Unilaterality and dry friction in the dynamics of rigid body collections

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UNILATERALITY AND DRY FRICTION IN THE DYNAMICS OF RIGID BODY COLLECTIONS

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1. INTRODUCTION

Dynamical evolution problems concerning collections of rigid bodies are *nonsmooth* in three respects :

- the mutual impenetrability of the considered bodies or their possible confinement by external boundaries with given motion impose on the configuration parameters $q = (q^1, \ldots, q^n)$, to remain in a region with corners and edges,
- in the event of a *collision*, velocity jumps are expected,
- the law of Coulomb, that we shall use for representing *dry friction* at possible contact points, consists of a nonsmooth relation between local velocities and contact forces.

Methods from "Nonsmooth Mechanics" [1] [2] may be used in formulating these problems. This generates computational algorithms which face nonsmoothness without resorting to any technique of regularization and prove efficient enough to treat the motion of large collections of rigid bodies by using only a microcomputer.

This text presents the main features of such a formulation and sketches the related algorithms. The lecture was illustrated with videoprojector simulations created by running programs or read from the microcomputer hard disk concerning :

- The two-dimensional motion of multistone buildings supported by quaking ground, including the possible phases of collapse. Stone blocks are viewed as superimposed without mortar, which is the case of ancient monuments [3].
- The motion of granular materials under various circumstances, in two or three dimensions. As in most models investigated in literature, grains are treated as rigid balls. Examples of the *fluidization* caused in a granular bed by the vibration of containing vessel were displayed, as well as the slow shear motion produced in compact grain assemblies by the displacement of boundaries.

Two drawings extracted from these simulations are annexed at the end :

Example 1 shows the progressive ruin of a wall supported by a ground affected with horizontal sinusoidal oscillation.

Example 2 shows the simulation of an experiment currently made with "Schneebeli materials", i.e. piles of parallel cylindrical pins. The lower and left boundaries are fixed. The right boundary moves with constant velocity. The upper boundary is free to move vertically but experiences a given external load.

This is a general fact that any application of Science to a real situation depends on the quantitative information one has been able - or willing - to collect about it. Such an information is always uncomplete ; the model one uses just defines the "format" in which the available data are recorded and processed and in which predictions are eventually expressed. Since it relies on fragmentary data, the model cannot be expected to generate exhaustive predictions.

These facts of life are conspicuous in the present subject matter. Real bodies are not rigorously rigid. It is only said, for the two above examples, that stone blocks or grains exhibit *very small deformability* ; the most objective way of treating this - rather vague - piece of information consists in using the model of a perfectly rigid body. The drawback is that, in case of multiple contacts, such a state of information leaves the system of contact forces underdetermined. If the values of these forces are needed for predicting the motion, our algorithms implicitely make choices which depend, for instance, of the numbering of bodies or other computational details.

The assertion of *dry friction* covers a physically intricate set of phenomena. Coulomb law, in spite of its limited precision, provides in very numerous cases the only practical framework for dealing with this sort of contact condition.

It is still more difficult to quantitatively apprehend *collisions* and the velocity jumps they generate. Here also one has to be content with a pretty crude description of reality. The systematization we present in Sec. 6 below proves logically consistent and computationally efficient. Its domain of physical validity has to be determined through experiments which now are only at their preliminary stage : experiments on blocks have been conducted, interactively with our programs, by M. Raous [4]. Comparaison is also under way with the experiments on collections of metal balls of J. Clément, J. Duran and J. Rajchenbach [5].

The existential study of solutions is not evoked in this paper. Only partial results have been obtained so far [6]-[11], but research still continues. Uniqueness of solution to an initial value problem should not be expected in general.

No comparison has yet been made between our "nonsmooth" computation techniques and the numerical results which would be produced from the same data by more conventional software, treating the impenetrability constraints through penalization methods. Several existing codes of this sort are related to the pioneering work of P. Cundall [12]. Recently, codes devised for Molecular Dynamics simulations on big computers have been applied to the dynamics of granular materials [13][14], through the approximation of impenetrability constraints by sufficiently steep interaction potentials.

2. ANALYTICAL SETTING

Let the possible configurations of the system be parametrized, at least locally, through generalized coordinates, say $q = (q^1, q^2,..., q^n)$. As usual such a reduction to finite freedom is assumed to result from (bilateral) ideal constraints, namely the strict rigidity of the various parts and the possible operation of frictionless linkages.

After constructing the parametrization, one additionally takes into account some *unilateral constraints* whose geometric effect is expressed by a finite set of inequalities

$$f_\alpha(t, q) \leq 0\,, \quad \alpha \in \{1, 2,..., \kappa\}, \tag{2.1}$$

where $f_1, f_2,...,f_\kappa$ are given functions. Commonly, such inequalities describe the mutual impenetrability of some parts of the system or the confinement of some of them by external boundaries with prescribed motion. Equality $f_\alpha = 0$ then corresponds to the occurrence of a *contact*.

In all the sequel, it will be assumed that each of the functions f_α is C^1, with $\partial f_\alpha/\partial q \neq 0$ at least in a neighborhhood of the hypersurface $f_\alpha = 0$ of $\mathbf{R}^{n+1}$.

For every imagined motion $t \rightarrow q(t)$ and for t such that the derivative $\dot{q}(t) \in \mathbf{R}^n$ exists, the kinetic energy has an expression of degree 2 in $\dot{q}$, say

$$\mathcal{E}_k(t,q,\dot q) = \frac{1}{2} A_{ij}(t,q)\, \dot q^i \dot q^j + B_i(t,q)\, \dot q^i + C(t,q), \tag{2.2}$$

where A is a symmetric positive definite n×n-matrix, $B \in \mathbf{R}^n$ and $C \in \mathbf{R}$.

