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THIN LAYERS FOR THE LANDAU-LIFSCHITZ EQUATION

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ABSTRACT. In this paper we study the behaviour of the magnetization in a thin layer of ferromagnetic material when the exchange coefficient is small. We explain the interaction between the boundary layer phenomenon and the thin layer phenomenon.

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1. INTRODUCTION

We consider here ferromagnetic media which are spontaneously magnetized. They are used for example in computers and in aeronautics for the skins of the planes. In the quasi-stationary case, the behaviour of the magnetization, denoted by \( u \), is described by the Landau-Lifschitz equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= u \wedge H_e - u \wedge (u \wedge H_e) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial \Omega, \\
u(0, \cdot) &= u_0 \quad \text{in} \quad \Omega,
\end{aligned}
\]

where \( \Omega \) is a domain filled with ferromagnetic material and immersed in the vacuum, \( n \) is the outward unitary normal to \( \partial \Omega \) and \( H_e \) is the effective magnetic field given by:

\[
H_e = \varepsilon^2 \Delta u + \mathcal{H}(u).
\]

Here, \( \varepsilon^2 \) is a small parameter called the exchange coefficient, \( \mathcal{H}(u) \) is called the demagnetizing field given by the magnetostatic equations (we assume that we always are at the electromagnetic equilibrium):

\[
\begin{aligned}
\mathcal{H}(u) &\in L^2(\mathbb{R}^3), \\
\text{div} (\mathcal{H}(u) + \bar{n}) &= 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3), \\
\text{curl} \mathcal{H}(u) &= 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3),
\end{aligned}
\]
where \(\overline{u}\) is the extension of \(u\) by 0 outside \(\Omega\).

When the exchange coefficient \(\varepsilon\) is fixed, G. Carbou and P. Fabrie have proved in [6] the existence and uniqueness of local regular solutions for the equations (1)-(3).

When the exchange coefficient \(\varepsilon\) goes to zero, G. Carbou, P. Fabrie and O. Guès have proved in [8] the following theorem on \(u^\varepsilon\), solution of the previous equations:

**Theorem 1.1** (G. Carbou, P. Fabrie, O. Guès). Let \(u_0 \in H^5(\Omega)\), \(|u_0| = 1\),
\[
\left( \frac{\partial u_0}{\partial n} \right)_{|\partial\Omega} = 0,
\]
there exists a function \(U^0 \in C([0, +\infty[; H^5(\Omega))\) solution of
\[
\begin{aligned}
\frac{\partial U^0}{\partial t} &= U^0 \wedge H(U^0) \wedge (U^0 \wedge H(U^0)) \quad \text{in } [0, +\infty[ \times \Omega, \\
U^0(0, x) &= u_0(x) \quad \text{on } \Omega,
\end{aligned}
\]
which, moreover, fulfills \(|U^0(t, x)| = 1\) for all \((t, x) \in [0, +\infty[ \times \Omega\). There also exists a function \(\tilde{U}^1 \in L^\infty_{\text{loc}}(R^+_t; H^4(\Omega) \otimes H^4(R^+))\) and a time \(T^\varepsilon > 0\) such that \(\lim_{\varepsilon \to 0} T^\varepsilon = +\infty\), so that the solution \(u^\varepsilon\) of the equations (1)-(3) can be written in the form
\[
u(t, x) = U^0(t, x) + \varepsilon \tilde{U}^1 \left( t, x, \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon v^\varepsilon(t, x), \quad t \in [0, T],
\]
\(\forall \ T < T^\varepsilon\) and with \(\varphi(x) = \text{dist}(x, \partial\Omega)\). Moreover \(v^\varepsilon\) is bounded in \(L^\infty(0, T; H^1(\Omega))\), and \(\varepsilon v^\varepsilon\) is bounded in \(L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega))\).

So the solution \(u^\varepsilon\) of the equations (1)-(3) admits an asymptotic expansion when \(\varepsilon\) goes to zero on a time interval \([0, T^\varepsilon]\) with \(T^\varepsilon \to +\infty\), whose first order term is the solution of the Landau-Lifschitz equation with \(\varepsilon = 0\) and whose second order term is a perturbation which lives in a neighbourhood of the boundary \(\partial\Omega\) and whose characteristic thickness is \(\varepsilon\).

