Statistical tests of heterogeneity for anisotropic multifractional Brownian fields

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Preprint to be published in Stochastic Processes and Their Applications

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January 22, 2020

Abstract In this paper, we deal with some anisotropic extensions of the multifractional Brownian fields that account for spatial phenomena whose properties of regularity and directionality may both vary in space. Our aim is to set statistical tests to decide whether an observed field of this kind is heterogeneous or not. The statistical methodology relies upon a field analysis by quadratic variations, which are averages of square field increments. Specific to our approach, these variations are computed locally in several directions. We establish an asymptotic result showing a linear Gaussian relationship between these variations and parameters related to regularity and directional properties of the model. Using this result, we then design a test procedure based on Fisher statistics of linear Gaussian models. Eventually we evaluate this procedure on simulated data.

Keywords isotropy · anisotropy · homogeneity · heterogeneity · increment · quadratic variations · statistical test · anisotropic fractional Brownian field · multifractional Brownian field

1 Introduction

The history of the Brownian fields dates back to the introduction of the fractional Brownian motion by Kolmogorov and Mandelbrot [26,27]. The spatial extension of this process has paved the way for a multitude of random field models to describe spatial phenomena of an irregular nature. These fields are roughly divided into two large families: anisotropic and heterogeneous fields. Anisotropic fields include anisotropic fractional Brownian fields [17], their extension in the framework of intrinsic fields [34], or operator-scaling fields [9,13,20]. They enable to model homogeneous spatial phenom-

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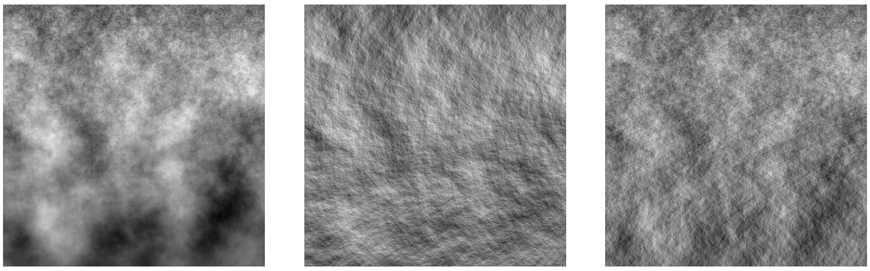
ena whose global properties vary according to directions. Heterogeneous fields mainly refer to the multifractional Brownian fields [8, 29] and its various extensions [2, 4], but also to heterogeneous operator-scaling fields [12]. They permit to model heterogeneous spatial phenomena whose regularity may vary in a local way. Most of these fields are essentially isotropic. In this paper, we deal with more generic heterogeneous fields accounting for spatial phenomena whose regularity and anisotropy may both vary.



(a) (b) (c)

Fig. 1: Realizations of different multifractional Brownian fields whose spatial properties vary along the vertical direction: (a) an isotropic one with a varying regularity, (b) an anisotropic one with varying pattern orientations, and (c) another anisotropic one with a varying degree of anisotropy.

The emergence of Brownian fields has raised many statistical issues. The main questions revolve around the estimation of parameters (possibly functional) of these fields (see, for instance, [25, 16, 34, 5, 22, 3]). Other questions concern tests of properties of these fields (anisotropy, heterogeneity, etc) [36, 34]. A widespread approach to analyzing Brownian fields is based on quadratic variations, which are averages of squared field increments. This approach has been developed both in oriented forms to analyze anisotropic fields [16, 34] or local forms to analyze heterogeneous fields [5, 22]. The specification of this approach is usually associated to a convergence study that aims at identifying asymptotic laws of model estimates. The knowledge of this law has served as a theoretical basis for the construction of statistical estimates of parameters and hypothesis tests. In this paper, we develop a quadratic variation approach for the statistical analysis of anisotropic heterogeneous fields. We construct local estimates of field properties, and identify their asymptotic law. We eventually design some tests to check if a field is spatially homogeneous or not.

The heterogeneity of fields we consider may have different aspects. As illustrated in Figure 1 (a), it may be due to spatial variations of field regularity. As shown in Figures 1 (b) and (c), it can also result from local variations of directional field properties. These variations may further have different origins: for instance, variations of the degree of anisotropy as in Figure 1 (b), variations of pattern orientations as in Figure 1 (c). Eventually, the heterogeneity can reflect a combination of several types of variations. The purpose of our statistical tests is to help decide whether properties of the field are the same at several space locations in terms of regularity, anisotropy or both.

2 Brownian field models

2.1 Previous models

Derived from the fractional Brownian motion [26,27], the fractional Brownian field is both isotropic and homogeneous. Its homogeneity is due to the stationarity of its increments, and provides it with a constant Hölder regularity all over the space.

In [17], Bonami and Estrade introduced some anisotropic extensions of the fractional Brownian field. Among them, the so-called anisotropic fractional Brownian field is a field with stationary increments whose spectral density is of the form

$$\forall \omega \in \mathbb{R}^d, \quad h_{\tau,\beta}(\omega) = \tau(\omega)|\omega|^{-2\beta(\omega)-d}, \tag{1}$$

where functions τ and β are called the topothesy and Hurst functions, respectively. By assumption, these functions are both homogeneous, *i.e.*

$$\forall\, w \in \mathbb{R}^d \backslash \{0\}, \tau_x(w) = \tau_x\left(\frac{w}{|w|}\right) \text{ and } \beta_x(w) = \beta_x\left(\frac{w}{|w|}\right).$$

In other words, their values only depend on the direction $\frac{w}{|w|}$ of the frequency vector w. When they are both constant, the spectral density of Equation (1) is radial. In such a case, the associated field is isotropic and corresponds to a fractional Brownian field. At contrary, when one of these functions is not constant, the spectral density is non-radial and the field is anisotropic. Although more generic than the fractional Brownian fields, these anisotropic fields are still homogeneous. On the one hand, their increments are stationary. On the other hand, they have the same regularity and directional properties all over the space.

Following [8], the multifractional Brownian field can be defined, for any position x, by a stochastic integral of the form

$$Z(x) = C_x \int_{\mathbb{R}^d} \frac{e^{i\langle x,\omega\rangle} - 1}{|\omega|^{H_x+d/2}}\, d\widehat{W}(\omega) \tag{2}$$

where $d\widehat{W}$ stands for a complex Brownian measure, and C_x and H_x are two positive constants that may vary according to the position x. The Hölder regularity of this field is not constant anymore. Its order varies depending on H_x. These variations were originally assumed to be Hölder so as to ensure that the field realization are continuous [8]. This assumption was subsequently weakened to allow sharper variations of the regularity [2,4]. At a position x, a multifractional Brownian field admits a tangent field $\tilde{Z}_x$ which is a fractional Brownian field with a Hurst index H_x. This tangent field is obtained as a limit [8]

$$(\tilde{Z}_x(u))_{u\in\mathbb{R}^d} = \lim_{t\to 0^+}\left(\frac{Z(x+tu)-Z(x)}{t^{H_x}}\right)_{u\in\mathbb{R}^d}.$$

According to this property, a multifractional Brownian field has the same local properties as a fractional Brownian field. Consequently, it is locally isotropic as its tangent field.

Some anisotropic extensions of the multifractional Brownian field were considered in [31,30]. At any fixed position x of $\mathbb{R}^d$, they were defined by a stochastic integral

$$Z(x) = \int_{\mathbb{R}^d} (e^{i\langle x,\omega\rangle}-1)\sqrt{h_{\tau_x,H_x}(\omega)}\, d\widehat{W}(\omega) \tag{3}$$

in which h_{τ_x,H_x} is a spectral density of the form (1) specified by a constant function H_x and an homogeneous function β_x that may both change from a position x to another. Polisano showed that such a field is tangent to an anisotropic fractional Brownian field with a spectral density h_{τ_x,H_x} at a position x [30]. In other words, locally, it has the same directional properties as an anisotropic fractional Brownian field.

2.2 An extended model

In this paper, we propose a more generic anisotropic multifractional Brownian field Z (AMFBF) defined, for some $A \geq 0$ and all positions $x \in \mathbb{R}^d$, by a stochastic integral

$$Z(x) = \int_{|w|>A} (e^{i\langle x,\omega\rangle}-1)\sqrt{f_x(\omega)}\, d\widehat{W}(\omega), \tag{4}$$

where $f_x = h_{\tau_x,\beta_x}$ is a spectral density of the form (1) specified by two homogeneous functions τ_x and β_x. In contrast with the work of [31,30], the Hurst functions β_x are no longer constant. Moreover, the integral is partially defined for $|w| > A$ where A is arbitrary. Therefore, the proposed field may not have the same lowest frequencies as a fractional Brownian field.