Then, as far as *smooth*, i.e. twice differentiable, motions are concerned, the system Dynamics is governed by Lagrange's equations, written below as an equality in $\mathbf{R}^n$

$$A(t,q)\, \ddot q = F(t,q,\dot q) + \sum_\alpha r^\alpha. \tag{2.3}$$

Expression F here comprises standard terms of Lagrange's equations and the covariant components, relative to the parametrization (q), of some applied forces supposed given as functions of time, position and velocity. The element r^α of $\mathbf{R}^n$ is made of the covariant components of the *contact forces* experienced by the system in case the contact $f_\alpha = 0$ holds.

The definition of the covariant components of a force is classically connected with the system kinematics.

Suppose first that inequality $f_\alpha \leq 0$ expresses the mutual impenetrability of some pair of rigid constituents of the system, say $\mathcal{B}$ and $\mathcal{B}'$, so that equality $f_\alpha = 0$ corresponds to these two bodies touching each other at some point of space denoted by M_α. This we shall assume to be an isolated contact point, but other contacts, corresponding to different values of α, may also be effective between the same bodies. For every imagined motion $t \to q(t)$ bringing the system into the investigated position, the velocities $\mathcal{V}_\alpha$ and $\mathcal{V}'_\alpha$ of the respective particles of $\mathcal{B}$ and $\mathcal{B}'$ passing at point M_α let themselves be expressed as affine functions of the possible value u of the derivative $\dot q$. The same is thus true for the *relative velocity* $\mathcal{U}_\alpha = \mathcal{V}_\alpha - \mathcal{V}'_\alpha$ of $\mathcal{B}$ with respect to $\mathcal{B}'$ at this point, say

$$\mathcal{U}_\alpha = G_\alpha u + \mathcal{W}_\alpha, \tag{2.4}$$

where $G_\alpha : \mathbf{R}^n \to \mathbf{R}^3$ denotes a linear mapping, depending on t and q. No attention is paid at this stage to the imagined motion preserving contact or not. The term $\mathcal{W}_\alpha \in \mathbf{R}^3$, a known function of t and q, vanishes in the usual case of a *scleronomic*, i.e. time-independent, parametrization.

Let $\mathcal{R}^\alpha$ denote the contact force that body $\mathcal{B}$ experiences at point M_α from body $\mathcal{B}'$; then $\mathcal{B}'$ experiences from $\mathcal{B}$ the force $-\mathcal{R}^\alpha$. Classically, the covariant component of this pair of forces are given by

$$r^\alpha = G^*_\alpha\, \mathcal{R}^\alpha, \tag{2.5}$$

whith $G^*_\alpha : \mathbf{R}^3 \to \mathbf{R}^n$ denoting the transpose of G_α.

Similar formulas hold if inequality $f_\alpha \leq 0$ expresses the confinement of a rigid part $\mathcal{B}$ of the system by some external boundary with prescribed motion. Assume that equality $f_\alpha = 0$ corresponds to contact taking place at some point,

here again denoted by M_α. The relative velocity, at this point, of $\mathcal{B}$ with respect to the boundary still has an expression of the form (2.4), where $\mathcal{W}_\alpha$ now takes into account the given velocity of the boundary. And r^α in (2.5) turns out to be the covariant components of the force $\mathcal{R}^\alpha$ only, acting on $\mathcal{B}$. Its counterpart $-\mathcal{R}^\alpha$, exerted by $\mathcal{B}$ upon the boundary, is no more in this case a force experienced the system.

In both cases, the following relationship is found [15] to hold between $\partial f_\alpha / \partial q$ and the normal unit vecteur $\mathbf{n}^\alpha$ at point M_α to the two contacting surfaces, *directed toward* $\mathcal{B}$

$$\exists \lambda \geq 0 \quad \text{such that} \quad G^*_\alpha\, \mathbf{n}^\alpha = -\lambda\, \partial f_\alpha / \partial q. \tag{2.6}$$

In all the sequel, we shall assume that the mapping G_α is *surjective* of $\mathbf{R}^n$ to $\mathbf{R}^3$; equivalently, the mapping G^*_α is *injective* of $\mathbf{R}^3$ to $\mathbf{R}^n$. Only some special positions of certain systems submitted to linkages may give rise to "wedging" effects which break this assumption.

3. CONTACT LAWS

As far as smooth motions are concerned, the system dynamics is governed by the differential equation (2.3) where the elements r^α are involved in the problem through (2.4), (2.5), $u = \dot q$, and through a system of *contact laws*

$$\text{law}_\alpha(t, q, \mathcal{U}_\alpha, \mathcal{R}^\alpha) = \text{true}. \tag{3.1}$$

The latter describes what, in physical space, happens at contact α. Strictly speaking, $\mathcal{U}_\alpha$, $\mathcal{R}^\alpha$, G_α make sense only in the case of effective contact, so the index α in conditions (2.3) to (3.1), would have to range through the subset $\{\alpha \in \{1,2,\ldots,\kappa\} : f_\alpha(t,q)=0\}$. Actually, in existential studies, as well as in computation, it proves convenient to make α range through the larger set

$$J(q) = \{\alpha \in \{1,2,\ldots,\kappa\} : f_\alpha(t,q) \geq 0\} \tag{3.2}$$

The matrix G_α which was so far defined only for q lying in the hypersurface $f_\alpha(t,q) = 0$ has then to be extended (in some smooth arbitrary way) to a neighbourhood of this hypersurface. One similarly extends the definition of the unit vector $\mathbf{n}^\alpha$, preferably with preservation of (2.6).

