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Abstract

In the literature, median type control charts have been widely investigated as easy and efficient means to monitor the process mean when observations are from a normal distribution. In this work, a Variable Sampling Interval (VSI) Exponentially Weighted Moving Average (EWMA) median control chart is proposed and studied. A Markov chain method is used to obtain optimal designs and evaluate the statistical performance of the proposed chart. Furthermore, practical guidelines and comparisons with the basic EWMA median control chart are provided. Results show that the proposed chart is considerably more efficient than the basic EWMA median control chart. Finally, the implementation of the proposed chart is illustrated with an example in the food production process.

Keywords: EWMA, VSI, Median, Control chart, Order statistics.

1 Introduction

Statistical Process Control (SPC) is a method of quality control which uses statistical methods in achieving process stability and improving capability through the reduction of variability, see Montgomery [1]. It's well known that control charts are the fundamental tool for SPC applications. There are numerous types of control charts, the most common ones are the Shewhart control charts, the cumulative sum (CUSUM) control charts and the exponentially weighted moving average (EWMA) control charts. The EWMA control charts have a "built in" mechanism for incorporating information from all previous subgroups by means of weights decreasing geometrically with the sample mean age. Thus EWMA type control charts are very effective for the detection of small or moderate process shifts, see Tran et al. [2]. Their properties and design stategies have been thoroughly investigated by many authors. For further details see, for instance, Robinson and Ho [3], Hunter [4], Crowder [5], Lucas and Saccucci [6], Tran et al. [2] to name a few.

In recent years, many researchers have focused on developing advanced control charts with various applications in manufacturing and service processes, for example, see Castagliola and Figueiredo [7], Huang [8], Da Costa Quinino et al. [9], Tran et al. [10], Castagliola et al. [11], Tran [12], Tran et al. [13] and Tran [14]. Among these control charts, median ($\tilde{X}$) type charts have been widely investigated as easy and efficient means to monitor the mean. The main advantages of median type charts are that they are simpler than mean ($\bar{X}$) charts and that they are robust against outliers, contamination or small deviations from normality, see Castagliola et al. [11].

In the SPC literature, the EWMA median chart was introduced by Castagliola [15] (EWMA-$\tilde{X}$) with fast detection of assignable causes. Then, a generally weighted moving average median (GWMA-$\tilde{X}$) control chart has been studied by Sheu and Yang [16] as a continuation to improve the statistical performance of median type control charts. When the parameters are estimated, Castagliola and Figueiredo [7] and Castagliola et al. [11] developed a Shewhart median chart and a EWMA-$\tilde{X}$ chart, respectively, with estimated control limits to monitor the mean value of a normal process. Very recently, Lin et al. [17] investigated the performances of the EWMA-$\tilde{X}$ control chart under several distributions. As a result, the EWMA-$\tilde{X}$ is always more efficient than the EWMA-$\bar{X}$ chart in detecting shifts in the process mean if the data follow a heavy-tailed distribution. Finally, Tran [18] proposed and studied the Run Rules Shewhart median control charts ($\text{RR}_{r,s}-\tilde{X}$ charts).

It is known that, the EWMA-$\tilde{X}$ control chart suggested by Castagliola [15] is a Fixed Sampling Interval (FSI) control chart. By definition, an adaptive control chart involves varying at least one of the chart's parameters, such as the sampling interval or the sample size. Variable Sampling Interval (VSI)

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control charts are adaptive control charts where the sampling intervals vary as a function of what is observed from the process. The VSI control charts are demonstrated to detect process changes faster than FSI control charts. The idea is that the time interval until the next sample should be short, if the position of the last plotted control statistic indicates a possible out-of-control situation; and long, if there is no indication of a change. Most work on developing VSI control charts has been done for the problem of monitoring the mean of the process (see Reynolds [19], Reynolds et al. [20] and Castagliola et al. [21]).

In this paper, we propose a VSI EWMA-$\tilde{X}$ control chart as a logical extension of the control chart developed by Castagliola [15]. The goal of this paper is to show how the VSI behaves with respect to the basic EWMA median control chart. The rest of this paper proceeds as follows: in Section 2, a brief review of the FSI EWMA-$\tilde{X}$ control chart is provided; Section 3 provides a VSI version of the FSI EWMA-$\tilde{X}$ control chart; in Section 4, the run length performances of proposed chart are defined by using the Markov Chain-based approach; in Section 5, the computational results and the tables reporting the optimal design parameters of the VSI EWMA-$\tilde{X}$ chart are presented. Section 6 presents an illustrative example and, finally, some concluding remarks and recommendations are made in Section 7.

2 The FSI EWMA-$\tilde{X}$ control chart

Let us assume that, at each sampling period $i = 1,2,\ldots$, we collect a sample of n independent random variables $\{X_{i,1},\ldots,X_{i,n}\}$. We assume that each $X_{i,j}$ follows a normal distribution $N(\mu_0 + \delta\sigma_0, \sigma_0)$, $j = 1,\ldots,n$, μ_0 is the in-control mean value, σ_0 is the in-control standard deviation and δ is the magnitude of the standardized mean shift. If $\delta = 0$ the process is in-control and, when $\delta \neq 0$, the process is out-of-control. Let $\tilde{X}_i$ be the sample median of subgroup i, i.e.

