The flow of a proof -Establishing a basis of agreement

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The notion of flow of a proof encapsulates mathematical, didactical and contextual aspects of proof presentation, related to the lecturer's choices regarding the presentation. We explore the relationship between mathematics teaching and rhetoric, suggesting Perelman's New Rhetoric (PNR) as theoretical framework to assess different rhetorical aspects of the flow of a proof. In this paper we relate particularly to the establishment of a shared basis of agreement between the lecturer and the students, and to potential fallacies in this basis. We present examples of analysis of the basis of agreement from a lesson in Number Theory, at the beginning undergraduate level.

Keywords: Proof teaching, flow of proof, Perelman's New Rhetoric, mathematical argumentation.

Theoretical background

Mathematics and Rhetoric - "Can two walk together, except they be agreed?"

Mathematics "possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature…" (Russell, 1917, p. 60) and is "independent of us and our thoughts" (ibid, p. 69). This perception of mathematics seems to stand in drastic contrast to rhetoric, the ancient art of persuasion, which over the centuries became mostly related to the study of the ostentatious and artificial aspects of discourse. Yet, over the last few decades, scholars have begun to discuss the mathematics- rhetoric separation and its consequences.

A pioneering effort of associating mathematics and rhetoric was made by Davis and Hersh (1987) who argued "that mathematics is not really the antithesis of rhetoric, but rather that rhetoric may sometimes be mathematical, and that mathematics may sometimes be rhetorical" (p. 54). Davis and Hersh challenged the opinion that mathematics establishes truth "by a unique mode of argumentation, which consists of passing from hypothesis to conclusion by…small logical steps…", and claimed that "mathematical proof has its rhetorical moments and its rhetorical elements" (ibid, pp. 59–60). They illustrated this by phrases that a college mathematics lecturer may use while presenting a proof (in addition to the expected logical transformations), such as: "It is easy to show …", "… simple computation, which I leave to the student, will verify that…"; they identified these phrases as rhetorical means in the service of proof. They acknowledged that the use of such phrases may be related to context, but rejected the myth that behind each theorem stands a flawless, logical proof. For them 'proof' is an amalgam of formality, of convincing arguments and of appeals to imagination and intuition.

Another example is the 'rhetoric of the sciences' movement, which studies the stylistic forms used by scientists in scientific texts (mathematics included), to persuade others that their claims are valid. So, as in the other sciences, the rhetoric of mathematics plays an essential role in maintaining its epistemological claims (Ernest, 1999). Ernest relates to another aspect of rhetoric in mathematics, namely the importance of persuasion for mathematics instruction.

Reyes (2014) asserts that it should be in the interest of rhetorical scholars to explore mathematics discourse, as it is the basis of techno-science. He analyses conceptual mathematical metaphors as an example for a mode of analysis of mathematics whose roots lie in rhetorical studies. Elsewhere, Reyes studies the rhetorical process during the invention of mathematics, and explores the introduction of infinitesimals by Newton and Leibniz as an example of the role of mathematical rhetoric in mathematical invention, in addition to its role in communicating the mathematics.

In conclusion, inquiry into relations between rhetoric and mathematics is growing in extent and richness. An increasing number of scholars explore the possibilities offered by the use of rhetorical concepts and ideas to gain better understanding of mathematics and mathematical education. Instead of viewing mathematics as a 'perfected, austere' product, they re-connect it to its 'human features', that in addition to formal logic utilizes persuasive argumentation and exploits rhetorical means.

Argumentation theory and 'The New Rhetoric'

Aberdein (2016) rejects the common thesis that mathematical reasoning is fundamentally different from everyday reasoning and that formal logic adequately models the practice of mathematical reasoning. Research in mathematical education uses argumentation theory to address aspects of mathematical argumentation other than formal logic, and for that purpose frequently uses Toulmin's model that permits schematic analysis of formal proofs as well as of arguments classified as deductively invalid. Toulmin's model has been shown to be an efficient framework to discuss local arguments as well as global argumentation structures (e.g. Knipping & Reid, 2013) and Inglis, Mejia-Ramos and Simpson (2007) claim that implementing Toulmin's full model (including rebuttals and qualifiers) should be used for this purpose. However, Toulmin's model has been criticized for not relating to the effect of the arguments on the audience, and for denigrating rhetoric in argumentation (Olbrechts-Tyteca, cited in Frank, 2004).