One is looking for motions verifying the κ inequalities (2.1) for every t. Instead of explicitely adjoining these to (2.3)-(3.1), *we prefer to rely on some adequate formulation of the contact laws* (2.7) *for securing them.*

DEFINITION. *A contact law, i.e. a relation of the form* (3.1), *is said* complete *if it involves the three following implications*

$$f_\alpha(t, q) < 0 \Rightarrow \mathcal{R}^\alpha = 0. \tag{3.3}$$
$$f_\alpha(t, q) \geq 0 \Rightarrow \mathbf{n}^\alpha.\mathcal{U}_\alpha \geq 0, \tag{3.4}$$
$$\mathbf{n}^\alpha.\mathcal{U}_\alpha > 0 \Rightarrow \mathcal{R}^\alpha = 0. \tag{3.5}$$

Let us comment on the importance of (3.4). Put

$$\mathcal{K}_\alpha(t, q) = \begin{cases} \{\mathcal{U} \in \mathbf{R}^3 : \mathbf{n}^\alpha.\mathcal{U} \geq 0\} & \text{if } f_\alpha(t, q) \geq 0 \\ \mathbf{R}^3 \text{ if } f_\alpha(t, q) < 0, \end{cases} \tag{3.6}$$

called the set of the *right-admissible* values for the relative velocity of the two concerned bodies at point M_α. The following is easily established [15]

PROPOSITION. *Let* I *be a time-interval with origin* t_0 *and let a motion* $q: I \to \mathbf{R}^n$ *be defined through a locally integrable velocity function* $u: I \to \mathbf{R}^n$ *by*

$$t \to q(t) = q(t_0) + \int_{t_0}^{t} u(s)\,ds. \tag{3.7}$$

If $\mathcal{U}_\alpha(t) = G_\alpha(t, q)u(t) + \mathcal{W}_\alpha(t, q)$ *belongs to* $\mathcal{K}_\alpha(t, q(t))$ *for almost every* t *and if inequality* $f_\alpha(t, q(t)) \leq 0$ *holds at the initial instant* t_0, *then this inequality holds for every* $t \in I$.

In other words, provided the initial position is correct, the impenetrability condition $f_\alpha \leq 0$ is automatically taken care of by (3.4). Observe that this statement is sensitive to the ordering of time. In the symmetric assertion involving, instead of the initial instant t_0, the possible final point of I, one should replace $\mathcal{K}_\alpha$ by $-\mathcal{K}_\alpha$, which may be viewed as the set of the *left-admissible* values of $\mathcal{U}_\alpha$.

The importance of (3.6) will only become apparent in further Sections, devoted to the study of collisions and to numerical algorithms.

EXAMPLE : *Frictionless contact.*

Classically, the contact at point M_α is said frictionless if

$$\exists \rho \geq 0 \qquad \text{such that} \qquad \mathcal{R}^\alpha = \rho\, \mathbf{n}^\alpha. \tag{3.8}$$

Let us incorporate this into a complete contact law.

The subset $\mathcal{K}_\alpha(t,q)$ of $\mathbf{R}^3$ defined in (3.6) is closed and convex. In Convex Analysis, a general definition is given for the *normal (outward) cone* to such a subset at any point x of $\mathbf{R}^3$. This cone is empty if and only if x does not belong to the considered subset ; it reduces to $\{0\}$ if x is an interior point. In what concerns $\mathcal{K}_\alpha(t, q)$, its normal cone at point x, that we shall denote by $\mathcal{N}^\alpha(x)$, essentially equals $\{0\}$ if $f_\alpha(t,q)<0$ since $\mathcal{K}_\alpha(t,q)=\mathbf{R}^3$ in that case. For $f_\alpha(t,q) \geq 0$, $\mathcal{K}_\alpha(t,q)$ is a closed half-space, so $\mathcal{N}^\alpha(x)=\{0\}$ if $\mathbf{n}^\alpha.x>0$ and $\mathcal{N}^\alpha(x)=\varnothing$ if $\mathbf{n}^\alpha.x<0$; otherwise, i.e. if x lies in the boundary of the half-plane, $\mathcal{N}^\alpha(x)$ consists of the half-line generated by $-\mathbf{n}^\alpha$.

So, by asserting

$$-\mathcal{R}^\alpha \in \mathcal{N}^\alpha(\mathcal{U}), \tag{3.9}$$

a complete contact law is formulated, which involves (3.8).

4. COULOMB FRICTION

Let us stipulate from start that the contact law we are to define is *complete*. Then $\mathcal{R}^\alpha$ can be nonzero only if $f_\alpha(t, q) \geq 0$ and $\mathbf{n}^\alpha.\mathcal{U}_\alpha = 0$; the latter means that $\mathcal{U}_\alpha$ belongs to T, the vector-plane tangent at point M_α to the contacting bodies. This is just the situation in which the law of Coulomb is classically considered.

For brevity, let us omit the index α. Friction data may be specified by giving the *Coulomb cone*, say C, at the considered contact point. This is a (closed, convex) conical region of $\mathbf{R}^3$, axisymmetric about $\mathbf{n}$, with half-angle equal to the *angle of friction*.