For any $x \in \mathbb{R}^d$, we will assume that the functions τ_x and β_x are both homogeneous and even, *i.e.*

$$\forall\, w \in \mathbb{R}^d, \tau_x(w) = \tau_x(-w) = \tau_x\left(\frac{w}{|w|}\right) \text{ and } \beta_x(w) = \beta_x(-w) = \beta_x\left(\frac{w}{|w|}\right).$$

This implies that both functions are determined by their values on the unit sphere $\mathcal{S} = \{u \in \mathbb{R}^d, |u| = 1\}$ of $\mathbb{R}^d$. We will further assume that τ and β are non-negative and satisfy

$$\begin{aligned}
\overline{\tau} &:= \sup_{x\in\mathbb{R}^d}\ \operatorname*{ess\,sup}_{s\in\mathcal{S}}\ \tau_x(s) < +\infty\\
\bar{\beta} &:= \sup_{x\in\mathbb{R}^d}\ \operatorname*{ess\,sup}_{s\in\mathcal{S}}\ \beta_x(s) < 1,\\
H &:= \inf_{x\in\mathbb{R}^d}\ \operatorname*{ess\,inf}_{s\in\mathcal{S}}\ \beta_x(s) > 0.
\end{aligned}$$

For any $x \in \mathbb{R}^d$, we will denote

$$H_x = \operatorname*{ess\,inf}_{s\in\mathcal{S}}\ \beta_x(s), \tag{5}$$

and assume that the set

$$E_x = \{s \in \mathcal{S}, \beta_x(s) = H_x \text{ and } \tau_x(s) > 0\} \tag{6}$$

is of positive measure on $\mathcal{S}$. Under this assumption, the parameter H_x, called the Hurst index, gives the order of the local Hölder regularity of the field at x [8]. By abusing the notation, H_x will also refer to a function defined on $\mathbb{R}^d$ and taking the constant value H_x.

We will also use a function δ_x defined, for all $w \in \mathbb{R}^d$, by

$$\delta_x(w) = \begin{cases} \tau_x(w) & \text{if } \beta_x(w) = H_x,\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Besides, we will assume that the Hurst function $x \to \beta_x$ is locally Hölder of an order $\eta > \bar{\beta}$ for the L^∞-norm on $\mathcal{S}$: *i.e.* for any $x \in \mathbb{R}^d$, there exist a positive constant K_x and a local neighborhood V_x of x such that

$$\forall\, y, z \in V_x,\ \sup_{s\in\mathcal{S}} |\beta_y(s) - \beta_z(s)| \leq K_x\, |y-z|^\eta. \tag{8}$$

This assumption (8) extends the one in [30] to non-constant Hurst functions.

We will also assume that the topothesy function $x \to \tau_x$ is locally Hölder of an order $\alpha > \bar{\beta}$ for the L^2-norm on $\mathcal{S}$: *i.e.* for any $x \in \mathbb{R}^d$, there exist a positive constant $\tilde{K}_x$ and a local neighborhood $\tilde{V}_x$ of x such that

$$\forall\, y, z \in \tilde{V}_x,\ \int_{\mathcal{S}} \left(\tau_y^{1/2}(s) - \tau_z^{1/2}(s)\right)^2 ds \leq \tilde{K}_x\, |y-z|^\alpha. \tag{9}$$

This assumption is also made in [30].

The Hölder assumptions made on the Hurst and topothesy functions constrain their order of regularity to be larger than a bound $\bar{\beta}$ which is above the largest local Hölder regularity of the field.

Under the assumption above, we found the expression of tangent fields of our anisotropic multifractional Brownian field, extending the result in [30].

Theorem 1 *Let Z be a generic anisotropic multifractional Brownian field satisfying assumptions of this section. For any fixed $x \in \mathbb{R}^d$, we denote by $\tilde{Z}_x$ the anisotropic fractional Brownian defined, for all $y \in \mathbb{R}^d$, by*

$$\tilde{Z}_x(y) = \int_{\mathbb{R}^d} (e^{i\langle y,\omega\rangle} - 1)\sqrt{h_{\delta_x,H_x}(\omega)}\, dW(\omega), \tag{10}$$

with a spectral density h_{δ_x,H_x} of the form (1) specified by the topothesy function δ_x of Equation (7) and the Hurst function H_x taking the constant value of Equation (5). Then,

$$(\tilde{Z}_x(u))_{u\in\mathbb{R}^d} = \lim_{t\to 0^+} \left(\frac{Z(x+tu) - Z(x)}{t^{H_x}} \right)_{u\in\mathbb{R}^d},$$

the convergence being in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets [24].

The field $\tilde{Z}_x$ is an homogeneous field which is called the tangent field of Z at x. It characterizes local properties of the field Z at x. Due to this property, Z is locally asymptotically self-similar (LASS) of order H_x at X [8].

Being zero-mean, an AMFBF can not model phenomena with large trends. To overcome this shortcoming, we will consider an additive random field model $Z = Z_L + Z_H$ composed of an AMFBF Z_H satisfying conditions of this section and an intrinsic random field Z_L of order $M \in \mathbb{N} \cup \{-1\}$ [34,28,19]. The field Z_L may include large polynomial trends of an order $M+1$. We will further assume that it has a spectral density g which fulfills the condition

$$|\omega| > B \Rightarrow g(\omega) \leq C|\omega|^{-2\bar{H}-d} \tag{11}$$

for some $B \geq 0$, $C > 0$, and $\bar{H} > \bar{\beta}$. This condition ensures that the intrinsic field is more regular than the AMFBF Z_H at any point x of $\mathbb{R}^d$, and that high-frequencies of Z_H are predominant in Z.

3 Analysis method

In this section, we propose a method to analyze AMFBF defined in Section 2.2. This method is based on localized quadratic variations that extend the ones used in [34]. We first define these quadratic variations. Then, we establish their asymptotic normality.

Throughout this section, we will assume that a field Z is observed at points $x = k/N$ for k varying on a grid $\mathcal{T}_N = [\![1,N]\!]^d$ and N fixed in $\mathbb{N}^*$. We will denote $Z^N[k]$ the observed field $Z(x)$ at position $x = k/N$.

3.1 Quadratic variations

Quadratic variations are common tools for the analysis of processes or fields derived from the fractional Brownian motion. They were initially used to identify the Hurst index of fractional Brownian motions and some related processes [7,6,21,25]. They were also used to estimate the Hurst function of multifractional Brownian motions or fields [5,22,25] and their generalizations [3,1,4]. Besides, they were used to estimate regularity and directional features of anisotropic fractional Brownian fields and related fields [18,36, 34,33,37].

Quadratic variations of a field Z are averages of square increments of Z. When observed on a grid, an increment $V^N[m]$ of Z is a linear combination of the form

$$V^N[m] = \sum_{k\in\mathbb{Z}^d} v[k] Z^N[m-k], \tag{12}$$

where v is a finite support kernel v such that $\sum_{k\in\mathbb{Z}^d} v[k]P(\frac{m-k}{N})$ vanishes for any polynomial P of a fixed order K [19,34]. A common kernel is defined by

$$v[k] = (-1)^{|k|}\binom{L_1}{k_1}\cdots\binom{L_d}{k_d}, \tag{13}$$

for $k \in [[0, L_1]] \times \cdots \times [[0, L_d]]$ and $v[k] = 0$ otherwise. Such a kernel is of order $K = |L_1| + \cdots + |L_d| - 1$. In what follows, we will denote $[[0, L]] = [[0, L_1]] \times \cdots \times [[0, L_d]]$ the support of the kernel v.

For the local analysis of multifractional Brownian motions or fields, quadratic variations were usually computed in a neighborhood of a point x of $\mathbb{R}^d$ (see [22] for instance). With our notations, this operation amounts to sum up some increments of the form

$$V^N[m] = \sum_{k\in\mathbb{Z}^d} v[k] Z^N[m-k+p_x^N], \tag{14}$$

specifically centered at a grid point p_x^N getting closer and closer to x as the resolution N tends to $+\infty$.