$$\tilde{X}_i = \begin{cases} X_{i,((n+1)/2)} & \text{if } n \text{ is odd} \\ \dfrac{X_{i,(n/2)} + X_{i,(n/2+1)}}{2} & \text{if } n \text{ is even} \end{cases} \tag{1}$$

where $\{X_{i,(1)}, X_{i,(2)},\ldots,X_{i,(n)}\}$ is the ordered i-th subgroup. In the rest of this paper, without loss of generality, we assume that the sample size n is an odd value. Let $Z_1, Z_2,\ldots$ be the EWMA sequence obtained from $\tilde{X}_1, \tilde{X}_2,\ldots$, i.e. for $i \in \{1,2,\ldots\}$,

$$Z_i = (1-\lambda)Z_{i-1} + \lambda\tilde{X}_i, \tag{2}$$

where $Z_0 = \mu_0$ and $\lambda \in (0,1]$ is a smoothing constant. If the in-control mean value μ_0 and the standard deviation σ_0 are assumed known, the control limits of the EWMA-$\tilde{X}$ chart for the median are simply equal to

$$LCL = \mu_0 - K\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0, \tag{3}$$

$$UCL = \mu_0 + K\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0, \tag{4}$$

where $K > 0$ is a constant that depends on n and on the desired in-control performance.

3 Implementation of the VSI EWMA-$\tilde{X}$ control chart

In this section, a VSI version of the FSI EWMA-$\tilde{X}$ control chart described in the previous section is presented (denoted as VSI EWMA-$\tilde{X}$). The control statistic Z_i for the VSI EWMA-$\tilde{X}$ control chart is given by (2). The upper (UCL) and lower (LCL) control limits of the VSI EWMA-$\tilde{X}$ control chart can be easily calculated as:

$$LCL = \mu_0 - K\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0, \tag{5}$$

$$UCL = \mu_0 + K\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0, \tag{6}$$

where $K \geq 0$ is a constant influencing the width of the control interval.

For the FSI control chart, the sampling interval is a fixed value h_0. As for the VSI control chart, the sampling interval depends on the current value of Z_i. A longer sampling interval h_L is used when the control statistic falls within region $R_L = [LWL, UWL]$ defined as:

$$LWL = \mu_0 - W\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0, \tag{7}$$

$$UWL = \mu_0 + W\sqrt{\frac{\lambda}{2-\lambda}}\sigma_0, \tag{8}$$

where W is the warning limit coefficient of the VSI EWMA-$\tilde{X}$ control chart that determines the proportion of times that the control statistic falls within the long and short sampling regions. On the other hand, the short sampling interval h_S is used when the control statistic falls within the region $R_S = [LCL, LWL) \cup (UWL, UCL]$. The process is considered out-of-control and action should be taken whenever Z_i falls outside the range of the control limits $[LCL, UCL]$. In order to evaluate the ARL and $SDRL$ of the VSI EWMA-$\tilde{X}$ control chart, we follow the discrete Markov chain approach originally proposed by Brook and Evans [22] . In Appendix, the discrete Markov chain approach for VSI EWMA-$\tilde{X}$ control chart is provided.

4 Optimal design of the VSI EWMA-$\tilde{X}$ control chart

In the literature, the Average Run Length (ARL), defined as the average number of samples before the control chart signals an out-of-control condition or issues a false alarm, and the Average Time to Signal (ATS), which is the expected value of the time between the occurrence of a special cause and a signal from the chart are used as the performance measures of control charts, see Castagliola et al. [21]. It is well known that, when the process is in-control, it is better to have a large ATS, since in this operating condition a signal represents a false alarm (in this case, the ATS will be denoted as ATS_0). On the other hand, after the parameter of the process under control has shifted, it is preferable to have an ATS that is as small as possible (in this case, the ATS will be denoted as ATS_1).

For a FSI model, the ATS is a multiple of the ARL since the sampling interval h_F is fixed. Thus, in this case we have the following expression:

$$ATS^{\text{FSI}} = h_F \times ARL^{\text{FSI}}. \tag{9}$$

For a VSI model, the ATS is defined as:

$$ATS^{\text{VSI}} = E(h) \times ARL^{\text{VSI}}. \tag{10}$$

where $E(h)$ is the expected sampling interval value.

According to Castagliola et al. [21], for VSI type control charts, we need to define them with the same in-control $ARL = ARL_0$ and the same in-control average sampling interval $E_0(h)$. For FSI-type control charts, the sampling interval is set equal to $h_S = h_L = h_F = 1$ time units. Then, the in-control expected sampling interval of the VSI chart is set equal to $E_0(h) = 1$ time unit to ensure $ATS_0 = ARL_0$ time unit for both FSI and VSI type control charts. The value of h_S represents the shortest feasible time interval between subgroups from the process, see Castagliola et al. [21] for more details. Then, in this paper we will consider the impact on the expected time until detection, using small but non-zero values of h_S.

The design procedure of VSI EWMA-$\tilde{X}$ control chart is implemented by finding out the optimal combination of parameters λ^*, K^* and h_L^* which minimize the out-of-control ATS for predefined values of δ, W, h_S, n and ATS_0, i.e., the optimization scheme of the VSI EWMA-$\tilde{X}$ consists in finding the optimal parameters λ^*, K^* and h_L^* such that

$$(\lambda^*, K^*, h_L^*) = \underset{(\lambda, K, h_L)}{\operatorname{argmin}} ATS(n, \lambda, K, W, h_L, h_S, \delta) \tag{11}$$

subject to the constraint

$$\begin{aligned} E_0(h) &= 1, \\ ATS(n, \lambda, K, W, h_L, h_S, \delta = 0) &= ATS_0. \end{aligned} \tag{12}$$

Similar to Tran and Tran [23], the choice of the optimal combination of parameters generally entails two steps:

1. Find the potential combinations (λ, K, h_L) such that $ATS = ATS_0$ and $E_0(h) = 1$.
2. Choose, among these potential combinations (λ, K, h_L), the one (λ^*, K^*, h_L^*) that allows for the best performance, i.e. the smallest "out-of-control" ATS value for a particular shift δ.

