A framework for classifying mathematical justification tasks

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The large corpus of research on mathematical reasoning and justification in the mathematics education literature has yielded a wide range of tasks that require a mathematical argument to be established. This paper presents the DIVINE framework that classifies justification tasks by their nature and purpose as well as the expected element to be provided in the justifications. The framework is then used as a theoretical basis for appraising justifications produced by mathematics teachers.

Keywords: Mathematical justification, classification framework, teacher competency.

Introduction

Mathematical reasoning plays a crucial role in mathematics learning at all grade levels. It is a useful tool for exploring, discovering and understanding new mathematical concepts, for applying mathematical ideas and procedures flexibly to other situations, and for reconstructing previous knowledge in order to generate new arguments (Ball & Bass, 2003). To probe into the mathematical reasoning of students, another tool is needed to make such reasoning visible – justification. With the emphasis in schools worldwide on developing a broad set of competencies that are believed to be an imperative for success in the workplaces in the 21st century, greater demands are therefore being placed on students to reason and justify in the learning of mathematics.

Mathematical reasoning and communication are two key process skills in the framework of the Singapore school mathematics curriculum (Ministry of Education (Singapore), 2012) that have been advocated for a long time. The notion of communication refers to the ability of using mathematical language to articulate mathematical ideas and arguments precisely, concisely and logically (Ministry of Education (Singapore), 2012). In this sense, mathematical justification is considered part of communication. But very little is known about the justification ability of Singapore mathematics teachers and students at the secondary level. I am thus interested to find out more about it and commenced the investigation with a survey of the various justification tasks that secondary school students had been tested in the national examinations over the past ten years. The survey has found that the justification tasks are of varied nature and can be classified into different categories.

This paper seeks to address the following questions: What are the different types of justification tasks given to secondary school students? How might justifications for the different types of tasks qualify as acceptable? What elements should be present in an acceptable justification? It presents a theoretical framework for classifying mathematical justification tasks and discusses the expectation required in each type of tasks. The structure of this paper broadly follows these strands of work: (a) a perspective of what justification encompasses, (b) a view of justification tasks and the elements expected in the justifications, and (c) a discussion of justifications produced by Singapore mathematics teachers.

Theoretical framework

Justification According to Simon and Blume (1996), mathematical justification involves "establishing validity [and] developing an argument that builds from the community's taken-as-shared knowledge" (p. 28). The notion of justification as a means of determining and explaining the truth of a mathematical conjecture or assertion resonates strongly with many other researchers. For instance, it is consistent with Balacheff's (1988) perception of justification as "the basis of the validation of the conjecture" (p. 225) – a view also supported by Huang (2005) as well. To Harel and Sowder (2007), justification for validation serves two different roles: to *ascertain* the truth of a conjecture, and to *persuade* others that the conjecture is true. Even these two roles have slightly dissimilar intention. In Ellis' (2007) view, ascertaining the truth is meant to remove one's own doubts whereas persuading is one's attempt to remove others' doubts. As the discussion reveals, expressing justification for the purpose of ascertaining truth is a cognitive process whilst convincing others of the truth is a social process.

The notion of justification focuses traditionally on the notion of proof from the primary to the high school and university levels in the research literature (see e.g., Jones, 2010; Stylianides, 2007). Thus proof is viewed as a type of justification in this regard. So I think the definitions of proof available in the literature can help to deepen our understanding of mathematical justification. A prime example that stands out is Stylianides' (2007) definition of proof as a mathematical argument made up of a connected sequence of assertions for or against a mathematical claim. This definition echoes Hanna's (1989) definition of proof as "an argument needed to validate a statement" (p. 20) and is considered by far the most comprehensive meaning of proof.

Mathematical justification encompasses a broad range of arguments besides proof. The types of arguments that students are expected to produce depend on at least two factors: *the cognitive abilities of students* and *the nature of the task*. For primary and secondary school students, particularly those in the lower secondary grades, a justification does not need to measure up to a formal proof. This is because providing a theoretical argument for a mathematical result is sometimes not required in the light of their cognitive level until they reach higher level of study (Hoyles & Healy, 1999). This is illustrated by the justification task on algebra asking lower secondary school students to explain why $2n - 1$ is an odd number for any positive integer n. This task presents a mathematical claim (i.e., $2n - 1$ is an odd number for any positive integer n) and requires the students to provide supporting evidence to show why the claim is true. In short, the nature of such a task is to *validate* the claim. Therefore a reasoned argument within the conceptual reach of the students of this grade level could take the form as follows: with n being any positive integer, forming two groups of n, which can be expressed as $2n$ in notation, thus generates an even number, therefore subtracting one from it will result in an odd number. This justification simply uses everyday language rather than formal mathematical language, and does not draw on any theorems as in a typical theoretical argument.