In 1958, Perelman and Olbrechts-Tyteca published 'The New Rhetoric' (PNR, translated in 1969), an argumentation theory based on the notion that argumentation aimed at justification of choices, decisions, and actions, is a rational activity complementing formal argument. PNR studies techniques used by an arguer to increase audience adherence to the arguer's theses and conditions that allow argumentation to begin and develop. PNR asserts that reducing an argument to its formal aspects undermines the rhetoric features that support its meaning; it recognizes the distance between dialectic and rhetoric but creates an alignment between them. This complex view at times produced an inherent ambiguity in definitions of some concepts. However, PNR adds meaningful layers of analysis beyond the analysis of argument structure and type achieved by using Toulmin's model.

PNR describes the 'threads that make the cloth of the argument': the starting points that establish a shared basis of agreement, the scope and organization of arguments, ways of creating presence to arguments, and different argumentation techniques. The audience plays a pivotal role in PNR since each 'thread', or aspect, is tied to what the arguer believes will deeply persuade the audience. This means that argumentation techniques should be adjusted to the audience's frame of reference, its previous knowledge, experiences, expectations, opinions and norms. So arguers construct arguments that they consider persuading for a particular audiences or convincing by a 'universal audience' (an arguer construct consisting of all reasonable humans) (van Eemeren et al., 2014).

In our study, we wish to analyze rhetorical aspects of proof presentation, in a scenario of a lecturer presenting a mathematical proof to a class of students. We use PNR as a theoretical framework as it incorporates aspects of rhetoric, argumentation and lecturer-classroom relations. Elsewhere (Gabel and Dreyfus, 2017), we demonstrate an analysis of other PNR aspects: scope and organization of the argumentation, and presence of proof elements. In this paper we address a different aspect: establishing a shared basis of agreement with the audience.

PNR's basis of agreement and its adaptation to proof teaching

According to PNR, argumentation can be successful if it advances from premises already accepted by the audience, i.e. the arguer established a shared basis of agreement with the audience. These premises are classified as follows: (1) *Premises relating to the real*: premises where the arguer claims recognition or acknowledgement of the universal audience. Those include: Facts, truths and presumptions. (2) *Premises relating to the preferable*: premises that have to do with the preferences of a particular audience. Those include: Values, value hierarchies and loci of preferable.

Facts and truths are statements already agreed to by the universal audience; they are considered to require no further justification. Truths stand for connections between facts. Presumptions are opinions or statements about what is the usual course of events which need not be proved, although adherence to them can be reinforced, and it is expected that at some point they will be confirmed. Values relate to the preference of one particular audience as opposed to another. They function as guidelines in making choices of the arguer (even though not all would accept them as good reasons). Values are normally arranged in value hierarchies, which are very important since different audiences may possess the same set of values arranged in different hierarchies. Values and value hierarchies generally remain implicit, but the arguer cannot simply ignore them. Loci of the preferable (aka commonplaces, Topoi) are premises used to justify values or hierarchies and express the preferences of a particular audience (e.g. quantity, quality, essence) (van Eemeren et al., 2014).

We have adapted PNR's classification of premises to the context of our study (analyzing proof presentation in class) as described in Table 1. We do not include in the table the loci of the preferable since they are highly abstract mental constructs which did not need adaptation.

		Adaptation to proof teaching
Premises relating to the real	Facts	Axioms, definitions, givens, previously consolidated symbols/results
	Truths	Lemmas, theorems, newly established symbols/results
	Presumptions	Statements or opinions about what previous knowledge to use, for example: mathematical techniques, proving methods, past theorems.
Premises relating to the preferable	Values	The preference or adaptability to a particular audience of a certain proving method or technique as opposed to another.
	Hierarchies of values	The hierarchies of values will affect audience preferences for choosing notation, proving method or mathematical technique.

Table 1: Adapting PNR types of premises to a proof teaching context

According to Perelman and Olbrechts-Tyteca, lack of agreement concerning the basis of agreement may occur at one or more of the following three levels:

a) *Status of premises*: e.g. if the arguer advances something as a fact but the audience wants to see it proven or if the arguer assumes a value hierarchy not accepted by the audience;
b) *Choice of premises*: e.g. if the arguer uses facts that the audience does not consider relevant to the argument or would have preferred not to see mentioned;
c) *Verbal presentation of premises*: e.g. if the arguer is presenting certain facts (acknowledged as relevant by the audience) in words which have connotations unacceptable to the audience.