In this study, like in Tran and Tran [23], in order to find these optimal combinations (λ^*, K^*, h_L^*) we simultaneously use a non-linear equation solver coupled to an optimization algorithm (developed with Scicoslab software).

5 Numerical results

Optimal designs were obtained for the FSI and VSI EWMA-$\tilde{X}$ control charts, for all combinations of $\delta \in [0.5, 2]$ and $n = \{3, 5, 7, 9\}$. The sampling interval h_F of the FSI charts has been set equal to 1 time unit. The shorter time interval h_S can assume the following values: 0.5 and 0.1 time units. The optimal combinations of design parameters (λ^*, K^*, h_L^*) have been selected by constraining the in-control ATS at the value $ATS_0 = 370.4$ and the in-control expected sampling interval of the VSI chart is set equal to $E_0(h) = 1$. To ensure a fair comparison, the ARL_0 of EWMA-$\tilde{X}$ chart is set as 370.4. The optimal combinations of design parameters (λ^*, K^*, h_L^*) of the VSI EWMA-$\tilde{X}$ control chart are presented in Tables 1-4. Some simple conclusions can be drawn from Tables 1-4:

$n = 3$					
$h_S = 0.5$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0513, 1.6750)	(0.0514, 1.6750)
	(1.08, 139.5)	(1.24, 135.9)	(1.81, 133.8)	(2.54, 134.7)	(4.68, 135.9)
0.2	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0514, 1.6750)
	(1.08, 50.0)	(1.24, 47.3)	(1.81, 46.2)	(2.40, 46.4)	(4.68, 48.6)
0.3	(0.0514, 1.6750)	(0.0518, 1.6767)	(0.0500, 1.6686)	(0.0517, 1.6763)	(0.0535, 1.684)
	(1.09, 26.1)	(1.26, 24.6)	(1.81, 24.3)	(2.55, 24.9)	(4.68, 26.9)
0.5	(0.0989, 1.8090)	(0.1095, 1.8273)	(0.1073, 1.8234)	(0.1046, 1.8190)	(0.1124, 1.8315)
	(1.10, 11.8)	(1.28, 11.2)	(1.89, 11.4)	(2.46, 11.8)	(4.49, 13.8)
0.7	(0.1605, 1.8883)	(0.1690, 1.8957)	(0.1563, 1.8844)	(0.1742, 1.9000)	(0.1798, 1.9043)
	(1.10, 6.9)	(1.28, 6.7)	(1.87, 7.0)	(2.58, 7.6)	(4.39, 9.4)
1.0	(0.2743, 1.9557)	(0.2773, 1.9569)	(0.2759, 1.9563)	(0.2783, 1.9572)	(0.2885, 1.9609)
	(1.10, 4.0)	(1.29, 3.9)	(1.91, 4.4)	(2.54, 5.0)	(4.31, 6.8)
1.5	(0.4746, 2.0017)	(0.4681, 2.0008)	(0.4685, 2.0008)	(0.4745, 2.0016)	(0.4293, 1.9950)
	(1.10, 2.2)	(1.28, 2.3)	(1.89, 2.9)	(2.51, 3.5)	(4.26, 5.2)
2.0	(0.6883, 2.0195)	(0.6890, 2.0196)	(0.5499, 2.0100)	(0.5498, 2.0100)	(0.4293, 1.9950)
	(1.11, 1.6)	(1.30, 1.7)	(1.89, 2.3)	(2.50, 2.9)	(4.26, 4.7)
$h_S = 0.1$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0500, 1.6686)
	(1.15, 134.2)	(1.44, 127.7)	(2.47, 124.0)	(3.52, 123.9)	(6.52, 125.5)
0.2	(0.0500, 1.6686)	(0.0500, 1.6686)	(0.0500, 1.6685)	(0.0500, 1.6686)	(0.0500, 1.6686)
	(1.15, 44.8)	(1.44, 40.0)	(2.47, 38.0)	(3.52, 38.5)	(6.52, 41.0)
0.3	(0.0515, 1.6752)	(0.0515, 1.6753)	(0.0577, 1.701)	(0.0502, 1.6693)	(0.0643, 1.7240)
	(1.15, 22.1)	(1.46, 19.4)	(2.57, 19.0)	(3.51, 19.8)	(7.53, 23.4)
0.5	(0.0987, 1.8090)	(0.1140, 1.8339)	(0.1278, 1.8530)	(0.1325, 1.8589)	(0.1394, 1.8669)
	(1.17, 9.3)	(1.50, 8.3)	(2.59, 8.5)	(3.90, 9.6)	(7.19, 12.8)
0.7	(0.1737, 1.8995)	(0.1952, 1.9154)	(0.2088, 1.9241)	(0.2142, 1.9273)	(0.2269, 1.9344)
	(1.18, 5.3)	(1.50, 4.8)	(2.67, 5.4)	3.81, 6.5)	(7.02, 9.6)
1.0	(0.3086, 1.9676)	(0.3239, 1.9721)	(0.3326, 1.9746)	(0.3415, 1.9769)	(0.3661, 1.9829)
	(1.19, 3.0)	(1.52, 2.9)	(2.63, 3.7)	(3.74, 4.8)	(6.90, 7.9)
1.5	(0.5211, 2.0072)	(0.5218, 2.0072)	(0.5366, 2.0087)	(0.5498, 2.0100)	(0.4025, 1.9903)
	(1.18, 1.7)	(1.50, 1.9)	(2.59, 2.9)	(3.70, 4.0)	(6.88, 7.2)
2.0	(0.7043, 2.0203)	(0.5498, 2.0100)	(0.5498, 2.0100)	(0.5498, 2.0100)	(0.4025, 1.9903)
	(1.19, 1.3)	(1.50, 1.6)	(2.59, 2.7)	(3.70, 3.8)	(6.88, 7.0)

Table 1: Optimal couples (λ^*, K^*) (first row of each block) and values of (h_L, ATS_1)(second row of each block) of the VSI EWMA-$\tilde{X}$ control chart for $n = 3$.