In [16] we study the case where the demagnetizing field is replaced by an anisotropic field \(\Psi(x, u)\) in \(H_e\).

**Remark 1.2.** We can study these equations with both the demagnetizing field and the anisotropic field but we ignore the latter for simplification.

According to the previous result, the perturbation occurs in a \(\varepsilon\)-thick neighbourhood of \(\partial\Omega\). If we now consider a domain with thickness \(\varepsilon\), there will be a competition between the behaviours due to the small thickness and the boundary layer that we will describe in the following: in a previous work (see [17]) we considered the case of a \(\varepsilon\)-thick periodic sheet of ferromagnetic material immersed in the vacuum. We will present here the case of a thin layer of ferromagnetic material spread on a non-flat perfect conductor. We then take into account the geometry of the domain (see [18] for the whole proof).
2. THIN LAYERS

Let us precise the notations. Let $\Omega$ be a regular bounded domain of $\mathbb{R}^3$. We denote $\Gamma = \partial \Omega$. For $x \in \Omega$ we introduce $\varphi(x) = \text{dist}(x, \Gamma)$ and $P_{\Gamma}(x)$ the orthogonal projection of $x$ onto $\Gamma$. We remark that since $\Gamma$ is a regular surface of $\mathbb{R}^3$, $\varphi$ and $P_{\Gamma}$ are regular in a neighbourhood of $\Gamma$.

We set

$$\omega_\varepsilon = \{x \in \Omega, \ 0 < \varphi(x) < \varepsilon\}.$$ 

![Diagram of a thin layer with a perfect conductor and vacuum.] 

We denote $\mathcal{U}^\varepsilon = \Omega \setminus \omega_\varepsilon$ the domain where lies the perfect conductor.

The equations we will consider are:

$$\begin{cases}
\frac{\partial u^\varepsilon}{\partial t} = u^\varepsilon \wedge (\varepsilon^2 \Delta u^\varepsilon + \mathcal{H}(u^\varepsilon)) - u^\varepsilon \wedge (u^\varepsilon \wedge (\varepsilon^2 \Delta u^\varepsilon + \mathcal{H}(u^\varepsilon))) & \text{in } \omega_\varepsilon, \\
\frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \partial \omega_\varepsilon, \\
u^\varepsilon(0, ) = u_0 & \text{in } \omega_\varepsilon,
\end{cases}$$

where the operator $\mathcal{H}$ fulfills now

$$\begin{cases}
\mathcal{H}(u^\varepsilon) \in L^2(\mathbb{R}^3 \setminus \mathcal{U}^\varepsilon), \\
\text{curl } \mathcal{H}(u^\varepsilon) = 0 \ \text{in } \mathbb{R}^3 \setminus \mathcal{U}^\varepsilon, \\
\text{div } (\mathcal{H}(u^\varepsilon) + \overline{u^\varepsilon}) = 0 \ \text{in } \mathbb{R}^3 \setminus \mathcal{U}^\varepsilon, \\
[\mathcal{H}(u^\varepsilon) \wedge n] = 0 \ \text{on } \Gamma, \\
\mathcal{H}(u^\varepsilon) \wedge n = 0 \ \text{on } \Gamma_\varepsilon, \quad (\text{Boundary condition for a perfect conductor}) \\
[(\mathcal{H}(u^\varepsilon) + \overline{u^\varepsilon}) \cdot n] = 0 \ \text{on } \Gamma, \\
(\mathcal{H}(u^\varepsilon) + \overline{u^\varepsilon}) \cdot n = 0 \ \text{on } \Gamma_\varepsilon, \quad (\text{Boundary condition for a perfect conductor})
\end{cases}$$

where $n$ denotes the outward unitary normal to $\partial \omega_\varepsilon$ and $[v]$ denotes the jump of $v$ at the interface $\partial \omega_\varepsilon$.

