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Abstract

The goal of the current paper is to provide information about the basins of attraction of the granular media equation when there are exactly three stable states. Indeed, it has been proved in our previous works [16, 17] that there is convergence. However, very few is known about the basins of attraction. We provide them with a small diffusion coefficient. The techniques that we use here are related to the ones about the exit-problem of the associated McKean-Vlasov diffusion.

Key words and phrases: Granular media equation ; Self-stabilizing diffusion ; Freidlin-Wentzell theory ; Non uniqueness of the invariant probabilities ; Basins of attraction

2000 AMS subject classifications: Primary: 35K55, 60J60; Secondary: 26D10, 60E15, 60F10

1 Introduction

Our goal is to deal in a probabilistic way with the following nonlinear equation

$$\frac{\partial}{\partial t} u^\sigma = \frac{\sigma^2}{2} \Delta u^\sigma + \text{div} \left\{ u^\sigma (\nabla V + \nabla F \ast u^\sigma) \right\},$$

(1)

where $u^\sigma(t, \cdot)$ is a probability measure, $\ast$ denotes the standard convolution operator and $V$ and $F$ are two potentials on $\mathbb{R}^d$. We assume that both $V$ and $F$ are convex at infinity.

This equation can be obtained as a simplification - proposed by Kac in 1959 - of the kinetic equation of Vlasov on the plasmas. This model corresponds to a mean-field system of interacting particles with an infinite number of such particles.
particles. By considering the law of probability of a representative particle, we know that this law is absolutely continuous with respect to the Lebesgue measure for any positive time. Moreover, the density of the law satisfies the so-called granular media equation (1).

We will not discuss the existence and the uniqueness of a solution to the equation.

One major problem is the behaviour as the time goes to infinity: existence and uniqueness of the steady state then convergence to this unique stable state. The question of the rate of convergence also arises as a very important one. However, we will not address it here.

The existence of a stable state has been obtained by Benachour, Roynette, Talay and Vallois (see [2]) in the one-dimensional case by assuming that the friction term $V$ is equal to 0 and that $F$ is a convex potential. Let us point out that in this particular setting, the center of mass is fixed. So despite there is an infinite number of stationary measures with total mass equal to 1, the identification of the limiting probability is obvious. In a subsequent article, see [3], the authors obtain the convergence towards the invariant probability measure. For the case in which $V$ is not identically equal to 0, let us mention the work [1]. The authors consider two uniformly strictly convex potentials and they obtain the convergence with an explicit exponential rate of convergence. In [5], Carrillo, McCann and Villani proceed with a more general type of equation and with a potential $V$ nonconvex. The assumptions are the synchronization (roughly speaking: the convexity of $F$ is stronger than the nonconvexity of $V$) and the center of mass is fixed. Up to our knowledge, there is no assumption on the initial condition which ensures this hypothesis of fixed center of mass, except if $V$ and $F$ are symmetrical (then the condition is to assume that the initial law is also symmetrical). The used techniques are analytical. About probabilistic approach, we refer to Malrieu ([11, 12]) and Cattiaux, Guillin and Malrieu ([4]), still in the case where both potentials are convex.

In the nonconvex case, the existence of stationary measures has been investigated in [8, 9, 10, 18, 19]. The main result is the nonuniqueness of the stationary measures. More precisely, under simple assumptions that are easy to satisfy, there are exactly three such invariant probability measures.

Thus, some questions arise: What is the limiting probability? What are the basins of attraction of each invariant probability?

However, one should first prove the convergence. In the nonconvex case, the convergence has been obtained in [16, 17]. More precisely, we assume that $V$ is nonconvex (but convex at infinity) and that the interacting potential $F$ is convex (albeit the case in which $F$ is also nonconvex could be solved by the same method). However, let us point out we use some compactness arguments in these two papers. Consequently, very few is obtained regarding to the basins of attraction. The present work is dedicated to the basins of attraction for the granular media equation in a setting in which there are several stable states.

To present the idea in the introduction, we choose to consider a simple case in dimension one: $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\alpha}{2} x^2$ with $\alpha > 0$. Let us now
present the probabilistic approach of this problem. The idea is to consider a stochastic process $X^\sigma$, which law at time $t$ is $u^\sigma(t,\cdot)$. It is the solution of the nonlinear stochastic differential equation

$$X^\sigma_t = X_0 + \sigma B_t - \int_0^t \nabla V(X^\sigma_s) \, ds - \alpha \int_0^t (X^\sigma_s - \mathbb{E}[X^\sigma_s]) \, ds,$$

(2)

$B$ being a Brownian motion. This kind of processes were introduced by McKean, see [14, 13].