- Given the values of δ, n and W, the value of ATS depends on h_S. In particular, with smaller values of h_S, the value of ATS_1 decreases. For example, when $\delta = 0.1$, $n = 3$, $W = 0.6$ we have $ATS_1 = 135.9$ for $h_S = 0.5$ and $ATS_1 = 127.7$ for $h_S = 0.1$, see Table 1.

- For a defined value of h_S, it is obvious that when W decreases the length of the long sampling interval h_L increases. For example, when $\delta = 0.1$, $n = 3$, $h_S = 0.5$ we have $h_L = 1.08$ for $W = 0.9$ and $h_L = 4.68$ for $W = 0.1$, see Table 1.

- The VSI EWMA-$\tilde{X}$ control chart is directly compared to the FSI EWMA-$\tilde{X}$ control chart, to evaluate the impact of the adaptive feature on the statistical performance of the original static chart. As expected, the results in Tables 1-4 clearly indicate that the VSI EWMA-$\tilde{X}$ chart is superior to the FSI EWMA-$\tilde{X}$ control chart. For example, when $\delta = 0.1$, $n = 3$, $W = 0.6$ and $h_S = 0.5$ we have $ATS_1 = 135.9$ for VSI EWMA-$\tilde{X}$ chart and $ARL = 146.1$ for FSI EWMA-$\tilde{X}$ control chart, see Table 3 in Castagliola [15].

$n = 5$					
$h_S = 0.5$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)
	(1.03, 106.1)	(1.14, 101.4)	(1.60, 98.0)	(2.05, 97.7)	(3.60, 98.6)
0.2	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)
	(1.03, 36.7)	(1.14, 33.7)	(1.60, 32.1)	(2.05, 32.3)	(3.60, 33.6)
0.3	(0.06134, 1.3705)	(0.0619, 1.3719)	(0.0689, 1.3899)	(0.0710, 1.3951)	(0.0733, 1.4002)
	(1.04, 19.6)	(1.15, 17.9)	(1.63, 17.3)	(2.11, 17.6)	(3.53, 18.9)
0.5	(0.1290, 1.4828)	(0.1388, 1.4921)	(0.1467, 1.4989)	(0.1483, 1.5002)	(0.1526, 1.5036)
	(1.04, 8.8)	(1.16, 8.1)	(1.63, 8.0)	(2.15, 8.4)	(3.83, 10.0)
0.7	(0.2175, 1.5423)	(0.2304, 1.5479)	(0.2325, 1.5487)	(0.2264, 1.5462)	(0.2400, 1.5517)
	(1.05, 5.1)	(1.17, 4.8)	(1.65, 5.0)	(2.12, 5.4)	(3.76, 7.0)
1.0	(0.3773, 1.5869)	(0.3721, 1.5860)	(0.3662, 1.5850)	(0.3684, 1.5854)	(0.3804, 1.5874)
	(1.05, 3.0)	(1.17, 2.8)	(1.64, 3.2)	(2.21, 3.7)	(3.71, 5.2)
1.5	(0.6405, 1.6119)	(0.6369, 1.6117)	(0.6400, 1.6119)	(0.6450, 1.6121)	(0.6579, 1.6127)
	(1.05, 1.7)	(1.17, 1.7)	(1.68, 2.2)	(2.19, 2.7)	(3.67, 4.2)
2.0	(0.8517, 1.6182)	(0.8540, 1.6182)	(0.8594, 1.6183)	(0.8634, 1.6184)	(0.8728, 1.6185)
	(1.05, 1.2)	(1.17, 1.3)	(1.67, 1.8)	(2.19, 2.3)	(3.66, 3.8)
$h_S = 0.1$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)
	(1.06, 102.7)	(1.25, 94.3)	(2.08, 88.1)	(2.89, 87.6, , 370.4)	(5.69, 89.1)
0.2	(0.0500, 1.3341	(0.0516, 1.3402)	(0.0500, 1.3341)	(0.0500, 1.3341)	(0.0500, 1.3341)
	(1.06, 33.7)	(1.27, 28.2, , 370.4)	(2.08, 25.5)	(2.89, 25.8, , 370.4)	(5.69, 28.2)
0.3	(0.0695, 1.3915)	(0.0644, 1.3788)	(0.0791, 1.4124)	(0.0837, 1.4212)	(0.0772, 1.4085)
	(1.07, 17.2)	(1.27, 14.2)	(2.11, 13.0)	(2.97, 13.5)	(5.53, 15.8)
0.5	(0.1573, 1.5075)	(0.1586, 1.5082)	(0.1543, 1.5050)	(0.1799, 1.5225)	(0.1902, 1.5286)
	(1.08, 7.2)	(1.31, 6.0)	(2.12, 5.9)	(3.05, 6.6)	(6.03, 9.4)
0.7	(0.2618, 1.5596)	(0.2629, 1.5598)	(0.2784, 1.5647)	(0.2264, 1.5462)	(0.3018, 1.5711)
	(1.09, 4.1)	(1.31, 3.5)	(2.16, 3.8)	(3.02, 4.6)	(5.92, 7.4)
1.0	(0.4059, 1.5914)	(0.4240, 1.5939)	(0.3768, 1.5868)	(0.4404, 1.5960)	(0.4721, 1.5996)
	(1.09, 2.3)	(1.32, 2.1)	(2.14, 2.7)	(3.17, 3.7)	(5.84, 6.4)
1.5	(0.6616, 1.6129)	(0.6621, 1.6129)	(0.3768, 1.5868)	(0.7056, 1.6146)	(0.7466, 1.6159)
	(1.09, 1.4)	(1.31, 1.5)	(2.14, 2.3)	(3.14, 3.3)	(5.79, 5.9)
2.0	(0.8505, 1.6182)	(0.8649, 1.6184)	(0.3768, 1.5868)	(0.9103, 1.6189)	(0.9343, 1.6191)
	(1.09, 1.1)	(1.31, 1.3)	(2.14, 2.2)	(3.13, 3.2)	(5.79, 5.8)

