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SEM-DG Approximation for elasto-acoustics

Hélène Barucq\textsuperscript{1}, Henri Calandra\textsuperscript{2}, Aurélien Citrain\textsuperscript{3,1}, Julien Diaz\textsuperscript{1} and Christian Gout\textsuperscript{3}

\textsuperscript{1} Team project Magique.3D, INRIA, E2S UPPA, CNRS, Pau, France.
\textsuperscript{2} TOTAL SA, CSTJF, Pau, France.
\textsuperscript{3} INSA Rouen-Normandie Université, LMI EA 3226, 76000, Rouen.

MATHIAS 2018 October 22-24

The authors thank the M2NUM project which is co-financed by the European Union with the European regional development fund (ERDF, HN0002137) and by the Normandie Regional Council.
Why using hybrid meshes?

- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water).
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.
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- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.
Elastodynamic system

\[ x \in \Omega \subset \mathbb{R}^d, \quad t \in [0, T], \quad T > 0 : \]

\[
\begin{aligned}
\rho(x) \frac{\partial v}{\partial t}(x, t) &= \nabla \cdot \sigma(x, t), \\
\frac{\partial \sigma}{\partial t}(x, t) &= C(x) \varepsilon(v(x, t)).
\end{aligned}
\]

With:

- \( \rho(x) \) the density,
- \( C(x) \) the elasticity tensor,
- \( \varepsilon(x, t) \) the deformation tensor,
- \( v(x, t) \), the wavespeed,
- \( \sigma(x, t) \) the strain tensor.
Elasticus software

Software written in **Fortran** for wave propagation simulation in the **time domain**

**Features**

**Simulation:**
- on various types of meshes (**unstructured triangles and tetrahedra**),
- on **heterogeneous** media (**acoustic, elastic and elasto-acoustic**).

- **Discontinuous Galerkin (DG)** based on **unstructured triangles and unstructured tetrahedra**,
- with various time-schemes: **Runge-Kutta (2 or 4), Leap-Frog**,
- with **multi-order** computation (**p-adaptivity**)…
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1 Numerical Methods
2 Comparison DG/SEM on structured quadrangle mesh
3 DG/SEM coupling
4 3D extension
Numerical Methods
- Discontinuous Galerkin Method (DG)
- Spectral Element Method (SEM)
- Advantages of each method
Discontinuous Galerkin Method

Use discontinuous functions:

- mesh
- continuous
- discontinuous

Degrees of freedom necessary on each cell:

- P1
- P2
- P3

$h$ adaptivity:

$p$ adaptivity:
Spectral Element Method

**General principle**

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature,
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on $L^2$-norm).

\[
\int f(x) \, dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j),
\]

\[
\varphi_i(\xi_j) = \delta_{ij}.
\]
Advantages of each method

**DG**
- Element per element computation (hp-adaptivity).
- Time discretization quasi explicit (block diagonal mass matrix).
- Simple to parallelize.
- Robust to brutal changes of physics and geometry

**SEM**
- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).
2 Comparison DG/SEM on structured quadrangle mesh
- Description of the test cases
- Comparative tables
Description of the test cases

Physical parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>P wavespeed</td>
<td>1000 m.s(^{-1})</td>
</tr>
<tr>
<td>Density</td>
<td>1 kg.m(^{-3})</td>
</tr>
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</table>

Second order **Ricker Source** in \(P\)wave  
(\(f_{\text{peak}} = 10\)Hz)

General context

- **Acoustic homogeneous** medium.
- Four different meshes: 10000 cells, 22500 cells, 90000 cells, 250000 cells.
- CFL computed using **power iteration** method.
- **Leap-Frog** time scheme.
- **Eight threads** parallel execution with **OpenMP**.
Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method.
- Three cases considered: DG without penalization terms, DG with penalization terms and SEM.