The convergence plays an important role in the exit problem for the self-stabilizing process. Indeed let us assume that we have the convergence of $u^\sigma_t$ towards a steady state $u^\infty_{\sigma}$ with a precise rate of exponential convergence which does not depend on the temperature $\frac{\sigma^2}{2}$. Then, the drift converges uniformly (with respect to the time) towards a linear drift as $\sigma$ goes to 0. As a consequence, the new process obtained by this drift is an exponential approximation of $X^\sigma$ (see [6]) so that $X^\sigma$ satisfies large deviations which are time-homogeneous. Furthermore, we can obtain easily the Kramers’ type law like Herrmann, Imkeller and Peithmann did in [7]. Here, the paradigm is the exact opposite. We will not get the exit-time by using the convergence and the rate of convergence. In fact, we will obtain the convergence (the basin of attraction more precisely) by techniques linked to the exit-time.

Up to our knowledge, the only results about basins of attraction are the ones in [15] and in [16]. In [16], it is stated that if the initial random variable is symmetrical, then the limiting probability is the unique symmetrical invariant probability. Furthermore, if the free-energy at time 0 is less than some quantity, then the limiting probability is either the one with positive expectation (if the initial random variable has a positive expectation) or the one with negative expectation (if the initial random variable has a negative expectation). In [15], the author proved that if the initial law is close to an invariant probability which second derivative of the free-energy is positive then $u^\sigma(t,\cdot)$ converges (exponentially fast) towards this invariant probability. Except these two settings, none is known - up to our knowledge - about the basins of attraction.

In the current work, we assume the synchronization that is $\alpha > 1$ (in the setting $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$). This means that the convexity of $F$ will compensate the nonconvexity of $V$. However, if $\sigma$ is small enough, there are three invariant probability measures for the dynamic: $u^\sigma_0$ (with a center of mass equal to 0), $u^\sigma_+$ (with a positive center of mass) and $u^\sigma_-$ (with a negative center of mass). We remind a result in [9, 10] that is the weak convergence of $u^\sigma_0$ (resp. $u^\sigma_+$ and $u^\sigma_-$) towards $\delta_0$ (resp. $\delta_1$ and $\delta_{-1}$) as $\sigma$ goes to 0 for any $i \in \{+,-,0\}$.

The paper is organized as follows. Next section gives the general assumptions of the paper. In Section three, the main result (Theorem 3.6) is stated. It concerns the probability measure $u^\sigma_0$ (which converges towards $\delta_0$, $\alpha$ being a local minimum of the confining potential). Some immediate corollaries are given: Corollary 3.7, Corollary 3.8 and Corollary 3.9. We also give Proposition 3.10 which shows that the result can not be extended for a probability measure which is centered around a local maximum of $V$. Finally, in a section four, we give the
proof of Theorem 3.6.

2 Assumptions of the paper

In the current work, we assume the following hypotheses on $V$, on $F$ and on $u_0$.

**Assumption 2.1.**
- The coefficient $\nabla V$ is locally Lipschitz, that is, for each $R > 0$ there exists $K_R > 0$ such that
  $$||\nabla V(x) - \nabla V(y)|| \leq K_R ||x - y||,$$
  for $x, y \in \{z \in \mathbb{R}^d : ||z|| < R\}$.
- The function $V$ is continuously differentiable.
- The potential $V$ is convex at infinity: $\lim_{||x|| \to +\infty} \nabla^2 V(x) = +\infty$.
- There exist $m \in \mathbb{N}$ and $C > 0$ such that $||\nabla V(x)|| \leq C ||x||^{2m-1}$ and $m \geq 2$.
- There exists $\alpha > 0$ such that $F(x) = \frac{\alpha}{2} ||x||^2$.
- The $8m^2$th moment of $u_0$ is finite.
- $\alpha > \theta := \sup_{\mathbb{R}^d} -\nabla^2 V$.

For some corollaries, we will also consider the following assumption.

**Assumption 2.2.** The measure $u_0$ is absolutely continuous with respect to the Lebesgue measure with a density of probability that we denote by $u_0$. Moreover, the entropy $\int_{\mathbb{R}^d} u_0(x) \log (u_0(x)) \, dx$ is finite.