Defining this subset of $\mathbf{R}^3$ is equivalent to giving its *indicator function* ψ_C (namely $\psi_C(x)=0$ if $x \in C$ and $+\infty$ otherwise), a lower-semicontinuous convex function on $\mathbf{R}^3$. The *subdifferential* $\partial\psi_C(x)$ equals the *normal cone* to C at point x, in the sense recalled in the preceding Section. We propose to formulate Coulomb law in the following form

$$-\mathcal{U} \in \operatorname{proj}_T \partial\psi_C(\mathcal{R}). \tag{4.1}$$

In fact this relation compels $\mathcal{R}$ to belong to C, since otherwise the right-hand side would be empty. If $\mathcal{R} \in \operatorname{int} C$, the right-hand side reduces to $\{0\}$, so $\mathcal{U}=0$. If $\mathcal{R}$ belongs to the boundary of C and is nonzero, the normal cone to C at this point equals the half-line, outward normal to this boundary. Elementary Geometry readily yields the traditional statement of Coulomb law corresponding to this case. Finally, if $\mathcal{R}=0$, the subdifferential $\partial\psi_C(\mathcal{R})$ equals the *polar cone* of C, whose projection onto T equals the whole of T, so (4.1) coincides with the traditional Coulomb law in this case too.

By using instead of C a non axissymetric convex cone, one obtains a plausible description of *anisotropic friction.*

Numerous variants may be derived from (4.1). For instance [16], one may define on $\mathbf{R}^3$ the lower-semicontinuous convex function

$$\mathcal{U} \to \theta(\mathcal{U}) = \frac{1}{2}\|\mathcal{U}\|^2 + \psi_T(\mathcal{U}). \tag{4.2}$$

and standard arguments of Convex Analysis yield the equivalence of (4.1) to

$$0 \in \partial\psi_{\mathcal{C}}(\mathcal{R}) + \partial\theta(\mathcal{U}). \tag{4.3}$$

G. de Saxcé [17] has recently proposed alternative formulations of Coulomb law, stipulating that the pair $(\mathcal{U}, \mathcal{R})$ should be a minimum point or a saddle point for some functions called *bipotentials*.

5. NONSMOOTH EVOLUTIONS

In the event of a *collision*, the velocity function u is expected to have a jump. Even without collision, such jumps have long been recognized to be possible in the dynamics of systems involving dry friction [18]. Their occurrence is due to paroxysms in the contact forces, similar to the *locking* effect commonly observed in the statics of the same systems. One may call them *frictional catastrophes*.

Mathematically, the velocity function $u: I \to \mathbf{R}^n$, connected through (3.7) with the evolution function $q: I \to \mathbf{R}^n$, can no more be a solution on the whole interval I to the differential equation (2.3), that one could equivalently write

$$A(t, q)u'_t = F(t, q, u) + \sum_\alpha r^\alpha. \tag{5.1}$$

The most natural framework allowing u to exhibit jumps consists in assuming that this function of the real interval I has *locally bounded variation*, i.e. it has bounded variation on every compact subinterval of I. Notation $u \in \mathrm{lbv}(I, \mathbf{R}^n)$. The reader may refer to [19] as an expository text on this subject. Classically, with every such u, an $\mathbf{R}^n$-valued measure on I is associated, called the *differential measure* or *Stieltjes measure* of u ; let us denote it by du.

Looking again at the smooth case, governed by the differential equation (5.1), let us recall that the existence question for solutions to this equation is usually addressed by transforming it into an integral equation. This supposes that all terms are locally integrable with respect to the Lebesgue measure dt on I, so that (5.1) may equivalently be viewed as the following *equality of* $\mathbf{R}^n$-*valued measures* on the interval I

$$A(t, q)u'_t\, dt = F(t, q, u)\, dt + \sum_\alpha r^\alpha\, dt. \tag{5.2}$$

Now the term $u'_t\, dt$, namely the $\mathbf{R}^n$-valued measure possessing the element u'_t of $L^1_{loc}(I, dt; \mathbf{R}^n)$ as *density function* relative to the Lebesgue measure dt, is nothing but the differential measure du of the function u. In fact, the latter being in this case locally absolutely continuous, belongs to $\mathrm{lbv}(I, \mathbf{R}^n)$

A natural extension of (5.2) therefore is

$$A(t, q)du = F(t, q, u)\, dt + \sum_\alpha dR^\alpha, \tag{5.3}$$

where the $\mathbf{R}^n$-valued measures dR^α denote the covariant components of the *contact impulsions*. This may be called a *measure differential equation*.

The traditional theory of *percussions* provides an intuitive introduction to this extension of Classical Dynamics. In fact, percussions occuring at discrete instants are $\mathbf{R}^n$-valued Dirac measures on the considered time-interval. They constitute *atoms* of the contact impulsion measures. The reader in want of a more elaborate theoretical background for the equations of *Nonsmooth Dynamics* might refer to [21].

In smooth motions, contact impulsion measures admit contact *forces* as density functions with regard to dt, the Lebesgue measure on I. In any case, there exists (non uniquely) a nonnegative real measure, say $d\lambda$, on the interval I, relative to which the measures du, dt, dR^α possess density functions, say $u'_\lambda \in L^1_{loc}(I, d\lambda; \mathbf{R}^n)$, $t'_\lambda \in L^1_{loc}(I, d\lambda; \mathbf{R})$ and $R'^\alpha_\lambda \in L^1_{loc}(I, d\lambda; \mathbf{R}^n)$ respectively. Therefore (5.3) is equivalent to this equality of elements of $\mathbf{R}^n$

$$A(t, q)\, u'_\lambda(t) = F(t, q, u)\, t'_\lambda(t) + \sum_\alpha R'^\alpha_\lambda(t), \tag{5.4}$$

holding for every t in I, with the possible exception of a $d\lambda$-negligible subset (equivalently one may assign null values to the respective density functions on such a subset, so as to make (5.4) hold *everywhere*).

