\ell K_{Z^r/W}), \quad h'_*(\ell K_{M'/W'}) = g''_*(\ell K_{Z''/W'}).$$

We can leave out a subvariety of codimension at least two in Y' supported on $D + S$ (which thus avoids y_0 by our construction) so that $\psi' : W' \to Y'$ becomes a *flat finite* morphism. As discussed at the beginning of the proof, there is also a natural map

$$\Upsilon' : H^0(Z'', \ell K_{Z''/W'} - \ell (\nu' \circ g'')^* A_W) \to H^0\big(X', \ell K_{X'/Y'} - \ell (\nu \circ f')^* A_Y\big) \tag{1.6.5}$$

as (1.6.2) by factorizing through M'.

Note that for $V := Y \setminus T \cup B$, $\nu : \nu^{-1}(V) \xrightarrow{\simeq} V$ is also an isomorphism, and thus the restriction of $X \to Y$ to V is isomorphic to that of $X' \to Y'$ to $\nu^{-1}(V)$. Hence by our construction,the restriction of $Z'^r \to W$ to $\psi^{-1}(V)$ is isomorphic to that of $Z'' \to W'$ to $(\nu\circ\psi')^{-1}(V) = (\nu'\circ\psi)^{-1}(V)$. In particular, under the above isomorphism, for the section $\sigma \in H^0\big(Z'^r, \ell K_{Z'^r/W} - \ell(g'^r)^*A_W\big)$ in (1.6.3) with $\Upsilon(\sigma)$ defining H_1, one has

$$\Upsilon(\sigma)_{\upharpoonright f^{-1}(V)} \simeq \Upsilon'(\mu^*\sigma)_{\upharpoonright(\nu\circ f')^{-1}(V)}.$$

where μ^* and Υ' are defined in (1.6.4) and (1.6.5). Denote by $\tilde{H}$ the zero divisor defined by

$$\Upsilon'(\mu^*\sigma) \in H^0\big(X', \ell K_{X'/Y'} - \ell(\nu\circ f')^*A_Y\big).$$

Recall that H_1 is smooth over V, then $\tilde{H}$ is also smooth over $\nu^{-1}(V)$.

Note that $\Upsilon'(\mu^*\sigma) \in H^0\big(Y', f'_*(\ell K_{X'/Y'}) \otimes \nu^*A_Y^{-\ell}\big)$ is only defined over a big open set of Y' containing $\nu^{-1}(V)$. Hence it extends to a global section

$$s \in H^0(X', \ell K_{X'/Y'} - \ell(\nu\circ f')^*A_Y + \ell E),$$

where E is an f'-exceptional effective divisor with $f'(E) \subset \operatorname{Supp}(D+S)$. Denote by H the hypersurface in X' defined by s. Hence $H_{\upharpoonright(\nu\circ f')^{-1}(V)} = \tilde{H}_{\upharpoonright(\nu\circ f')^{-1}(V)}$, which is smooth over $\nu^{-1}(V) = Y'\setminus D\cup S \simeq V \ni y_0$. Note that the property of effective parametrization is invariant under fiber product. The theorem follows. □

Based on Theorem 1.23, one can apply the same methods in Theorem 1.24 to obtain the following result.

Theorem 1.25. *Let $f_U : U \to V$ be the smooth projective morphism as in Theorem 1.23. Then there exists a positive integer $r \gg 0$, a birational morphism from a smooth quasi-projective variety $\nu : \tilde{V} \to V$, a smooth projective compactification $f : X \to Y$ of $U^r \times_V \tilde{V} \to \tilde{V}$*

$$\begin{tikzcd}
U^r \arrow[d, "f_U"] & U^r \times_V \tilde{V} \arrow[l] \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\
V & \tilde{V} \arrow[l, "\nu"'] \arrow[r, hook] & Y
\end{tikzcd}$$

and a big and nef line bundle $\mathscr{L}$ over Y so that there is a hypersurface $H \in |mK_{X/Y} - mf^*\mathscr{L} + mE|$ satisfying the following conditions.

(i) *The boundary $D := Y \setminus \tilde{V}$ is a simple normal crossing divisor.*
(ii) *The hypersurface H is smooth over some Zariski open set $V_0 \subset \tilde{V}$, and $D + S := Y \setminus V_0$ is a simple normal crossing divisor.*
(iii) *The divisor E is effective and f-exceptional divisor with $f(E) \cap V_0 = \varnothing$.*
(iv) *The augmented base locus $\mathbf{B}_+(\mathscr{L}) \cap V_0 = \varnothing$.*

Here $U^r \to V$ is the r-fold fiber product of $f_U : U \to V$. □

2. Construction of negatively curved Finsler metric

To begin with, let us introduce the definition of *Viehweg-Zuo Higgs bundles* over quasi-projective manifolds in an abstract way following [VZ03, PTW18]. Then we prove a generic local Torelli for VZ Higgs bundles. Next we establish an algorithm to construct Finsler metrics whose holomorphic sectional curvatures are bounded above by a negative constant via VZ Higgs bundles. By our construction and generic local Torelli, those Finsler metrics are positively definite over a Zariski open set, and by the Ahlfors-Schwarz lemma, we prove that a quasi-projective manifold is pseudo Kobayashi hyperbolic once it is equipped with a VZ Higgs bundle.

2.1. **Abstract Viehweg-Zuo Higgs bundles.** The definition we present below follows from the formulation in [PTW18, Proposition 2.7], which relaxes the monodromy condition in [VZ03] and is thus less restrictive.

Definition 2.1 (Abstract Viehweg-Zuo Higgs bundles)**.** Let V be a quasi-projective manifold, and let $Y \supset V$ be a projective compactification of V with the boundary $D := Y \setminus V$ simple normal crossing. A *Viehweg-Zuo Higgs bundle on V* is a logarithmic Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ over Y consisting of the following data:

(i) a divisor S on Y so that $D + S$ is simple normal crossing,
(ii) a big and nef line bundle $\mathscr{L}$ over Y with $\mathbf{B}_+(\mathscr{L}) \subset D \cup S$,
(iii) a Higgs bundle $(\mathscr{E}, \theta) := \big(\bigoplus_{q=0}^{n} E^{n-q,q}, \bigoplus_{q=0}^{n} \theta_{n-q,q}\big)$ induced by the lower canonical extension of a polarized VHS defined over $Y \setminus (D \cup S)$,
(iv) a sub-Higgs sheaf $(\mathscr{F}, \eta) \subset (\tilde{\mathscr{E}}, \tilde{\theta})$,

which satisfy the following properties.

(1) The Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta}) := (\mathscr{L}^{-1} \otimes \mathscr{E}, \mathbb{1} \otimes \theta)$. In particular, $\tilde{\theta} : \tilde{\mathscr{E}} \to \tilde{\mathscr{E}} \otimes \Omega_Y\big(\log(D+S)\big)$, and $\tilde{\theta} \wedge \tilde{\theta} = 0$.
(2) The sub-Higgs sheaf $(\mathscr{F}, \eta)$ has log poles only on the boundary D, that is, $\eta : \mathscr{F} \to \mathscr{F} \otimes \Omega_Y(\log D)$.
(3) Write $\tilde{\mathscr{E}}_k := \mathscr{L}^{-1} \otimes E^{n-k,k}$, and denote by $\mathscr{F}_k := \tilde{\mathscr{E}}_k \cap \mathscr{F}$. Then the first stage $\mathscr{F}_0$ of $\mathscr{F}$ is an *effective line bundle*. In other words, there exists a non-trivial morphism $\mathscr{O}_Y \to \mathscr{F}_0$.

As shown in [VZ02], by iterating η for k-times, we obtain

$$\mathscr{F}_0 \xrightarrow{\overbrace{\eta \circ \cdots \circ \eta}^{k \text{ times}}} \mathscr{F}_k \otimes \big(\Omega_Y(\log D)\big)^{\otimes k}.$$

Since $\eta \wedge \eta = 0$, the above morphism factors through $\mathscr{F}_k \otimes \mathrm{Sym}^k \Omega_Y(\log D)$, and by (3) one thus obtains

$$\mathscr{O}_Y \to \mathscr{F}_0 \to \mathscr{F}_k \otimes \mathrm{Sym}^k \Omega_Y(\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k} \otimes \mathrm{Sym}^k \Omega_Y(\log D).$$

Equivalently, we have a morphism

$$\tau_k : \mathrm{Sym}^k \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k}. \tag{2.1.1}$$

It was proven in [VZ02, Corollary 4.5] that τ_1 is always non-trivial. We say that a VZ Higgs bundle satisfies the *generic local Torelli* if $\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1}$ in (2.1.1) is generically injective. As we will see in § 2.3, in Theorem F we prove that the generic local Torelli holds for any VZ Higgs bundles.

2.2. **Proper metrics for logarithmic Higgs bundles.** We adopt the same notations as Definition 2.1 in the rest of § 2. As is well-known, $\mathscr{E}$ can be endowed with the Hodge metric h induced by the polarization, which may blow-up around the simple normal crossing boundary $D + S$. However, according to the work of Schmid and Cattani-Schmid-Kaplan [Sch73, CKS86], h has *mild singularities* (at most logarithmic singularities), and as proved in [VZ03, §7] (for unipotent monodromies) and [PTW18, §3] (for quasi-unipotent monodromies), one can take a proper singular metric g_α on $\mathscr{L}$ such that the induced singular hermitian metric $g_\alpha^{-1} \otimes h$ on $\tilde{\mathscr{E}} := \mathscr{L}^{-1} \otimes \mathscr{E}$ is locally bounded from above. Before we summarize the above-mentioned results in [PTW18, §3], we introduce some notations in *loc. cit.*

Write the simple normal crossing divisor $D = D_1 + \cdots + D_k$ and $S = S_1 + \cdots + S_\ell$. Let $f_{D_i} \in H^0\big(Y, \mathscr{O}_Y(D_i)\big)$ and $f_{S_i} \in H^0\big(Y, \mathscr{O}_Y(S_i)\big)$ be the canonical section defining D_i and S_i. We fix smooth hermitian metrics g_{D_i} and g_{S_i} on $\mathscr{O}_Y(D_i)$ and $\mathscr{O}_Y(S_i)$. Set

$$r_{D_i} := -\log\|f_{D_i}\|^2_{g_{D_i}}, \quad r_{S_i} := -\log\|f_{S_i}\|^2_{g_{S_i}},$$

and define

$$r_D := \prod_{i=1}^{k} r_{D_i}, \quad r_S := \prod_{i=1}^{\ell} r_{S_i}.$$

Let g be a singular hermitian metric with analytic singularities of the big and nef line bundle $\mathscr{L}$ such that g is smooth on $Y\setminus\mathbf{B}_+(\mathscr{L}) \supset Y\setminus D\cup S$, and the curvature current $\sqrt{-1}\Theta_g(\mathscr{L}) \geqslant \omega$ for some smooth Kähler form ω on Y. For $\alpha \in \mathbb{N}$, define

$$g_\alpha := g\cdot(r_D\cdot r_S)^\alpha$$

The following proposition is a slight variant of [PTW18, Lemma 3.1, Corollary 3.4].

Proposition 2.2 ([PTW18]). *When $\alpha \gg 0$, after rescaling f_{D_i} and f_{S_i}, there exists a continuous, positively definite hermitian form ω_α on $\mathscr{T}_Y(-\log D)$ such that*

(i) *over $V_0 := Y \setminus D \cup S$, the curvature form*

$$\sqrt{-1}\Theta_{g_\alpha}(\mathscr{L})_{\restriction V_0} \geqslant r_D^{-2}\cdot\omega_{\alpha\restriction V_0}.$$

(ii) *The singular hermitian metric $h_g^\alpha := g_\alpha^{-1}\otimes h$ on $\mathscr{L}^{-1}\otimes\mathscr{E}$ is locally bounded on Y, and smooth outside $(D+S)$. Moreover, h_g^α is degenerate on $D+S$.*

(iii) *The singular hermitian metric $r_D^2 h_g^\alpha$ on $\mathscr{L}^{-1}\otimes\mathscr{E}$ is also locally bounded on Y.* □

Remark 2.3. It follows from Proposition 2.2 that both h_g^α and $r_D^2 h_g^\alpha$ can be seen as Finsler metrics on $\mathscr{L}^{-1}\otimes\mathscr{E}$ which are degenerate on $\mathrm{Supp}(D+S)$, and positively definite on V_0.

Although the last statement of Proposition 2.2.(ii) is not explicitly stated in [PTW18], it can be easily seen from the proof of [PTW18, Corollary 3.4]. Proposition 2.2 mainly relies on the asymptotic behavior of the Hodge metric for *lower canonical extension* of a variation of Hodge structure (cf. Theorem 2.4 below) when the monodromy around the boundaries are only quasi-unipotent.

Theorem 2.4 ([PTW18, Lemma 3.2]). *Let $\mathcal{H} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^N \supset 0$ be a variation of Hodge structures defined over $(\Delta^*)^p\times\Delta^q$, where Δ (resp. Δ^*) is the (resp. punctured) unit disk. Consider the lower canonical extension ${}^l\mathcal{F}^\bullet$ over $\Delta^{p+q} \supset (\Delta^*)^p\times\Delta^q$, and denote by $(\mathscr{E},\theta)$ the associated Higgs bundle. Then for any holomorphic section $s \in \Gamma(U,\mathscr{E})$, where $U \subsetneq \Delta^{p+q}$ is a relatively compact open set containing the origin, one has the following norm estimate*

$$|s|_{\mathrm{hod}} \leqslant C\big((-\log|t_1|)\cdot(-\log|t_2|)\cdots(-\log|t_p|)\big)^\alpha, \tag{2.2.1}$$

where α is some positive constant independent of s, and $t = (t_1,\ldots,t_{p+q})$ denotes to be the coordinates of Δ^{p+q}.