Thanks to Assumption 2.1, there exists a unique strong solution $X^\sigma$ to the McKean-Vlasov diffusion

$$X^\sigma_t = X_0 + \sigma B_t - \int_0^t \nabla V(X^\sigma_s) \, ds - \alpha \int_0^t (X^\sigma_s - \mathbb{E}[X^\sigma_s]) \, ds,$$

see [7, Theorem 2.13] for a proof. Moreover, for any $p \in [1; 4m^2]$, we have:

$$\sup_{t \in \mathbb{R}_+} \mathbb{E} \left\{ ||X^\sigma_t||^{2p} \right\} < \infty.$$

3 Main results

From now on, $a$ is a local minimum of $V$ such that $\nabla^2 V(a)$ is strictly positive. Let us give a last assumption.
Assumption 3.1. There exists $\kappa_0 > 0$ and $\sigma_0 > 0$ such that there exists a unique invariant probability measure $u_\sigma^a$ satisfying

$$W_2(u_\sigma^a; \delta_a)^2 = \int_{\mathbb{R}^d} ||x - a||^2 u_\sigma^a(dx) \leq \kappa_0^2$$

for any $\sigma \leq \sigma_0$.

One could object to this assumption that it is not easy to verify. However, thanks to [8, 9, 10, 18, 19], we know some cases in which the local uniqueness of the invariant probability measure around $a$ is satisfied.

We now define some set of interest.

Definition 3.2. For any $\rho > 0$, set

$$S_\rho(a) := \left\{ x \in \mathbb{R}^d : \langle \nabla V(x); x - a \rangle \geq \rho ||x - a||^2 \right\}.$$  

Definition 3.3. For any $\rho > 0$, by $S_\rho(a)$, we denote the path-connected subset of $S_\rho(a)$ which contains $a$.

Remark 3.4. We know that $S_\rho(a)$ is not empty for $\rho$ sufficiently small thanks to the hypothesis $\nabla^2 V(a) > 0$.

The quantity of interest is the following:

Definition 3.5. We put

$$\xi(t) := E\left[ ||X_\sigma^a - a||^2 \right].$$

We present the main result for an invariant probability measure centered around $a$.

Theorem 3.6. We assume that $u_\sigma$ has a compact support included into $S_\rho(a)$ for some $\rho > 0$. Then, for any $\kappa > 0$, there exists a time $T_\kappa \geq 0$ and a positive real $\sigma_0$ such that $\sup_{0 < \sigma < \sigma_0, \sup_{t \geq T_\kappa}} \xi(t) \leq \kappa^2$.

The proof is postponed in Section 4. We give some immediate corollaries.

Corollary 3.7. We here assume Assumption 2.1, Assumption 2.2 and Assumption 3.1. Then $u^\sigma(t, \cdot)$ converges weakly towards $u_\sigma^a$ as $t$ goes to infinity providing that $\sigma$ is smaller than $\sigma_0$ (defined in Theorem 3.6).

Let us point out that the proof of Corollary 3.7 is immediate thanks to the results in [16, 17]. Let us point out that the diffusion coefficient does depend on the compact. However, it does not depend on the measure $u_0$. Indeed, the previous results in [16] about the basins of attraction imply that the diffusion coefficient does depend on the initial probability measure.

Let us give some corollary implied by Corollary 3.7.
Corollary 3.8. We here assume Assumption 2.2, \( d = 1 \), \( V(x) = \frac{x^4}{4} - \frac{x^2}{2} \) with \( \alpha > 1 \). Then, if \( u_0 \) has compact support in \([0; +\infty[\) (respectively in \([-\infty; 0[\)), there exists \( \sigma_0 > 0 \) such that for any \( \sigma < \sigma_0 \), \( u^\sigma(t, \cdot) \) converges weakly towards \( u^\sigma_+ \) (respectively \( u^\sigma_- \)) - the unique invariant probability with positive (respectively negative) expectation - as \( t \) goes to infinity.

The proof is immediate thanks to the results in [18] about the thirdness of the invariant probabilities if \( \sigma \) is small enough.

We now give some results in the case where \( u_0 \) is a Dirac measure (which of course violates Assumption 2.2).

Corollary 3.9. We assume \( d = 1 \), \( V(x) = \frac{x^4}{4} - \frac{x^2}{2} \) and \( F(x) = \alpha^2 x^2 \) with \( \alpha > 1 \). We put \( u_0 := \delta_{x_0} \) with \( x_0 > 0 \). Then, for any \( \kappa > 0 \), there exists a time \( T_\kappa \geq 0 \) which does not depend on \( \sigma \) such that \( E\{||X^\sigma_t - 1||^2\} \) is less than \( \kappa^2 \) for any \( t \geq T_\kappa \) providing that \( \sigma \) is sufficiently small.