The use of density functions makes clear the calculation of the $\mathbf{R}^n$-valued measure dR^α as the "covariant component" of the corresponding contact impulsion. The latter is a $\mathbf{R}^3$-valued measure on I that we shall denote by $d\mathcal{S}^\alpha$. The base measure $d\lambda$ can be chosen in such a way that $d\mathcal{S}^\alpha$ possess a density function $\mathcal{S}'^\alpha_\lambda \in L^1_{loc}(I, d\lambda; \mathbf{R}^3)$. Then, similarly to (2.5),

$$R'^\alpha_\lambda(t) = G^*_\alpha(t,q)\, \mathcal{S}'^\alpha_\lambda(t).$$

In particular, the considered contact develops a percussion at time t_c if and only if the measure $d\lambda$ possesses at the point t_c of I an atom, with mass $\lambda_c > 0$, such that $\mathcal{S}'^\alpha_\lambda(t_c) \neq 0$. The value of this percussion equals the vector $\lambda_c \mathcal{S}'^\alpha_\lambda(t_c)$.

Any $u \in \mathrm{lbv}(I, \mathbf{R}^n)$ possesses at every point t of I a *right-limit* and a *left-limit* respectively denoted by $u^+(t)$ and $u^-(t)$ (by convention the left-limit at the initial instant t_0 is interpreted as $u(t_0)$ and the symmetric convention should be applied to the possible final point of I). Typically, for every compact subinterval [a, b] of I, one has

$$\int_{[a,b]} du = u^+(b) - u^-(a).$$

This applies in particular to the case a = b : in other words the integral of du over the singleton {a} equals the jump of u at point a. Thus the point a carries an atom of the measure du if and only if the jump is nonzero.

This formula shows that the values that u *may take at its discontinuity points bear no relationship with* du. *These values are also immaterial in relation* (3.7), which connects u with the evolution $q:I \rightarrow \mathbf{R}^n$, since the discontinuity points of an lbv function classically make a countable subset of I, hence Lebesgue-negligible.

In case of a collision affecting the system at time t_c, every effective contact (including of course those suddenly introduced by the collision) is susceptible to exhibit a percussion. One is tempted to introduce the percussion vector $\lambda_c \mathcal{S}'^{\alpha}_{\lambda}(t_c)$ in the place of $\mathcal{R}^\alpha$ into a law of the form (3.1). *Assuming this law positively homogeneous with regard to* $\mathcal{R}^\alpha$ *makes the choice of the base measure* $d\lambda$ *indifferent*. But then, which value of $\mathcal{U}_\alpha = G_\alpha u + \mathcal{W}_\alpha$ should be considered? The most prudent answer to this question would be that the percussion vector conveys only an average of very large interaction forces arising during some very short, but intricate, episode. The process of interaction might appear too complex to be described by a relation involving this only vector, whatever value of $\mathcal{U}$ is adopted.

Finite element simulations of collisions, with body deformations completely taken into account, were presented during the lecture.

Anyway, since the data needed for investigating deformations are, in most practical situations, unavailable we are going to propose a pragmatic procedure leading to plausible calculation.

6. DISSIPATION INDEX

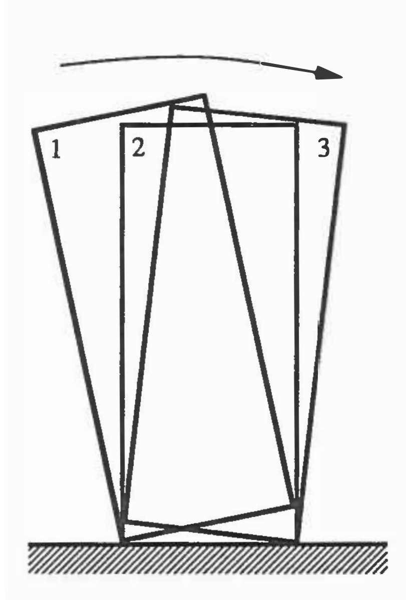
A collision consists of a short contact episode. The more rigid the involved bodies are, the shorter this episode should be. Even if the material, the bodies are made of, is assumed perfectly elastic, energy conservation cannot be expected. In fact, disturbances are likely to propagate from the collision locus to the whole system and also, if the latter is linked with some external support, to the outside world. After contact recedes, a state of vibration should persist. At the macroscopic observation level, this does not contradict the rigidity assertion of the system bodies, but the energy involved in such a microscopic agitation could not be negligible. So the collision macroscopically appears as a *dissipative process*. This is all the more true if material dissipation also occurs, due to friction at the contact locus or to permanent deformation and internal damping affecting the bodies.

In the Rational Mechanics of the past hundred years, authors have tended to localize the collision mechanism in the immediate vicinity of the contact locus, the affected bodies being otherwise considered as perfectly rigid. In that restricted framework, some plausible analysis may be developed. As early as 1880, G. Darboux [22] (see also [24]) applied to rigid body collisions a *multiple scaling* method : the very short duration of the contact episode is parametrized through a micro-time, say τ. Using the equations of rigid

dynamics one may study the velocities of the involved bodies as functions of τ, while their positions are treated as constant. Coulomb law can in that way be invoked, so as to relate, for every τ, the contact force with the sliding velocity calculated at the contact point. In our opinion, the probable global microdeformation of the bodies limits the applicability of this approach.

Double scaling in time has also been used in [23] for analysing the collision of two bodies which, in the vicinity of the impact point, are assumed to admit some very slight viscoelastic deformability.