For the analysis of anisotropic fractional Brownian fields, quadratic variations were often associated to a directional transform of the field. In [36], they were computed on increments of the process resulting from a Radon projection of the field. In [34,33], they involved oriented increments obtained by applying to the field some combinations T of rescaling and rotation of $\mathbb{Z}^d$ into itself. These increments were of the form

$$V^N[m] = \sum_{k\in\mathbb{Z}^d} v[k] Z^N[m-Tk]. \tag{15}$$

In dimension 2, the transform T were defined, for $u \in \mathbb{Z}^2$, as

$$T_u = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} = |u| \begin{pmatrix} \cos(\arg(u)) & -\sin(\arg(u)) \\ \sin(\arg(u)) & \cos(\arg(u)) \end{pmatrix},$$

which corresponds to a rescaling of factor $|u|$ and a rotation of angle $\arg(u)$.

More generally, in dimension $d \geq 2$, the transform can be expressed as $T = \rho R$ where $\rho > 0$ is a rescaling factor and R is a rotation matrix in $\mathbb{R}^d$ satisfying $R^T R = I$. Such a matrix R can be decomposed as a product of $d-1$ matrices $(R^{(j)}, j \in [[1, d-1]])$ of $\mathbb{R}^d$ accounting for rotations on successive planes (e_1, e_{j+1}). To R, we can associate a multi-dimensional angle $\varphi = (\varphi^{(1)}, \cdots, \varphi^{(d-1)})$ where $\varphi^{(j)}$ corresponds to the angle of the rotation $R^{(j)}$ in the plane (e_1, e_{j+1}).

In this paper, we propose to use increments that are both localized and oriented. These increments are of the form

$$V^N[m] = \sum_{k \in \mathbb{Z}^d} v[k] Z^N[m - Tk + p_x^N]. \tag{16}$$

We assume that the sequence $(p_x^N)_N$ of grid points is chosen in such a way that

$$\left| \frac{p_x^N}{N} - x \right|_\infty = \underset{N \to +\infty}{O}(N^{-1}), \tag{17}$$

taking, for instance, $p_x^N = \lfloor Nx \rfloor$ for $k \in [[1, d]]$.

We select a finite set of transforms T indexed in $\mathcal{F}$ and a finite set of positions x indexed in $\mathcal{G}$. For $(a, x) \in \mathcal{F} \times \mathcal{G}$, we will denote by $V_{a,x}^N$ the increment field obtained with an ath transform and at an xth point. We will also denote T_a, $\tilde{\rho}_a$, φ_a the ath transform, the square of its rescaling factor and its rotation angle, respectively.

Next, for a fixed N, we define quadratic variations on a local neighborhood of each position x of $\mathcal{G}$ as

$$W_{a,x}^N = \frac{1}{n_\epsilon} \sum_{m \in \mathcal{V}_N} (V_{a,x}^N[m])^2, \tag{18}$$

where $\mathcal{V}_N = [[-N_\epsilon, N_\epsilon]]^d$ is a set of n_ϵ indices $m \in \mathbb{Z}^d$ such that, for all $(a, x) \in \mathcal{F} \times \mathcal{G}$, and $k \in [[1, L]]$, $m - T_a k + p_x^N$ is on the observation grid $\mathcal{T}_N$, and

$$\left| \frac{m - T_a k + p_x^N}{N} - x \right| \leq \epsilon_N. \tag{19}$$

The parameter ϵ_N determines the precision at which x is approached by grid points $\frac{m - T_a k + p_x^N}{N}$ of $\mathcal{V}_N$. It has to satisfy three conditions:

$$\begin{aligned} &(i) \lim_{N \to +\infty} \epsilon_N = 0, \\ &(ii) \lim_{N \to +\infty} N \epsilon_N = +\infty, \\ &(iii) \lim_{N \to +\infty} \log(N) \epsilon_N^\alpha = 0, \forall\, 0 < \alpha < 1. \end{aligned} \tag{20}$$

Condition (i) makes the precision increase with N while condition (ii) enables to increase the cardinality of $\mathcal{V}_N$ to $+\infty$. Condition (iii) is required to ensure the asymptotic normality of quadratic variations (see Theorem 2 and its proof for details). To fulfill the three conditions, the parameter ϵ_N can be of the form

$$\epsilon_N = KN^{-b}\log(N)^c,$$

for any $K > 0$, $0 < b < 1$ and $c \in \mathbb{R}$.

For $(a,x) \in \mathcal{F} \times \mathcal{G}$, we also define log-variations $Y^N_{a,x}$ and log-scales ρ_a as

$$Y^N_{a,x} = \log(W^N_{a,x}) \text{ and } \rho_a = \log(\tilde{\rho}_a), \text{respectively.} \tag{21}$$

3.2 Asymptotic normality

Next, we state a theorem about asymptotic properties of the log-variation random vector $Y^N = (Y^N_{a,x})_{(a,x)\in\mathcal{F}\times\mathcal{G}}$ defined above.

Theorem 2 *Let $Z = Z_L + Z_H$ be the sum of two independent fields: an anisotropic multifractional Brownian field Z_H and an intrinsic random field Z_L of order M satisfying assumptions of Section 2.2. Consider a log-variation vector Y^N constructed using increments of an order $K \geq \max(M, \frac{d}{4}, 1)$.*

Let ζ^N be a vector with terms

$$\forall a \in \mathcal{F}, x \in \mathcal{G},\ \zeta^N_{a,x} = \rho_a H_x + B^N_x(\varphi_a), \tag{22}$$

where B^N_x is defined, for any angle φ, by

$$B^N_x(\varphi) = \log\left(\frac{C_x(\varphi)}{N^{2H}}\right),$$

and

$$C_x(\varphi) = \frac{1}{(2\pi)^d}\int_0^{+\infty}\int_{E_x} |\hat{v}(\rho R'_\varphi s)|^2 \delta_x(s)\rho^{-2H_x-1}\, ds d\rho, \tag{23}$$

R_φ standing for the matrix of rotation of angle φ.

Then,

$$\sqrt{n_\epsilon}(Y^N - \zeta^N) \underset{N\to+\infty}{\overset{f.d.d.}{\longrightarrow}} \mathcal{N}(0,\Sigma) \tag{24}$$

for some covariance matrix Σ whose terms are given, for all $(a,x),(b,y) \in \mathcal{F}\times\mathcal{G}$, by

$$\Sigma_{a,x;b,y} = \frac{2(2\pi)^d \int_{[0,2\pi]^d} |\tilde{h}_{a,x;b,y}(\omega)|^2\, d\omega}{\tilde{\rho}_a^{2H_x} C_x(\varphi_a)\tilde{\rho}_b^{2H_x} C_y(\varphi_b)}, \tag{25}$$

where $\tilde{h}_{a,x;b,y}(\omega)$ is defined, for any $\omega \in [0,2\pi]^d$, by

$$\hat{v}(T'_a\omega)\bar{\hat{v}}(T'_b\omega)\sum_{q\in\mathbb{Z}^d} \delta_x^{\frac{1}{2}}(\omega+2q\pi)\delta_y^{\frac{1}{2}}(\omega+2q\pi)|\omega+2q\pi|^{-H_x-H_y-d}. \tag{26}$$

Proof The proof is divided into two parts. In a first part, we establish an asymptotic normality of quadratic variations of tangent fields of Z. Then, in a second part, we extend this result to quadratic variations of the field Z itself using a Slutsky lemma.

Part 1. The tangent field $\tilde{Z}_x$ of $Z(x)$ at position x is given by Equation (10) of Theorem 1. We define a vector-valued increment fields $\tilde{V}^N$ of $\tilde{Z}_x$ whose terms at positions $m \in \mathbb{Z}^d$ are given by

$$\forall\,(a,x) \in \mathcal{F} \times \mathcal{G},\ \tilde{V}^N_{a,x}[m] = \sum_k v[k] \tilde{Z}_x \left(\frac{p^N_x + m - T_a k}{N} \right). \tag{27}$$

Some properties of this field are stated in the following lemma.

Proposition 1 *Assume that the field $\tilde{V}^N$ is obtained with a kernel of order $K \geq \frac{d}{4}$. Then, it is zero-mean, second-order stationary with an auto-covariance whose terms are defined, for any $m, n \in \mathbb{Z}^d$, by an integral of the form*

$$\mathrm{Cov}\left(\tilde{V}^N_{a,x}[m], \tilde{V}^N_{b,y}[n]\right) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} e^{i\langle m-n, w\rangle} \tilde{h}^N_{a,x;b,y}(\omega) d\omega.$$

The function $\tilde{h}^N_{a,x;b,y}$ is the multivariate spectral density of $\tilde{V}^N$. It is given, for any $(a,x),(b,y) \in \mathcal{F} \times \mathcal{G}$, by

$$\tilde{h}^N_{a,x;b,y}(w) = N^{-H_x - H_y} \tilde{h}_{a,x;b,y}(w) e^{i\langle p^N_x - p^N_y, w\rangle},$$

where $\tilde{h}_{a,x;b,y}$ is defined by Equation (26). It is in $L^2([0,2\pi]^d)$.