Table 2: Optimal couples (λ^*, K^*) (first row of each block) and values of (h_L, ATS_1) (second row of each block) of the VSI EWMA-$\tilde{X}$ control chart for $n = 5$.

$n = 7$					
$h_S = 0.5$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0500, 1.1427)
	(1.01, 87.0)	(1.09, 81.8)	(1.46, 77.5)	(1.88, 77.0)	(3.25, 77.7)
0.2	(0.0507, 1.1449)	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0505, 1.1443)
	(1.01, 30.2)	(1.09, 27.0)	(1.46, 25.3)	(1.88, 25.4)	(3.25, 26.5)
0.3	(0.0870, 1.2223)	(0.0767, 1.2060)	(0.0860, 1.2208)	(0.0886, 1.2247)	(0.0913, 1.2286)
	(1.02, 16.2)	(1.10, 14.4)	(1.49, 13.6)	(1.91, 13.9)	(3.15, 14.9)
0.5	(0.1593, 1.2921)	(0.1723, 1.2998)	(0.1819, 1.3049)	(0.1831, 1.3055)	(0.1873, 1.3076)
	(1.02, 7.2)	(1.11, 6.5)	(1.49, 6.3)	(1.94, 6.6)	(3.37, 8.0)
0.7	(0.2678, 1.3370)	(0.2845, 1.3412)	(0.2847, 1.3413)	(0.2788, 1.3398)	(0.2912, 1.3428)
	(1.02, 4.2)	(1.12, 3.8)	(1.51, 3.9)	(1.91, 4.3)	(3.32, 5.7)
1.0	(0.4864, 1.3705)	(0.4590, 1.3681)	(0.4507, 1.3673)	(0.4527, 1.3675)	(0.4646, 1.3686)
	(1.03, 2.4)	(1.11, 2.3)	(1.50, 2.6)	(1.98, 3.0)	(3.28, 4.3)
1.5	(0.7609, 1.3831)	(0.7604, 1.3831)	(0.7643, 1.3831)	(0.7679, 1.3832)	(0.7766, 1.3834)
	(1.02, 1.4)	(1.12, 1.4)	(1.53, 1.8)	(1.97, 2.3)	(3.26, 3.5)
2.0	(0.9354, 1.3853)	(0.9377, 1.3853)	(0.9419, 1.3853)	(0.9448, 1.3853)	(0.9513, 1.3854)
	(1.02, 1.1)	(1.12, 1.2)	(1.53, 1.6)	(1.97, 2.0)	(3.25, 3.3)
$h_S = 0.1$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0500, 1.1427)	(0.0500, 1.1427)
	(1.02, 85.2)	(1.17, 75.8)	(1.82, 68.1)	(2.59, 67.2)	(5.05, 68.5)
0.2	(0.0557, 1.1596)	(0.0500, 1.1427)	(0.0505, 1.1443)	(0.0553, 1.1583)	(0.0599, 1.1702)
	(1.03, 28.6)	(1.17, 22.9)	(1.82, 19.7)	(2.57, 19.8)	(4.99, 21.8)
0.3	(0.0870, 1.2224)	(0.0927, 1.2306)	(0.0986, 1.2383)	(0.1046, 1.2455)	(0.0996, 1.2395)
	(1.03, 14.8)	(1.19, 11.5)	(1.87, 10.1)	(2.61, 10.4)	(4.84, 12.3)
0.5	(0.1593, 1.2921)	(0.1946, 1.3111)	(0.2161, 1.3201)	(0.2212, 1.3221)	(0.2330, 1.3264)
	(1.04, 6.2)	(1.20, 4.9)	(1.94, 4.6)	(2.67, 5.1)	(5.22, 7.6)
0.7	(0.3249, 1.3500)	(0.3204, 1.3491)	(0.3368, 1.3521)	(0.3426, 1.3532)	(0.3617, 1.3563)
	(1.04, 3.6)	(1.21, 2.9)	(1.91, 3.0)	(2.79, 3.8)	(5.14, 6.1)
1.0	(0.4890, 1.3707)	(0.5066, 1.3721)	(0.4709, 1.3692)	(0.5203, 1.3731)	(0.5548, 1.3753)
	(1.05, 2.0)	(1.21, 1.8)	(1.90, 2.3)	(2.76, 3.1)	(5.08, 5.4)
1.5	(0.7655, 1.3832)	(0.7675, 1.3832)	(0.4709, 1.3692	(0.8106, 1.3840)	(0.8430, 1.3845)
	(1.04, 1.2)	(1.21, 1.3)	(1.90, 2.0)	(2.74, 2.8)	(5.06, 5.1)
2.0	(0.9324, 1.3853)	(0.9459, 1.3853)	(0.4709, 1.3692)	(0.0626, 1.1768)	(0.9825, 1.3854)
	(1.04, 1.1)	(1.21, 1.2)	(1.90, 1.9)	(2.56, 2.7)	(5.06, 5.1)

Table 3: Optimal couples (λ^*, K^*) (first row of each block) and values of (h_L, ATS_1) (second row of each block) of the VSI EWMA-$\tilde{X}$ control chart for $n = 7$.