Let us mention that the estimates of Hodge metric for *upper canonical extension* were obtained by Peters [Pet84] in one variable, and by Catanese-Kawamata [CK17] in several variables, based on the work [Sch73,CKS86]. We provide a slightly different proof of Theorem 2.4 for completeness sake, following closely the approaches in [Pet84, CK17].

Proof of Theorem 2.4. The fundamental group $\pi_1\big((\Delta^*)^p\times\Delta^q\big)$ is generated by elements $\gamma_1,\ldots,\gamma_p$, where γ_j may be identified with the counter-clockwise generator of the fundamental group of the j-th copy of Δ^* in $(\Delta^*)^p$. Set T_j to be the monodromy transformation with respect to γ_j, which pairwise commute and are known to be quasi-unipotent; that is, for any multivalued section $\underline{v}(t_1,\ldots,t_{p+q})$ of $\mathcal{H}$, one has

$$\underline{v}(t_1,\ldots,e^{2\pi i}t_j,\ldots,t_{p+q}) = T_j\cdot\underline{v}(t_1,\ldots,t_{p+q})$$

and $[T_j, T_k] = 0$ for any $j, k = 1, \ldots, p$. Set $T_j = D_j \cdot U_j$ to be the (unique) Jordan-Chevally decomposition, so that D_j diagonalizable and U_j is unipotent with $[D_j, U_j] = 0$. Since T_j is quasi-unipotent by the theorem of Borel, all the eigenvalues of D_j are thus the roots of unity. Set $N_j := \frac{1}{2\pi i}\sum_{k>0}(I - U_j)^k/k$. If $D_j = \mathrm{diag.}(d_{j\ell})$ then we set $S_j = \mathrm{diag.}(\lambda_{j\ell})$ with $\lambda_{j\ell} \in (-2\pi i, 0]$ and $\exp(\lambda_{j\ell}) = d_{j\ell}$. Since $[T_j, T_k] = 0$, Jordan-Chevally decomposition implies that

$$[S_j, S_k] = [S_j, N_k] = [N_j, N_k] = 0. \tag{2.2.2}$$

Fix a point $t_0 \in (\Delta^*)^p \times \Delta^q$, and take a basis $v_1, \ldots, v_r \in V_{t_0}$ so that $S_1, \ldots, S_p$ are simultaneously diagonal, that is, one has

$$S_j(v_\ell) = \lambda_{j\ell}. \tag{2.2.3}$$

Let us define $\underline{v}_1(t), \ldots, \underline{v}_r(t)$ to be the induced multivalued flat sections. Then

$$e_j(t) := \exp\big(-\frac{1}{2\pi i}\sum_{i=1}^{p}(S_i + N_i)\cdot\log t_i\big)\underline{v}_j(t)$$

is single-valued and forms a basis of holomorphic sections for the lower canonical extension ${}^l\mathcal{H}$.

Recall that $d_{j\ell}$ are all roots of unity. One thus can take a positive integer m so that $m_{j\ell} := -m\lambda_{j\ell}/2\pi i$ are all *non-negative integers*. Equivalently, each T_j^m is unipotent. Define a ramified cover

$$\begin{aligned}\pi : \Delta^{p+q} &\longrightarrow \Delta^{p+q}\\ (w_1, \ldots, w_{p+q}) &\mapsto (w_1^m, \ldots, w_p^m, w_{p+1}, \ldots, w_{p+q})\end{aligned}$$

and set π' to be the restriction of π to $(\Delta^*)^p\times\Delta^q$. Then $\pi'^*\mathcal{F}^\bullet$ is a variation of Hodge structure on $(\Delta^*)^p \times \Delta^q$ with unipotent monodromy, and we define ${}^c\pi'^*\mathcal{H}$ the canonical extension of $\pi'^*\mathcal{H}$. Set $\underline{u}_j(w) = \pi'^*\underline{v}_j$ which are multivalued sections for the local system $\pi'^*\mathcal{H}$. Then

$$\underline{u}_j(w_1, \ldots, e^{2\pi i}w_j, \ldots, w_{p+q}) = T_j^m \cdot \underline{u}_j(w_1, \ldots, w_{p+q}).$$

Define

$$\tilde{e}_j(w) := \exp\big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i\cdot\log w_i\big)\underline{u}_j(w) \tag{2.2.4}$$

which forms a basis of ${}^c\pi'^*\mathcal{H}$. Based on the work of [Sch73, CKS86], it was shown in [VZ03, Claim 7.8] that one has the upper bound of norms

$$|\tilde{e}_j(w)|_{\mathrm{hod}} \leqslant C_0\big((-\log|w_1|)\cdot(-\log|w_2|)\cdots(-\log|w_p|)\big)^\alpha \tag{2.2.5}$$

for some positive constants C_0 and α. One the other hand, we have

$$\begin{aligned}\pi'^*e_j(w) &= \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p}(S_i+N_i)\cdot\log w_i^m\Big)\pi'^*\underline{v}_j(w)\\ &\overset{(2.2.2)}{=} \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i\cdot\log w_i\Big)\cdot\exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mS_i\log w_i\Big)\pi'^*\underline{v}_j(w)\\ &\overset{(2.2.3)}{=} \exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i\cdot\log w_i\Big)\cdot\exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} m\lambda_{ij}\log w_i\Big)\pi'^*\underline{v}_j(w)\\ &= \prod_{i=1}^{p} w_i^{m_{ij}}\cdot\exp\Big(-\frac{1}{2\pi i}\sum_{i=1}^{p} mN_i\cdot\log w_i\Big)\cdot\underline{u}_j(w)\\ &\overset{(2.2.4)}{=} \prod_{i=1}^{p} w_i^{m_{ij}}\cdot\tilde{e}_j(w).\end{aligned}$$

By the definition of lower canonical extension, m_{ij} are all non-negative integers, and thus

$$\begin{aligned}\pi'^*|e_j|_{\mathrm{hod}}(w) = |\pi'^*e_j(w)|_{\mathrm{hod}} &= \prod_{i=1}^{p}|w_i|^{m_{ij}}|\tilde{e}_j(w)|_{\mathrm{hod}}\\ &\overset{(2.2.5)}{\leqslant} C_0\big((-\log|w_1|)\cdot(-\log|w_2|)\cdots(-\log|w_p|)\big)^{\alpha}.\end{aligned}$$

Hence

$$|e_j|_{\mathrm{hod}}(t)\leqslant\frac{C_0}{m^p}\big((-\log|t_1|)\cdot(-\log|t_2|)\cdots(-\log|t_p|)\big)^{\alpha}.$$

Note that ${}^{l}\mathcal{H}\overset{\mathscr{C}^\infty}{\simeq}\mathscr{E}$. Therefore, for any holomorphic section $s\in\Gamma(U,\mathscr{E})$, there exist smooth functions $f_1,\ldots,f_r\in\mathscr{O}(U)$ so that $s=\sum_{j=1}^{r}f_je_j$. This shows the estimate (2.2.1). □

Remark 2.5. For the Hodge metric of upper canonical extension, one makes the choice that $\lambda_{j\ell}\in[0,2\pi i)$ instead of $\lambda_{j\ell}\in(-2\pi i,0]$ in the proof of Theorem 2.4. Then the same computation as above can easily show that

$$|e_j|_{\mathrm{hod}}(t)\leqslant\prod_{i=1}^{p}|t_i|^{-\frac{\lambda_{ij}}{2\pi i}}\frac{C}{m^p}\big((-\log|t_1|)\cdot(-\log|t_2|)\cdots(-\log|t_p|)\big)^{\alpha},$$

which were obtained in [CK17].

2.3. **A generic local Torelli for VZ Higgs bundle.** In this section we prove that the generic local Torelli holds for any VZ Higgs bundle, which is a crucial step in the proofs of Theorems A and B.

Theorem F (Generic local Torelli)**.** *For the abstract Viehweg-Zuo Higgs bundles defined in Definition 2.1, the morphism $\tau_1:\mathscr{T}_Y(-\log D)\to\mathscr{L}^{-1}\otimes E^{n-1,1}$ defined in (2.1.1) is generically injective.*

Proof of Theorem F. By Definition 2.1, the non-zero morphism $\mathscr{O}_Y\to\mathscr{F}_0\to\mathscr{L}^{-1}\otimes E^{n,0}$ induces a global section $s\in H^0(Y,\mathscr{L}^{-1}\otimes E^{n,0})$, which is *generically* non-vanishing over $V_0:=Y\setminus D\cup S$. Set

$$V_1:=\{y\in V_0\mid s(y)\neq0\}\tag{2.3.1}$$

which is a non-empty Zariski open set of V_0. For the first stage of VZ Higgs bundle $\mathscr{L}^{-1}\otimes E^{n,0}$, we equip it with a singular metric $h_g^{\alpha}:=g_{\alpha}^{-1}\otimes h$ as in Proposition 2.2, so that Propositions 2.2.(i) and 2.2.(ii) are satisfied. Note that h_g^{α} is smooth over V_0. Let us denote D' to be the $(1,0)$-part of its Chern connection over V_0, and Θ_0 to be its curvature form. Then by the Griffiths curvature formula of Hodge bundles (see [GT84]), over V_0 we have

$$\begin{aligned}\Theta_0 &= -\Theta_{\mathscr{L},g_\alpha}\otimes \mathbb{1} + \mathbb{1}\otimes \Theta_h(E^{n,0})\\ &= -\Theta_{\mathscr{L},g_\alpha}\otimes \mathbb{1} - \mathbb{1}\otimes (\theta_{n,0}^*\wedge\theta_{n,0})\\ &= -\Theta_{\mathscr{L},g_\alpha}\otimes \mathbb{1} - \tilde{\theta}_{n,0}^*\wedge\tilde{\theta}_{n,0}, \end{aligned}\tag{2.3.2}$$

where we set $\tilde{\theta}_{n-k,k} := \mathbb{1}\otimes\theta_{n-k,k} : \mathscr{L}^{-1}\otimes E^{n-k,k}\to \mathscr{L}^{-1}\otimes E^{n-k-1,k+1}\otimes\Omega_Y\big(\log(D+S)\big)$, and define $\tilde{\theta}_{n,0}^*$ to be the adjoint of $\tilde{\theta}_{n,0}$ with respect to the metric h_g^α. Hence over V_1 one has

$$\begin{aligned}-\sqrt{-1}\partial\bar{\partial}\log|s|^2_{h_g^\alpha} &= \frac{\big\{\sqrt{-1}\Theta_0(s),s\big\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}} + \frac{\sqrt{-1}\{D's,s\}_{h_g^\alpha}\wedge\{s,D's\}_{h_g^\alpha}}{|s|^4_{h_g^\alpha}} - \frac{\sqrt{-1}\{D's,D's\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}}\\ &\leqslant \frac{\big\{\sqrt{-1}\Theta_0(s),s\big\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}}\end{aligned}\tag{2.3.3}$$

thanks to the Lagrange's inequality

$$\sqrt{-1}|s|^2_{h_g^\alpha}\cdot\{D's,D's\}_{h_g^\alpha}\geqslant\sqrt{-1}\{D's,s\}_{h_g^\alpha}\wedge\{s,D's\}_{h_g^\alpha}.$$

Putting (2.3.2) to (2.3.3), over V_1 one has

$$\sqrt{-1}\Theta_{\mathscr{L},g_\alpha} - \sqrt{-1}\partial\bar{\partial}\log|s|^2_{h_g^\alpha}\leqslant -\frac{\big\{\sqrt{-1}\tilde{\theta}_{n,0}^*\wedge\tilde{\theta}_{n,0}(s),s\big\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}} = \frac{\sqrt{-1}\big\{\tilde{\theta}_{n,0}(s),\tilde{\theta}_{n,0}(s)\big\}_{h_g^\alpha}}{|s|^2_{h_g^\alpha}}\tag{2.3.4}$$

where $\tilde{\theta}_{n,0}(s)\in H^0\Big(Y,\mathscr{L}^{-1}\otimes E^{n-1,1}\otimes\Omega_Y\big(\log(D+S)\big)\Big)$. By Proposition 2.2.(ii), for any $y\in D\cup S$, one has

$$\lim_{y'\in V_0, y'\to y}|s|^2_{h_g^\alpha}(y') = 0.$$

Therefore, it follows from the compactness of Y that there exists $y_0\in V_0$ so that $|s|^2_{h_g^\alpha}(y_0)\geqslant |s|^2_{h_g^\alpha}(y)$ for any $y\in V_0$. Hence $|s|^2_{h_g^\alpha}(y_0)>0$, and by (2.3.1), $y_0\in V_1$. Since $|s|^2_{h_g^\alpha}$ is smooth over V_0, $\sqrt{-1}\partial\bar{\partial}\log|s|^2_{h_g^\alpha}(y_0)$ is semi-negative. By Proposition 2.2.(i), $\sqrt{-1}\Theta_{\mathscr{L},g_\alpha}$ is strictly positive at y_0. By (2.3.4) and $|s|^2_h(y_0)>0$, we conclude that $\sqrt{-1}\big\{\tilde{\theta}_{n,0}(s),\tilde{\theta}_{n,0}(s)\big\}_{h_g^\alpha}$ is strictly positive at y_0. In particular, for any non-zero $\xi\in\mathscr{T}_{Y,y_0}$, $\tilde{\theta}_{n,0}(s)(\xi)\neq 0$. For

$$\tau_1:\mathscr{T}_Y(-\log D)\to\mathscr{L}^{-1}\otimes E^{n-1,1}$$

in (2.1.1), over V_0 it is defined by $\tau_1(\xi):=\tilde{\theta}_{n,0}(s)(\xi)$, which is thus *injective at* $y_0\in V_1$. Hence τ_1 is *generically injective*. The theorem is thus proved. □

Remark 2.6. Let us stress here that, we cannot give a precise description of the loci where τ_1 is injective, for our method in proving Theorem F relies on the global aspects of the VZ Higgs bundles. Roughly speaking, the bigness of $\mathscr{L}$ forces τ_1 to be injective at least one point, which is analogous to Demailly's (weak) holomorphic Morse inequality [Dem12, §8.2(a)].