In order to perform an asymptotic expansion and get the profiles we deal with the rescaled equation. We assume the following conditions are fulfilled:
(H1) The initial data \( u_0^\varepsilon \) satisfies \( \frac{\partial u_0^\varepsilon}{\partial n} = 0 \) on \( \partial \omega_\varepsilon \) and can be written in the form:

\[
 u_0^\varepsilon(x) = U_0^0(P(x)) + \varepsilon U_1^1 \left( P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^2 U_2^2 \left( P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^3 U_3^3 \left( P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^3 r_0^\varepsilon(x), \quad \forall x \in \omega_\varepsilon,
\]

(H2) \( |u_0^\varepsilon(x)| = 1 \) for all \( x \in \omega_\varepsilon \),

(H3) \( U_0^0 \in \mathcal{W}^{7,\infty}(\Gamma) \) such that \( |U_0^0| \equiv 1 \),

(H4) \( U_0^1 \in \mathcal{H}^6(\Gamma) \otimes \mathcal{H}^5(0,1) \),

(H5) \( U_0^2 \in \mathcal{H}^5(\Gamma) \otimes \mathcal{H}^5(0,1) \),

(H6) \( U_0^3 \in \mathcal{H}^4(\Gamma) \otimes \mathcal{H}^5(0,1) \),

(H7) \( r_0^\varepsilon \in \mathcal{H}^2(\omega_\varepsilon) \) is the initial data for the remainder term \( r^\varepsilon \) and fulfills the additional condition:

\[
\|r_0^\varepsilon\|^2_{L^2(\omega_\varepsilon)} + \|\nabla_\Gamma r_0^\varepsilon\|^2_{L^2(\omega_\varepsilon)} + \|\varepsilon \partial_\varepsilon r^\varepsilon\|^2_{L^2(\omega_\varepsilon)} + \varepsilon^2 \left( \|\Delta_\Gamma r_0^\varepsilon\|^2_{L^2(\omega_\varepsilon)} + \|\partial_\varepsilon^2 r_0^\varepsilon\|^2_{L^2(\omega_\varepsilon)} \right) \leq C,
\]

where \( C \) is an \( \varepsilon \)-independent constant, \( \nabla_\Gamma \) and \( \Delta_\Gamma \) are the tangential parts of the gradient and Laplace operators.

**Remark 2.1.** We work with Sobolev spaces adapted to the anisotropy of the problem:

\[
\mathbb{L}^p(\omega_\varepsilon) = \left\{ u \in \mathcal{D}(\omega_\varepsilon), \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} |u|^p \, dx \right\}, \quad \mathbb{H}^0(\omega_\varepsilon) = L^2(\omega_\varepsilon),
\]

\[
\mathbb{H}^{p+1}(\omega_\varepsilon) = \left\{ u \in L^2(\omega_\varepsilon), \nabla_\Gamma u \in \mathbb{H}^p(\omega_\varepsilon) \text{ and } \varepsilon \partial_\varepsilon u \in \mathbb{H}^p(\omega_\varepsilon) \right\},
\]

and with the natural norms on these spaces.

Our main result is the following:

**Theorem 2.2.** Under the assumptions (H1)-(H7), if \( T^\varepsilon \) is the maximum time of existence of the regular solution \( u^\varepsilon \) of the equation (5) then \( \lim_{\varepsilon \to 0} T^\varepsilon = +\infty \) and there exists some profiles \( U_0^0, U_1^1, U_2^2, U_3^3 \) defined on \( \mathbb{R}^+_t \times \Gamma \times (0,1) \) such that for all \( T < T^\varepsilon, x \in \omega_\varepsilon \) and \( t < T \),

\[
u(t, x) = u^0(t, P(x)) + \varepsilon U_1^1(t, P(x), \frac{\varphi(x)}{\varepsilon}) + \varepsilon^2 U_2^2(t, P(x), \frac{\varphi(x)}{\varepsilon}) + \varepsilon^3 U_3^3(t, P(x), \frac{\varphi(x)}{\varepsilon}) + \varepsilon^3 r^\varepsilon(t, x),
\]

where for all \( T > 0 \),

- \( U_0^0 \in C^\infty(\mathbb{R}^+_t; \mathcal{W}^{7,\infty}(\Gamma)) \) is solution of the equation (5) with \( \varepsilon = 0 \), i.e.

