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SEM-DG Approximation for elasto-acoustics

Hélène Barucq\textsuperscript{1}, Henri Calandra\textsuperscript{2}, Aurélien Citrain\textsuperscript{3,1}, Julien Diaz\textsuperscript{1} and Christian Gout\textsuperscript{3}

\textsuperscript{1} Team project Magique.3D, INRIA, E2S UPPA, CNRS, Pau, France.
\textsuperscript{2} TOTAL SA, CSTJF, Pau, France.
\textsuperscript{3} INSA Rouen-Normandie Université, LMI EA 3226, 76000, Rouen.

MATHIAS 2018 October 22-24

The authors thank the M2NUM project which is co-financed by the European Union with the European regional development fund (ERDF, HN0002137) and by the Normandie Regional Council.
Why using hybrid meshes?

- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water).
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.
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- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.
Elastodynamic system

\( x \in \Omega \subset \mathbb{R}^d, \ t \in [0, T], \ T > 0 : \)

\[
\begin{cases}
\rho(x) \frac{\partial v}{\partial t}(x, t) = \nabla \cdot \sigma(x, t), \\
\frac{\partial \sigma}{\partial t}(x, t) = C(x)\epsilon(v(x, t)).
\end{cases}
\]

With:
- \( \rho(x) \) the density,
- \( C(x) \) the elasticity tensor,
- \( \epsilon(x, t) \) the deformation tensor,
- \( v(x, t) \), the wavespeed,
- \( \sigma(x, t) \) the strain tensor.
Elasticus software

Software written in **Fortran** for wave propagation simulation in the **time domain**

**Features**

Simulation:
- on various types of meshes (**unstructured triangles and tetrahedra**),
- on **heterogeneous** media (**acoustic, elastic and elasto-acoustic**).

- **Discontinuous Galerkin (DG)** based on **unstructured triangles and unstructured tetrahedra**, with various time-schemes: **Runge-Kutta (2 or 4), Leap-Frog**, with **multi-order** computation (**p-adaptivity**)...
1 Numerical Methods
2 Comparison DG/SEM on structured quadrangle mesh
3 DG/SEM coupling
4 3D extension
Numerical Methods
- Discontinuous Galerkin Method (DG)
- Spectral Element Method (SEM)
- Advantages of each method
Discontinuous Galerkin Method

Use discontinuous functions:

- mesh
- continuous
- discontinuous

Degrees of freedom necessary on each cell:

- $h$ adaptivity:
- $p$ adaptivity:

- $P1$
- $P2$
- $P3$
Spectral Element Method

General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature,
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on $L^2$-norm).

\[ \int f(x) \, dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j), \]

\[ \varphi_i(\xi_j) = \delta_{ij}. \]
Advantages of each method

DG
- Element per element computation (hp-adaptivity).
- Time discretization quasi explicit (block diagonal mass matrix).
- Simple to parallelize.
- Robust to brutal changes of physics and geometry

SEM
- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).
Comparison DG/SEM on structured quadrangle mesh

- Description of the test cases
- Comparative tables
Description of the test cases

Physical parameters

- **P wavespeed**: 1000 m.s\(^{-1}\)
- **Density**: 1 kg.m\(^{-3}\)

Second order **Ricker Source** in \(P\)wave (\(f_{\text{peak}} = 10\)Hz)

General context

- **Acoustic homogeneous** medium.
- Four different meshes: 10000 cells, 22500 cells, 90000 cells, 250000 cells.
- CFL computed using **power iteration** method.
- **Leap-Frog** time scheme.
- **Eight threads** parallel execution with **OpenMP**.
Error computed as the difference between an analytical and a numerical solution for each method.

Three cases considered: DG without penalization terms, DG with penalization terms and SEM.

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- Simple to parallelize.
- Robust to brutal changes of physics and geometry

**SEM**
- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).
- Reduces the computational costs on structured quadrangle cells in comparison with DG
DG/SEM coupling
- Hybrid meshes structures
- Variational formulation
Aim at coupling $P_k$ and $Q_k$ structures.

Need to extend or split some structures (e.g. neighbour indexes).