In particular, for any $(a,x) \in \mathcal{F} \times \mathcal{G}$, the variance terms of $\tilde{V}^N$ are

$$\mathrm{Var}(\tilde{V}^N_{a,x}[m]) = \left(\frac{\tilde{\rho}_a}{N} \right)^{2H_x} C_x(\varphi_a), \tag{28}$$

where C_x is defined by Equation (23).

The proof of this proposition can be found in [34] (Supplementary materials, Lemma 1). For $(a,x) \in \mathcal{F} \times \mathcal{G}$, let $\tilde{X}^N_{a,x}$ be the normalized random field defined, for any $m \in \mathbb{Z}^d$, by

$$\tilde{X}^N_{a,x}[m] = \frac{N^{H_x} \tilde{V}^N_{a,x}[m]}{\tilde{\rho}_a^{H_x} \sqrt{C_x(\varphi_a)}}.$$

According to Proposition 1, it is zero-mean, second order stationary with spectral density $\frac{\tilde{h}_{a,x;a,x}}{\tilde{\rho}_a^{H_x} \sqrt{C_x(\varphi_a)}}$ in $L^2([0,2\pi]^d)$. Now, let

$$\tilde{S}^N_{a,x} = \frac{1}{\sqrt{n_\epsilon}} \sum_{m \in \mathcal{V}_N} \left((\tilde{X}^N_{a,x}[m])^2 - 1 \right). \tag{29}$$

Applying a multivariate version of the Breuer Major Theorem established in [11] (Theorem 3.2), we obtain

$$(\tilde{S}^N_{a,x})_{a \in \mathcal{F}} \underset{N \to +\infty}{\overset{f.d.d.}{\longrightarrow}} \mathcal{N}(0, \Sigma^{(x)}), \tag{30}$$

where the covariance matrix $\Sigma^{(x)} = (\Sigma_{a,x;b,x})_{a,b \in \mathcal{F}}$ is defined with terms $\Sigma_{a,x;b,x}$ of Equation (25). Now, to establish the convergence of the whole random vector $(\tilde{S}^N_{a,x})_{(a,x) \in \mathcal{F} \times \mathcal{G}}$, it suffices to show the convergence of the covariance terms $\mathrm{Cov}(\tilde{S}^N_{a,x}, \tilde{S}^N_{b,y})$. This is stated in the following lemma proved in Appendix A.

Lemma 1 *Let $\tilde{S}^N$ be defined by Equation (29) and $K > \frac{d}{4}$. Then*

$$\lim_{N\to+\infty} \mathrm{Cov}(\tilde{S}^N_{a,x}, \tilde{S}^N_{b,y}) = \Sigma_{a,x;b,y},$$

where the term $\Sigma_{a,x;b,y}$ is given by Equation (25).

Part 2. Let now

$$S^N_{a,x} = \frac{1}{\sqrt{n_\epsilon}} \sum_{m\in\mathcal{V}_N} \left(\left(\frac{N^{H_x} V^N_{a,x}[m]}{\tilde{\rho}_a^{H_x} \sqrt{C_x(\varphi_a)}} \right)^2 - 1 \right) = \sqrt{n_\epsilon} \left(\frac{N^{2H_x} W^N_{a,x}}{\tilde{\rho}_a^{2H_x} C_x(\varphi_a)} - 1 \right),$$

where $W^N_{a,x}$ are quadratic variations associated to the field Z. We are going to establish the asymptotic normality

$$S^N \underset{N\to+\infty}{\overset{f.d.d.}{\longrightarrow}} \mathcal{N}(0,\Sigma). \tag{31}$$

Assume for a moment that this convergence holds. Then, the asymptotic normality (24) of log-quadratic variations Y^N can be obtained with the Δ-method by applying the logarithm transform to $\frac{N^{2H_x} W^N_{a,x}}{\tilde{\rho}_a^{2H_x} C_x(\varphi_a)}$ and observing that

$$\log\left(\frac{N^{2H_x} W^N_{a,x}}{\tilde{\rho}_a^{2H_x} C_x(\varphi_a)} \right) = Y^N_{a,x} - \zeta^N_{a,x}.$$

To establish the convergence (31), we use a Slutsky lemma. We observe that $S^N = \tilde{S}^N + T^N$ where, for $(a,x) \in \mathcal{F}\times\mathcal{G}$,

$$T^N_{a,x} = \frac{1}{\tilde{\rho}_a^{2H_x} C_x(\varphi_a)} \frac{1}{\sqrt{n_\epsilon}} \sum_{m\in\mathcal{V}_N} \left(N^{2H_x} (V^N_{a,x}[m])^2 - N^{2H_x} (\tilde{V}^N_{a,x}[m])^2 \right).$$

The convergence in distribution of $\tilde{S}^N$ was shown in Part 1. Hence, it suffices to show that T^N converges to 0 in probability.

Let $\epsilon > 0$. Using Markov and triangular inequalities, we get

$$\mathbb{P}(|T^N| \geq \epsilon) \leq \frac{c_\epsilon}{n_\epsilon} \sum_{a,x} \sum_{m\in\mathcal{V}_N} N^{4H_x} E^N_{a,x}[m],$$

with $c_\epsilon = \epsilon^{-2} \max_{a,x} (\tilde{\rho}_a^{2H_x} C_x(\varphi_a))^{-2}$, and

$$E^N_{a,x}[m] = \mathbb{E}\left(\left((V^N_{a,x}[m])^2 - (\tilde{V}^N_{a,x}[m])^2 \right)^2 \right).$$

Furthermore, due to moment properties of Gaussian variables,

$$E^N_{a,x}[m] = 2\left((\mathbb{E}((V^N_{a,x}[m])^2)^2 + (\mathbb{E}(\tilde{V}^N_{a,x}[m])^2)^2 - 2(\mathbb{E}(V^N_{a,x}[m]\tilde{V}^N_{a,x}[m]))^2 \right).$$

We conclude the proof using the following lemma proved in Appendix A.

Lemma 2 *Take the same conditions as in Theorem 2. There exists a sequence $(\zeta_N)_N$ which tends to 0 as N tends to $+\infty$, such that, for large N*

$$|N^{2H_x}\mathbb{E}((V_{a,x}^N[m])^2) - \tilde{\rho}_a^{2H_x} C_x(\varphi_a)| \leq \zeta_N, \tag{32}$$

and

$$|N^{2H_x}\mathbb{E}(V_{a,x}^N[m]\tilde{V}_{a,x}^N[m]) - \tilde{\rho}_a^{2H_x} C_x(\varphi_a)| \leq \zeta_N, \tag{33}$$

hold for any $m \in \mathcal{V}_N$, $(a,x) \in \mathcal{F} \times \mathcal{G}$.

Indeed, due to this lemma and Equation (28), $N^{4H_x}|E_{a,x}^N[m]| \leq c\zeta_N^2$ for some $c > 0$, and any $m \in \mathcal{V}_N$, $(a,x) \in \mathcal{F} \times \mathcal{G}$. Hence, $\mathbb{P}(|T^N| \geq \epsilon) \leq \tilde{c}\, \zeta_N^2$ for some $\tilde{c} > 0$. Therefore, T^N converges to 0 in probability, and Assertion (31) holds.

4 Tests of heterogeneity

4.1 Setting

In this section, we aim at testing whether an observed AMFBF is spatially homogeneous or not. With regard to its definition in Section 2.2, an AMFBF will be considered as heterogeneous whenever the topothesy functions τ_x or the Hurst functions β_x characterizing its density vary with the position x.

Spatial heterogeneities of fields may appear in their local regularity, their directional properties, or both. The regularity of a field is heterogeneous whenever the Hurst index H_x (see Equation (5)) is not constant. Its directional properties are heterogeneous when, from a position to another, the topothesy or Hurst functions change. Field heterogeneities can be further analyzed through the function B_x^N defined by Equation (23). In the case when H_x is constant, variations of B_x^N reveal some directional heterogeneities. In the general case, they may also reflect a regularity heterogeneity as B_x^N depends on H_x.