Let us point out that we have the same result with a sum of Dirac measures. We can wonder if Corollary 3.7 can be extended to a local maximum. We now answer negatively to the question.

Proposition 3.10. We assume \( d = 1 \), \( V(x) = \frac{x^4}{4} - \frac{x^2}{2} \) and \( F(x) = \alpha^2 x^2 \) with \( \alpha > 1 \). Then for any \( \kappa > 0 \), there exists a probability measure \( u_0 \) satisfying \( W_2(u_0; \delta_0) \leq \kappa \) and such that \( u^\sigma(t, \cdot) \) converges weakly towards \( u^\sigma_+ \) as \( \sigma \) is small enough.

Proof. It is sufficient to consider \( u_0 \) with compact support included in \([\kappa^2; 2\kappa^2]\) (which is a subset of \([0; +\infty[\)) for \( \kappa \) sufficiently small then to apply Theorem 3.6.

\[ \square \]

4 Proof of Theorem 3.6

We first give the following lemma (which is in fact [20, Lemma 4.1]).

Lemma 4.1. For any \( t \geq 0 \), we have:

\[ \xi'(t) \leq - 2\rho \xi(t) + \sigma^2 + K \sqrt{\mathbb{P}(X_t \not\in S_\rho(a))}, \]  

\( K \) being a positive constant.

The proof is already in [20] but we give it for consistency.

Proof. By Itô formula, we have:

\[ ||X^\sigma_t - a||^2 = ||X_0 - a||^2 + 2\sigma \int_0^t \langle X^\sigma_s - a; dB_s \rangle - 2 \int_0^t \langle X^\sigma_s - a; \nabla V(X^\sigma_s) \rangle ds \]

\[ - 2\alpha \int_0^t \langle X^\sigma_s - a; X^\sigma_s - \mathbb{E}[X^\sigma_s] \rangle ds + \sigma^2 t. \]
However, we know, that
\[
\mathbb{E} \{ (X_t^\sigma - a) ; X_t^\sigma - \mathbb{E} [X_t^\sigma] \} = \text{Var} (X_t^\sigma - a) \geq 0.
\]

We take the expectation then we take the derivative. We thus obtain:
\[
\frac{d}{dt} \xi (t) \leq -2\mathbb{E} [ (X_t^\sigma - a ; \nabla V (X_t^\sigma)) ] + \sigma^2.
\]

We use the following trick:
\[
(X_t^\sigma - a ; \nabla V (X_t^\sigma)) = (X_t^\sigma - a ; \nabla V (X_t^\sigma)) \mathbb{1}_{X_t^\sigma \in S_\rho (a)} + (X_t^\sigma - a ; \nabla V (X_t^\sigma)) \mathbb{1}_{X_t^\sigma \notin S_\rho (a)}.
\]

Consequently, we have:
\[
\frac{d}{dt} \xi (t) \leq -2\rho \xi (t) + \sigma^2 + 2\mathbb{E} \left\{ \rho ||X_t^\sigma - a||^2 - (X_t^\sigma - a ; \nabla V (X_t^\sigma)) \right\} \mathbb{1}_{X_t^\sigma \notin S_\rho (a)}
\]

According to Assumption 2.1, we have \(||\nabla V (X_t^\sigma)|| \leq C \||X_t^\sigma||^{2m-1}\) so that
\[
\left| \rho ||X_t^\sigma - a||^2 - (X_t^\sigma - a ; \nabla V (X_t^\sigma)) \right| \leq C' \||X_t||^{2m},
\]
\(C'\) being a positive constant. Cauchy-Schwarz inequality yields
\[
\frac{d}{dt} \xi (t) \leq -2\rho \xi (t) + \sigma^2 + C' \sqrt{\mathbb{E} \left[ ||X_t^\sigma||^{4m} \right]} \sqrt{\mathbb{P} (X_t^\sigma \notin S_\rho (a))}.
\]

The uniform boundedness of the moments (see [7]) implies the existence of a positive constant \(K\) such that (3) holds, which achieves the proof.