2.4. **Finsler metric and (pseudo) Kobayashi hyperbolicity.** Throughout this subsection X will denoted to be a complex manifold of dimension n.

Definition 2.7 (Finsler metric). Let $\mathscr{E}$ be a holomorphic vector bundle on X. A *Finsler metric*[3] on $\mathscr{E}$ is a real non-negative continuous function $F : \mathscr{E} \to [0, +\infty[$ such that

$$F(av) = |a|F(v)$$

for any $a \in \mathbb{C}$ and $v \in \mathscr{E}$. The Finsler metric$F$ is *positively definite* at some subset $S \subset X$ if for any $x \in S$ and any non-zero vector $v \in \mathscr{E}_x$, $F(v) > 0$.

When F is a Finsler metric on $\mathscr{T}_X$, we also say that F is a *Finsler metric on* X.

Let $\mathscr{E}$ and $\mathscr{G}$ be two locally free sheaves on X, and suppose that there is a morphism

$$\varphi : \mathrm{Sym}^m \mathscr{E} \to \mathscr{G}$$

Then for any Finsler metric F on $\mathscr{G}$, φ induces a pseudo metric $(\varphi^* F)^{\frac{1}{m}}$ on $\mathscr{E}$ defined by

$$(\varphi^* F)^{\frac{1}{m}}(e) := F\big(\varphi(e^{\otimes m})\big)^{\frac{1}{m}} \tag{2.4.1}$$

for any $e \in \mathscr{E}$. It is easy to verify that $(\varphi^* F)^{\frac{1}{m}}$ is also a Finsler metric on $\mathscr{E}$. Moreover, if over some open set U, φ is an injection as a morphism between vector bundles, and F is positively definite over U, then $(\varphi^* F)^{\frac{1}{m}}$ is also positively definite over U.

Definition 2.8. (i) The *Kobayashi-Royden infinitesimal pseudo-metric* of X is a length function $\kappa_X : \mathscr{T}_X \to [0, +\infty[$, defined by

$$\kappa_X(\xi) = \inf_{\gamma} \big\{\lambda > 0 \mid \exists \gamma : \mathbb{D} \to X, \gamma(0) = x, \lambda \cdot \gamma'(0) = \xi\big\} \tag{2.4.2}$$

for any $x \in X$ and $\xi \in \mathscr{T}_X$, where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

(ii) The *Kobayashi pseudo distance* of X, denoted by $d_X : X \times X \to [0, +\infty[$, is

$$d_X(p, q) = \inf_{\ell} \int_0^1 \kappa_X\big(\ell'(\tau)\big) d\tau$$

for every pair of points $p, q \in X$, where the infimum is taken over all differentiable curves $\ell : [0, 1] \to X$ joining p to q.

(iii) Let $\Delta \subsetneq X$ be a closed subset. A complex manifold X is *Kobayashi hyperbolic modulo* Δ if $d_X(p, q) > 0$ for every pair of distinct points $p, q \in X$ not both contained in Δ. When Δ is an empty set, the manifold X is *Kobayashi hyperbolic*; when Δ is proper and Zariski closed, the manifold X is *pseudo Kobayashi hyperbolic.*

By definition it is easy to show that if X is Kobayashi hyperbolic (resp. pseudo Kobayashi hyperbolic), then X is Brody hyperbolic (resp. algebraically degenerate). Brody's theorem says that when X is compact, X is Kobayashi hyperbolic if it is Brody hyperbolic. However unlike the case of Kobayashi hyperbolicity, no criteria is known for pseudo Kobayashi hyperbolicity of a compact complex space in terms of entire curves. Moreover, there are many examples of complex (quasi-projective) manifolds which are Brody hyperbolic but not Kobayashi hyperbolic.

For any holomorphic map $\gamma : \mathbb{D} \to X$, the Finsler metric F induces a continuous Hermitian pseudo-metric on $\mathbb{D}$

$$\gamma^* F^2 = \sqrt{-1}\lambda(t) dt \wedge d\bar{t},$$

where $\lambda(t)$ is a non-negative continuous function on $\mathbb{D}$. The *Gaussian curvature* $K_{\gamma^* F^2}$ of the pseudo-metric $\gamma^* F^2$ is defined to be

$$K_{\gamma^* F^2} := -\frac{1}{\lambda} \frac{\partial^2 \log \lambda}{\partial t \partial \bar{t}}. \tag{2.4.3}$$

[3]This definition is a bit different from the definition in [Kob98], which requires *convexity* or *triangle inequality*, and the Finsler metric there can be upper-semi continuous.

Definition 2.9. Let X be a complex manifold endowed with a Finsler metric F.

(i) For any $x \in X$, and $v \in \mathscr{T}_{X,x}$, let $[v]$ denote the complex line spanned by v. We define the holomorphic sectional curvature $K_{F,[v]}$ in the direction of $[v]$ by

$$K_{F,[v]} := \sup K_{\gamma^* F^2}(0)$$

where the supremum is taken over all $\gamma : \mathbb{D} \to X$ such that $\gamma(0) = x$ and $[v]$ is tangent to $\gamma'(0)$.

(ii) We say that F is *negatively curved* if there is a positive constant c such that $K_{F,[v]} \leqslant -c$ for all $v \in \mathscr{T}_{X,x}$ for which $F(v) > 0$.

(iii) A point $x \in X$ is a *degeneracy point* of F if $F(v) = 0$ for some nonzero $v \in \mathscr{T}_{X,x}$, and the set of such points is denoted by Δ_F.

As mentioned in § 0, our negatively curved Finsler metrics are only constructed on birational models of the base spaces in Theorems A and C, we thus have to establish bimeromorphic criteria for (pseudo) Kobayashi hyperbolicity to prove the main theorems.

Lemma 2.10 (Bimeromorphic criteria for pseudo Kobayashi hyperbolicity)**.** *Let $\mu : X \to Y$ be a bimeromorphic morphism between complex manifolds. If there exists a Finsler metric F on X which is negatively curved in the sense of Definition 2.9.(ii), then X is Kobayashi hyperbolic modulo Δ_F, and Y is Kobayashi hyperbolic modulo $\mu\big(\mathrm{Ex}(\mu)\cup\Delta_F\big)$, where $\mathrm{Ex}(\mu)$ is the exceptional locus of μ. In particular, when Δ_F is a proper analytic subvariety of X, both X and Y are pseudo Kobayashi hyperbolic.*

Proof. The first statement is a slight variant of [Kob98, Theorem 3.7.4]. By normalizing F we may assume that $K_F \leqslant -1$. By the Ahlfors-Schwarz lemma, one has $F \leqslant \kappa_X$. Let $\delta_F : X \times X \to [0, +\infty[$ be the distance function on X defined by F in a similar way as d_X:

$$\delta_F(p, q) := \inf_{\ell} \int_0^1 F\big(\ell'(\tau)\big) d\tau$$

for every pair of points $p, q \in X$, where the infimum is taken over all differentiable curves $\ell : [0, 1] \to X$ joining p to q. Since F is continuous and positively definite over $X \setminus \Delta_F$, for any $p \in X \setminus \Delta_F$, one has $d_X(p, q) \geqslant \delta_F(p, q) > 0$ for any $q \neq p$, which proves the first statement.

Let us denote by $\mathrm{Hol}(Y, y)$ to be the set of holomorphic maps $\gamma : \mathbb{D} \to Y$ with $\gamma(0) = y$. Pick any point $y \in U := Y \setminus \mu\big(\mathrm{Ex}(\mu)\big)$, then there is a unique point $x \in X$ with $\mu(x) = y$. Hence μ induces a bijection between the sets

$$\mathrm{Hol}(X, x) \xrightarrow{\simeq} \mathrm{Hol}(Y, y)$$

defined by $\tilde{\gamma} \mapsto \mu \circ \tilde{\gamma}$. Indeed, observe that $\mu^{-1} : Y \dashrightarrow X$ is a meromorphic map, so is $\mu^{-1} \circ \gamma$ for any $\gamma \in \mathrm{Hol}(Y, y)$. Since $\dim \mathbb{D} = 1$, the map $\mu^{-1} \circ \gamma$ is moreover holomorphic. It follows from (2.4.2) that

$$\kappa_X(\xi) = \kappa_Y\big(\mu_*(\xi)\big)$$

for any $\xi \in \mathscr{T}_{X,x}$. Hence one has

$$\mu^* \kappa_Y|_{\mu^{-1}(U)} = \kappa_X|_{\mu^{-1}(U)} \geqslant F|_{\mu^{-1}(U)}.$$

Let $G : \mathscr{T}_U \to [0, +\infty[$ be the Finsler metric on U so that $\mu^* G = F|_{\mu^{-1}(U)}$. Then G is continuous and positively definite over $U \setminus \mu(\Delta_F)$, and one has

$$\kappa_Y|_U \geqslant G.$$

Therefore, for any $y \in Y \setminus \mu\big(\Delta_F \cup \mathrm{Ex}(\mu)\big)$, one has $d_Y(y, z) > 0$ for any $z \neq y$, which proves the second statement. □

The above criteria can be refined further to show the Kobayashi hyperbolicity of the complex manifold.

Lemma 2.11 (Bimeromorphic criteria for Kobayashi hyperbolicity)**.** *Let X be a complex manifold. Assume that for each point $p \in X$, there is a bimeromorphic morphism $\mu : \tilde{X} \to X$ with $\tilde{X}$ equipped with a negatively curved Finsler metric F such that $p \notin \mu\big(\Delta_F \cup \mathrm{Ex}(\mu)\big)$. Then X is Kobayashi hyperbolic.*

Proof. It suffices to show that $d_X(p, q) > 0$ for every pair of distinct points $p, q \in X$. We take the bimeromorphic morphism $\mu : \tilde{X} \to X$ in the lemma with respect to p. By Lemma 2.10, X is Kobayashi hyperbolic modulo $\mu\big(\Delta_F \cup \mathrm{Ex}(\mu)\big)$, which shows that $d_X(p, q) > 0$ for any $q \neq p$. The lemma follows. □

2.5. **Curvature formula.** Let $(\tilde{\mathscr{E}}, \tilde{\theta})$ be the VZ Higgs bundles on a quasi-projective manifold V defined in § 2.1. In the next two subsections, we will construct a negatively curved Finsler metric on V via $(\tilde{\mathscr{E}}, \tilde{\theta})$. Our main result is the following.

Theorem 2.12 (Existence of negatively curved Finsler metrics)**.** *Same notations as Definition 2.1. Assume that τ_1 is injective over a non-empty Zariski open set $V_1 \subsetneq Y \setminus D \cup S$. Then there exists a Finsler metric F (see (2.6.6) below) on $\mathscr{T}_Y(-\log D)$ such that*

(i) *it is positively definite over V_1.*
(ii) *When F is seen as a Finsler metric on $V = Y \setminus D$, it is negatively curved in the sense of Definition 2.9.(ii).*

Let us first construct the desired Finsler metric F, and we then proved the curvature property. By (2.1.1), for each $k = 1, \ldots, n$, there exists

$$\tau_k : \mathrm{Sym}^k \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-k,k}. \tag{2.5.1}$$

Then it follows from Proposition 2.2.(ii) that the Finsler metric h_g^α on $\mathscr{L}^{-1} \otimes E^{n-k,k}$ induces a Finsler metric F_k on $\mathscr{T}_Y(-\log D)$ defined as follows: for any $e \in \mathscr{T}_Y(-\log D)_y$,

$$F_k(e) := (\tau_k^* h_g^\alpha)^{\frac{1}{k}}(e) = h_g^\alpha\big(\tau_k(e^{\otimes k})\big)^{\frac{1}{k}} \tag{2.5.2}$$

For any $\gamma : \mathbb{D} \to V$, one has

$$d\gamma : \mathscr{T}_{\mathbb{D}} \to \gamma^* \mathscr{T}_V \hookrightarrow \gamma^* \mathscr{T}_Y(-\log D)$$

and thus the Finsler metric F_k induces a continuous Hermitian pseudo-metric on $\mathbb{D}$, denoted by

$$\gamma^* F_k^2 := \sqrt{-1} G_k(t) dt \wedge d\bar{t}. \tag{2.5.3}$$

In general, $G_k(t)$ may be identically equal to zero for all k. However, if we further assume that $\gamma(\mathbb{D}) \cap V_1 \neq \varnothing$, by the assumption in Theorem 2.12 that the restriction of τ_1 to V_1 is injective, one has $G_1(t) \not\equiv 0$. Denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector fields in $\mathbb{D}$, and $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. Set $C := \gamma^{-1}(V_1)$, and note that $\mathbb{D} \setminus C$ is a discrete set in $\mathbb{D}$.

Lemma 2.13. *Assume that $G_k(t) \not\equiv 0$ for some $k > 1$. Then the* Gaussian curvature *K_k of the continuous pseudo-hermitian metric $\gamma^* F_k^2$ on C satisfies that*

$$K_k := -\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} / G_k \leqslant \frac{1}{k}\Big(-\big(\frac{G_k}{G_{k-1}}\big)^{k-1} + \big(\frac{G_{k+1}}{G_k}\big)^{k+1}\Big) \tag{2.5.4}$$

over $C \subset \mathbb{D}$.