Let $\mathcal{O} = \{\arg(u), u \in \mathcal{F}\}$ be the set of rotation angles of all transforms of $\mathcal{F}$. We express the presence of an heterogeneity at positions of $\mathcal{G}$ in terms of two hypotheses:

$$\begin{aligned}
&\mathcal{H}_1^{(1)}: \ \exists\, x,y \in \mathcal{G}, H_x \neq H_y.\\
&\mathcal{H}_1^{(2)}: \ \exists\, \varphi \in \mathcal{O}, \exists\, x,y \in \mathcal{G}, H_x \neq H_y \text{ or } B_x^N(\varphi) \neq B_y^N(\varphi).
\end{aligned}$$

The hypothesis $\mathcal{H}_1^{(1)}$ accounts for a regularity heterogeneity, and $\mathcal{H}_1^{(2)}$ for an heterogeneity concerning either the regularity or directional properties. When the field regularity is presumed constant (*ie* $\forall x \in \mathcal{G}, H_x = H$), $\mathcal{H}_1^{(2)}$ may only account for a directional heterogeneity. The negation of these hypotheses leads to two homogeneity hypotheses :

$$\begin{aligned}
&\mathcal{H}_0^{(1)}: \ \forall\, x \in \mathcal{G}, H_x = H. \qquad (34)\\
&\mathcal{H}_0^{(2)}: \ \forall\, \varphi \in \mathcal{O}, \forall\, x \in \mathcal{G}, H_x = H \text{ and } B_x^N(\varphi) = B(\varphi).
\end{aligned}$$

In what follows, we develop statistical tests to validate $\mathcal{H}_0^{(j)}$ against $\mathcal{H}_1^{(j)}$ for $j = 1, 2$.

4.2 Fisher tests

The construction of statistical tests is grounded in Theorem 2. This theorem implies that log-variations $Y_{a,x}^N$ are linked to log-scales ρ_a by a linear model of the form

$$\forall a \in \mathcal{F}, x \in \mathcal{G},\ Y_{a,x}^N = \rho_a H_x + B_x^N(\varphi_a) + \varepsilon_{a,x}, \tag{35}$$

where the distribution of the vector $(\varepsilon_{a,x})_{(a,x)\in\mathcal{F}\times\mathcal{G}}$ is close to the one of a centered Gaussian vector with covariance matrix Σ. In this relationship, the variables of interest H_x and $B_x^N(\varphi_a)$ appear as model parameters. Their values can be estimated and tested using classical techniques of generalized Gaussian linear models.

In particular, we can design some Fisher tests to check $\mathcal{H}_0^{(j)}$ against $\mathcal{H}_1^{(j)}$ for $j = 1, 2$. To define this test, we first express the model in a matricial form. For that, we create a multi-index (i, j, k), $i \in [[1, I]]$ referring to a ith position x_i in $\mathcal{G}$, $j \in [[1, J]]$ to a jth rotation angle φ_j of $\mathcal{O}$, and $k \in [[1, K_j]]$ to a rescaling factor ρ_{jk} of a kth transform of $\mathcal{G}$, say a_{jk}, among those with a rotation of angle φ_j. Using this multi-index, we define two column random vectors $\mathbf{Y} = (Y_{a_{jk},x_i}^N)_{ijk}$ and $\epsilon = (\varepsilon_{a_{jk},x_i})_{ijk}$ of size IK, where $K = K_1 + \cdots + K_J$ is the cardinality of $\mathcal{F}$. We also form a parameter vector θ of size $I(J+1)$ whose terms are defined as

$$\theta_{ml} = \begin{cases} H_{x_m} & \text{if } l = 1, \\ B_{x_m}^N(\varphi_{l-1}) & \text{if } l > 1. \end{cases}$$

using a multi-index (m, l) in $[[1, I]] \times [[1, J+1]]$. We eventually set a design matrix X of size $IK \times I(J+1)$ having terms:

$$X_{ijk,ml} = \begin{cases} \rho_{jk} & \text{if } m = i \text{ and } l = 1, \\ 1 & \text{if } m = i \text{ and } l = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

With these notations, Model (35) is equivalent to

$$\mathbf{Y} = X\theta + \epsilon, \epsilon \sim \mathcal{N}(0, \Sigma). \tag{36}$$

In this model, the parameter θ belongs to a vectorial space $\Theta = ((0,1) \times \mathbb{R}^J)^I$. When one of the assumptions $H_0^{(l)}$ given by Equations (34) holds, θ belongs to a subspace $\Theta_0^{(l)}$ of Θ of the form

$$\Theta_0^{(l)} = \{\theta \in \Theta, L^{(l)}\theta = 0\},$$

for some matrix $L^{(l)}$. For $l=1$, $L^{(1)}$ is the matrix of size $(I-1)\times I(J+1)$ whose terms are

$$L^{(1)}_{m,ij} = \begin{cases} 1 & \text{if } i=1 \text{ and } j=1,\\ -1 & \text{if } i=m+1 \text{ and } j=1,\\ 0 & \text{otherwise.}\end{cases}$$

for $m \in [[1, I-1]]$ and $(i,j) \in [[1,I]] \times [[1, J+1]]$. Likewise, for $l=2$, $L^{(2)}$ is the matrix of size $(I-1)(J+1)\times I(J+1)$ whose terms are

$$L^{(2)}_{mn,ij} = \begin{cases} 1 & \text{if } i=1 \text{ and } j=n,\\ -1 & \text{if } i=m+1 \text{ and } j=n,\\ 0 & \text{otherwise.}\end{cases}$$

for $(m,n) \in [[1,I-1]] \times [[1,J+1]]$ and $(i,j) \in [[1,I]] \times \in [[1,J+1]]$. Let $l=1,2$. The construction of a log-likelihood ratio test for the hypothesis $\mathcal{H}_0^{(l)}$ leads to a so-called Fisher statistics

$$F^{(l)} = \frac{I(K-J-1)}{I(J+1)-\mathrm{rank}(L^{(l)})} \frac{\hat{\theta}' L^{(l)\prime}(L^{(l)}(X'\Sigma^{-1}X)^{-1}L^{(l)\prime})^{-1}L^{(l)}\hat{\theta}}{(Y-X\hat{\theta})'\Sigma^{-1}(Y-X\hat{\theta})}, \tag{37}$$

where $\hat{\theta}$ is the generalized least square estimate of θ in Model (36)

$$\hat{\theta}^N = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y^N. \tag{38}$$

Setting $l=1,2$, the decision of rejecting (resp. accepting) the null hypothesis $\mathcal{H}_0^{(l)}$ is taken if $F^{(l)} \geq s_\alpha$ (resp. $\leq s_\alpha$). Under the null hypothesis, the statistic $F^{(l)}$ has a Fisher law distribution with degrees of freedom $(I(J+1) - \mathrm{rank}(L^{(l)}), I(K-(J+1)))$. Given $\alpha \in (0,1)$, the rejection threshold s_α can be set accordingly so as to ensure that

$$\mathbb{P}_{\mathcal{H}_0^{(l)}}(F^{(l)} \geq s_\alpha) \leq \alpha,$$

meaning that the probability of wrong rejection remains below α.

To implement this test, we simply use an ordinary least square estimate of θ and replace Σ by an empirical covariance matrix computed on neighborhoods of positions of $\mathcal{G}$.

5 Numerical study

So as to evaluate the test procedure presented in Section 4, we made some experiments on synthetic data. **This data was simulated using the method developed in [14].**

This method is based on the principles of the so-called turning-band method: given a set of orientations $\Theta = \{\theta_i, i = [[1,n]]\}$ in $\mathbb{R}^2$ and a set of positive weights $\Lambda = \{\lambda_i, i = [[1,n]]\}$, the random field Z to be simulated is approximated by an appropriate combination of independent random processes Y_i

$$Z_{\Theta,\Lambda}(x) = \sum_{i=1}^{n} \lambda_i Y_i(\langle u(\theta_i), x\rangle),$$

where $u(\theta_i)$ stands for a unit vector in the direction θ_i. Using this approximation, the simulation of the field Z reduces to the simulation of n random processes Y_i. In [14], we specified the turning-band method for the simulation of AFBF. In particular, we showed that the band processes Y_i should be a fractional Brownian motions with a Hurst index corresponding to the value of the Hurst function of the AFBF in the direction θ_i. We also designed a dynamic programming algorithm to select some optimal directions θ_i where processes Y_i could be exactly and quickly simulated using a circulant embedding method.

Here, we propose to extend the application of this method to AMFBF in an algorithmic way. Given an AMFBF Z and a set of simulation points $\{x_k, k = [\![1,K]\!]\}$, a realization of Z at a point x_k is obtained by simulating at position x_k the tangent field $\tilde{Z}_{x_k}$ of Z. Each tangent field is generated using a same pseudo-number sequence so as to maintain a coherent realization of Z accross different simulation points. This approach has not received any theoretical background, but was sufficient to test our method in the context of the paper.