The ability of creating a shared basis of agreement with the audience is crucial to the success of argumentation. Arguers should therefore carefully consider the status they ascribe to premises, the selection of premises, and the wording of explicit premises (van Eemeren et al., 2014). Examine, for example, two possible values related to proof teaching: (1) Certainty: every argument in the proof should be proved formally or at least justified; (2) Pedagogy: parts pf the proof should be left for the students. A lecturer may choose to leave parts of the proof as homework because her/his value hierarchy places (2) over (1). However, if the students have an opposite value hierarchy, this implies that the lecturer had a fallacy in the shared basis of agreement at the level of the status of his value hierarchy, which might consequently weaken students' persuasion.

The study – description and methods

Our research concerns the notion of 'The flow of a proof' (Gabel and Dreyfus, 2017) which encapsulates various aspects of the proof presentation. The flow is an outcome of the choices made by the lecturer regarding presentation of: (i) the logical structure of the proof (arranging the proof of the theorem into claims, which are proved in a specific order); (ii) informal features and considerations of the proof and proving process (e.g. examples, intuitions), and (iii) mathematical and instructional contextual factors. One aim of the research was to analyze global and local aspects the flow of the proof, in particular to examine rhetorical aspects of the proof presentation.

The research took place in a Number Theory course, given by the same lecturer to prospective mathematics teachers in two consecutive years. Each year, three lessons including the same suitable proofs (length, richness) were observed and audio-recorded. The three proofs were unrelated to each other. After each lesson students answered a questionnaire relating to cognitive and affective aspects; also, a reflective interview with the lecturer and individual interviews with students were conducted. The post-lesson interviews conducted with the lecturer in Year 1 were analyzed and interpreted, and a list of the lecturer's general considerations regarding proof teaching was produced. In this paper we relate to the second lesson in each year, in which the following theorem related to linear Diophantine equations $ax+by=d\ ,\ x, y \in Z$ was formulated and proved:

<u>*Theorem:*</u> *The greatest common divisor (gcd) of two integers* a, b*, at least one of which is not 0, equals the smallest natural number of the form* $ma+nb$*, where* m, n *are integers:* $\gcd(a,b) = \min\{ma+nb > 0 : m, n \in Z\}$.

The full proof of this theorem requires the use of previously proven results. In the next section we present a partial analysis of the shared basis of agreement, demonstrating the different types of lecturer premises (in the PNR sense) reflected in the proof presentation, consider potential fallacies in these premises and demonstrate the lecturer's attempt to fix these fallacies.

Examples of analysis of the basis of agreement

All post lesson interviews conducted with the lecturer in Year 1 were analyzed and interpreted as two sets of lecturer considerations (Gabel and Dreyfus, 2017). One of the sets contains general considerations for proof teaching. In the current paper we relate to three of these general considerations: (a) A proof should be mathematically complete and exact; (b) Some of the proof elements should be left for students to prove by themselves; and (c) Proof structure should be clear. One aspect of the clarity of proof structure was exhibited when the lecturer referred to the myth about Ariadne's thread: "I use… Ariadne's thread many times since mathematical proofs are built in such a way that you need to find the tip of the thread and just follow it…" We relate to these lecturer considerations as values that affect the lecturer's choice and status of premises.

Our examples stem from the last part of the proof as presented by the lecturer, and we will address lecturer premises as reflected in his arguments. In Year 1, just after proving that d is a divisor of a, leaving the (almost identical) proof that d is a divisor of b to the students, the lecturer said:

Lecturer: The same way we showed that d is a divisor of a it follows that d is a divisor of b, so d is a common divisor of a,b. Now, it can't be smaller than the gcd, yes? Because once I write the equation $ax+by=d$ then…like we said in the beginning of the lesson, we said that this d must be divisible by $\gcd(a,b)$ … so it can't be smaller then $\gcd(a,b)$ and that means it is equal to $\gcd(a,b)$.