Proof. For $i = 1, \ldots, n$, let us write $e_i := \tau_i\big(d\gamma(\partial_t)^{\otimes i}\big)$, which can be seen as a section of $\gamma^*(\mathscr{L}^{-1} \otimes E^{n-i,i})$. Then by (2.5.2) one observes that

$$G_i(t) = \|e_i\|_{h_g^\alpha}^{2/i}. \tag{2.5.5}$$

Let $\mathscr{R}_k = \Theta_{h_g^\alpha}(\mathscr{L}^{-1}\otimes E^{n-k,k})$ be the curvature form of $\mathscr{L}^{-1}\otimes E^{n-k,k}$ on $V_0 := Y\setminus D\cup S$ induced by the metric $h_g^\alpha = g_\alpha^{-1}\cdot h$ defined in Proposition 2.2.(ii), and let D' be the $(1,0)$-part of the Chern connection D of $(\mathscr{L}^{-1}\otimes E^{n-k,k}, h_g^\alpha)$. Then for $k=1,\ldots,n$, one has

$$\begin{aligned}-\sqrt{-1}\partial\bar{\partial}\log G_k &= \frac{1}{k}\Big(\frac{\big\{\sqrt{-1}\mathscr{R}_k(e_k),e_k\big\}_{h_g^\alpha}}{\|e_k\|^2_{h_g^\alpha}}+\frac{\sqrt{-1}\{D'e_k,e_k\}_{h_g^\alpha}\wedge\{e_k,D'e_k\}_{h_g^\alpha}}{\|e_k\|^4_{h_g^\alpha}}-\frac{\sqrt{-1}\{D'e_k,D'e_k\}_{h_g^\alpha}}{\|e_k\|^2_{h_g^\alpha}}\Big)\\ &\leqslant \frac{1}{k}\frac{\big\{\sqrt{-1}\mathscr{R}_k(e_k),e_k\big\}_{h_g^\alpha}}{\|e_k\|^2_{h_g^\alpha}}\end{aligned}$$

thanks to the Lagrange's inequality

$$\sqrt{-1}\|e_k\|^2_{h_g^\alpha}\cdot\{D'e_k,D'e_k\}_{h_g^\alpha}\geqslant\sqrt{-1}\{D'e_k,e_k\}_{h_g^\alpha}\wedge\{e_k,D'e_k\}_{h_g^\alpha}.$$

Hence

$$-\frac{\partial^2\log G_k}{\partial t\partial\bar{t}}\leqslant\frac{1}{k}\cdot\frac{\big\langle\mathscr{R}_k(e_k)(\partial_t,\bar{\partial}_t),e_k\big\rangle_{h_g^\alpha}}{\|e_k\|^2_{h_g^\alpha}}. \tag{2.5.6}$$

Recall that for the logarithmic Higgs bundle $(\bigoplus_{k=0}^n E^{n-k,k},\bigoplus_{k=0}^n\theta_{n-k,k})$, the curvature Θ_k on $E^{n-k,k}_{\upharpoonright V_0}$ induced by the Hodge metric h is given by

$$\Theta_k = -\theta^*_{n-k,k}\wedge\theta_{n-k,k}-\theta_{n-k+1,k-1}\wedge\theta^*_{n-k+1,k-1},$$

where we recall that $\theta_{n-k,k}: E^{n-k,k}\to E^{n-k-1,k+1}\otimes\Omega_Y\big(\log(D+S)\big)$. Set $\tilde{\theta}_{n-k,k}:=\mathbb{1}\otimes\theta_{n-k,k}:\mathscr{L}^{-1}\otimes E^{n-k,k}\to\mathscr{L}^{-1}\otimes E^{n-k-1,k+1}\otimes\Omega_Y\big(\log(D+S)\big)$, and one has

$$\begin{tikzcd} \mathscr{L}^{-1}\otimes E^{n-k+1,k-1} \arrow[r, bend left, "\tilde{\theta}_{n-k+1,k-1}(\partial_t)"] & \mathscr{L}^{-1}\otimes E^{n-k,k} \arrow[l, bend left, "\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)"] \arrow[r, bend left, "\tilde{\theta}_{n-k,k}(\partial_t)"] & \mathscr{L}^{-1}\otimes E^{n-k-1,k+1} \arrow[l, bend left, "\tilde{\theta}^*_{n-k,k}(\bar{\partial}_t)"] \end{tikzcd}$$

where $\tilde{\theta}^*_{n-k,k}$ is the adjoint of $\tilde{\theta}_{n-k,k}$ with respect to the metric h_g^α over $Y\setminus D\cup S$. Here we also write ∂_t (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$) abusively. Then over V_0, we have

(2.5.7)
$$\mathscr{R}_k = -\Theta_{\mathscr{L},g_\alpha}\otimes\mathbb{1}+\mathbb{1}\otimes\Theta_k = -\Theta_{\mathscr{L},g_\alpha}\otimes\mathbb{1}-\tilde{\theta}^*_{n-k,k}\wedge\tilde{\theta}_{n-k,k}-\tilde{\theta}_{n-k+1,k-1}\wedge\tilde{\theta}^*_{n-k+1,k-1}.$$

By the definition of τ_k in (2.1.1), for any $k=2,\ldots,n$ one has

$$e_k = \tilde{\theta}_{n-k+1,k-1}(\partial_t)(e_{k-1}), \tag{2.5.8}$$

and we can derive the following curvature formula

$$\begin{aligned}
\langle \mathscr{R}_k(e_k)(\partial_t,\bar{\partial}_t), e_k\rangle_{h_g^\alpha} &= -\Theta_{\mathscr{L},g_\alpha}(\partial_t,\bar{\partial}_t)\|e_k\|^2_{h_g^\alpha}+\\
&\quad \left\langle \tilde{\theta}^*_{n-k,k}(\bar{\partial}_t)\circ\tilde{\theta}_{n-k,k}(\partial_t)(e_k)-\tilde{\theta}_{n-k+1,k-1}(\partial_t)\circ\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k), e_k\right\rangle_{h_g^\alpha}\\
&\leqslant \left\langle \tilde{\theta}^*_{n-k,k}(\bar{\partial}_t)\circ\tilde{\theta}_{n-k,k}(\partial_t)(e_k), e_k\right\rangle_{h_g^\alpha}\\
&\quad -\left\langle \tilde{\theta}_{n-k+1,k-1}(\partial_t)\circ\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k), e_k\right\rangle_{h_g^\alpha}\\
&\overset{(2.5.8)}{=} \|e_{k+1}\|^2_{h_g^\alpha}-\|\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k)\|^2_{h_g^\alpha}\\
&\leqslant \|e_{k+1}\|^2_{h_g^\alpha}-\frac{|\langle\tilde{\theta}^*_{n-k+1,k-1}(\bar{\partial}_t)(e_k), e_{k-1}\rangle_{h_g^\alpha}|^2}{\|e_{k-1}\|^2_{h_g^\alpha}} \quad \text{(Cauchy-Schwarz inequality)}\\
&= \|e_{k+1}\|^2_{h_g^\alpha}-\frac{|\langle e_k, \tilde{\theta}_{n-k+1,k-1}(\partial_t)(e_{k-1})\rangle_{h_g^\alpha}|^2}{\|e_{k-1}\|^2_{h_g^\alpha}}\\
&\overset{(2.5.8)}{=} \|e_{k+1}\|^2_{h_g^\alpha}-\frac{\|e_k\|^4_{h_g^\alpha}}{\|e_{k-1}\|^2_{h_g^\alpha}}\\
&\overset{(2.5.5)}{=} G_{k+1}^{k+1}-\frac{G_k^{2k}}{G_{k-1}^{k-1}}
\end{aligned}$$

Putting this into (2.5.6), we obtain (2.5.4). □

Remark 2.14. For the final stage $E^{0,n}$ of the Higgs bundle $(\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$. We make the convention that $G_{n+1}\equiv 0$. Then the Gaussian curvature for G_n in (2.5.6) is always semi-negative, which is similar as the Griffiths curvature formula for Hodge bundles in [GT84].

When $k=1$, by (2.5.6) one has

$$\begin{aligned}
-\frac{\partial^2\log G_1}{\partial t\partial\bar{t}}/G_1 &\leqslant \frac{\langle\mathscr{R}_1(e_1)(\partial_t,\bar{\partial}_t), e_1\rangle_{h_g^\alpha}}{\|e_1\|^4_{h_g^\alpha}}\\
&\overset{(2.5.7)}{=} \frac{-\Theta_{\mathscr{L},g_\alpha}(\partial_t,\bar{\partial}_t)}{\|e_1\|^2_{h_g^\alpha}}+\\
&\quad \frac{\langle\tilde{\theta}^*_{n-1,1}(\bar{\partial}_t)\circ\tilde{\theta}_{n-1,1}(\partial_t)(e_1)-\tilde{\theta}_{n,0}(\partial_t)\circ\tilde{\theta}^*_{n,0}(\bar{\partial}_t)(e_1), e_1\rangle_{h_g^\alpha}}{\|e_1\|^4_{h_g^\alpha}}\\
&\overset{(2.5.8)}{\leqslant} \frac{-\Theta_{\mathscr{L},g_\alpha}(\partial_t,\bar{\partial}_t)\|e_1\|^2_{h_g^\alpha}+\|e_2\|^2_{h_g^\alpha}}{\|e_1\|^4_{h_g^\alpha}}\\
&= \frac{-\Theta_{\mathscr{L},g_\alpha}(\partial_t,\bar{\partial}_t)}{\|e_1\|^2_{h_g^\alpha}}+\left(\frac{G_2}{G_1}\right)^2
\end{aligned}$$

We need the following lemma to control the negative term in the above inequality.

Lemma 2.15. *When $\alpha \gg 0$, there exists a* universal *constant $c > 0$, such that for any $\gamma : \mathbb{D} \to V$ with $\gamma(\mathbb{D}) \cap V_0 \neq \varnothing$, one has*

$$\frac{\Theta_{\mathscr{L},g_\alpha}(\partial_t, \bar{\partial}_t)}{\|e_1\|^2_{h_g^\alpha}} \geqslant c.$$

In particular,

$$-\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}}/G_1 \leqslant -c + (\frac{G_2}{G_1})^2$$

Proof. By Proposition 2.2.(ii), it suffices to prove that

$$\frac{\gamma^*(r_D^{-2} \cdot \omega_\alpha)(\partial_t, \bar{\partial}_t)}{\|e_1\|^2_{h_g^\alpha}} \geqslant c. \tag{2.5.9}$$

Note that

$$\frac{\gamma^*(r_D^{-2} \cdot \omega_\alpha)(\partial_t, \bar{\partial}_t)}{\|e_1\|^2_{h_g^\alpha}} = \frac{\gamma^*(\omega_\alpha)(\partial_t, \bar{\partial}_t)}{\gamma^*(r_D^2) \cdot \|e_1\|^2_{h_g^\alpha}} = \frac{\gamma^*\omega_\alpha(\partial_t, \bar{\partial}_t)}{\gamma^*\tau_1^*(r_D^2 \cdot h_g^\alpha)(\partial_t, \bar{\partial}_t)},$$

where $\tau_1^*(r_D^2 \cdot h_g^\alpha)$ is the Finsler metric on $\mathscr{T}_Y(-\log D)$ defined by (2.4.1). By Proposition 2.2.(iii), ω_α is a positively definite Hermitian metric on $\mathscr{T}_Y(-\log D)$. Since Y is compact, there exists a *uniform constant $c > 0$* such that

$$\omega_\alpha \geqslant c\tau_1^*(r_D^2 \cdot h_g^\alpha).$$

We thus obtained the desired inequality (2.5.9). □

In summary, we have the following curvature estimate for the Finsler metrics $F_1, \ldots, F_n$ defined in (2.5.2), which is similar as [Sch17b, Lemma 9] for the Weil-Petersson metric.

Proposition 2.16. *For any $\gamma : \mathbb{D} \to V$ such that $\gamma(\mathbb{D}) \cap V_1 \neq \varnothing$. Assume that $G_k \not\equiv 0$ for $k = 1, \ldots, q$, and $G_{q+1} \equiv 0$ (thus $G_j \equiv 0$ for all $j > q+1$). Then $q \geqslant 1$, and over $C := \gamma^{-1}(V_1)$, which is a complement of a discrete set in $\mathbb{D}$, one has*

$$-\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}}/G_1 \leqslant -c + (\frac{G_2}{G_1})^2 \tag{2.5.10}$$

$$-\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}}/G_k \leqslant \frac{1}{k}\Big(- (\frac{G_k}{G_{k-1}})^{k-1} + (\frac{G_{k+1}}{G_k})^{k+1}\Big) \quad \forall 1 < k \leqslant q. \tag{2.5.11}$$

Here the constant $c > 0$ does not depend on the choice of γ.

2.6. **Construction of the Finsler metric.** By Proposition 2.16, we observe that none of the Finsler metrics $F_1, \ldots, F_n$ defined in (2.5.2) is negatively curved. Following the similar strategies in [TY15, Sch17b, BPW17], we construct a new Finsler metric F (see (2.6.6) below) by defining a convex sum of all $F_1, \ldots, F_n$, to cancel the positive terms in (2.5.10) and (2.5.11) by negative terms in the next stage. By Remark 2.14, we observe that the highest last order term is always semi-negative. We mainly follow the computations in [Sch17b], and try to make this subsection as self-contained as possible. Let us first recall the following basic inequalities by Schumacher.