Define new face matrices:

$$M_{ij}^{K,L} = \int_{K \cap L} \phi^K_i \phi^L_j, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi^K_i \psi^L_j, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi^K_i \psi^L_j.$$
Domain in two parts: $\Omega_{h,1}$ (structured quadrangles + SEM), $\Omega_{h,2}$ (unstructured triangles + DG).
Variational formulation

SEM variational formulation:

\[
\begin{align*}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 &= -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 n_1) \cdot w_1, \\
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 &= -\int_{\Omega_{h,1}} (\nabla (C \xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C \xi_1 n_1) \cdot v_1.
\end{align*}
\]

DG variational formulation:

\[
\begin{align*}
\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 &= -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot n_2, \\
\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= -\int_{\Omega_{h,2}} (\nabla (C \xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C \xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[[C \xi_2]] \cdot n_2.
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Variational formulation

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\begin{align*}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 &= -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 \\
+ \int_{\Gamma_{out,1}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{1/2}} [[\sigma w]] \cdot n,
\end{align*}
\]

\[
\begin{align*}
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= -\int_{\Omega_{h,1}} (\nabla (C\xi_1)) \cdot v_1 - \int_{\Omega_{h,2}} (\nabla (C\xi_2)) \cdot v_2 \\
+ \int_{\Gamma_{out,1}} (C\xi_1 n_1) \cdot v_1 + \int_{\Gamma_{out,2}} (C\xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{1/2}} [((C\xi)v)] \cdot n.
\end{align*}
\]
\[
\begin{align*}
&\int_{\Omega_{h,1}} \rho \frac{\partial}{\partial t} v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \frac{\partial}{\partial t} v_2 \cdot w_2 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 \\
&+ \int_{\Gamma_{out,1}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2 \\
&+ \int_{\Gamma_{1/2}} \{\sigma\}[[w]] \cdot n + [[\sigma]]\{\{w\}\} \cdot n,
\end{align*}
\]
\[
\begin{align*}
&\int_{\Omega_{h,1}} \frac{\partial}{\partial t} \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \frac{\partial}{\partial t} \sigma_2 : \xi_2 = -\int_{\Omega_{h,1}} (\nabla(C \xi_1)) \cdot v_1 - \int_{\Omega_{h,2}} (\nabla(C \xi_2)) \cdot v_2 \\
&+ \int_{\Gamma_{out,1}} (C \xi_1 n_1) \cdot v_1 + \int_{\Gamma_{out,2}} (C \xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2 \\
&+ \int_{\Gamma_{1/2}} [[C \xi]]\{\{v\}\} \cdot n + \{\{C \xi\}\}[[v]] \cdot n.
\end{align*}
\]
Variational formulation

\[
\begin{align*}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 &= -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 \\
+ \int_{\Gamma_{\text{out},1}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{\text{out},2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{\text{int}}} \{\{\sigma_2\}\} [[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{\text{out},1}} \{\{\sigma\}\} [[[w]]] \cdot n + \{\{\sigma\}\} \{\{w\}\} \cdot n,
\end{align*}
\]

\[
\begin{align*}
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= -\int_{\Omega_{h,1}} (\nabla (C \xi_1)) \cdot v_1 - \int_{\Omega_{h,2}} (\nabla (C \xi_2)) \cdot v_2 \\
+ \int_{\Gamma_{\text{out},1}} (C \xi_1 n_1) \cdot v_1 + \int_{\Gamma_{\text{out},2}} (C \xi_2 n_2) \cdot v_2 + \int_{\Gamma_{\text{int}}} \{\{\sigma_2\}\} [[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{\text{int}}} [[[C \xi_1][[v]]]} \cdot n + \{\{C \xi_1\}\} [[[v]]]} \cdot n.
\end{align*}
\]
Only deal with a simple case of 3D hybrid meshes: one hexahedron has only two tetrahedra as neighbour.

Extend SEM in 3D (basis functions...).

Require introducing a new matrix which handles the rotation cases between two elements.
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General settings

Figure: Hexa/Tet boundary configuration

- Only deal with a simple case of 3D hybrid meshes: one hexahedron has only two tetrahedra as neighbour.
- Require introducing a new matrix which handles the rotation cases between two elements.
Conclusion and perspectives

Conclusion

1. SEM is more efficient on structured quadrangle mesh than DG
2. Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
3. Show the utility of using hybrid meshes and method coupling (reduce computational cost, ...)

Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Develop PML in the hexahedral part
- Add a local time-stepping scheme
Thank you for your attention!

Questions?
Figure: $L^2$-error comparison on a 10000 cells mesh
Figure: $L^2$-error comparison on a 10000 cells mesh

<table>
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<tr>
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<th>CFL</th>
<th>L2-error</th>
<th>CPU-time</th>
<th>Nb of time steps</th>
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<td>SEM(order five)</td>
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Figure: SEM and DG comparison with fixed error