Concerning the collision of two otherwise unconstrained bodies, Newton's *restitution coefficient* can be experimentally identified only for simple geometry, such as a pair of balls. It can also reasonably be applied to the collision of a ball with the boundary of a massive obstacle.



Anyway, the concept is not valid in case several contacts are present at the collision instant. This is demonstrated by the example of the *rocking* of a slender rectangular block on a fixed horizontal ground with zero alleged restitution coefficient. Assume a slight concavity of the lower edge so that, when the block attains its vertical position, contact takes place through the two lower corners. Clearly, if the assumption of zero restitution coefficient is applied to both points, no rocking is found.

One of us has previously proposed [15] a consistent formalism for specifying the "degree of bounciness" in systems with an arbitrary number of contacts. This consists in asserting that, even for nonsmooth motions, a contact law of the form (3.1) relates, for every α and every t, the density of contact impulsion $\mathcal{S}'^{\alpha}_{\lambda}(t)$ to the *average velocity* defined as the weighted mean of the right- and left- limits of the local velocity $\mathcal{U}_\alpha = G_\alpha u + \mathcal{W}_\alpha$, namely

$$\mathcal{U}^a_\alpha = \frac{1-\delta_\alpha}{2}\,\mathcal{U}^-_\alpha + \frac{1+\delta_\alpha}{2}\,\mathcal{U}^+_\alpha. \qquad (6.1)$$

Here δ_α denotes an element of [0,1] called the *dissipation index* of the considered contact. The name is justified by the following formula, concerning the case where all δ_α have the same value δ. One calculates the *decrease* of kinetic energy at the time of a collision

$$\mathcal{E}_k^- - \mathcal{E}_k^+ = \frac{1}{2} A_{ij}(u_i^+ - u_i^-)(u_j^+ - u_j^-)\delta - \sum_{\alpha} \mathcal{U}_\alpha^a . \mathcal{S}^\alpha. \qquad (6.2)$$

The first term on the right-hand side is nonnegative since A_{ij} is a positive definite matrix and $\delta \geq 0$. All terms after the Σ symbol are nonpositive if the contact laws assumed to hold at the various contact points all are *dissipative* (such are, in particular, the law of Coulomb or the law of frictionless contact as it was formulated in Sec. 3).

The above formula is actually a special case of the expression of the *differential measure* of the function $t \mapsto \mathcal{E}_k$ in Nonsmooth Dynamics, obtained through the *Differential Calculus of functions with locally bounded variation* [15][19][20].

Let us now specify that the condition $\mathrm{law}_\alpha(\mathcal{U}_\alpha^a, \mathcal{S}^\alpha) = \mathrm{true}$, assumed to hold at contact α at the time of a collision, is a *complete contact law*. In view of (3.4) and (3.5) this assumption involves the implication

$$\mathcal{S}^\alpha \neq 0 \qquad \Rightarrow \qquad \mathbf{n}^\alpha . \mathcal{U}_\alpha^a = 0$$

Due to the definition (6.1) of $\mathcal{U}_\alpha^a$, the latter equality is equivalent to

$$\mathbf{n}_\alpha . \mathcal{U}_\alpha^+ = -\frac{1-\delta_\alpha}{1+\delta_\alpha} \mathbf{n}_\alpha . \mathcal{U}_\alpha^- .$$

Therefore $1-\delta_\alpha / 1+\delta_\alpha$ *coincides in most cases with the restitution coefficient.* For that reason δ_α must not exceed 1 ; otherwise, the above would yield a negative restitution coefficient, which is kinematically inacceptable.

But equality $\mathbf{n}^\alpha . \mathcal{U}_\alpha^a = 0$ may not hold if $\mathcal{S}^\alpha = 0$: *this allows the block to rock* !

REMARK 1. The case where the *complete law of frictionless contact*, such as it was formulated in (3.9), is assumed to hold at all contacts, *with dissipation index equal to* 1, corresponds to the *Standard Inelastic Shocks*, introduced in an earlier paper [25].

REMARK 2. The above way of formulating Dynamics in the event of a velocity jump also applies to the frictional catastrophes referred to in Sec. 5 (see discussion in [15]).

7. OUTLINE OF AN ALGORITHM

Among numerous variants experimented so far, here is the sketch of a time-discretization algorithm directly derived from the foregoing.

Let $[t_I, t_F]$, $t_F = t_I + h$, denote an interval of the discretization. Starting with q_I, u_I, approximate values of q and u at time t_I, one has to calculate the final approximate values q_F, u_F.

Identification of contacts.

Using the middle time $t_M = t_I + \frac{1}{2} h$ and the test position $q_M = q_I + \frac{1}{2} h\, u_I$, the set of the contacts to be treated as active is estimated as

$$J = \{\alpha : f_\alpha(t_M, q_M) \leq 0\}.$$

Discretization of the measure differential equation (5.3).

$$A(q_M)(u_F - u_I) = h\, F(t_M, q_M, u_I) + \sum_{\beta \in J} G^{\beta *}(t_M, q_M)\, \mathcal{S}^\beta,$$

i.e. in short

$$u_F = u_I + hA^{-1}F + A^{-1} \sum_{\beta \in J} G^{\beta *} \mathcal{S}^\beta. \qquad (7.1)$$

Contact laws.

$$\forall \alpha \in J : \qquad \mathrm{law}_\alpha(\mathcal{U}_\alpha^a, \mathcal{S}^\alpha) = \mathrm{true}, \qquad (7.2)$$

where the average local velocity is estimated through (2.4) and (6.1), with u_F and u_I playing the roles of u^+ and u^- respectively,

$$\mathcal{U}_\alpha^a = \frac{\rho_\alpha}{1+\rho_\alpha} G^\alpha u_I + \frac{1}{1+\rho_\alpha} G^\alpha u_F + \mathcal{W}_\alpha . \qquad (7.3)$$

Here $\rho_\alpha = (1-\delta_\alpha)/(1+\delta_\alpha)$ denotes the restitution coefficient attributed to the contact α. If the contact α takes place between a body of the system and some external boundary with prescribed motion, the term $\mathcal{W}_\alpha$ equals the negative of the boundary velocity vector at time t_M and at the estimated contact point. For a contact between two members of the system, $\mathcal{W}_\alpha$ vanishes in the usual case of a scleronomic parametrization.