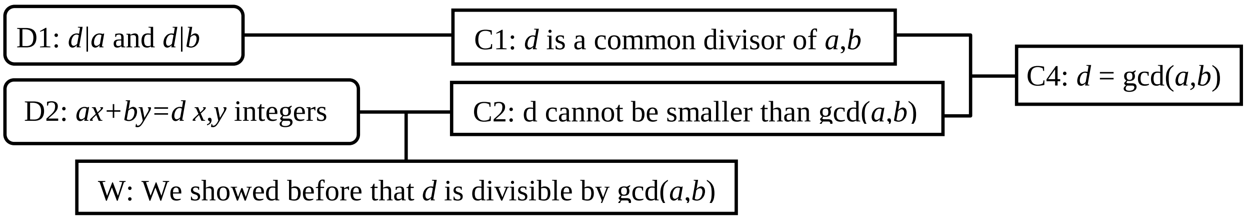

Figure 1: Toulmin's scheme representation of 1st explanation

The argumentation in this excerpt is represented by the Toulmin's scheme in Figure 1. We suggest a possible interpretation of the lecturer's explicit and implicit premises reflected in this explanation. For the lecturer this argumentation requires very little justification (if any) and he presents it as a chain of facts (D1, C1, D2 and possibly C2) that does not need to be discussed, and whose connection results in the conclusion (C4) in a self-evident way. The lecturer implicit presumption is that in order to prove that $d=\gcd(a,b)$ one needs to show two inequalities: $(d \le \gcd(a,b)$ and $d \ge \gcd(a,b)) \Rightarrow d=\gcd(a,b)$; he believes that this presumption does not need to be made explicit and that he and the students share this presumption. As for the values reflected in this explanation and their hierarchy, since the lecturer chose to leave some of the proof (that $d|b$) to the students, in this case the pedagogical value of leaving some of the proof elements for students was placed above the value regrading proof completeness. In addition, we recognize another implicit value: for the lecturer the 'tip of Ariadne's thread' here is to realize that d is a common divisor of a,b from which the rest of the proof just unfolds.

However, the students had difficulties following this first explanation and asked the lecturer to repeat it. A possible reason for this difficulty is that the premises that the lecturer considered as facts were not considered as facts by the students and required further justification. For example, the students probably needed an explicit justification for the argument "if d is a common divisor of a,b then $d \leq \gcd(a,b)$". Moreover, the lecturer's implicit presumption regarding the natural proving technique (the two inequalities) is not necessarily clear and natural to the students. In PNR language, there was a lack of agreement about the status and choice of the lecturer's premises, which caused a fallacy in establishing a shared basis of agreement. So, following the students' request, the lecturer instantly explained again the argument in Year 1 lesson as follows:

Lecturer: We said that d, as a minimal element of this set [$\{ma + nb > 0 : m,n \in Z\}$], is of the form $a \cdot$integer+$b \cdot$integer. Now the first thing I have shown today is that in such a situation, actually this is a result of theorem 1 that we have used before,… it follows that d must be divisible by $\gcd(a,b)$, yes? Once I can write a number as a linear combination of two numbers a,b, with integer multipliers m,n, this d must be divisible by $\gcd(a,b)$. On one hand it must be divisible by $\gcd(a,b)$; on the other hand…it is a divisor of a,b. It can't be smaller then $\gcd(a,b)$ so it can only be equal to it. Because $\gcd(a,b)$ is the greatest common divisor, yes? And that finishes the proof… d is a common divisor of a,b that also has to be divisible by $\gcd(a,b)$ so we conclude that $d = \gcd(a,b)$…

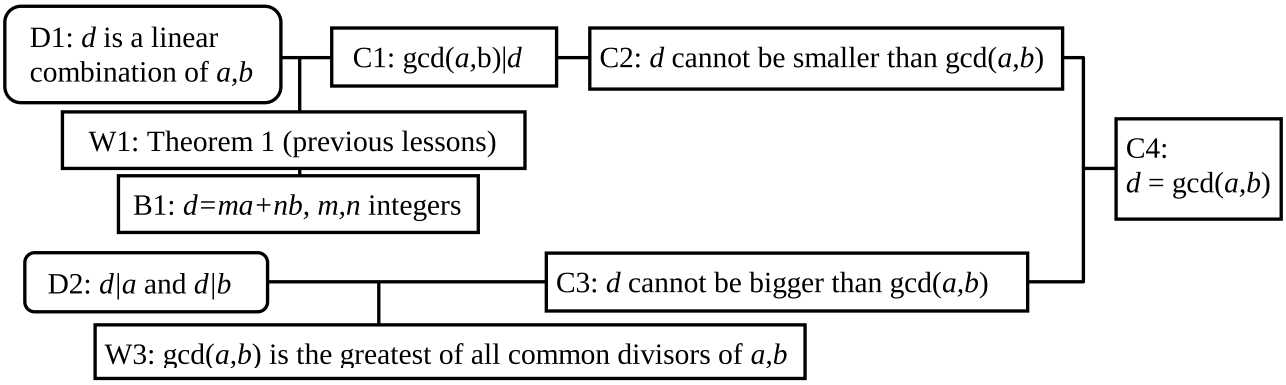

Figure 2: Toulmin's scheme representation of 2nd explanation

The argumentation in this excerpt is represented by Toulmin's scheme in Figure 2. In the second explanation the lecturer added some justification (W1, B1, W3) to the conclusions C1, C2 and C3; we interpret they were not presented as facts but rather as truths, i.e. the lecturer changed the status of the premises to establish a stronger basis of agreement with the students. However, his presumption still remained implicit – a point which we will revisit shortly.