Lemma 2.17 ([Sch12, Lemma 8])**.** *Let V be a complex manifold, and let $G_1, \ldots, G_n$ be non-negative $\mathscr{C}^2$ functions on V. Then*

$$\sqrt{-1}\partial\bar{\partial}\log(\sum_{i=1}^n G_i) \geqslant \frac{\sum_{j=1}^n G_j\sqrt{-1}\partial\bar{\partial}\log G_j}{\sum_{i=1}^n G_i} \tag{2.6.1}$$

Lemma 2.18 ([Sch17b, Lemma 17]). *Let $\alpha_j > 0$ for $j = 1, \ldots, n$. Then for all $x_j \geqslant 0$*

$$\sum_{j=2}^{n}(\alpha_j x_j^{j+1} - \alpha_{j-1}x_j^{j})x_{j-1}^2 \cdot \ldots \cdot x_1^2$$

$$\geqslant \frac{1}{2}\left(- \frac{\alpha_1^3}{\alpha_2^2}x_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}x_n^2 \cdot \ldots \cdot x_1^2 + \sum_{j=2}^{n-1}\left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\right)x_j^2 \cdot \ldots \cdot x_1^2\right) \tag{2.6.2}$$

Set $x_j = \frac{G_j}{G_{j-1}}$ for $j = 2, \ldots, n$ and $x_1 := G_1$ where $G_j \geqslant 0$ for $j = 1, \ldots, n$. Put them into (2.6.2) and we obtain

$$\sum_{j=2}^{n}\left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^{j}}{G_{j-1}^{j-2}}\right)$$

$$\geqslant \frac{1}{2}\left(- \frac{\alpha_1^3}{\alpha_2^2}G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2 + \sum_{j=2}^{n-1}\left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\right)G_j^2\right) \tag{2.6.3}$$

The following technical lemma is crucial in constructing our negatively curved Finsler metric F.

Lemma 2.19 ([Sch17b, Lemma 10]). *Let $F_1, \ldots, F_n$ be Finsler metrics on a complex space X, with the holomorphic sectional curvatures denoted by $K_1, \ldots, K_n$. Then for the Finsler metric $F := (F_1^2 + \ldots + F_n^2)^{1/2}$, its holomorphic sectional curvature*

$$K_F \leqslant \frac{\sum_{j=1}^{n} K_j F_j^4}{F^4}. \tag{2.6.4}$$

Proof. For any holomorphic map $\gamma : \mathbb{D} \to X$, we denote by $G_1, \ldots, G_n$ the semi-positive functions on $\mathbb{D}$ such that

$$\gamma^* F_i^2 = \sqrt{-1}G_i dt \wedge d\bar{t}$$

for $i = 1, \ldots, n$. Then

$$\gamma^* F^2 = \sqrt{-1}(\sum_{i=1}^{n} G_i)dt \wedge d\bar{t},$$

and it follows from (2.4.3) that the Gaussian curvature of $\gamma^* F^2$

$$\begin{aligned} K_{\gamma^* F^2} &= -\frac{1}{\sum_{i=1}^{n} G_i}\frac{\partial^2 \log(\sum_{i=1}^{n} G_i)}{\partial t \partial \bar{t}} \\ &\overset{(2.6.1)}{\leqslant} -\frac{1}{(\sum_{i=1}^{n} G_i)^2}\sum_{j=1}^{n} G_j \frac{\partial^2 \log G_j}{\partial t \partial \bar{t}} \\ &\leqslant \frac{\sum_{j=1}^{n} K_j G_j^2}{(\sum_{i=1}^{n} G_i)^2}. \end{aligned}$$

The lemma follows from Definition 2.9.(i). □

For any $\gamma : \mathbb{D} \to V$ with $C := \gamma^{-1}(V_1) \neq \varnothing$, we define a Hermitian pseudo-metric $\sigma := \sqrt{-1}H(t)dt \wedge d\bar{t}$ on $\mathbb{D}$ by taking convex sum in the following form

$$H(t) := \sum_{k=1}^{n} k\alpha_k G_k(t),$$

where G_k is defined in (2.5.3), and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ are some *universal constants* which will be fixed later. Following the similar estimate in [Sch17b, Proposition 11], one can choose those constants properly such that the Gaussian curvature K_σ of σ is uniformly bounded.

Proposition 2.20. *There exists* universal *constants* $0 < \alpha_1 \leqslant \ldots \leqslant \alpha_n$ *and* $K > 0$ *(independent of* $\gamma : \mathbb{D} \to V$*) such that the Gaussian curvature*

$$K_\sigma \leqslant -K.$$

on C.

Proof. It follows from (2.6.4) that

$$K_\sigma \leqslant \frac{1}{H^2}\sum_{j=1}^{n} j\alpha_j K_j G_j^2$$

and

$$K_j := -\frac{\partial^2 \log G_j}{\partial t \partial \bar{t}} / G_j.$$

By Proposition 2.16, one has

$$\begin{aligned}
K_\sigma &\leqslant \frac{\alpha_1 G_1^2}{H^2}\Big(-c + \Big(\frac{G_2}{G_1}\Big)^2\Big) + \frac{1}{H^2}\sum_{j=2}^{n}\alpha_j G_j^2\Big(-\Big(\frac{G_j}{G_{j-1}}\Big)^{j-1} + \Big(\frac{G_{j+1}}{G_j}\Big)^{j+1}\Big)\\
&\leqslant \frac{1}{H^2}\Big(-c\alpha_1 G_1^2 - \sum_{j=2}^{n}\Big(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^j}{G_{j-1}^{j-2}}\Big)\Big)\\
&\overset{(2.6.3)}{\leqslant} \frac{1}{H^2}\Big(\Big(-c + \frac{1}{2}\frac{\alpha_1^2}{\alpha_2^2}\Big)\alpha_1 G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1}\Big(\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} - \frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}\Big)G_j^2 - \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\Big)\\
&=: -\frac{1}{H^2}\sum_{j=1}^{n}\beta_j G_j^2
\end{aligned}$$

One can take $\alpha_1 = 1$, and choose the further $\alpha_j > \alpha_{j-1}$ inductively such that $\min_j \beta_j > 0$. Set $\beta_0 := \min_j \frac{\beta_j}{(j\alpha_j)^2}$. Then

$$\begin{aligned}
K_\sigma &\leqslant -\frac{1}{H^2}\beta_0 \sum_{j=1}^{n}(j\alpha_j G_j)^2\\
&\leqslant -\frac{\beta_0}{nH^2}\Big(\sum_{j=1}^{n} j\alpha_j G_j\Big)^2\\
&= -\frac{\beta_0}{n} =: -K.
\end{aligned}$$

Note that $\alpha_1, \ldots, \alpha_n$ and K is universal. The lemma is thus proved. □

It follows from Proposition 2.20 and (2.4.3) that one has the following estimate

$$\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} \geqslant K H(t) \geqslant 0 \tag{2.6.5}$$

over the Zariski dense open set $C \subseteq \mathbb{D}$, and in particular $\log H(t)$ is a subharmonic function over C. Since $H(t) \in [0, +\infty[$ is continuous (in particular locally bounded from above) over $\mathbb{D}$, $\log H(t)$ is a subharmonic function over $\mathbb{D}$, and the estimate (2.6.5) holds over the whole $\mathbb{D}$.

In summary, we construct a *negatively curved Finsler metric* F on $Y \setminus D$, defined by

$$F := \Big(\sum_{k=1}^{n} k\alpha_k F_k^2\Big)^{1/2}, \tag{2.6.6}$$

where F_k is defined in (2.5.2), such that $\gamma^*F^2 = \sqrt{-1}H(t)dt \wedge d\bar{t}$ for any $\gamma : \mathbb{D} \to V$. Since we assume that τ_1 is injective over V_0, the Finsler metric F_1 is positively definite on V_0, and *a fortiori* F. Therefore, we finish the proof of Theorem 2.12.

2.7. **Existence of Viehweg-Zuo Higgs bundles.** For the smooth family $U \to V$ in Theorem A, it was shown in [PTW18, Proposition 2.7] that there is a VZ Higgs bundle over some birational model $\tilde{V}$ of V. Indeed, using the deep theory of mixed Hodge modules, they can even construct VZ Higgs bundles over the bases of maximal variational smooth families whose geometric generic fiber admits a good minimal model. In this subsection we provide a construction of VZ Higgs bundles over the base space V (up to a birational model and a projective compactification) in Theorem A combining the methods in [PTW18] and [VZ02] without using the tools of mixed Hodge modules for completeness sake. In § 3, we show how to refine this construction to prove Theorem C.

Theorem 2.21 (Popa-Taji-Wu)**.** *Let $U \to V$ be the smooth family in Theorem A. Then after replacing V by a birational model $\tilde{V}$, there is a smooth compactification $Y \supset \tilde{V}$ and a VZ Higgs bundle over $\tilde{V}$.*

Proof. By Theorem 1.25, one can take a birational morphism $\nu : \tilde{V} \to V$ and a smooth compactification $f : X \to Y$ of $U^r \times_V \tilde{V} \to \tilde{V}$ so that there exists a hypersurface

$$H \in |\ell\Omega^n_{X/Y}(\log \Delta) - \ell f^*\mathscr{L} + \ell E|, \quad n := \dim X - \dim Y \tag{2.7.1}$$

with $\mathscr{L}$ a big and nef line bundle over Y satisfying that

(1) the complement $D := Y \setminus \tilde{V}$ is simple normal crossing.
(2) The hypersurface H is smooth over some Zariski open set $V_0 \subset \tilde{V}$ with $D + S := Y \setminus V_0$ simple normal crossing.
(3) The divisor E is effective and f-exceptional divisor with $f(E) \cap V_0 = \varnothing$.
(4) The augmented base locus $\mathbf{B}_+(\mathscr{L}) \cap V_0 = \varnothing$.

Here we denote by $\Delta := f^{-1}(D)$ so that $(X, \Delta) \to (Y, D)$ is a log morphism. Within this basic setup, let us first introduce two Higgs bundles in the theorem following [VZ02, §4]. Leaving out a codimension two subvariety of Y supported on $D + S$, we assume that

- the morphism f is flat, and E in (2.7.1) disappears.
- The divisor $D+S$ is smooth. Moreover, both Δ and $\Sigma = f^{-1}S$ are relative normal crossing.

Set $\mathcal{L} := \Omega^n_{X/Y}(\log \Delta)$. Let $\delta : W \to X$ be a blow-up of X with centers in $\Delta + \Sigma$ such that $\delta^*(H + \Delta + \Sigma)$ is a normal crossing divisor. One thus obtains a cyclic covering of δ^*H, by taking the ℓ-th root out of δ^*H. Let Z to be a strong desingularization of this covering, which is smooth over V_0 by (2). We denote the compositions by $h : W \to Y$ and $g : Z \to Y$, whose restrictions to V_0 are both smooth. Write $\Pi := g^{-1}(S \cup D)$ which can be assumed to be normal crossing. Leaving out codimension two subvariety supported $D + S$ further, we assume that h and g are also flat, and both $\delta^*(H + \Delta + \Sigma)$ and Π are relative normal crossing. Set

$$F^{n-q,q} := R^q h_*\Big(\delta^*\big(\Omega^{n-q}_{X/Y}(\log \Delta)\big) \otimes \delta^*\mathcal{L}^{-1} \otimes \mathscr{O}_W\big(\lfloor \frac{\delta^*H}{\ell} \rfloor\big)\Big)/\text{torsion}.$$

It was shown in [VZ02, §4] that there exists a natural edge morphism

$$\tau_{n-q,q} : F^{n-q,q} \to F^{n-q-1,q+1} \otimes \Omega_Y(\log D), \tag{2.7.2}$$

which gives rise to the first Higgs bundle $\big(\bigoplus_{q=0}^n F^{n-q,q}, \bigoplus_{q=0}^n \tau_{n-q,q}\big)$ defined over a big open set of Y containing V_0.

Write $Z_0 := Z \setminus \Pi$. Then the local system $R^n g_* \mathbb{C}_{\restriction Z_0}$ extends to a locally free sheaf $\mathcal{V}$ on Y (here Y is projective rather than the big open set!) equipped with the logarithmic connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_Y\big(\log(D + S)\big),$$

whose eigenvalues of the residues lie in $[0,1)$ (the so-called *lower canonical extension*). By [Sch73, CKS86, Kol86], the Hodge filtration of $R^n g_* \mathbb{C}_{\restriction Z_0}$ extends to a filtration $\mathcal{V} := \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n$ of *subbundles* so that their graded sheaves $E^{n-q,q} := \mathcal{F}^{n-q}/\mathcal{F}^{n-q+1}$ are also locally free, and there exists

$$\theta_{n-q,q} : E^{n-q,q} \to E^{n-q-1,q+1} \otimes \Omega_Y(\log D + S).$$

This defines the second Higgs bundle $\big(\bigoplus_{q=0}^n E^{n-q,q}, \theta_{n-q,q}\big)$. As observed in [VZ02, VZ03], $E^{n-q,q} = R^q g_* \Omega^{n-q}_{Z/Y}(\log \Pi)$ over a big open set of Y by the theorem of Steenbrink [Ste77, Zuc84]. By the construction of the cyclic cover Z, this in turn implies the following commutative diagram over a big open set of Y:

$$\begin{tikzcd} \mathscr{L}^{-1} \otimes E^{n-q,q} \arrow[r, "\mathbb{1}\otimes\theta_{n-q,q}"] & \mathscr{L}^{-1} \otimes E^{n-q-1,q+1} \otimes \Omega_Y\big(\log(D+S)\big) \\ F^{n-q,q} \arrow[u, "\rho_{n-q,q}"] \arrow[r, "\tau_{n-q,q}"] & F^{n-q-1,q+1} \otimes \Omega_Y(\log D) \arrow[u, "\rho_{n-q-1,q+1}\otimes\iota"'] \end{tikzcd} \tag{2.7.3}$$

as shown in [VZ03, Lemma 6.2] (cf. also [VZ02, Lemma 4.4]).