Final position.

$$q_F = q_M + \frac{1}{2} h\, u_F .$$

The heaviest part of the computation lies in the resolution of (7.1),(7.2). Contact laws considered in the foregoing Sections were positively homogeneous with regard to velocities. This allows one to replace (7.2),(7.3) by

$$\forall \alpha \in J : \qquad \mathrm{law}_\alpha(\rho_\alpha G^\alpha u_I + G^\alpha u_F + (1+\rho_\alpha)\mathcal{W}_\alpha, \mathcal{S}^\alpha) = \mathrm{true},$$

to be joined with (7.1). Here is a relaxation technique, amounting to treat a succession of one-contact problems.

Let an estimated solution u_F^{esti}, $\mathcal{S}_{esti}^\beta$, β running through J, be obtained, with (7.1) verified. One attempts to construct an improved estimate, say u_F^{corr}, $\mathcal{S}_{corr}^\beta$, by altering only $\mathcal{S}^\alpha$, i.e. $\mathcal{S}_{corr}^\beta = \mathcal{S}_{esti}^\beta$ for $\beta \neq \alpha$. The new estimate is astrained to verify (7.1), i.e. since the old estimate verifies the same,

$$u_F^{\mathrm{corr}} = u_F^{\mathrm{esti}} + A^{-1}G^{\alpha *}(\mathcal{S}^{\alpha}_{\mathrm{corr}} - \mathcal{S}^{\alpha}_{\mathrm{esti}}), \tag{7.4}$$

and to satisfy the law of contact α. By applying G^α to both members of (7.4) one gives to this contact law the following forrn

$$\mathrm{law}_\alpha(\mathcal{U}^{\mathrm{const}}_\alpha + G^\alpha u_F^{\mathrm{esti}} + H_\alpha(\mathcal{S}^{\alpha}_{\mathrm{corr}} - \mathcal{S}^{\alpha}_{\mathrm{esti}}), \mathcal{S}^{\alpha}_{\mathrm{corr}}) = \text{true}. \tag{7.5}$$

Here the expressions

$$\mathcal{U}^{\mathrm{const}}_\alpha = \rho_\alpha G^\alpha u_I + (1+\rho_\alpha)\mathcal{W}_\alpha,$$

an element of $\mathbf{R}^3$, and $H_\alpha = G^\alpha A^{-1} G^{\alpha *}$, a symmetric positive definite 3×3 matrix, have constants values in the whole iterative process.

Solving (7.5) with regard to the unknown $\mathcal{S}^{\alpha}_{\mathrm{corr}}$ is, in usual cases, easy. The above computation will be iterated, with α ranging cyclically through J. The decision of stopping iterations may be taken on observing the magnitude of $\mathcal{S}^{\alpha}_{\mathrm{corr}}-\mathcal{S}^{\alpha}_{\mathrm{esti}}$ or on checking the precision at which each $\mathcal{U}^{a}_{\alpha}$ satisfies the corresponding contact law. Convergence in this sense can always been obtained. Observe that, provided convergence is checked, the operator H_α in (7.5) may be replaced by any other mapping with zero limit at the origin ; this generates tricks for accelerating convergence.

Clearly, the above algorithm tolerates a certain amount of violation of the impenetrability constraints. By adjusting the step-length and the stopping criterium, one may keep these errors arbitrarily small and prevent their accumulation.

The iterated calculation is very simple, but needs to be repeated many times in case of numerous contacts. Since equation (7.1) is only preserved from one iteration to the next through the conservation condition (7.4), one should think of the possible accumulation of arithmetic errors. For safety, one may refresh u_F^{esti} from time to time, by returning to (7.1) ; this proves useful only for motions involving thousands of contacts.

Technically, let us also observe that in many usual applications, the n×n matrix A is constant and diagonal. G^α is a 3×n matrix, but only the elements corresponding to the two bodies involved in contact α are nonzero. So the treatment of large collections of bodies does not require the handling of large matrices.

REMARK. At every time-step, the above algorithm is ready to face *velocity jumps*, whould they result from collisions or arise as *frictional catastrophes* [15].

The possible breaking of some contacts is also automatically taken care of. Recall that, though contact breaks are usually smooth, their analytical treatment is not completely trivial. The traditional approach consists in calculating the motion under the tentative assumption that all contacts present

at some instant remain effective. If the calculation of contact forces in the course of such a motion yields, at a further instant, an unfeasible direction, one concludes that some contacts should break, so the consequent motion has to be calculated differently. But contacts which break are not necessarily those for which unfeasible contact forces were just found (for the frictionless case, see [26][27]).