Using this approach, we simulated AMFBF on a square grid of size 768×728. We divided this grid into three regions: two regions of size 256×768 at the top and bottom of the grid, $[\![1,257]\!]\times[\![1,768]\!]$ and $[\![512,768]\!]\times[\![1,768]\!]$, respectively, and the region in between. To apply the test procedure, we focused on the top and bottom regions. In each of these regions, the simulated AMFBF had a same tangent field.

Tangent fields of these regions were defined using two topothesy functions τ_0 and τ_1, two constant Hurst functions $\eta_0 \equiv H_0$ and $\eta_1 \equiv H_1$, and a translation parameter δ. For each experiment, these parameters were randomly and independently set: Hurst indices H_0 and H_1 were sampled from uniform distributions on $(0.05, 0.95)$ and $(H_0, 0.95)$, respectively. The translation parameter was sampled from a uniform distribution in $(0,\pi)$. We expanded topothesy functions

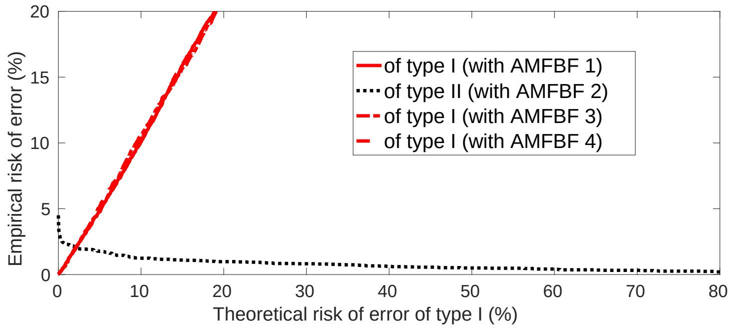
$$\tau_j(s) = \alpha_0 + \sum_{n=1}^{20} \alpha_{jn}^{(1)} \cos(2s) + \alpha_{jn}^{(2)} \sin(2s),$$

and, for $j \in [\![1,20]\!]$ and $u = 1,2$, sampled coefficients $\alpha_{jn}^{(u)}$ independently from centered normal distributions with a variance of $1/(1+n^2)$; we set $\alpha_0 = \sum_{j=1}^{20} |\alpha_{jn}^{(1)}|+|\alpha_{jn}^{(2)}|$ so as to ensure the positivity of functions. The first tangent field had parameters (τ_0,η_0), the second (τ_0,η_1), the third $(\tau_0(\cdot+\delta),\eta_0)$, and the fourth (τ_1,η_0).

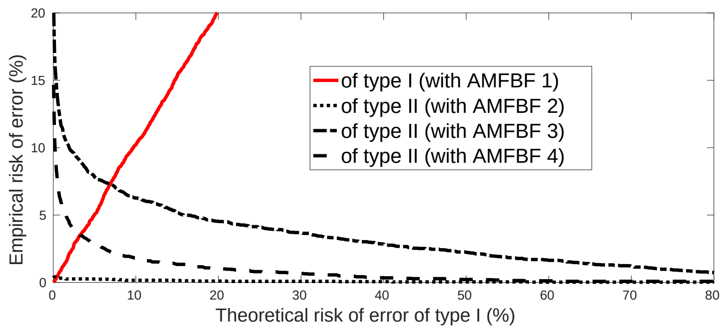
We considered four AMFBF that were tangent to the first AFBF in the top region and to one of the four AFBF in the bottom region: AMFBF 1 to 4, the number referring to the tangent AMBF on the bottom region. The AMFBF 1 was actually homogeneous while AMFBF 2, 3, 4 accounted for heterogeneous fields with variations of regularity, orientations, and anisotropy forms, respectively. By construction, AMFBF 1, 3, and 4 were

homogeneous in terms of regularity, and fulfilled Hypothesis $\mathcal{H}_0^{(1)}$ of Equation (34). By contrast, only AMFBF 1 satisfied Hypothesis $\mathcal{H}_0^{(2)}$.

We made 5000 experiments, each containing realizations of the four AMFBF. To each realizations, we applied the test procedure to the check the homogeneity of the top and bottom regions. We analyzed both regions with quadratic variations using the filter v of order 1 defined by Equation (13) with $(L_1, L_2) = (0,2)$. These variations were computed for all transformations T_u associated to $u \in \mathbb{Z}^2, \sqrt{2} \leq |u| \leq 4.3$. Hence, we had rotations of angles 0 ($u \propto (1,0)$), $\pi/4$ ($u \propto (1,1)$), $\pi/2$ ($u \propto (0,1)$) or $3\pi/4$ ($u \propto (1,-1)$), and rescalings of factors $|u|$ between $\sqrt{2}$ and 4.3. In particular, we had three quadratic variations in each direction.



(a) Test of $\mathcal{H}_0^{(1)}$ against $\mathcal{H}_1^{(1)}$.



(b) Test of $\mathcal{H}_0^{(2)}$ against $\mathcal{H}_1^{(2)}$.

Fig. 2: Empirical risks of error for the two heterogeneity tests.

Next, we computed Fisher statistics $F^{(1)}$ and $F^{(2)}$ for testing hypotheses $\mathcal{H}_0^{(1)}$ and $\mathcal{H}_0^{(2)}$ of Equations (34), respectively. For each test and type of simulated AMFBF, we assessed rates of wrong decisions taken over the dataset for different values of rejection threshold s_α. In the case when the AMFBF fulfilled $\mathcal{H}_0^{(u)}$, such empirical rates gave us estimates of the risk of error of type I (error made when rejecting $\mathcal{H}_0^{(u)}$) whereas, in the opposite case, they were estimates of the risk of error of type II (error made when rejecting $\mathcal{H}_1^{(u)}$). In Figure 2, these estimated risks are plotted in percent as a function of the theoretical risk of error of type I associated to the threshold s_α.

For both tests, we notice that red lines representing estimated risks of error of type I with respect to the theoretical ones were close to the identity line. This means that empirical distributions of Fisher statistics under the null hypotheses approximately matched the theoretical ones. It also suggests that our simulation and estimation procedures were well-suited to the theoretical model.

Besides, we observe that estimated risks of error of type II depended on the heterogeneity nature of simulated AMFBF. The regularity heterogeneity of AMFBF 2 was better detected than directional heterogeneities of AMFBF 3 and 4. For AMFBF 2, the risk of missing a regularity heterogeneity was about 3.5% (with the first test) and 0.5% (with the second test) while the risk of error of type I was only about 0.05%. By contrast, for AMFBF 3 and 4, the risk of missing a directional heterogeneity with the second test was above 15% for the same level of risk of error of type I. Heterogeneities resulting from a change of the topothesy orientation (AMFBF 3) was particularly difficult to detect. When the risk of error of type I was about 1%, the risk of missing such heterogeneities was about 10%.

6 Discussion

We proposed statistical tests to decide whether an observed random field is heterogeneous or not, the heterogeneity resulting from spatial variations of the regularity of the field, its directional properties, or both. The test procedure was based on quadratic variations specifically designed for a local and oriented analysis of the observed field. For developing the theory, we set a framework of anisotropic multifractional Brownian field heterogeneity. In this framework, we established the asymptotic normality of quadratic variations. This result highlighted a Gaussian linear relationship between quadratic variations and parameters related to the regularity and directional features of the model. From this relationship, we designed our tests using Fisher statistics of linear Gaussian models. We evaluated the test procedure using data simulated with a turning-band method. Results showed the feasability of the procedure: applied to datasets mixing realizations of isotropic and anisotropic fields, it could correctly detect anisotropic fields with error rates ranging from 0.5 % to 10 % while maintaining a low error rate (below 1%) of detecting isotropic fields. Errors of detecting anisotropic fields varied depending on the type of heterogeneity. Regularity heterogeneities were the easiest to detect while those due to orientation changes were the most difficult.

Random fields derived from the fractional Brownian motion has proved to be useful models for the analysis of rough textures in the domain of image processing. In particular, anisotropic fractional Brownian fields could be applied to analyze textures appearing in images as various as bone radiographs [10], mammographs [15,36,32,33] , or photographic films [35]. Accounting for directional properties of textures, they enabled to achieve various image classification tasks. However, as they applied homogeneous textures, they were only suitable for global processing tasks. The introduction of an anisotropic heterogeneous Brownian model enables to overcome this limitation and widen the scope of applications of Brownian models. It notably provides practionners with a model which may account for the presence of several textures within a single image. It also sets a theoretical framework for image segmentation, task which consists of delimiting regions where these different textures are. In such a task, heterogeneity tests can be used to decide if textures contained in different regions are similar or not.