\[
\begin{cases}
\frac{\partial U_0^0}{\partial t} = U_0^0 \wedge H_0^0 - U_0^0 \wedge (U_0^0 \wedge H_0^0) \text{ in } \Gamma, \\
U_0^0(0,.) = U_0^0 \text{ in } \Gamma, 
\end{cases}
\]

where \( H_0^0 = -U_3^3 \varepsilon^3 \) becomes a local operator,
\( U^1 \in \bigcap_{k=0}^{2} \left[ \mathbb{W}^{k,\infty} (0, T; \mathbb{H}^6 (\Gamma) \otimes \mathbb{H}^{6-2k} (0, 1)) \right] \cap \mathbb{H}^k (0, T; \mathbb{H}^6 (\Gamma) \otimes \mathbb{H}^{6-2k} (0, 1)) \),

\( U^2 \in \bigcap_{k=0}^{2} \left[ \mathbb{W}^{k,\infty} (0, T; \mathbb{H}^5 (\Gamma) \otimes \mathbb{H}^{5-2k} (0, 1)) \right] \cap \mathbb{H}^k (0, T; \mathbb{H}^5 (\Gamma) \otimes \mathbb{H}^{5-2k} (0, 1)) \),

\( U^3 \in \bigcap_{k=0}^{2} \left[ \mathbb{W}^{k,\infty} (0, T; \mathbb{H}^4 (\Gamma) \otimes \mathbb{H}^{4-2k} (0, 1)) \right] \cap \mathbb{H}^k (0, T; \mathbb{H}^4 (\Gamma) \otimes \mathbb{H}^{4-2k} (0, 1)) \),

for all \( T < T^\epsilon \) the function \( r^\epsilon \) is bounded in \( L^\infty (0, T; \mathbb{H}^1 (\omega_\epsilon)) \) and the function \( \varepsilon r^\epsilon \) is bounded in \( L^\infty (0, T; \mathbb{H}^2 (\omega_\epsilon)) \).

We have then obtained a first order asymptotic expansion valid on the time interval \( (0, T^\epsilon) \) with \( T^\epsilon \to +\infty \) as \( \varepsilon \to 0 \) and whose first order term is solution of the previous equations taken with \( \varepsilon = 0 \), i.e. solution of the 2D Landau-Lifschitz equation with a null exchange coefficient. Also the magnetic field in the 2D Landau-Lifshitz equation is no longer a solution of equation (6), it is now a local operator orthogonal to the surface \( \Gamma \) (in accordance with the physics). Moreover the second order perturbation lies in the whole domain contrary to the boundary layer case.

**Sketch of the proof.**

- Since we can prove that the norm of \( u \) is conserved, the Landau-Lifschitz equation is equivalent to the following equation (see [6]):

\[
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial t} - \varepsilon^2 \Delta u^\varepsilon &= \varepsilon^2 |\nabla u^\varepsilon|^2 u^\varepsilon + \varepsilon^2 u^\varepsilon \wedge \Delta u^\varepsilon + u^\varepsilon \wedge \mathcal{H}(u^\varepsilon) \\
&\quad - u^\varepsilon \wedge (u^\varepsilon \wedge \mathcal{H}(u^\varepsilon)) \quad \text{in } \mathbb{R}_t^+ \times \omega_\varepsilon, \\
\frac{\partial u^\varepsilon}{\partial n} &= 0 \quad \text{on } \mathbb{R}_t^+ \times (\Gamma \cup \Gamma_\varepsilon), \\
u^\varepsilon(0, \cdot) &= u^\varepsilon_0 \quad \text{in } \omega_\varepsilon,
\end{aligned}
\]

since \( u \cdot \Delta u = -|\nabla u|^2 \) when \( |u| \equiv 1 \).