$n = 9$					
$h_S = 0.5$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0500, 1.0152)	(0.0500, 1.0152)	(0.0500, 1.0152)	(0.0500, 1.0152)	(0.0500, 1.0152)
	(1.00, 74.4)	(1.06, 69.4)	(1.38, 64.3)	(1.75, 63.7)	(2.97, 64.2)
0.2	(0.0548, 1.0279)	(0.0524, 1.0220)	(0.0540, 1.0258)	(0.0569, 1.0330)	(0.0598, 1.0395)
	(1.01, 26.3)	(1.06, 23.1)	(1.37, 21.2)	(1.74, 21.2)	(2.94, 22.1)
0.3	(0.1013, 1.1030))	(0.0954, 1.0964)	(0.1017, 1.1034)	(0.1048, 1.1066)	(0.1077, 1.1096)
	(1.01, 14.1))	(1.07, 12.3)	(1.39, 11.4)	(1.76, 11.5)	(2.85, 12.4)
0.5	(0.1878, 1.1617)	(0.2030, 1.1679)	(0.2141, 1.1719)	(0.2145, 1.1720)	(0.2182, 1.1733)
	(1.01, 6.3)	(1.08, 5.5)	(1.42, 5.3)	(1.78, 5.5)	(3.03, 6.7)
0.7	(0.3139, 1.1970)	(0.3334, 1.2003)	(0.3318, 1.2000	(0.2408, 1.1804)	(0.3367, 1.2008)
	(1.01, 3.7)	(1.08, 3.3)	(1.41, 3.3)	(1.77, 3.7)	(2.99, 4.8)
1.0	(0.5404, 1.2203,)	(0.5410, 1.2203)	(0.5335, 1.2199)	(0.5354, 1.2200)	(0.5456, 1.2206)
	(1.01, 2.1)	(1.08, 2.0)	(1.43, 2.2)	(1.81, 2.6)	(2.96, 3.7)
1.5	(0.8434, 1.2289)	(0.8442, 1.2289)	(0.8480, 1.2289)	(0.8507, 1.2289)	(0.8570, 1.2290)
	(1.01, 1.2)	(1.08, 1.2)	(1.42, 1.6)	(1.81, 2.0)	(2.94, 3.1)
2.0	(0.9757, 1.2297)	(0.9772, 1.2297)	(0.9802, 1.2297)	(0.9822, 1.2297)	(0.9859, 1.2297)
	(1.01, 1.0)	(1.08, 1.1)	(1.42, 1.4)	(1.80, 1.8)	(2.94, 3.0)
$h_S = 0.1$					
δ	$W = 0.9$	$W = 0.6$	$W = 0.3$	$W = 0.2$	$W = 0.1$
0.1	(0.0504, 1.0163)	(0.0500, 1.0152)	(0.0500, 1.0152)	(0.0500, 1.0152)	(0.0500, 1.0152)
	(1.01, 73.6)	(1.11, 64.7)	(1.68, 55.6)	(2.36, 54.4)	(4.55, 55.4)
0.2	(0.0599, 1.0397)	(0.0598, 1.0397)	(0.0599, 1.0398)	(0.0649, 1.0501)	(0.0711, 1.0616)
	(1.01, 25.4)	(1.11, 19.9)	(1.71, 16.3)	(2.32, 16.2)	(4.45, 17.9)
0.3	(0.1014, 1.1031)	(0.0964, 1.0975)	(0.1171, 1.1183)	(0.1106, 1.1124)	(0.1027, 1.1044)
	(1.01, 13.2)	(1.13, 10.0)	(1.75, 8.3,)	(2.36, 8.5)	(4.35, 10.3)
0.5	(0.2224, 1.1747)	(0.2192, 1.1737)	(0.2524, 1.1836)	(0.2359, 1.1789)	(0.2701, 1.1880)
	(1.02, 5.6)	(1.13, 4.3)	(1.75, 3.8)	(2.39, 4.3)	(4.61, 6.4)
0.7	(0.3240, 1.1987)	(0.3737, 1.2060)	(0.3496, 1.2028)	(0.3916, 1.2082)	(0.4117, 1.2104)
	(1.02, 3.3)	(1.15, 2.5)	(2.6, 1.0)	(2.48, 3.2)	(4.55, 5.2)
1.0	(0.5801, 1.2223)	(0.5775, 1.2221)	(0.5783, 1.2222)	(0.5896, 1.2227)	(0.6235, 1.2240)
	(1.02, 1.8)	(1.14, 1.6)	(1.77, 2.0)	(2.46, 2.7)	(4.51, 4.7)
1.5	(0.8417, 1.2289)	(0.8441, 1.2289)	(0.8686, 1.2291)	(0.8829, 1.2292)	(0.9070, 1.2294)
	(1.02, 1.1)	(1.14, 1.2)	(1.76, 1.8)	(2.45, 2.5)	(4.50, 4.5)
2.0	(0.9741, 1.2297)	(0.9868, 1.2297)	(0.3496, 1.2028)	(0.9941, 1.2297)	(0.9967, 1.2297)
	(1.02, 1.0)	(1.14, 1.1)	(1.73, 1.7)	(0.92.45, 2.5)	(4.50, 4.5)

Table 4: Optimal couples (λ^*, K^*) (first row of each block) and values of (h_L, ATS_1) (second row of each block) of the VSI EWMA-$\tilde{X}$ control chart for $n = 9$.