Let us point out that we have \(\mathbb{P} (X_t^\sigma \notin S_\rho (a)) \leq \mathbb{P} (\tau_\rho (\sigma) \leq t)\) for any \(t \geq 0\) where \(\tau_\rho (\sigma)\) is the first exit-time from \(S_\rho (a)\) of diffusion \(X^\sigma\). It is the control that has been done in [20]. However, it is a bad idea since, as \(t\) goes to infinity, the right hand side converges to 1 as the one in the left may be small. The key of the present paper is to deal in another way with the term \(\mathbb{P} (X_t^\sigma \notin S_\rho (a))\).

We now take \(\gamma > 0\) sufficiently small such that \(\mathbb{B} (a ; \gamma) \subset S_\rho (a)\) with the hypothesis \(d (\mathbb{B} (a ; \gamma) ; S_\rho (a)) > 0\).

**Lemma 4.2.** For any \(\kappa > 0\), there exist \(\sigma_0\) and \(T_\kappa\) such that for any \(\sigma < \sigma_0\), we have \(\xi (T_\kappa) \leq \frac{4\rho^2 \gamma^2}{K^2} \left( \frac{\kappa}{2} \right)^4\).

**Proof.** It is a straightforward consequence of previous lemma.

We remark that if \(\kappa\) is small enough, \(\frac{4\rho^2 \gamma^2}{K^2} \left( \frac{\kappa}{2} \right)^4 < \kappa^2\). 

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Definition 4.3. We put \( \zeta_\kappa(\sigma) := \inf \{ t \geq T_\kappa : \xi(t) \geq \kappa^2 \} \) with the convention \( \inf \emptyset = +\infty \).

Let us proceed a reductio ad absurdum by assuming that there exists a decreasing sequence \( (\sigma_l)_l \) with \( \sigma_\infty = 0 \) such that \( \zeta_\kappa(\sigma_l) < \infty \) for any \( l \in \mathbb{N} \).

We now make a coupling.

Definition 4.4. We consider the diffusion \( Y^{\sigma_l} := (Y_t^{\sigma_l})_{t \geq T_\kappa} \) defined by

\[
Y_{T_\kappa + t}^{\sigma_l} = X_{T_\kappa}^{\sigma_l} + \sigma_l (B_{T_\kappa + t} - B_{T_\kappa}) - \int_{T_\kappa}^{T_\kappa + t} \nabla V (Y_s^{\sigma_l}) \, ds - \alpha \int_{T_\kappa}^{T_\kappa + t} (Y_s^{\sigma_l} - a) \, ds.
\]

Lemma 4.5. For any \( \xi > 0 \) and \( l \in \mathbb{N} \), we have:

\[
P \left\{ \sup_{t \in [T_\kappa; \zeta_\kappa(\sigma_l)]} ||X_t^{\sigma_l} - Y_t^{\sigma_l}|| \geq \xi \right\} = 0
\]

if \( \kappa < \xi \left( 1 - \frac{\theta}{\alpha} \right) \) is small enough.

Proof. Differential calculus provides

\[
d ||X_t^{\sigma_l} - Y_t^{\sigma_l}||^2 = -2 \left\langle X_t^{\sigma_l} - Y_t^{\sigma_l}; \nabla W_{u_t^{\sigma_l}} (X_t^{\sigma_l}) - \nabla W_{\delta_a} (Y_t^{\sigma_l}) \right\rangle dt,
\]

where \( W_u(x) := V(x) + F \ast u(x) \) and \( u_t^{\sigma_l} := \mathcal{L} (X_t^{\sigma_l}) \).

For any \( T_\kappa \leq t \leq \zeta_\kappa(\sigma_l) \), we have:

\[
d ||X_t^{\sigma_l} - Y_t^{\sigma_l}||^2 = -2 \left\langle X_t^{\sigma_l} - Y_t^{\sigma_l}; \nabla W_{u_t^{\sigma_l}} (X_t^{\sigma_l}) - \nabla W_{u_t^{\sigma_l}} (Y_t^{\sigma_l}) \right\rangle dt
- 2 \left\langle X_t^{\sigma_l} - Y_t^{\sigma_l}; \nabla W_{u_t^{\sigma_l}} (Y_t^{\sigma_l}) - \nabla W_{\delta_a} (Y_t^{\sigma_l}) \right\rangle dt.
\]