- We look for solutions in the form of an asymptotic expansion in \( \varepsilon \) where the profiles are defined on \( \mathbb{R}_t^+ \) and on the new coordinates \( \Gamma \times (0, 1) \),

\[
u^\varepsilon(t, x) = U^0\left(t, P_T(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon U^1\left(t, P_T(x), \frac{\varphi(x)}{\varepsilon}\right) + \ldots
\]

\[
H^\varepsilon(t, x) = H^0\left(t, P_T(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon H^1\left(t, P_T(x), \frac{\varphi(x)}{\varepsilon}\right) + \ldots
\]

where \( P_T(x) \) is the orthogonal projection of \( x \) onto \( \Gamma \) and \( \varphi(x) = \text{dist}(x, \Gamma) \). In the vacuum outside \( \omega \) we can prove that there is no boundary layer, so for all \( x \in \mathbb{R}^3 \setminus \omega \),

\[
H^\varepsilon(t, x) = H^0(t, x) + \varepsilon H^1(t, x) + \ldots
\]

We substitute to \( u \) and \( H \) their asymptotic expansion in (6)-(7) and we obtain

- transmission conditions for the profiles \( H^i \) deduced from the transmission condition written in equation (6),
- the Neumann conditions written in the new coordinates for the profiles,
- the equations fulfilled by the magnetic field outside $\omega_\varepsilon$ and which can be solved by Lax-Milgram theorem,
- the equations fulfilled by the profiles of $u$ and $H$ in the thin layer. Thanks to the introduction of the new coordinates, the classical differential operators split into two parts, a tangential operator and a normal operator (see [5] and [18]) which allows us to perform the calculus in a simpler way. For example:

If $u : \omega_\varepsilon \to \mathbb{R}$ or $\mathbb{R}^3$, we have, setting $\tilde{u}(\sigma, z) = u(x)$ with $\sigma = P_\Gamma(x)$, $z = \varphi(x)$:

$$\nabla u(x) = \frac{\partial \tilde{u}}{\partial z}(\sigma, z)n(\sigma) + (\nabla_{\Gamma_z} \tilde{u})(\sigma, z),$$

where $\Gamma_z = \{x \in \Omega, \text{dist}(x, \Gamma) = z\}$, $\nabla_{\Gamma_z}$ is the tangential gradient on $\Gamma_z$ when $\Gamma_z$ is parameterized by $\Gamma$.

With this formalism we obtain the equations fulfilled by $H^0$:

$$\begin{align*}
\frac{\partial}{\partial z} [(H^0 + U^0) \cdot n] &= 0, \\
n \wedge \frac{\partial}{\partial z} H^0 &= 0.
\end{align*}$$

We can solve them thanks to the transmission conditions and the solution we obtained outside $\omega_\varepsilon$. We obtain $H^0 = -(U^0 \cdot n)n$, i.e. $H$ behaves itself as a local operator at order $\varepsilon^0$.

Next $U^0$ fulfilled the following equation:

$$\frac{\partial U^0}{\partial t} - U^0_{zz} = |U^0_z|^2 U^0 + U^0 \wedge U^0_{zz} + U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) \quad \text{in} \quad \Gamma \times (0, 1).$$

According to the assumption (H3) we can prove that $U^0$ does not depend on $z$, so $U^0$ fulfills

$$\frac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) \quad \text{in} \quad \Gamma \times (0, 1).$$

By the same way we get the equations fulfilled by $H^i$, $U^i$. We obtain for $i = 1, 2, 3$,

$$H^i = -(U^i \cdot n)n + a(U^1, \ldots, U^{i-1}),$$

where $a$ is a linear operator, and $U^i$ fulfills

$$\frac{\partial U^i}{\partial t} - U^i_{zz} = U^0 \wedge U^1_{zz} + F_i(U^0, \ldots, U^{i-1}, U^i) + G_i(U^0, \ldots, U^{i-1}),$$

where $F_i$ is a linear operator in its last variable $U^i$. We can then obtain the existence of the profiles and the properties we announced above.