6 Illustrative example

In this Section, we discuss the implementation of the VSI EWMA-$\tilde{X}$ chart. The context of the example presented here is similar as the one introduced in Castagliola et al. [11], i.e, a production process of 500 ml milk bottles where the quality characteristic X of interest is the capacity (in ml) of each bottle. Like in Castagliola et al. [11], we have $\mu_0 = 500.0230$ and $\sigma_0 = 0.9616$. In fact, according to the process engineer experience, a shift $\delta = 0.5$ should be interpreted as a signal that something is going wrong in the production process. Thus, for $n = 5$, $\delta = 0.5$ and $ATS_0 = 370.4$ the VSI EWMA-$\tilde{X}$ parameters are chosen to be $h_S = 0.5$, $h_L = 1.63$, $\lambda = 0.1467$, $K = 1.4989$ and $W = 0.3$. This yields the following control limits for the VSI EWMA-$\tilde{X}$ chart:

$$LCL = 500.023 - 1.4989\sqrt{\frac{0.1467}{2 - 0.1467}} \times 0.9616 = 499.617,$$

$$UCL = 500.023 + 1.4989\sqrt{\frac{0.1467}{2 - 0.1467}} \times 0.9616 = 500.429.$$

and the warning control limits for the VSI EWMA-$\tilde{X}$ chart:

$$LWL = 500.023 - 0.3\sqrt{\frac{0.1467}{1 - 0.1467}} \times 0.9616 = 499.942,$$

$$UWL = 500.023 + 0.3\sqrt{\frac{0.1467}{1 - 0.1467}} \times 0.9616 = 500.104.$$

The first 10 subgroups are supposed to be in-control while the last 10 subgroups are supposed to have a lower milk capacity, and thus, to be out-of-control. The corresponding sample median values $\tilde{X}_i$ and the EWMA sequence Z_i for VSI EWMA-$\tilde{X}$

control chart are both presented in Table 5 and plotted in Figure 1. This figure confirms that from sample #15 onwards, the process is clearly out-of-control.

	Sampling interval	Total time	Phase II ($X_{i,j}$)					$\tilde{X}_i$	Z_i
1	0.5	0.5	500.01	499.78	498.24	501.29	500.64	500.01	500.021
2	1.63	2.13	499.41	500.95	499.53	498.72	502.81	499.53	499.949
3	1.63	3.76	501.66	500.03	500.23	500.70	500.57	500.57	500.040
4	1.63	5.39	499.67	499.26	501.28	500.21	498.89	499.67	499.986
5	1.63	7.02	499.71	500.36	500.28	499.63	500.45	500.28	500.029
6	1.63	8.65	499.63	499.44	500.94	501.23	501.26	500.94	500.163
7	0.5	9.15	498.32	498.54	499.88	500.58	499.59	499.59	500.079
8	1.63	10.78	500.12	500.62	501.02	499.46	500.09	500.12	500.085
9	1.63	12.41	500.05	499.99	500.64	500.81	501.04	500.64	500.166
10	0.5	12.91	500.79	498.70	501.02	501.04	498.41	500.79	500.258
11	0.5	13.41	500.00	499.07	501.40	499.15	500.70	500.00	500.220
12	0.5	13.91	499.90	500.62	499.81	500.67	501.39	500.62	500.279
13	0.5	14.41	500.04	500.86	501.00	500.15	499.82	500.15	500.260
14	0.5	14.91	501.03	500.42	501.36	502.33	499.83	501.03	500.373
15	0.5	15.41	501.66	501.24	500.26	502.87	501.43	501.43	500.528
16	0.5	15.91	498.44	499.96	500.45	500.47	500.36	500.36	500.503
17	0.5	16.41	498.52	500.45	500.41	501.06	500.54	500.45	500.495
18	0.5	16.91	500.09	500.05	501.02	499.78	500.47	500.09	500.436
19	0.5	17.41	499.88	498.91	500.96	499.65	498.20	499.65	500.321
20	0.5	17.91	500.31	500.48	499.78	499.56	502.04	500.31	500.319

Table 5: Illustrative Phase II dataset

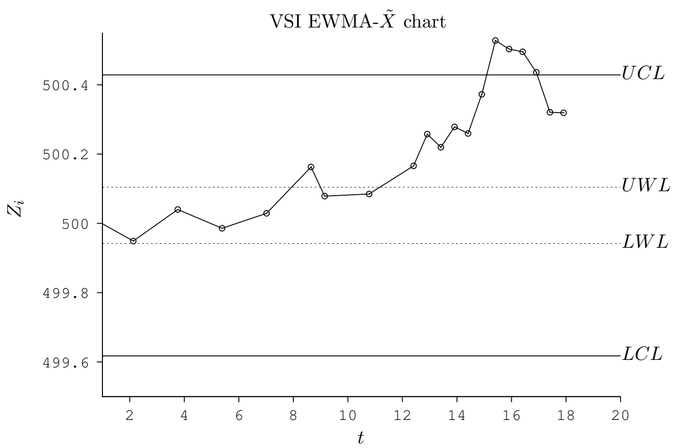

Figure 1: VSI EWMA-$\tilde{X}$ control chart corresponding to Phase II data set in Table 5.

7 Concluding remarks

In this paper, we have investigated a VSI EWMA-$\tilde{X}$ control chart for monitoring process median. We have also studied the statistical properties of the VSI EWMA-$\tilde{X}$ control chart and optimized their parameters for several shift sizes. For fixed values of the shift size δ, several tables are provided for presenting the out-of-control ATS_1 corresponding to many different scenarios. Also, the numerical comparison with the performance of the EWMA-$\tilde{X}$ control chart proposed by Castagliola [15] shows that the detection ability of the proposed control chart are better than the EWMA-$\tilde{X}$ control chart. Thus, the proposed chart can be used as a best alternative method.