The first term can be bounded like so:

\[
-2 \left\langle X_t^{\sigma_l} - Y_t^{\sigma_l}; \nabla W_{u_t^{\sigma_l}} (X_t^{\sigma_l}) - \nabla W_{u_t^{\sigma_l}} (Y_t^{\sigma_l}) \right\rangle \leq -2 (\alpha - \theta) ||X_t^{\sigma_l} - Y_t^{\sigma_l}||^2,
\]

thanks to the synchronization. We now bound the second term:

\[
-2 \left\langle X_t^{\sigma_l} - Y_t^{\sigma_l}; \nabla W_{u_t^{\sigma_l}} (Y_t^{\sigma_l}) - \nabla W_{\delta_a} (Y_t^{\sigma_l}) \right\rangle \leq 2\alpha ||X_t^{\sigma_l} - Y_t^{\sigma_l}|| \times ||\mathbb{E} [X_t^{\sigma_l}] - a||
\leq 2\alpha \kappa ||X_t^{\sigma_l} - Y_t^{\sigma_l}||,
\]

since, for any \( t \in [T_\kappa; \zeta_\kappa(\sigma_l)] \), \( \xi(t) \leq \kappa^2 \). We deduce the inequality

\[
\frac{d}{dt} ||X_t^{\sigma_l} - Y_t^{\sigma_l}||^2 \leq -2 (\alpha - \theta) ||X_t^{\sigma_l} - Y_t^{\sigma_l}||^2 + 2\alpha \kappa ||X_t^{\sigma_l} - Y_t^{\sigma_l}||.
\]

However, \( X_{T_\kappa}^{\sigma_l} = Y_{T_\kappa}^{\sigma_l} \). Hence, for any \( t \in [T_\kappa; \zeta_\kappa(\sigma_l)] \), we have:

\[
||X_t^{\sigma_l} - Y_t^{\sigma_l}|| \leq \frac{\alpha}{\alpha - \theta} \kappa.
\]

Taking \( \kappa < \frac{\alpha - \theta}{\alpha} \xi \) yields the result.

\[
\square
\]
We now apply Lemma 4.5 with $\xi := d(\mathbb{B}(a;\gamma);S_\rho(a)^c)$. Then, we deduce
\[
P(X_t^\sigma \notin S_\rho(a)) \leq P(Y_t^\sigma \notin \mathbb{B}(a;\gamma)),
\]
providing that $\kappa$ is sufficiently small.
By using Markov inequality, $P(Y_t^\sigma \notin \mathbb{B}(a;\gamma)) \leq \frac{1}{\gamma} \mathbb{E}\{||Y_t^\sigma - a||^2\} =: \tau(t)$.
Now, Itô formula implies
\[
\tau'(t) \leq \sigma^2 - 2\mathbb{E}\{\langle Y_t^\sigma - a; \nabla V(Y_t^\sigma) \rangle\} \leq \sigma^2 - 2(\alpha - \theta)\tau(t).
\]
We immediately deduce $\tau(t) \leq \max\left\{\tau(T_\kappa); \frac{\sigma^2}{2(\alpha - \theta)}\right\}$. As $\tau(T_\kappa) = \xi(T_\kappa) \leq \frac{4\rho^2}{K^2} \left(\frac{\kappa}{2}\right)^4$, we immediately obtain $\frac{1}{\tau}(t) \leq \frac{4\rho^2}{K^2} \left(\frac{\kappa}{2}\right)^4$ if $\sigma_t$ is small enough.
As a conclusion, for any $t \in [T_\kappa; \zeta_\kappa(\sigma_t)]$,
\[
P(X_t^\sigma \notin S_\rho(a)) \leq P(Y_t^\sigma \notin \mathbb{B}(a;\gamma)) \leq \frac{4\rho^2}{K^2} \left(\frac{\kappa}{2}\right)^4.
\]
We remind the main result of Lemma 4.1 that is Inequality (3):
\[
\xi'(t) \leq -2\rho \xi(t) + \sigma^2 + K \sqrt{P(X_t^\sigma \notin S_\rho(a))} \leq -2\rho \xi(t) + \sigma_t^2 + \frac{2\rho}{K} \left(\frac{\kappa}{2}\right)^2,
\]
if $t \in [T_\kappa; \zeta_\kappa(\sigma_t)]$. We immediately obtain that
\[
\xi(\zeta_\kappa(\sigma_t)) \leq \left(\frac{\kappa}{2}\right)^2 + \frac{1}{2\rho} \sigma_t^2 < \kappa^2
\]
by taking $\sigma_t$ sufficiently small. This is absurd so we deduce that $\zeta_\kappa(\sigma) = +\infty$ as $\sigma$ is small enough. This provides the existence of a value $T_\kappa > 0$ such that for any $\sigma$ small enough, we have $\xi(t) \leq \kappa^2$ for any $t \geq T_\kappa$, which concludes the proof.

References