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EXAMPLE 1 : WALL ON QUAKING GROUND

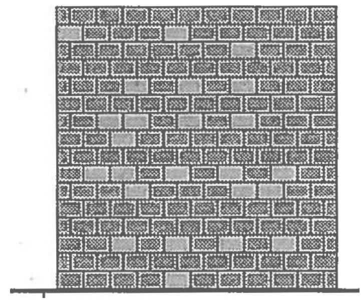
Ground motion :

Horizontal sinusoidal oscillation
Frequency : 2 hertz
Total amplitude : 25 cm

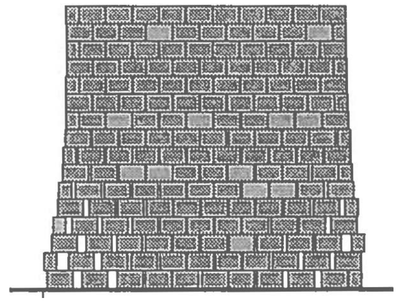
Wall :

Initial height : 640 cm
Initial width : 630 cm
Number of blocks : 176
(sizes : 60x40 cm and 30x40 cm*)*

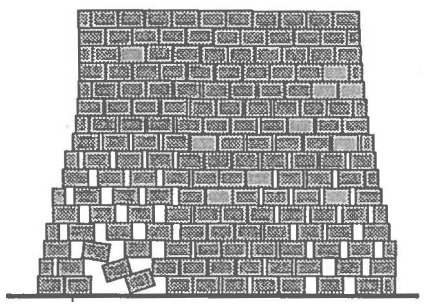
Friction coefficient : 0.3 everywhere
Restitution coefficient : 0 everywhere



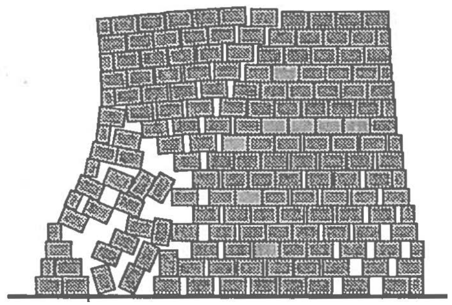
Time : 0 s



Time : 8 s

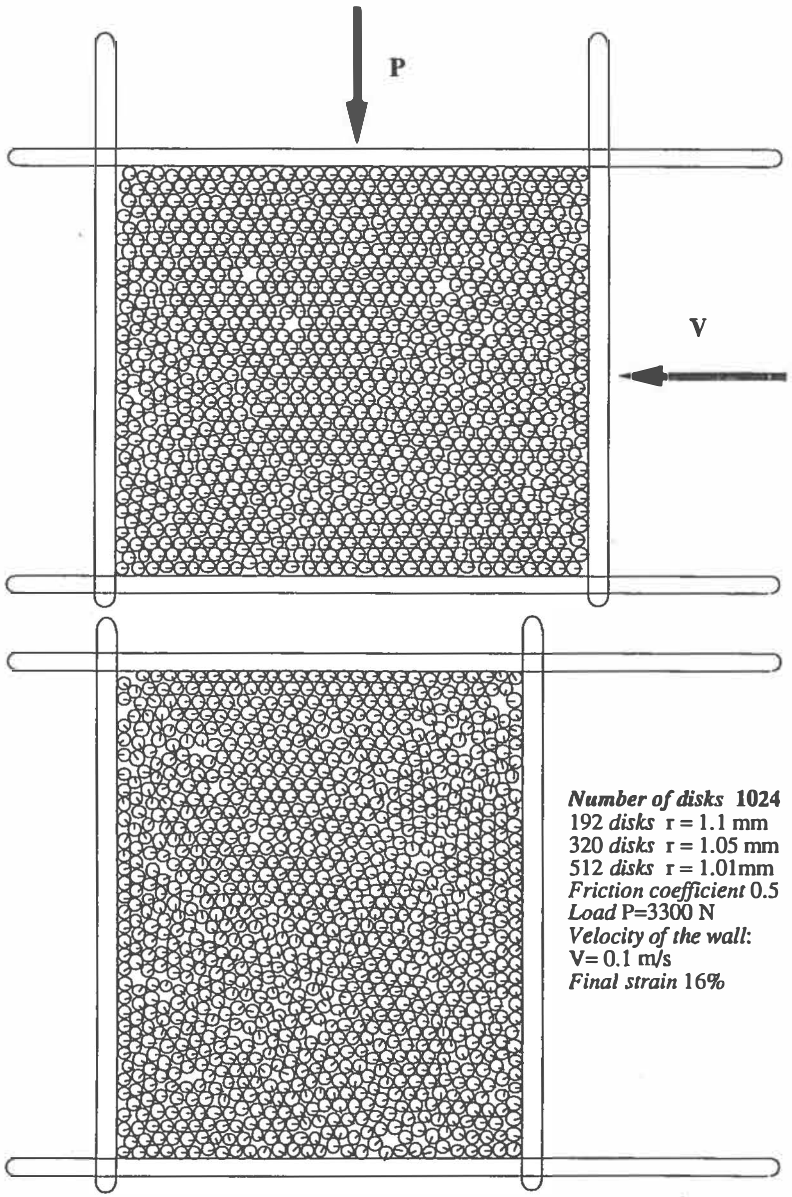


Time : 26.7 s



Time : 28 s

EXAMPLE 2 : BIDIRECTIONAL STRAIN-STRESS EXPERIMENT



***Number of disks* 1024**
192 *disks* r = 1.1 mm
320 *disks* r = 1.05 mm
512 *disks* r = 1.01mm
Friction coefficient 0.5
Load P=3300 N
Velocity of the wall:
V= 0.1 m/s
Final strain 16%