In this paper, the analysis of the field was focused on testing its heterogeneity. But, it could also be used to characterize the field heterogeneity. Following an approach that is similar to the one developed in [32,33,34] for AFBF, estimates involved in our field analysis could serve for defining some measures of the heterogeneity of AMFBF. Moreover, the approach proposed in [35] to estimate the so-called asymptotic topothesy of AFBF could also be extended to the one of tangent fields of AMFBF. This would path the way to the local estimation of the whole topothesy and Hurst functions of AMFBF.

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A Proof of supporting lemmas and propositions

We first show a main technical lemma.

Lemma 3 **(Main lemma)** *Let Z be an anisotropic multifractional Brownian field which fulfills all assumptions of Section 2.2. Consider an increment kernel v of order $K \geq 0$ with a support $[\![0, L]\!]$ for some $L \in \mathbb{N}^d$, as well as two transforms T_1 and T_2 composed of a rotation and a rescaling. Fix a position $x \in \mathbb{R}^d$. Choose a positive sequence $(\varepsilon_t)_{t>0}$ such that*

$$\lim_{t \to 0^+} |\log(t)| \varepsilon_t^{\eta} = 0. \tag{39}$$

For any $t > 0$, set the local neighborhood of x

$$\mathcal{V}_t = \{y \in \mathbb{R}^d, |y - tT_j k - x| \leq \varepsilon_t, \forall\, k \in [\![0, L]\!], j = 1, 2\}.$$

and, for $y \in \mathbb{R}^d$ and $j \in \{1, 2\}$, denote $V_{tj}(y) = \sum\limits_{k \in [\![0,L]\!]} v[k] Z(y - tT_j k)$.

For any $y \in \mathbb{R}^d$, define

$$I(y) = \int_{\mathbb{R}^d} \hat{v}(T_1' w) \overline{\hat{v}(T_2' w)} e^{i\langle y, w\rangle} \delta_x(w) |w|^{-2H_x - d} dw, \tag{40}$$

where H_x and δ_x are defined by Equations (5) and (7), respectively.

Then there exists a positive sequence $(\gamma_t)_{t>0}$ converging to 0 as $t \to 0^+$ such that, for any small $t > 0$,

$$\forall y_1, y_2 \in \mathcal{V}_t, \left| t^{-2H_x} \mathbb{E}(V_{t1}(y_1) V_{t2}(y_2)) - I(t^{-1}(y_1 - y_2)) \right| \leq \gamma_t. \tag{41}$$

We also state a variant of this lemma whose proof can be established in a same way.

Lemma 4 **(A variant of the main lemma)** *Take the same conditions as in Lemma 3. Set $x \in \mathbb{R}^d$. Let $\tilde{Z}_x$ be the tangent field of Z at x defined by Equation (10).*

For $y \in \mathbb{R}^d$, let $V_{t1}(y) = \sum\limits_{k \in [\![0,L]\!]} v[k] Z(y - tT_1 k)$, and $V_{t2}(y) = \sum\limits_{k \in [\![0,L]\!]} v[k] \tilde{Z}_x(y - tT_2 k)$. Then there exists a positive sequence $(\gamma_t)_{t>0}$ converging to 0 as $t \to 0^+$ such that, for any small $t > 0$, Equation (41) holds for the integral I defined by Equation (40).

Proof (Lemma 3)

We first show the result in the case of an elementary kernel v of order 0. Then, we extend it to the case of a general kernel of order $K \geq 0$.

So, let us begin by assuming that, for some $i \in [\![1, d]\!]$, $v[0] = 1$, $v[e_i] = -1$, and $v[k] = 0$ for all other $k \in \mathbb{Z}^d$ (e_i denotes the unit vector of $\mathbb{R}^d$ whose ith components is 1 and others are 0). Then, for $t > 0$ and $y_1, y_2 \in \mathcal{V}_t$,

$$\mathbb{E}(V_{t1}(y_1) V_{t2}(y_2)) = \int_{|w| \geq A} R_{t1}(w) \overline{R_{t2}(w)} dw, \tag{42}$$

where, for $j \in \{1, 2\}$, $R_{tj}(w) = (e^{i\langle y_j, w\rangle} - 1)\sqrt{f_{y_j}(w)} - (e^{i\langle y_j - t_j, w\rangle} - 1)\sqrt{f_{y_j - t_j}(w)}$, with $t_j = tT_j e_i$. Specifying f_x with its expression (1), we can further write $R_{tj}(w) = \sum_{k=1}^{3} R_{tjk}(w)$ with

$$\begin{aligned}
R_{tj1}(w) &= e^{i\langle y_j, w\rangle} (1 - e^{-i\langle t_j, w\rangle}) \sqrt{\tau_{y_j - t_j}(w)}\, |w|^{-\beta_{y_j - t_j}(w) - \frac{d}{2}}, \\
R_{tj2}(w) &= (e^{i\langle y_j, w\rangle} - 1) \left(\sqrt{\tau_{y_j}(w)} - \sqrt{\tau_{y_j - t_j}(w)} \right) |w|^{-\beta_{y_j}(w) - \frac{d}{2}}, \\
R_{tj3}(w) &= (e^{i\langle y_j, w\rangle} - 1) \sqrt{\tau_{y_j - t_j}(w)} \left(|w|^{-\beta_{y_j}(w) - \frac{d}{2}} - |w|^{-\beta_{y_j - t_j}(w) - \frac{d}{2}} \right).
\end{aligned} \tag{43}$$

Now, let

$$D_t = t^{-2H_x}\mathbb{E}(V_{t1}(y_1)V_{t2}(y_2)) - I(t^{-1}(y_1-y_2)). \tag{44}$$

After a variable change in the integral I, we get $D_t = t^{-2H_x}(D_{t1} - D_{t2} + D_{t3})$ where

$$\begin{aligned}
D_{t1} &= \int_{|\omega|>A} R_{t11}(w)\,\overline{R_{t21}(w)} - e^{i\langle y_1-y_2,w\rangle}(e^{-i\langle t_1,w\rangle}-1)(e^{i\langle t_2,w\rangle}-1)\delta_x(w)|w|^{-2H_x-d}\,d\omega,\\
D_{t2} &= \int_{|\omega|\le A} e^{i\langle y_1-y_2,w\rangle}(e^{-i\langle t_1,w\rangle}-1)(e^{i\langle t_2,w\rangle}-1)\,\delta_x(w)|w|^{-2H_x-d}\;d\omega,\\
D_{t3} &= \sum_{(k,l)\ne(1,1)}\int_{|w|>A} R_{t1k}(w)\,\overline{R_{t2l}(w)}dw.
\end{aligned}$$

Using $|e^{iu}-1| = 2|\sin(\frac{u}{2})| = \underset{u\to 0}{O}(|u|)$, we obtain that

$$|D_{t2}| \le ct^2\int_{|w|\le A}|w|^{2-2H_x-d}dw,$$

where $c>0$ and the integral on the right-hand side is finite since $0<H_x<1$. Thus,

$$t^{-2H_x}D_{t2} = \underset{t\to 0^+}{O}(t^{2(1-H_x)}). \tag{45}$$

Due to the Cauchy-Schwarz inequality, $D_{t3} \le \sum_{(k,l)\ne(1,1)}\sqrt{\Delta_{t1k}}\sqrt{\Delta_{t2l}}$, where

$$\Delta_{tjk} = \int_{|w|\ge A}|R_{tjk}(w)|^2 d. \tag{46}$$

To deal with D_{t3}, we establish the following lemma.