Finally, possible enhancements and future work about VSI EWMA-$\tilde{X}$ control chart include the investigation of the effect of the parameters estimation and of the measurement errors on their statistical properties.

Appendix

The Markov chain approach of Brook and Evans [22] and Lucas and Saccucci [6] is modified to evaluate the Run Length properties of the VSI EWMA-$\tilde{X}$ chart. This procedure involves dividing dividing the interval $[LCL, UCL]$ into $2m+1$ subintervals $(H_j-\Delta, H_j+\Delta]$, $j \in \{-m,\ldots,0,\ldots,+m\}$, centered at $H_j = \frac{LCL+UCL}{2} + 2j\Delta$ where $2\Delta = \frac{UCL-LCL}{(2m+1)}$. Each subinterval $(H_j - \Delta, H_j+\Delta]$, $j \in \{-m,\ldots,0,\ldots,+m\}$, represents a transient state of a Markov chain. If $Z_i \in (H_j-\Delta, H_j+\Delta]$ then the Markov chain is in the transient state $j \in \{-m,\ldots,0,\ldots,+m\}$ for sample i. If $Z_i \notin (H_j-\Delta, H_j+\Delta]$ then the Markov chain reached the absorbing state $(-\infty, LCL] \cup [UCL, +\infty)$. We assume that H_j is the representative value of state $j \in \{-m,\ldots,0,\ldots,+m\}$. Let $\mathbf{Q}$ be the $(2m+1, 2m+1)$ sub-matrix of probabilities $Q_{j,k}$ corresponding to the $2m+1$ transient states defined above, i.e.

$$\mathbf{Q} = \begin{pmatrix} Q_{-m,-m} & \cdots & Q_{-m,-1} & Q_{-m,0} & Q_{-m,+1} & \cdots & Q_{-m,+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{-1,-m} & \cdots & Q_{-1,-1} & Q_{-1,0} & Q_{-1,+1} & \cdots & Q_{-1,+m} \\ Q_{0,-m} & \cdots & Q_{0,-1} & Q_{0,0} & Q_{0,+1} & \cdots & Q_{0,+m} \\ Q_{+1,-m} & \cdots & Q_{+1,-1} & Q_{+1,0} & Q_{+1,+1} & \cdots & Q_{+1,+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{+m,-m} & \cdots & Q_{+m,-1} & Q_{+m,0} & Q_{+m,+1} & \cdots & Q_{+m,+m} \end{pmatrix}.$$

By definition, we have $Q_{j,k} = P(Z_i \in (H_k-\Delta, H_k+\Delta] | Z_{i-1} = H_j)$ or, equivalently, $Q_{j,k} = P(Z_i \leq H_k+\Delta | Z_{i-1} = H_j) - P(Z_i \leq H_k - \Delta | Z_{i-1} = H_j)$. Replacing $Z_i = (1-\lambda)Z_{i-1} + \lambda\tilde{X}_i$, $Z_{i-1} = H_j$ and isolating $\tilde{X}_i$ gives

$$\begin{aligned} Q_{j,k} &= P\left(\tilde{X}_i \leq \frac{H_k+\Delta-(1-\lambda)H_j}{\lambda}\right) - P\left(\tilde{X}_i \leq \frac{H_k-\Delta-(1-\lambda)H_j}{\lambda}\right), \\ &= F_{\tilde{X}}\left(\frac{H_k+\Delta-(1-\lambda)H_j}{\lambda}\middle| n\right) - F_{\tilde{X}}\left(\frac{H_k-\Delta-(1-\lambda)H_j}{\lambda}\middle| n\right), \end{aligned}$$

where $F_{\tilde{X}}(\ldots|n)$ is the c.d.f. (cumulative distribution function) of the sample median $\tilde{X}_i$, $i \in \{1,2,\ldots\}$, i.e.

$$F_{\tilde{X}}(y|n) = F_\beta\left(\Phi\left(\frac{y-\mu_0}{\sigma_0} - \delta\right)\middle| \frac{n+1}{2}, \frac{n+1}{2}\right), \qquad (13)$$

where $\Phi(x)$ is the c.d.f. of the standard normal distribution and $F_\beta(x|a,b)$ is the c.d.f. of the beta distribution with parameters (a,b). Here $a = b = \frac{n+1}{2}$. Let $\mathbf{q} = (q_{-m},\ldots,q_0,\ldots,q_m)^T$ be the $(2m+1, 1)$ vector of initial probabilities associated with the $2m+1$ transient states, where

$$q_j = \begin{cases} 0 & \text{if } Z_0 \notin (H_j-\Delta, H_j+\Delta] \\ 1 & \text{if } Z_0 \in (H_j-\Delta, H_j+\Delta] \end{cases}. \qquad (14)$$

The ATS_1 can be evaluated through the following expression:

$$ATS_1 = \mathbf{q}^T(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{g} \qquad (15)$$

where $\mathbf{g}$ is the vector of sampling intervals corresponding to the discretized states of the Markov chain and the jth element g_j of the vector $\mathbf{g}$ is the sampling interval when the control statistic is in state j (represented by H_j), i.e.

$$g_j = \begin{cases} h_L & \text{if } LWL \leq H_j \leq UWL, \\ h_S & \text{otherwise.} \end{cases} \qquad (16)$$

The average sampling interval of the VSI EWMA-$\tilde{X}$ chart is given as:

$$E(h) = \frac{\mathbf{q}^T(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{g}}{\mathbf{q}^T(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{1}} \tag{17}$$

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