Before the lesson in Year 2 the lecturer was informed by the researcher about some student difficulties that were found in the post lesson students' questionnaires of Year 1; in particular, the questionnaires reflected that the last part of the proof, where combining the inequalities $d \leq \gcd(a,b)$ and $d \geq \gcd(a,b)$ leads to the equality $d = \gcd(a,b)$ was not trivial to the students. For lack of space we will concentrate on demonstrating the change in the lecturer's presumptions and his value hierarchy between Years 1 and 2. The lecturer took the reported students' difficulty into account and during the lesson in Year 2, just before the last part of the proof he explained:

Lecturer: Here we are doing something similar to what we already had in the past, when we wanted to prove that two numbers are equal…

Student: We assume that they are unequal…

Lecturer: No, we should prove two inequalities, right? Or refute two inequalities, right? I remind you, we already used it: when we wanted to show that two numbers are equal then we need to show that it is impossible that a is smaller than b … it is impossible that a is bigger than b, or in other words … to show that a is not smaller than b is actually showing that $a \geq b$, and instead of showing that a is not bigger than b we'll show that $a \leq b$. If I want to show that $a = b$, I need to show that $a \geq b$, i.e. not smaller than b, and that $a \leq b$, meaning that a is not bigger than b. Once I will show these two inequalities I am done, I've shown that $a = b$.

Here, the lecturer consciously makes his presumption explicit to the students, justifies the choice of this presumption and makes it relevant. By explicating his presumption the lecturer also caused a change of value hierarchies: he enhanced the clarity of the proof structure, making it more explicit before going into the details of the proof. Indeed, the lecturer also explicitly declared:

Lecturer: It remains to prove the other inequality: $\gcd(a,b) \geq d$. In fact, I will show you that this… minimum of the set, *d*, is a divisor of *a,b*… and this will end the story…

So before formally proving that *d* is a divisor of *a,b,* the lecturer spread out his proving plan, identifying "the tip of Ariadne's thread" and explained exactly why this "will end the story".

We conclude this short example by stressing that while Toulmin's model enables the presentation and analysis of the argumentation structure, PNR complements it by relating to other argumentation qualities, such as the adaptability to the intended audience. The fallacies that have been mentioned in the example were related to the status and choice of premises.

Concluding remarks

Weber and Mejia-Ramos (2014) demonstrated that mathematics students and mathematicians have different perceptions regrading students' responsibilities when reading a mathematical proof: the students believe that reading a good proof is quite a passive process, one in which they are not expected to construct justifications, diagrams or sub-proofs by themselves, and they may simply follow each and every step. Mathematicians believe the opposite. This tension between the students' and their teachers' beliefs supports our interpretation regarding the different value hierarchies that the lecturer and the students have. But beyond the identification of the difference, we argue that PNR has the potential to explain the consequences of that difference on the effectiveness of argumentation; in other words, PNR provides a suitable framework to identify ways to increase argumentation effectiveness, for instance by referring to the shared basis of agreement.

Moreover, PNR relates to other rhetorical and dialectical aspects of argumentation. Some of these aspects (scope and organization and presence) have been studied in Gabel and Dreyfus (2017); others, namely argumentation techniques and the manner by which PNR complements the use of Toulmin's scheme, need further study. One advantage of PNR is that because of its theoretical scope, all these aspects can be studied within a single unifying theoretical framework.

Although Perelman perceived PNR as a complement of formal logic and focused on disputes in which values play a part (van Eemeren et al., 2014), we argue that PNR can be adapted to be a productive theoretical framework in the context of proof teaching, particularly the flow of a proof: firstly, Perelman was much inspired by formal logic (mainly the work of Frege), and secondly, the context of argumentation in the mathematics classroom resembles PNR's context of persuading an audience. Thus PNR is a comprehensive argumentation theory that can broaden and enrich researchers' perspectives regarding different aspects of mathematics classroom argumentation.

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