Clearly not all justification tasks require a theoretical argument. Some lend themselves well to experiential justification, which is mainly supported by specific examples and illustrations. Consider asking students to justify why the rule $a^m \times a^n = a^{m+n}$ is true for any positive integers *a*, *m* and *n*. The students can rely on intuitive reasoning using several concrete numerical examples in the justification. This mode of argument may be rejected as an adequate and valid justification of the rule because it does not cover all cases of the variables *a*, *m* and *n*. Although such an experiential justification does not involve any theorems and somewhat lacks mathematical sophistication, it does convey to some extent student understanding of why the mathematical claim is true, albeit a far less formal argument than a typical mathematical deductive proof (Becker & Rivera, 2009). But it is such justification that is valued because it "explains rather than simply convinces" (Lannin, 2005, p. 235).

Aside from presenting an explanation for or against a mathematical claim, a justification can also take the form of an *elaboration* of how a mathematical result is obtained, as pointed out by Becker and Rivera (2009). Consider, for instance, the topic of pattern generalisation. Becker and Rivera (2009) and Stylianides (2015) had asked students to justify how they established their general rules for figural patterns. The nature of this type of justification task expects the students to illuminate clearly the method used in rule construction. Like the *validation* task described previously, the justification for the *elaboration* task can also be articulated in two different modes: written as in paper-and-pencil tests and verbalised as in face-to-face interviews. Both modes were evident in Stylianides' (2015) study.

Justification tasks Different types of justification tasks are gleaned from the literature on mathematical reasoning, proof and argumentation. Justification tasks require individuals to make mathematical arguments, a process which is integral to mathematics learning in order for the individuals to make sense of the mathematical concepts and procedures, and learn mathematics with understanding. Additionally, these tasks provide insight into their thinking and reasoning as well. Justification tasks can be classified into what I call *elaboration*, *validation* and *making decision* tasks.

Elaboration justification tasks are very popular in the literature and have been widely used in research studies by many researchers, including Becker and Rivera (2009), Lannin (2005) and Stylianides (2015). Such tasks (for e.g., *Pizza Sharing* in Lannin (2005)) require individuals to elaborate the approach that was used to obtain a mathematical result. *Validation* justification tasks are questions that seek arguments to support or refute a mathematical claim. This kind of tasks (for e.g., *Mr. Right Triangle* in Chua (2016)) is used to gain insight into how individuals reason about a mathematical claim. *Making decision* justification tasks offer options for a mathematical situation and individuals have to exercise decision-making power to pick one of the options so as to answer the question. The geometry test item from the study by Küchemann and Hoyles (2006) is a case in point.

Apart from the three types of justification tasks discussed thus far, there is one more type which is seemingly less common in research studies but popular in the Singapore national examinations for secondary school students. Consider the algebra task in Figure 1 that requires individuals to make sense of the given context and then infer the significance of the positive solution of the quadratic equation from the context. Such a task exemplifies what I call an *inference* justification task. It is normally set in a real-world context and seeks an interpretation of a mathematical result.

A stone was thrown from the top of a vertical tower. Its position during the flight is represented by the equation $y = 50 + 21x - x^2$, where y metres is the height of the stone above the ground and x metres is its horizontal distance from the tower.

Explain what the positive solution of the equation $0 = 50 + 21x - x^2$ represents.

Figure 1: Inference task on algebra

In summary, this sub-section has highlighted four distinct types of justification tasks. All these tasks share a common objective, which is to elicit from someone a mathematical argument for a mathematical claim or result. As they vary in nature from one type to another, the essential elements to be expected in the argument for each type of task are therefore also not the same. In the next section, I introduce the *DIVINE* framework that classifies justification tasks by nature and purpose as well as the expected element to be provided in the justifications, and describe its usefulness. *DIVINE* is the acronym of the four types of justification tasks: making **D**ecision, **I**nference, **V**al**I**datio**N**, and **E**laboration.

The *DIVINE* framework

The conceptualisation and development of the *DIVINE* framework in Table 1 was informed by the literature on mathematical proof, reasoning and justification in the field of mathematics education, by analysis of justifications produced by students and mathematics teachers that I had encountered in the course of my teaching in recent years, and by my own disciplinary knowledge. It describes the *nature* and *purpose* of the justification tasks, and the expected *element* to be provided by individuals in their attempt to produce a correct justification.