Note that all the objects are defined on a big open set of Y except for $\big(\bigoplus_{q=0}^n E^{n-q,q}, \theta_{n-q,q}\big)$, which are defined on the whole Y. Following [VZ03, §6], for every $q = 0, \ldots, n$, we define $F^{n-q,q}$ to be the reflexive hull, and the morphisms $\tau_{n-q,q}$ and $\rho_{n-q,q}$ extend naturally.

To conclude that $\big(\bigoplus_{q=0}^n \mathscr{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n \mathbb{1} \otimes \theta_{n-q,q}\big)$ is a VZ Higgs bundle as in Definition 2.1, we have to introduce a sub-Higgs sheaf with log poles supported on D. Write $\tilde{\theta}_{n-q,q} := \mathbb{1} \otimes \theta_{n-q,q}$ for short. Following [VZ02, Corollary 4.5] (cf. also [PTW18]), for each $q = 0, \ldots, n$, we define a coherent torsion-free sheaf $\mathscr{F}_q := \rho_{n-q,q}(F^{n-q,q}) \subset E^{n-q,q}$. By $F^{n,0} \supset \mathscr{O}_Y$, $\mathscr{F}_0 \supset \mathscr{O}_Y$. By (2.7.2) and (2.7.3), one has

$$\tilde{\theta}_{n-q,q} : \mathscr{F}_q \to \mathscr{F}_{q+1} \otimes \Omega_Y(\log D),$$

and let us by η_q the restriction of $\tilde{\theta}_{n-q,q}$ to $\mathscr{F}_q$. Then $(\mathscr{F}, \eta) := \big(\bigoplus_{q=0}^n \mathscr{F}_q, \bigoplus_{q=0}^n \eta_q\big)$ is a sub-Higgs bundle of $(\tilde{\mathscr{E}}, \tilde{\theta}) := \big(\bigoplus_{q=0}^n \mathscr{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n \tilde{\theta}_{n-q,q}\big)$. □

Remark 2.22. The methods we presented in the above proof were originally established in [VZ02] for the construction of Viehweg-Zuo sheaf.

2.8. **Proofs of Theorems A and B.**

Proof of Theorem A. By Theorem 2.21, there is a VZ Higgs bundle over some birational model $\tilde{V}$ of V. By Theorem F and Theorem 2.12, we can associate this VZ Higgs bundle with a negatively curved Finsler metric which is positively definite over some Zariski dense open set of $\tilde{V}$. The theorem follows directly from the bimeromorphic criteria for pseudo Kobayashi hyperbolicity in Lemma 2.10. □

A standard inductive arguments in [VZ03,PTW18] can easily show that Theorem A implies Theorem B.

Proof of Theorem B. We will proceed by contradiction. Suppose that there exists a non-constant holomorphic map $\gamma : \mathbb{C} \to V$. By Theorem A, γ cannot be Zariski dense. Let $Z := \overline{\gamma(\mathbb{C})}^{\text{Zar}}$ be its Zariski closure, which is an irreducible quasi-projective variety. Take a desingularization $\pi : Z' \to Z$, and the entire curve γ can be lifted to a Zariski dense curve in Z', denoted by $\gamma' : \mathbb{C} \to Z'$. Note that the moduli map $\varphi_{W'} : Z' \to P_h$ associated with $(W' := U \times_V Z' \to Z', \mathscr{H}) \in \mathscr{P}_h(Z')$ is the composition of the morphism $Z' \to V$ and the quasi-finite moduli map $\varphi_U : V \to P_h$ associated with $(f : U \to V, \mathscr{H}) \in \mathscr{P}_h(V)$. Therefore, the morphism $\varphi_{W'}$ is generically finite, which implies that the smooth family $U \times_V Z' \to Z'$

is of maximal variation. By Theorem A again, Z' must be algebraically degenerate. This is a contradiction. □

3. Kobayashi hyperbolicity of the moduli spaces

In this section, for effectively parametrized smooth family of minimal projective manifolds of general type, we refine the Viehweg-Zuo Higgs bundles in Theorem 2.21 so that we can apply Theorem 2.12 and the bimeromorphic criteria for Kobayashi hyperbolicity in Lemma 2.11 to prove Theorem C.

Theorem 3.1. *Let $U \to V$ be an effectively parametrized smooth family of minimal projective manifolds of general type over the quasi-projective manifold V. Then for any given point $y \in V$, there exists a smooth projective compactification Y for a birational model $\nu : \tilde{V} \to V$, and a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta}) \supset (\mathscr{F}, \eta)$ over Y satisfying the following properties:*

(i) *there is a Zariski open set V_0 of V containing y so that $\nu : \nu^{-1}(V_0) \to V_0$ is an isomorphism.*
(ii) *Both $D := Y \setminus \tilde{V}$ and $D + S := Y \setminus \nu^{-1}(V_0)$ are simple normal crossing divisors in Y.*
(iii) *The Higgs bundle $(\tilde{E}, \tilde{\theta})$ has log poles supported on $D \cup S$, that is, $\tilde{\theta} : \tilde{E} \to \tilde{E} \otimes \big(\log(D+S)\big)$.*
(iv) *The morphism*

$$\tau_1 : \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1} \otimes E^{n-1,1} \tag{3.0.1}$$

induced by the sub-Higgs sheaf $(\mathscr{F}, \eta)$ is injective over V_0.

Proof. The proof is a continuation of that of Theorem 2.21, and we will adopt the same notations.

We first prove that for any $y \in V$, the set of $z \in V$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite. Take a polarization $\mathscr{H}$ for $U \to V$ with the Hilbert polynomial h. Denote by $\mathscr{P}_h(V)$ the set of such pairs $(U \to V, \mathscr{H})$, up to isomorphisms and up to fiberwise numerical equivalence for $\mathscr{H}$. By [Vie95, Section 7.6], there exists a coarse quasi-projective moduli scheme P_h for $\mathscr{P}_h$, and thus the family induces a morphism $V \to P_h$. By the assumption that the family $U \to V$ is effectively parametrized, the induced morphism $V \to P_h$ is quasi-finite, which in turn shows that the set of $z \in V$ with X_z isomorphic to X_y is finite. Note that a projective manifold of general type has finitely many minimal models. Hence the set of $z \in V'$ with $X_z \overset{\text{bir}}{\sim} X_y$ is finite as well.

Now we will choose the hypersurface in (2.7.1) carefully so that the cyclic cover construction in Theorem 2.21 can provide the desired refined VZ Higgs bundle. Let $Y' \supset V$ be the smooth compactification in Corollary A.2. By Theorem 1.24, for any given point $y \in V$ and any sufficiently ample line bundle $\mathscr{A}$ on Y', there exists a birational morphism $\nu : Y \to Y'$ and a new algebraic fiber space $f : X \to Y$ so that one can find a hypersurface

$$H \in |\ell\Omega^n_{X/Y}(\log\Delta) - \ell(\nu \circ f)^*\mathscr{A} + \ell E|, \quad n := \dim X - \dim Y \tag{3.0.2}$$

satisfying that

- the inverse image $D := \nu^{-1}(Y' \setminus V)$ is a simple normal crossing divisor.
- There exists a reduced divisor S so that $D + S$ is simple normal crossing, and $H \to Y$ is smooth over $V_0 := Y \setminus (D \cup S)$.
- The restriction $\nu : \nu^{-1}(V_0) \to V_0$ is an isomorphism.
- The given point y is contained in V_0.
- The divisor E is effective and f-exceptional with $f(E) \subset \operatorname{Supp}(D + S)$.
- For any $z \in V := \nu^{-1}(V')$, the canonical bundle of the fiber $X_z := f^{-1}(z)$ is big and nef.
- The restricted family $f^{-1}(V_0) \to V_0$ is smooth and effectively parametrized.

Here we set $\Delta := f^*D$ and $\Sigma := f^*S$. Write $\mathscr{L} := \nu^*\mathscr{A}$. Now we take the cyclic cover with respect to H in (3.0.2) instead of that in (2.7.1), and perform the same construction of VZ

Higgs $(\tilde{\mathscr{E}},\tilde{\theta}) \supset (\mathscr{F},\tau)$ bundle as in Theorem 2.21. Theorems 3.1.(i) to 3.1.(iii) can be seen directly from the properties of H and the cyclic construction.

Theorem 3.1.(iv) has already appeared in [PTW18, Proposition 2.11] implicitly, and we give a proof here for the sake of completeness. Recall that both Z and H are smooth over V_0. Denote by $H_0 := H \cap f^{-1}(V_0)$, $f_0 : X_0 = f^{-1}(V_0) \to V_0$, and $g_0 : Z_0 = g^{-1}(V_0) \to V_0$. We have

$$\begin{aligned} F^{n,0}_{\upharpoonright V_0} &= f_*\big(\Omega^n_{X/Y}(\log\Delta)\otimes\mathcal{L}^{-1}\big)_{\upharpoonright V_0} = \mathscr{O}_{V_0}\\ E^{n-1,1}_{\upharpoonright V_0} &= R^1(g_0)_*(\Omega^{n-1}_{Z_0/V_0}) = R^1(f_0)_*\big(\Omega^{n-1}_{X_0/V_0}\oplus\bigoplus_{i=1}^{\ell-1}\Omega^{n-1}_{X_0/V_0}(\log H_0)\otimes(K_{X_0/V_0}\otimes f_0^*\mathscr{L}^{-1})^{-i}\big)\\ F^{n-1,1}_{\upharpoonright V_0} &= R^1f_*\big(\Omega^{n-1}_{X/Y}(\log\Delta)\otimes\mathcal{L}^{-1}\big)_{\upharpoonright V_0} = R^1(f_0)_*\big(\Omega^{n-1}_{X_0/V_0}\otimes K^{-1}_{X_0/V_0}\big)\simeq R^1(f_0)_*(\mathscr{T}_{X_0/V_0}). \end{aligned} \tag{3.0.3}$$

Hence $\tau_{1\upharpoonright V_0}$ factors through

$$\begin{aligned}\tau_{1\upharpoonright V_0}:\mathscr{T}_{V_0}\xrightarrow{\rho} R^1(f_0)_*(\mathscr{T}_{X_0/V_0}) \xrightarrow{\simeq} R^1(f_0)_*\big(\Omega^{n-1}_{X_0/V_0}\otimes K^{-1}_{X_0/V_0}\big)\to\\ R^1(f_0)_*\big(\Omega^{n-1}_{X_0/V_0}(\log H_0)\otimes K^{-1}_{X_0/V_0}\big)\to R^1(g_0)_*(\Omega^{n-1}_{Z_0/V_0})\otimes\mathscr{L}^{-1},\end{aligned}$$

where ρ is the Kodaira-Spencer map. Although the intermediate objects in the above factorization might not be locally free, the induced $\mathbb{C}$-linear map by the sheaf morphism $\tau_{1\upharpoonright V_0}$ at the $z\in V_0$

$$\tau_{1,z}:\mathscr{T}_{Y,z}\to(\mathscr{L}^{-1}\otimes E^{n-1,1})_z$$

coincides with the following composition of $\mathbb{C}$-linear maps between finite dimensional complex vector spaces

$$\begin{aligned}\tau_{1,z}:\mathscr{T}_{Y,z}\xrightarrow{\rho_z} H^1(X_z,\mathscr{T}_{X_z})\xrightarrow{\simeq} H^1(X_z,\Omega^{n-1}_{X_z}\otimes K^{-1}_{X_z})\xrightarrow{j_z}\\ H^1\big(X_z,\Omega^{n-1}_{X_z}(\log H_z)\otimes K^{-1}_{X_z}\big)\to H^1\big(Z_z,\Omega^{n-1}_{Z_z}\big).\end{aligned}\tag{3.0.4}$$

To prove Theorem 3.1.(iv), it then suffices to prove that each linear map in (3.0.4) is injective for any $z\in V_0$.

By the effective parametrization assumption, ρ_z is injective. The map j_z in (3.0.4) is the same as the H^1-cohomology map of the short exact sequence

$$0\to K^{-1}_{X_z}\otimes\Omega^{n-1}_{X_z}\to K^{-1}_{X_z}\otimes\Omega^{n-1}_{X_z}(\log H_z)\to K^{-1}_{X_z\upharpoonright H_z}\otimes\Omega^{n-2}_{H_z}\to 0.$$

Observe that $K_{X_z\upharpoonright H_z}$ is big. Indeed, this follows from that

$$\mathrm{vol}(K_{X_z\upharpoonright H_z}) = c_1(K_{X_z\upharpoonright H_z})^{n-1} = c_1(K_{X_z})^{n-1}\cdot H_z = \ell c_1(K_{X_z})^n = \ell\,\mathrm{vol}(K_{X_z}) > 0.$$

Hence j_z injective by the Bogomolov-Sommese vanishing theorem

$$H^0\big(H_z, K^{-1}_{X_z\upharpoonright H_z}\otimes\Omega^{d-2}_{H_z}\big)=0,$$

as observed in [PTW18]. Since $\psi_z : Z_z\to X_z$ is the cyclic cover obtained by taking the ℓ-th roots out of the smooth hypersurface $H_z\in|\ell K_{X_z}|$, the morphism ψ is finite. It follows from the degeneration of the Leray spectral sequence that

$$H^1(Z_z,\Omega^{n-1}_{Z_z})\simeq H^1\big(X_z,(\psi_z)_*\Omega^{n-1}_{Z_z}\big) = H^1\big(X_z,\Omega^{n-1}_{X_z}\big)\oplus\bigoplus_{i=1}^{\ell-1}H^1\big(X_z,\Omega^{n-1}_{X_z}(\log H_z)\otimes K^{-i}_{X_z}\big).\tag{3.0.5}$$

The last map in (3.0.4) is therefore injective, for the cohomology group $H^1\big(X_z,\Omega^{n-1}_{X_z}(\log H_z)\otimes K^{-1}_{X_z}\big)$ is a direct summand of $H^1(Z_z,\Omega^{n-1}_{Z_z})$ by (3.0.5). As a consequence, the composition $\tau_{1,z}$ in (3.0.4) is injective at each point $z\in V_0$. Theorem 3.1.(iv) is thus proved. □

Remark 3.2. When the condition of *effective parametrization* in Theorem 3.1 is replaced by the quasi-finiteness of the morphism from the base to coarse moduli space $V \to P_h$ as in [VZ03, PTW18], all the statements in Theorem 3.1 hold true except Theorem 3.1.(iv). Indeed, it is easy to construct an example of smooth family $U \to V$ so that $V \to P_h$ is quasi-finite but the Kodaira-Spencer map is degenerate somewhere.