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Figure: DG not penalized and SEM comparison on the 10000 cells case

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DG
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SEM
- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).
- Reduces the computational costs on structured quadrangle cells in comparison with DG
DG/SEM coupling

- Hybrid meshes structures
- Variational formulation
Aim at coupling $P_k$ and $Q_k$ structures.
Need to extend or split some structures (e.g. neighbour indexes).
Define new face matrices:

\[
M_{ij}^{K,L} = \int_{K \cap L} \phi^K_i \phi^L_j, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi^K_i \psi^L_j, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi^K_i \psi^L_j.
\]
Global context

- Domain in two parts: $\Omega_{h,1}$ (structured quadrangles + SEM), $\Omega_{h,2}$ (unstructured triangles + DG).
Variational formulation

SEM variational formulation:

\[
\begin{align*}
\int_{\Omega_{h,1}} \rho \frac{\partial t}{\partial t} v_1 \cdot w_1 &= -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 n_1) \cdot w_1, \\
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 &= -\int_{\Omega_{h,1}} (\nabla (C \xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C \xi_1 n_1) \cdot v_1.
\end{align*}
\]

DG variational formulation:

\[
\begin{align*}
\int_{\Omega_{h,2}} \rho \frac{\partial t}{\partial t} v_2 \cdot w_2 &= -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2, \\
\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= -\int_{\Omega_{h,2}} (\nabla (C \xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C \xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{v_2\}[[C \xi_2]] \cdot n_2.
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\[ \begin{align*}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 &= -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 \\
+ \int_{\Gamma_{out,1}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{1/2}} [[\sigma w]] \cdot n,
\end{align*} \]

\[ \begin{align*}
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= -\int_{\Omega_{h,1}} (\nabla(C \xi_1)) \cdot v_1 - \int_{\Omega_{h,2}} (\nabla(C \xi_2)) \cdot v_2 \\
+ \int_{\Gamma_{out,1}} (C \xi_1 n_1) \cdot v_1 + \int_{\Gamma_{out,2}} (C \xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{1/2}} [[(C \xi) v]] \cdot n.
\end{align*} \]
Variational formulation

\[
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2
\]

\[
+ \int_{\Gamma_{out,1}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2
\]

\[
+ \int_{\Gamma_{1/2}} \{\{\sigma\}\}[w] \cdot n + [[\sigma]][\{w\}] \cdot n,
\]

\[
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,1}} (\nabla (C\xi_1)) \cdot v_1 - \int_{\Omega_{h,2}} (\nabla (C\xi_2)) \cdot v_2
\]

\[
+ \int_{\Gamma_{out,1}} (C\xi_1 n_1) \cdot v_1 + \int_{\Gamma_{out,2}} (C\xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\sigma_2\}[[w_2]] \cdot n_2
\]

\[
+ \int_{\Gamma_{1/2}} [[C\xi]][\{v\}] \cdot n + \{\{C\xi\}[[v]] \cdot n.
\]
Variational formulation

\[
\begin{align*}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 &= -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 \\
+ \int_{\Gamma_{\text{out},1}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{\text{out},2}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{\text{int}}} \{\sigma_2\} [[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{1/2}} \{\sigma\} [[w]] \cdot n + \{\sigma\} \{w\} \cdot n,
\end{align*}
\]

\[
\begin{align*}
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 &= -\int_{\Omega_{h,1}} (\nabla (C \xi_1)) \cdot v_1 - \int_{\Omega_{h,2}} (\nabla (C \xi_2)) \cdot v_2 \\
+ \int_{\Gamma_{\text{out},1}} (C \xi_1 n_1) \cdot v_1 + \int_{\Gamma_{\text{out},2}} (C \xi_2 n_2) \cdot v_2 + \int_{\Gamma_{\text{int}}} \{\sigma_2\} [[w_2]] \cdot n_2 \\
+ \int_{\Gamma_{1/2}} [C \xi] \{\xi\} \cdot n + \{[C \xi]\} [[\xi]] \cdot n.
\end{align*}
\]
Only deal with a simple case of 3D hybrid meshes: one hexahedron has only two tetrahedra as neighbour.

Extend SEM in 3D (basis functions...).

Require introducing a new matrix which handles the rotation cases between two elements.
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Conclusion

1. SEM is more efficient on structured quadrangle mesh than DG
2. Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
3. Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Develop PML in the hexahedral part
- Add a local time-stepping scheme
Thank you for your attention!

Questions?
Figure: $L^2$-error comparison on a 10000 cells mesh
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<table>
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<th>CFL</th>
<th>L2-error</th>
<th>CPU-time</th>
<th>Nb of time steps</th>
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<td>SEM(order five)</td>
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Figure: SEM and DG comparison with fixed error