- the main difficulty lies in the estimates on the remainder term. Except the technical part of these estimates, there are two important points: we first need to estimate the
remainder term $Q^\varepsilon$ of $H^\varepsilon$ with the one of $u^\varepsilon (r^\varepsilon)$ in the same spaces: $Q^\varepsilon$ solves
\begin{equation}
\begin{cases}
Q^\varepsilon \in L^2(\omega_\varepsilon), \\
curl Q^\varepsilon + A(H^0, \overline{U^0}, H^1, \overline{U^1}, H^2, \overline{U^2}, H^3, \overline{U^3}) = 0 \text{ in } \omega_\varepsilon, \\
div (Q^\varepsilon + r^\varepsilon) + B(H^0, \overline{U^0}, H^1, \overline{U^1}, H^2, \overline{U^2}, H^3, \overline{U^3}) = 0 \text{ in } \omega_\varepsilon,
\end{cases}
\end{equation}
where $A$ and $B$ are linear functions of their arguments, with the following transmission conditions
\begin{equation}
\begin{cases}
[Q^\varepsilon \wedge n] = 0 \text{ on } \Gamma, \\
[(Q^\varepsilon + r^\varepsilon) \cdot n] = 0 \text{ on } \Gamma, \\
Q^\varepsilon \wedge n = 0 \text{ on } \Gamma_\varepsilon, \\
(Q^\varepsilon + r^\varepsilon) \cdot n = 0 \text{ on } \Gamma_\varepsilon.
\end{cases}
\end{equation}

**Lemma 2.3.** If $r^\varepsilon \in \mathbb{H}^p$, there exists an unique $Q^\varepsilon = R^\varepsilon + S^\varepsilon$ and a constant $c_0$ such that $S^\varepsilon$ is regular, $R^\varepsilon$ is linear in $r^\varepsilon$ and:
\begin{equation}
\|R^\varepsilon\|_{\mathbb{H}^p(\omega_\varepsilon)} \leq c_0 \|r^\varepsilon\|_{\mathbb{H}^p(\omega_\varepsilon)}.
\end{equation}

Secondly we need $\varepsilon$-independent estimates on the remainder term of $u$. As we use Sobolev embeddings we need to precise the dependence on $\varepsilon$ of the Sobolev constants. Following the work of R. Temam and M. Ziane [19], we obtain some other estimates (See [17], [18]). For example we have in the anisotropic spaces (see Remark 2.1):

**Lemma 2.4.** There exists $c_0$ independent of $\varepsilon$ such that $\forall u \in \mathbb{H}^1(\omega_\varepsilon)$ and $\forall 2 \leq p \leq 6$,
\begin{equation}
\|u\|_{L^p(\omega_\varepsilon)} \leq c_0 \|u\|_{L^2(\omega_\varepsilon)}^{\frac{3}{2} - \frac{1}{p}} \|u\|_{H^1(\omega_\varepsilon)}^{\frac{3}{2} - \frac{3}{p}}
\end{equation}

and $\forall u \in \mathbb{H}^2(\omega_\varepsilon)$, $\forall 2 \leq p \leq +\infty$,
\begin{equation}
\|u\|_{L^p(\omega_\varepsilon)} \leq c_0 \|u\|_{L^2(\omega_\varepsilon)}^{\frac{1}{2} + \frac{1}{p}} \|u\|_{H^2(\omega_\varepsilon)}^{\frac{1}{2} - \frac{1}{p}}.
\end{equation}

**Remark 2.5.** We have studied here the behaviour of Landau-Lifschitz equations with an exchange coefficient $\varepsilon^2$ in a thin layer of thickness $\varepsilon$. That is the case where the thickness of the boundary layer (if there were any, cf [8]) and the domain's thickness are of the same size. If we instead consider a thin layer with a thickness $\varepsilon^\alpha$ ($\alpha > 0$, $\alpha = \frac{p}{q} \in \mathbb{Q}$), we perform now an asymptotic expansion at the scale $\varepsilon^{\frac{1}{2}}$ and we see two different behaviours. If $\alpha > 1$ we get the same type of behaviour than that of the case $\alpha = 1$, i.e. the behaviour of a thin layer and the localization of $H$ at the low orders. If $\alpha < 1$, the layer's thickness is bigger than the characteristic thickness of the boundary layer. We then obtain a boundary layer in the thin layer but with a smaller thickness ($\varepsilon^{1-\alpha}$) and we also get at the low orders the localization of $H$. 

\[\square\]
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