Pick a smooth family of projective manifolds $U \to V$ so that $V \to P_h$ is quasi-finite. Fix any smooth hypersurface $S \subset V$ which is sufficiently ample, so that we can take a cyclic cover of degree $\ell \geq 2$ along S to obtain V'. Then $\varphi : V' \to V$ is a finite covering ramified over S. Perform the base change to obtain another smooth family

$$f' : U' = U \times_V V' \to V'.$$

Hence $V' \to P_h$ is still quasi-finite. We will show that the Kodaira-Spencer map $\rho_{V'} : V' \to R^1 f'_*(\mathscr{T}_{U'/V'})$ degenerates at the ramified locus $\varphi^{-1}(S)$.

Pick any point $y' \in \varphi^{-1}(S)$, and set $y := \varphi(y')$. Then there exists non-zero $\xi \in \mathscr{T}_{V',y'}$ such that $\varphi_*(\xi) = 0$. As is well-known, the Kodaira-Spencer map is *invariant under base change* (see [Man05, Theorem I.34]). One thus has

$$\rho_{y'}(\xi) = \rho_y\big(\varphi_*(\xi)\big) = 0,$$

where ρ_y and $\rho_{y'}$ are the Kodaira-Spencer maps defined in (0.1.1) at $y \in V$ and $y' \in V'$.

Let us explain how Lemma 2.11 and Theorems 2.12 and 3.1 imply our main theorem.

Proof of Theorem C. We first take a smooth compactification $Y \supset V$ as in Corollary A.2,. By Theorem 3.1, for any given point $y \in V$, there exists a birational morphism $\nu : Y' \to Y$ which is isomorphic at y, so that $D := Y' \setminus \nu^{-1}(V)$ is a simple normal crossing divisor, and there exists a VZ Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ whose log pole $D + S$ avoids $y' := \nu^{-1}(y)$. Moreover, by Theorem 3.1.(iv), τ_1 is injective at y'. Applying Theorem 2.12, we can associate $(\tilde{\mathscr{E}}, \tilde{\theta})$ a Finsler metric F on $\mathscr{T}_{Y'}(-D)$ which is positively definite at y'. Moreover, if we think of F as a Finsler metric on $\nu^{-1}(V)$, it is negatively curved in the sense of Definition 2.9.(ii). Hence the base V satisfies the conditions in Lemma 2.11, and we conclude that V is Kobayashi hyperbolic. □

Appendix A. $\mathbb{Q}$-mild reductions (by Dan Abramovich)

Let us work over complex number field $\mathbb{C}$.

The main result in this appendix is the following:

Theorem A.1. *Let $f_0 : S_0 \to T_0$ be a projective family of smooth varieties with T_0 quasi-projective.*

(i) *There are compactifications $S_0 \subset \mathcal{S}$ and $T_0 \subset \mathcal{T}$, with $\mathcal{S}$ and $\mathcal{T}$ Deligne-Mumford stacks with projective coarse moduli spaces, and a projective morphism $f : \mathcal{S} \to \mathcal{T}$ extending f_0 which is a Kollár family of slc varieties.*
(ii) *Given a finite subset $Z \subset T_0$ there is a projective variety W and finite surjective lci morphism $\rho : W \to \mathcal{T}$, unramified over Z, such that $\rho^{-1}\mathcal{T}^{sm} = W^{sm}$.*

Here the notion of Kollár family refers to the condition that the sheaf $\omega^{[m]}_{\mathcal{S}/\mathcal{T}}$ is flat and its formation commutes with arbitrary base change for each m. We refer the readers to [AH11, Definition 5.2.1] for further details.

Note that the pullback family $\mathcal{S} \times_{\mathcal{T}} W \to W$ is a Kollár family of slc varieties compactifying the pullback $S_0 \times_{T_0} W_0 \to W_0$ of the original family to $W_0 := W \times_{\mathcal{T}} T_0$.

This is applied in the present paper, where some mild regularity assumption on T_0 and W is required:

Corollary A.2 ($\mathbb{Q}$-mild reduction). *Assume further T_0 is smooth. For any given finite subset $Z \subset T_0$, there exist*

(i) *a compactification* $T_0 \subset \underline{\mathcal{T}}$ *with* $\underline{\mathcal{T}}$ *a regular projective scheme,*
(ii) *a simple normal crossings divisor* $D \subset \underline{\mathcal{T}}$ *containing* $\underline{\mathcal{T}} \setminus T_0$ *and disjoint from* Z*,*
(iii) *a finite morphism* $W \to \underline{\mathcal{T}}$ *unramified outside* D*, and*
(iv) *A Kollár family* $S_W \to W$ *of slc varieties extending the given family* $S_0 \times_{\underline{\mathcal{T}}} W$*.*

The significance of these extended families is through their $\mathbb{Q}$-mildness property. Recall from [AK00] that a family $S \to T$ is $\mathbb{Q}$*-mild* if whenever $T_1 \to T$ is a dominant morphism with T_1 having at most Gorenstein canonical singularities, then the total space $S_1 = T_1 \times_S T$ has canonical singularities. It was shown by Kollár–Shepherd-Barron [KSB88, Theorem 5.1] and Karu [Kar00, Theorem 2.5] that Kollár families of slc varieties whose generic fiber has at most Gorenstein canonical singularities are $\mathbb{Q}$-mild.

The main result is proved using moduli of Alexeev stable maps.

Let V be a projective variety. A morphism $\phi : U \to V$ is a *stable map* if U is slc and K_U is ϕ-ample. More generally, given $\pi : U \to T$, a morphism $\phi : U \to V$ is a *stable map over* T or a *family of stable maps parametrized by* T if π is a Kollár family of slc varieties and $K_{U/T}$ is $\phi \times \pi$-ample. Note that this condition is very flexible and does not require the fibers to be of general type, although key applications in Theorems 1.24 and 1.21.(iii) require some positivity of the fibers.

Theorem A.3 ([DR18, Theorem 1.5])**.** *Stable maps form an algebraic stack* $M(V)$ *locally of finite type over* $\mathbb{C}$*, each of whose connected components is a proper global quotient stack with projective coarse moduli space.*

The existence of an algebraic stack satisfying the valuative criterion for properness was known to Alexeev, and can also be deduced directly from the results of [AH11], which presents it as a global quotient stack. The work [DR18] shows that the stack has bounded, hence proper components, admitting projective course moduli spaces. An algebraic approach for these statements is provided in [Kar00, Corollary 1.2].

Proof of Theorem A.1. (i) Let $T_0 \subset T$ and $S_0 \subset S$ be projective compactifications with $\pi : S \to T$ extending f_0. The family $S_0 \to T_0$ with the injective morphism $\phi : S_0 \to S$ is a family of stable maps into S, providing a morphism $T_0 \to M(S)$ which is in fact injective. Let $\mathcal{T}$ be the closure of T_0. Since $M(S)$ is proper, $\mathcal{T}$ is proper. Let $\mathcal{S}$ be the pullback of the universal family along $\mathcal{T} \to M(S/T)$. Then $\mathcal{S} \supset S_0$ is a compactification as needed.
(ii) The existence of W follows from the main result of [KV04]. □

Proof of Corollary A.2. Consider the coarse moduli space $\underline{\mathcal{T}}$ of the stack $\mathcal{T}$ provided by the first part of the main result. This might be singular, but by Hironaka's theorem we may replace it by a resolution of singularities such that $D_\infty := \underline{\mathcal{T}} \setminus T_0$ is a simple normal crossings divisor. Thus condition (i) is satisfied.

For each component $D_i \subset D_\infty$ denote by m_i the ramification index of $\mathcal{T} \to \underline{\mathcal{T}}$. In particular any covering $W \to \underline{\mathcal{T}}$ whose ramification indices over D_i are divisible by m_i lifts along the generic point of D_i to $\mathcal{T}$.

Choosing a Kawamata covering package [AK00] disjoint from Z we obtain a simple normal crossings divisor D as required by (ii), and finite covering $W \to \underline{\mathcal{T}}$ as required by (iii), such that $W \to \underline{\mathcal{T}}$ factors through $\mathcal{T}$ at every generic point of D_i.

By the Purity Lemma [AV02, Lemma 2.4.1] the morphism $W \to \mathcal{T}$ extends over all of W, hence we obtain a family $S_W \to W$ as required by (iv). □

References

[AH11] Dan Abramovich and Brendan Hassett, *Stable varieties with a twist*, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 1–38. ↑ 45, 46

[AK00] Dan Abramovich and Kalle Karu, *Weak semistable reduction in characteristic 0*, Invent. Math. **139** (2000), no. 2, 241–273, https://doi.org/10.1007/s002229900024. ↑ 5, 46

[Ara04] Donu Arapura, *Frobenius amplitude and strong vanishing theorems for vector bundles*, Duke Math. J. **121** (2004), no. 2, 231–267, https://doi.org/10.1215/S0012-7094-04-12122-0, With an appendix by Dennis S. Keeler. ↑ 24

[AV02] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75, https://doi.org/10.1090/S0894-0347-01-00380-0. ↑ 46

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468, https://doi.org/10.1090/S0894-0347-09-00649-3. ↑ 6, 13, 17

[BD18] Sébastien Boucksom and Simone Diverio, *A note on Lang's conjecture for quotients of bounded domains*, arXiv e-prints (2018), arXiv:1809.02398. ↑ 3

[BD19] Damian Brotbek and Ya Deng, *Kobayashi hyperbolicity of the complements of general hypersurfaces of high degree*, Geometric and Functional Analysis (2019), https://doi.org/10.1007/s00039-019-00496-2. ↑ 3

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248, https://doi.org/10.1090/S1056-3911-2012-00574-8. ↑ 7, 14, 15

[BHPS13] Bhargav Bhatt, Wei Ho, Zsolt Patakfalvi, and Christian Schnell, *Moduli of products of stable varieties*, Compos. Math. **149** (2013), no. 12, 2036–2070, https://doi.org/10.1112/S0010437X13007288. ↑ 23

[Bou04] Sébastien Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76, https://doi.org/10.1016/j.ansens.2003.04.002. ↑ 7, 15, 16

[BP08] Bo Berndtsson and Mihai Păun, *Bergman kernels and the pseudoeffectivity of relative canonical bundles*, Duke Math. J. **145** (2008), no. 2, 341–378, https://doi.org/10.1215/00127094-2008-054. ↑ 9, 11, 17

[BP10] Bo Berndtsson and Mihai Păun, *Bergman kernels and subadjunction*, ArXiv e-prints (2010). ↑ 17, 18

[BPW17] Bo Berndtsson, Mihai Paun, and Xu Wang, *Algebraic fiber spaces and curvature of higher direct images*, arXiv e-prints (2017), arXiv:1704.02279. ↑ 3, 4, 38

[Bro17] Damian Brotbek, *On the hyperbolicity of general hypersurfaces*, Publ. Math. Inst. Hautes Études Sci. **126** (2017), 1–34, https://doi.org/10.1007/s10240-017-0090-3. ↑ 3

[Cao16] Junyan Cao, *Albanese maps of projective manifolds with nef anticanonical bundles*, arXiv e-prints (2016), arXiv:1612.05921. ↑ 14, 16, 17

[Cao18] ———, *Kodaira dimension of algebraic fiber spaces over surfaces*, Algebr. Geom. **5** (2018), no. 6, 728–741, https://doi:10.14231/AG-2018-020. ↑ 14

[CDG19] Benoît Cadorel, Simone Diverio, and Henri Guenancia, *On subvarieties of singular quotients of bounded domains*, arXiv e-prints (2019), arXiv:1905.04212. ↑ 3

[CDM17] Junyan Cao, Jean-Pierre Demailly, and Shin-ichi Matsumura, *A general extension theorem for cohomology classes on non reduced analytic subspaces*, Sci. China Math. **60** (2017), no. 6, 949–962, https://doi.org/10.1007/s11425-017-9066-0. ↑ 18

[CH17] Junyan Cao and Andreas Höring, *Manifolds with nef anticanonical bundle*, J. Reine Angew. Math. **724** (2017), 203–244, https://doi.org/10.1515/crelle-2014-0073. ↑ 16

[CK17] Fabrizio Catanese and Yujiro Kawamata, *Fujita decomposition over higher dimensional base*, arXiv e-prints (2017), arXiv:1709.08065. ↑ 29, 31

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), no. 3, 457–535, https://doi.org/10.2307/1971333. ↑ 28, 29, 30, 42

[CP15a] Frédéric Campana and Mihai Păun, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*, arXiv:1508.02456 (2015). ↑ 2

[CP15b] ———, *Orbifold generic semi-positivity: an application to families of canonically polarized manifolds*, Ann. Inst. Fourier (Grenoble) **65** (2015), no. 2, 835–861, http://aif.cedram.org/item?id=AIF_2015__65_2_835_0. ↑ 2