Nature of justification tasks	Purpose of justification tasks	Expected element in the justification
Making Decision	Explain whether… Explain which…	a decision about the mathematical claim with evidence to support or refute the claim
Inference	Explain what…	the meaning of the mathematical result, with the key words in the task addressed
Validation	Explain why…	a reason or evidence to support or refute the mathematical claim
Elaboration	Explain how…	a clear description of the method or strategy used to obtain the mathematical result

Table 1: The *DIVINE* framework

The term *nature* can be described as the cognitive process that an individual undertakes when doing the justification task. The nature of the tasks places slightly different demands on thinking and reasoning. Making decision, inference, validation and elaboration are the four kinds of cognitive processes that have been identified in this paper. The *purpose* of a justification task refers to the reason for making the mathematical argument. Finally, the expected *element* is used to refer to the details that an individual is supposed to provide in order to give a correct justification.

It should be pointed out that although the expected element in a justification indicates what needs to be given for a particular type of justification task, the resulting justification may not necessarily be accepted as correct. For the justification to be judged as correct, I think it is imperative to also examine three other elements of a mathematical argument: the *mathematics* presented, the *clarity* in the argument and what Stylianides (2007) termed as the *modes of argumentation*. The mathematics presented refers to the mathematical concepts and procedures used in the justification, including the definitions and theorems that are used, the calculation that is shown and so on. The clarity in the argument means presenting the argument in a clear, easy-to-follow, and unambiguous way. The mode of argumentation concerns how a justification is developed. In other words, the form of the justification (such as a logical deduction, a proof by contradiction, exposition) has to be taken into consideration. A brief discussion of the potentiality of the *DIVINE* framework will now follow.

Usefulness of the framework Recognising whether a mathematical justification is correct is a vital task for teachers because they often have to evaluate the validity of students' justifications. But as Chua (2016) had noted, this task is fraught with difficulties as the teachers might not be clear about the rigour of justification. They may accept justifications as correct even when certain elements are missing. Teachers therefore need guidance in teaching justification. So the *DIVINE* framework shows them what essential elements to look out for so that they know whether certain details are still lacking in the justification. Teachers can also discuss the three components of the framework for the various types of justification tasks with the students to enrich their learning and appreciation of justification. In this way, students can develop a deeper understanding of constructing mathematical justification and become more confident in doing it. This pedagogical approach is particularly useful for those students who do not already have the justifying skill and struggle with justification. Additionally, for those who get stuck when attempting a justification task, the framework offers a structure for them to rely on and get unstuck instead of seeking immediate help from their mathematics teachers.

In the remaining sections, examples of justifications by both pre-service and in-service mathematics teachers will be discussed to demonstrate the rigour of the *DIVINE* framework as it currently stands. The pre-service teachers were Year 2 undergraduates undergoing their first course in mathematics pedagogy to prepare them to teach secondary school mathematics. The course content covers problem solving, learning theories and teaching strategies for a range of mathematics topics, including arithmetic, algebra, probability and statistics. The in-service teachers were from the same secondary school who attended my professional development workshop. A vast majority of them have taught mathematics for at least 5 years. The justifications were collected from the various classwork given to the teachers in my lessons. The names of the teachers are changed to protect their privacy. The discussion focuses specifically on *making decision*, *inference* and *validation* types of justification tasks. No *elaboration* task will be illustrated because the teachers were not given such tasks to do in my lessons.

Making Decision task: The justifications of Angel, Betty and Carl

The number pattern item in Figure 2 was given to the pre-service mathematics teachers. Before administering this item, the teachers had learnt the various generalising strategies for deriving the general rule for both numerical and figural patterns, but not how to deal with justification tasks. This item was therefore given to see how they would handle and justify a *making decision* task.

The first four terms of a sequence are 5, 9, 13 and 17.
(a) Find an expression, in terms of *n*, for the *n*th term of the sequence.
(b) Explain whether 207 is a term in the sequence.

Figure 2: *Making decision* task on number pattern

Part (a) was answered correctly by all the teachers. They established $4n + 1$ as the general rule of the sequence. However, the responses for part (b) were more varied, and the justifications produced by Angel, Betty and Carl are described below.

Angel began with the supposition $4n + 1 = 207$ and then solved the equation to obtain $n = 51.5$. He concluded: *Since n has to be a positive integer, then 207 is not a term*. Betty worked out the difference between 207 and the first term 5 to get 202. Then she wrote: *No. All terms in the sequence are divisible by 4 after being subtracted by 5. 202 is not divisible by 4*. For Carl, he started with the same supposition as Angel and found the value of *n*. He then stated: *n must be a whole number for the given number to be a term in the sequence*. The justifications of Angel and Betty, but not that of Carl, were considered fully correct. Their justifications contain all the vital elements for a *making decision* task: that is, a conclusion supported by evidence. Carl's justification is missing the conclusion, thus judged as partially correct. In all the three examples, the justifications are logical and easy to follow, and the mathematics is correct. Carl's case is a perfect example to illustrate the importance of the *DIVINE* framework. If he had known about the essential elements that he had to show in his justification, he would have constructed a complete and correct justification.