[CP16] ———, *Positivity properties of the bundle of logarithmic tensors on compact Kähler manifolds*, Compos. Math. **152** (2016), no. 11, 2350–2370, https://doi.org/10.1112/S0010437X16007442. ↑ 2

[CP17] Junyan Cao and Mihai Păun, *Kodaira dimension of algebraic fiber spaces over abelian varieties*, Invent. Math. **207** (2017), no. 1, 345–387, https://doi.org/10.1007/s00222-016-0672-6. ↑ 7, 11, 14, 15, 18, 19

[CRT19] Benoît Cadorel, Erwan Rousseau, and Behrouz Taji, *Hyperbolicity of singular spaces*, J. Éc. polytech. Math. **6** (2019), 1–18, https://doi.org/10.5802/jep.85. ↑ 3

[Dem82] Jean-Pierre Demailly, *Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 457–511, http://www.numdam.org/item?id=ASENS_1982_4_15_3_457_0. ↑ 9, 10, 11

[Dem97] ———, *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 285–360. ↑ 4

[Dem12] ———, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012. ↑ 32

[Dem16] ———, *Extension of holomorphic functions defined on non reduced analytic subvarieties*, The legacy of Bernhard Riemann after one hundred and fifty years. Vol. I, Adv. Lect. Math. (ALM), vol. 35, Int. Press, Somerville, MA, 2016, pp. 191–222. ↑ 18

[Dem18] ———, *Recent results on the Kobayashi and Green-Griffiths-Lang conjectures*, arXiv:1801.04765 (2018). ↑ 3

[Den17] Ya Deng, *Applications of the Ohsawa-Takegoshi Extension Theorem to Direct Image Problems*, arXiv e-prints (2017), arXiv:1703.07279. ↑ 17, 22

[Den18a] ———, *Kobayashi hyperbolicity of moduli spaces of minimal projective manifolds of general type (with the appendix by Dan Abramovich)*, arXiv e-prints (2018), arXiv:1806.01666. ↑ 7

[Den18b] ———, *Pseudo Kobayashi hyperbolicity of base spaces of families of minimal projective manifolds with maximal variation*, arXiv e-prints (2018), arXiv:1809.05891. ↑ 7

[Den19] ———, *Hyperbolicity of bases of log Calabi-Yau families*, arXiv e-prints (2019), arXiv:1901.04423. ↑ 2

[DR18] Ruadhaí Dervan and Julius Ross, *Stable maps in higher dimensions*, Mathematische Annalen (2018), https://doi.org/10.1007/s00208-018-1706-8. ↑ 46

[ELM+09] Lawrence Ein, Robert Lazarsfeld, Mircea Mustată, Michael Nakamaye, and Mihnea Popa, *Restricted volumes and base loci of linear series*, Amer. J. Math. **131** (2009), no. 3, 607–651, https://doi.org/10.1353/ajm.0.0054. ↑ 9

[Fuj16] Osamu Fujino, *Direct images of relative pluricanonical bundles*, Algebr. Geom. **3** (2016), no. 1, 50–62, https://doi.org/10.14231/AG-2016-003. ↑ 12

[GT84] Phillip Griffiths and Loring Tu, *Curvature properties of the Hodge bundles*, Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), Ann. of Math. Stud., vol. 106, Princeton Univ. Press, Princeton, NJ, 1984, pp. 29–49. ↑ 32, 37

[Iwa18] Masataka Iwai, *A characterization of weakly positive vector bundles by using singular Hermitian metrics*, ArXiv e-prints (2018). ↑ 12

[Kar00] Kalle Karu, *Minimal models and boundedness of stable varieties*, J. Algebraic Geom. **9** (2000), no. 1, 93–109. ↑ 46

[Kaw85] Yujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46, https://doi.org/10.1515/crll.1985.363.1. ↑ 7, 17

[KK08a] Stefan Kebekus and Sándor J. Kovács, *Families of canonically polarized varieties over surfaces*, Invent. Math. **172** (2008), no. 3, 657–682, https://doi.org/10.1007/s00222-008-0128-8. ↑ 2

[KK08b] ———, *Families of varieties of general type over compact bases*, Adv. Math. **218** (2008), no. 3, 649–652, https://doi.org/10.1016/j.aim.2008.01.005. ↑ 2

[KK10] ———, *The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties*, Duke Math. J. **155** (2010), no. 1, 1–33, https://doi.org/10.1215/00127094-2010-049. ↑ 2

[Kob98] Shoshichi Kobayashi, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 318, Springer-Verlag, Berlin, 1998, https://doi.org/10.1007/978-3-662-03582-5. ↑ 33, 34

[Kol86] János Kollár, *Higher direct images of dualizing sheaves. II*, Ann. of Math. (2) **124** (1986), no. 1, 171–202, https://doi.org/10.2307/1971390. ↑ 42

[Kol87] ———, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 361–398. ↑ 6, 20

[Kol90] János Kollár, *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268, http://projecteuclid.org/euclid.jdg/1214445046. ↑ 12, 25

[Kol13] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, https://doi.org/10.1017/CBO9781139547895, With a collaboration of Sándor Kovács. ↑ 13

[KP17] Sándor J. Kovács and Zsolt Patakfalvi, *Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension*, J. Amer. Math. Soc. **30** (2017), no. 4, 959–1021, https://doi.org/10.1090/jams/871. ↑ 23

[KSB88] János Kollár and Nicholas I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338, https://doi.org/10.1007/BF01389370. ↑ 46

[KV04] Andrew Kresch and Angelo Vistoli, *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map*, Bull. London Math. Soc. **36** (2004), no. 2, 188–192, https://doi.org/10.1112/S0024609303002728. ↑ 46

[Lan91] Serge Lang, *Number theory. III*, Encyclopaedia of Mathematical Sciences, vol. 60, Springer-Verlag, Berlin, 1991, https://doi.org/10.1007/978-3-642-58227-1, Diophantine geometry. ↑ 2

[Leh13] Brian Lehmann, *Comparing numerical dimensions*, Algebra Number Theory **7** (2013), no. 5, 1065–1100, https://doi.org/10.2140/ant.2013.7.1065. ↑ 14

[Man05] Marco Manetti, *Lectures on deformations of complex manifolds*, arXiv preprint math/0507286 (2005). ↑ 45

[Mor87] Shigefumi Mori, *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331. ↑ 8

[Nak04] Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004. ↑ 9

[Pat12] Zsolt Patakfalvi, *Viehweg's hyperbolicity conjecture is true over compact bases*, Adv. Math. **229** (2012), no. 3, 1640–1642, https://doi.org/10.1016/j.aim.2011.12.013. ↑ 2

[Pet84] C. A. M. Peters, *A criterion for flatness of Hodge bundles over curves and geometric applications*, Math. Ann. **268** (1984), no. 1, 1–19, https://doi.org/10.1007/BF01463870. ↑ 29

[PS14] Mihnea Popa and Christian Schnell, *Kodaira dimension and zeros of holomorphic one-forms*, Ann. of Math. (2) **179** (2014), no. 3, 1109–1120, https://doi.org/10.4007/annals.2014.179.3.6. ↑ 12

[PS17] ———, *Viehweg's hyperbolicity conjecture for families with maximal variation*, Invent. Math. **208** (2017), no. 3, 677–713, https://doi.org/10.1007/s00222-016-0698-9. ↑ 2, 3, 5

[PT18] Mihai Păun and Shigeharu Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images*, J. Algebraic Geom. **27** (2018), no. 2, 211–272, https://doi.org/10.1090/jag/702. ↑ 8, 9, 11, 12, 16, 17

[PTW18] Mihnea Popa, Behrouz Taji, and Lei Wu, *Brody hyperbolicity of base spaces of certain families of varieties*, arXiv:1801.05898 (2018). ↑ 3, 4, 5, 6, 7, 25, 26, 27, 28, 29, 41, 42, 44, 45

[PX17] Zsolt Patakfalvi and Chenyang Xu, *Ampleness of the CM line bundle on the moduli space of canonically polarized varieties*, Algebr. Geom. **4** (2017), no. 1, 29–39, https://doi.org/10.14231/AG-2017-002. ↑ 25

[Rau15] Hossein Raufi, *Singular hermitian metrics on holomorphic vector bundles*, Ark. Mat. **53** (2015), no. 2, 359–382, https://doi.org/10.1007/s11512-015-0212-4. ↑ 9

[Rou16] Erwan Rousseau, *Hyperbolicity, automorphic forms and Siegel modular varieties*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 1, 249–255, https://doi.org/10.24033/asens.2281. ↑ 3

[Roy75] H. L. Royden, *Intrinsic metrics on Teichmüller space*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 2 (1975), 217–221. ↑ 3

[Sch73] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319, https://doi.org/10.1007/BF01389674. ↑ 28, 29, 30, 42

[Sch08] Georg Schumacher, *Positivity of relative canonical bundles for families of canonically polarized manifolds*, arXiv preprint arXiv:0808.3259 (2008). ↑ 3

[Sch10] ———, *Curvature of higher direct images and applications*, arXiv preprint arXiv:1002.4858 (2010). ↑ 3

[Sch12] ———, *Positivity of relative canonical bundles and applications*, Invent. Math. **190** (2012), no. 1, 1–56, https://doi.org/10.1007/s00222-012-0374-7. ↑ 3, 4, 38

[Sch17a] Christian Schnell, *On a theorem of Campana and Păun*, Épijournal de Géométrie Algébrique **Volume 1** (2017), https://epiga.episciences.org/3871. ↑ 2

[Sch17b] Georg Schumacher, *Moduli of canonically polarized manifolds, higher order Kodaira-Spencer maps, and an analogy to Calabi-Yau manifolds*, arXiv e-prints (2017), arXiv:1702.07628. ↑ 3, 4, 38, 39

[Siu86] Yum Tong Siu, *Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class*, Contributions to several complex variables, Aspects Math., E9, Friedr. Vieweg, Braunschweig, 1986, pp. 261–298. ↑ 3

[Siu15] Yum-Tong Siu, *Hyperbolicity of generic high-degree hypersurfaces in complex projective space*, Invent. Math. **202** (2015), no. 3, 1069–1166, http://dx.doi.org/10.1007/s00222-015-0584-x. ↑ 3

[Ste77] Joseph Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563. ↑ 42

[Tak16] Shigeharu Takayama, *Singularities of Narasimhan-Simha type metrics on direct images of relative pluricanonical bundles*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 2, 753–783, http://aif.cedram.org/item?id=AIF_2016__66_2_753_0. ↑ 12

[TY15] Wing-Keung To and Sai-Kee Yeung, *Finsler metrics and Kobayashi hyperbolicity of the moduli spaces of canonically polarized manifolds*, Ann. of Math. (2) **181** (2015), no. 2, 547–586, https://doi.org/10.4007/annals.2015.181.2.3. ↑ 3, 4, 38

[TY18] ______, *Augmented Weil-Petersson metrics on moduli spaces of polarized Ricci-flat Kähler manifolds and orbifolds*, Asian J. Math. **22** (2018), no. 4, 705–727, https://doi.org/10.4310/AJM.2018.v22.n4.a6. ↑ 3

[Vie83] Eckart Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 329–353. ↑ 8, 9

[Vie89] ______, *Weak positivity and the stability of certain Hilbert points*, Invent. Math. **96** (1989), no. 3, 639–667, https://doi.org/10.1007/BF01393700. ↑ 12

[Vie90] ______, *Weak positivity and the stability of certain Hilbert points. II*, Invent. Math. **101** (1990), no. 1, 191–223, https://doi.org/10.1007/BF01231501. ↑ 6, 7, 8, 12, 21, 24

[Vie95] ______, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995, https://doi.org/10.1007/978-3-642-79745-3. ↑ 2, 18, 19, 43

[VZ01] Eckart Viehweg and Kang Zuo, *On the isotriviality of families of projective manifolds over curves*, J. Algebraic Geom. **10** (2001), no. 4, 781–799. ↑ 2

[VZ02] ______, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328. ↑ 2, 5, 6, 7, 20, 28, 41, 42

[VZ03] ______, *On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds*, Duke Math. J. **118** (2003), no. 1, 103–150, https://doi.org/10.1215/S0012-7094-03-11815-3. ↑ 2, 3, 4, 5, 6, 7, 20, 23, 27, 28, 30, 42, 45

[Wol86] Scott A. Wolpert, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85** (1986), no. 1, 119–145, https://doi.org/10.1007/BF01388794. ↑ 3

[Yam19] Katsutoshi Yamanoi, *Pseudo Kobayashi hyperbolicity of subvarieties of general type on abelian varieties*, J. Math. Soc. Japan **71** (2019), no. 1, 259–298, https://doi.org/10.2969/jmsj/75817581. ↑ 3

[Zuc84] Steven Zucker, *Degeneration of Hodge bundles (after Steenbrink)*, Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), Ann. of Math. Stud., vol. 106, Princeton Univ. Press, Princeton, NJ, 1984, pp. 121–141. ↑ 42

[Zuo00] Kang Zuo, *On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications*, Asian J. Math. **4** (2000), no. 1, 279–301, https://doi.org/10.4310/AJM.2000.v4.n1.a17, Kodaira's issue. ↑ 3

Université de Strasbourg, Institut de Recherche Mathématique Avancée, 7 Rue René-Descartes, 67084 Strasbourg, France

Current address: Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, Sweden

E-mail address: yade@chalmers.se dengya.math@gmail.com

URL: https://sites.google.com/site/dengyamath

Department of Mathematics, Box 1917, Brown University, Providence, RI, 02912, U.S.A

E-mail address: abrmovic@math.brown.edu

URL: http://www.math.brown.edu/abrmovic/