Inference task: The justifications of David and Eve

The algebra item in Figure 1 was administered to the in-service mathematics teachers. The item tested them on their understanding of the significance of the positive solution of the quadratic equation in the given context. I expected the teachers to explain what the following three parts mean in the context: (i) y = 0, which in this context means that the stone has hit the ground, (ii) positive, which represents the forward direction of the throw, and (iii) the numerical value of the solution, which refers to the horizontal distance from the tower. However, expecting all three parts was too demanding, so a reasonable justification should address at least (i) and (iii). The mathematics teachers were told to construct the justification that would get them the best mark because they were experienced in-service teachers. The justifications of David and Eve are illustrated below.

David: *x* metres is the distance of the stone from the tower, when $y = 0$ (at ground level).

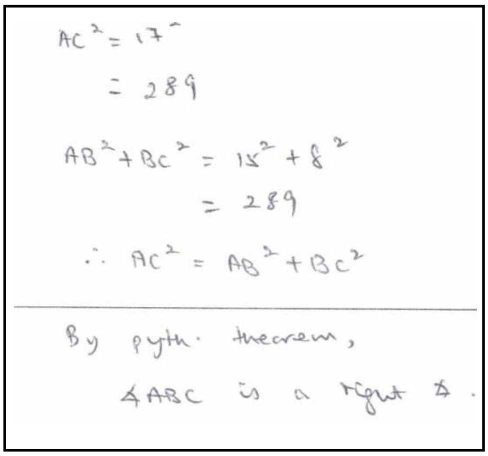
Eve: when $y = 0$, height above ground = 0, $\therefore$ stone is lying on ground.

David and Eve showed evidence of their attempt to explain the meaning of the positive solution. David's argument was regarded as correct because he justified (i) and (iii) correctly. For Eve, her justification was not deemed correct since she justified only (i). Her case again underscores the importance of knowing the critical elements that are needed in the justification, thus manifesting the usefulness of the *DIVINE* framework.

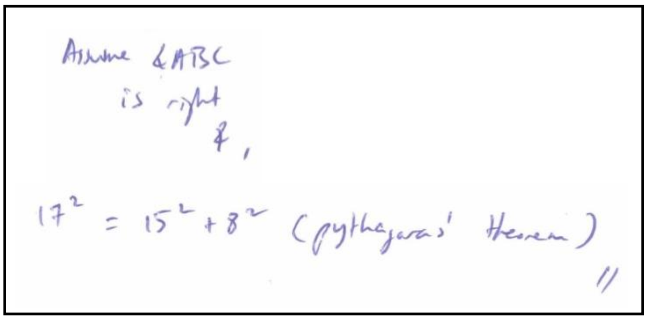
Validation task: The justifications of Faith and George

A geometry item involving a triangle with all three sides provided (15 cm, 8 cm and 17 cm) was given to the same group of in-service mathematics teachers mentioned above. They had to justify why the angle opposite the 17-cm side is a right angle. Figure 3 presents the justifications of Faith and George.

Faith established the condition $AC^2 = AB^2 + BC^2$ by separately working out the values of AC^2 and $AB^2 + BC^2$, and noticing that both values were equal (see Figure 3a). Subsequently, she inferred that angle *ABC* is a right angle. The mode of argumentation is correct, the justification is logical and easy to understand, but there is a mathematical flaw. The correct warrant to use should be the *converse* of Pythagoras' theorem and *not* Pythagoras' theorem. On the other hand, the mode of argumentation of George's justification (see Figure 3b) was wrong because he began with the wrong supposition by assuming angle *ABC* is a right angle, which was what he had to prove. So Faith's justification was judged as partially correct whereas George's justification was wrong.



(a) Faith



(b) George

Figure 3: Teachers' justifications for *Validation* task on geometry

What's next and conclusion

The *DIVINE* framework introduced in this paper is still emerging and will need further testing and refinement. For instance, it remains to be seen whether the framework can be put into use with student justifications and justification tasks in other mathematical topics. Furthermore, how do mathematics teachers judge what qualifies as a correct justification? What elements do they expect to see in the justifications? How would their judgement differ from peers and mathematics experts? Such evidence is needed to make the *DIVINE* framework more robust.

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