Lemma 5 *We have the following inequalities.*

$$\begin{aligned}
t^{-2H_x}\Delta_{tj1} &\le c\max(|T_j|^{2H}, |T_j|^{2\bar\beta}), && (47)\\
t^{-2H_x}\Delta_{tj2} &= |T_j|^{2\alpha}\underset{t\to 0^+}{O}(t^{2\alpha-2H_x}), && (48)\\
t^{-2H_x}\Delta_{tj3} &= |T_j|^{2\eta}\underset{t\to 0^+}{O}(t^{2\eta-2H_x}). && (49)
\end{aligned}$$

In polar coordinates,

$$\Delta_{tj1} = \int_{\mathcal{S}}\int_A^{+\infty}|1-e^{-i\rho\langle t_j,s\rangle}|^2\tau_{y_j-t_j}(s)\rho^{-2\beta_{y_j-t_j}(s)-1}d\rho ds.$$

Since τ is bounded and, for all w, $0<H\le H_x\le\beta_x(w)\le\bar\beta$, a variable change $r = tT_j'\rho$ in Δ_{tj1} yields

$$\Delta_{tj1} \le c\max(|T_j|^{2H}, |T_j|^{2\bar\beta})\int_{\mathcal{S}}(\int_{r<1}|1-e^{-ir\langle e_j,s\rangle}|^2 r^{-2\bar\beta-1}dr + \int_{r>1}r^{-2H-1}dr)J_0(s)ds$$

where

$$J_0(s) = \int_{\mathcal{S}} t^{2\beta_{y_j-t_j}(s)-2H_x}ds,$$

for some $c>0$. In the right-hand side of the inequality, both integrals over r are finite. Hence,

$$\Delta_{tj1} \le c\max(|T_j|^{2H}, |T_j|^{2\bar\beta})\int_{\mathcal{S}}J_0(s)ds.$$

Observe now that

$$\begin{aligned} t^{2\beta_{y_j-t_j}(s)-2H_x} &= t^{2\beta_x(s)-2H_x}\, t^{2\beta_{y_j-t_j}(s)-2\beta_x(s)} \\ &= t^{2\beta_x(s)-2H_x} \exp\Big(\log(t)[2\beta_{y_j-t_j}(s)-2\beta_x(s)]\Big). \end{aligned}$$

Using Assumption (8),

$$|2\beta_{y_j-t_j}(s)-2\beta_x(s)| \leq 2\epsilon_t^\eta.$$

Due to Assumption (39), we then deduce the limit:

$$\lim_{t\to 0^+} |\log(t)|[2\beta_{x_0+tv}(s)-2\beta_{x_0}(s)] = 0. \tag{50}$$

Consequently, $\lim_{t\to 0^+} t^{2\beta_{y_j-t_j}(s)-2\beta_x(s)} = 1$. Therefore, we get

$$J_0(s) \leq c \int_{\mathcal{S}} t^{2\beta_x(s)-2H_x}\, ds,$$

since $\beta_x(s) \geq H_x, \forall s \in \mathcal{S}$, $J_0(s)$ is finite. Hence, Equation (47) holds.

In polar coordinates,

$$\Delta_{tj2} = \int_{\mathcal{S}} \int_A^{+\infty} \rho^{-2\beta_{y_j}(s)-1} |1-e^{i\rho\langle y_j,s\rangle}|^2 \left(\sqrt{\tau_{y_j}(s)} - \sqrt{\tau_{y_j-t_j}(s)}\right)^2 d\rho\, ds.$$

Using the variable change $r = \rho\langle y_j, s\rangle$, we obtain

$$\Delta_{tj2} \leq c \int_0^{+\infty} |1-e^{-ir}|^2\, r^{-2\beta_{y_j}(s)-1} \int_{\mathcal{S}} \left(\sqrt{\tau_{y_j}(s)} - \sqrt{\tau_{y_j-t_j}(s)}\right)^2 ds\, dr.$$

Using Assumption (9), we get

$$\Delta_{tj2} \leq c\,|T_j|^{2\alpha}\, t^{2\alpha} \left(\int_0^1 |1-e^{-ir\rangle}|^2 r^{-2\bar{\beta}-1}\, d\rho + \int_1^{+\infty} r^{-2H-1}\, dr \right),$$

for small t and some $c > 0$. The integrals on the right-hand side are finite. Hence, Equation (48) holds.

Applying a mean value inequality to the function $u \to \rho^u$, we obtain that

$$|\rho^{-\beta_{y_j}(s)} - \rho^{-\beta_{y_j-t_j}(s)}| \leq c|\beta_{y_j}(s)-\beta_{y_j-t_j}(s)||\log(\rho)|\max(\rho^{-H}, \rho^{-\bar{\beta}}).$$

Hence, using Assumption (8), we get

$$|\rho^{-\beta_{y_j}(s)} - \rho^{-\beta_{y_j-t_j}(s)}| \leq c\,|T_j|^\eta\, t^\eta |\log(\rho)|\max(\rho^{-H}, \rho^{-\bar{\beta}}),$$

for small t and some $c > 0$. Therefore, in polar coordinates, we have

$$\Delta_{tj3} \leq c\,|T_j|^{2\eta}\, t^{2\eta} \left(\int_0^1 |1-e^{i\rho\langle y_j,s\rangle}|^2 \rho^{-2\bar{\beta}-1}\, |\log(\rho)| d\rho + \int_1^{+\infty} \rho^{-2H-1}\, |\log(\rho)| d\rho \right).$$

Since, for any $u > 0$, $|\log(t)|t^u$ (resp. $|\log(t)|t^{-u}$) tends to 0 as t tends to 0 (resp. $+\infty$), the integrals on the right-hand side are bounded by a constant proportional to a sum

$$\int_0^\nu \rho^{2-2(\bar{\beta}+\epsilon)-1}\, d\rho + \int_\nu^1 \rho^{-2\bar{\beta}}\, d\rho + \int_1^B \rho^{-2H-1}\, d\rho + \int_B^{+\infty} \rho^{-2(H-\epsilon_2)-1}\, d\rho \tag{51}$$

of finite integrals defined for some $0 < \nu < 1$, $B > 1$, and $0 < \epsilon < \min(H, 1-\bar{\beta})$. Hence, Equation (49) holds.

Eventually, using Lemma 5, we get

$$t^{-2H_x} D_{t3} = \underset{t\to 0^+}{O}(t^{2(\min(\alpha,\eta)-H_x)}). \tag{52}$$

Besides, $D_{t1} \leq D_{t11} + D_{t12}$, where, in polar coordinates,

$$\begin{aligned}
D_{t1j} &= \int_A^{+\infty}\int_{\mathcal{S}} |e^{-i\rho\langle t_1,s\rangle}-1||e^{i\rho\langle t_2,s\rangle}-1| d_{t1j}(\rho,s)dsd\rho,\\
d_{t1}(\rho,s) &= \delta_x(s)\rho^{-2H_x-1}\left|\rho^{2H_x-\beta_{y_1-t_1}(s)-\beta_{y_2-t_2}(s)}-1\right|,\\
d_{t2}(\rho,s) &= \left|\sqrt{\tau_{y_1-t_1}(s)}\sqrt{\tau_{y_2-t_2}(s)}-\delta_x(s)\right|\rho^{-\beta_{y_1-t_1}(s)-\beta_{y_2-t_2}(s)-1}.
\end{aligned}$$

Consider the set E_x defined by Equation (6). Due to the definition of δ_x in Equation (7), the integrand $d_{t11}(s)$ vanishes for $s\in E_x^c$. Applying the mean value theorem to the function $u\to\rho^u$, and using Assumption (8), we obtain that, for $s\in E_x$,

$$|\rho^{2H_x-\beta_{x_k^t}(s)-\beta_{x_l^t}(s)}-1| \leq 2|\log(\rho)|\max(\rho^{-2\varepsilon_t^\eta},\rho^{2\varepsilon_t^\eta})\varepsilon_t^\eta.$$

Moreover, due to Assumption (39), ε_t^η, $t^{2\varepsilon_t^\eta}$ and $t^{-2\varepsilon_t^\eta}$ are bounded for small t. Therefore, $\max(\rho^{-2\varepsilon_t^\eta},\rho^{2\varepsilon_t^\eta})\leq c\max((t\rho)^{-2\epsilon},(t\rho)^{2\epsilon})$ for some $\epsilon<\min(H,1-\bar{\beta})$ and small t. On the other hand, $|\log(\rho)|\leq\left(1+\left|\log(t\rho)\right|\right)|\log(t)|$ for small t. Therefore, we have

$$d_{t1}(\rho,s)\leq c(t\rho)^{-2H_x-1}\max((t\rho)^{-2\epsilon},(t\rho)^{2\epsilon})\left(1+\left|\log(t\rho)\right|\right)t^{2H_x+1}|\log(t)|\varepsilon_t^\eta.$$

Hence, after a variable change $r=t\rho$ in the integral, we get $D_{t11}\leq \underset{t\to 0^+}{O}\left(t^{2H_x}|\log(t)|\varepsilon_t^\eta\right)J_1$, where J_1 is equal to

$$\int_{\mathcal{S}}\int_0^{+\infty}|e^{-i\rho\langle T_1e_i,s\rangle}-1||e^{i\rho\langle T_2e_i,s\rangle}-1|\left(1+\left|\log(\rho)\right|\right)\max\left(\rho^{-2(H-\epsilon)-1},\rho^{-2(\bar{\beta}+\epsilon)-1}\right)d\rho ds$$

, an integral bounded by a sum of finite integrals of the form (51). Hence,

$$t^{-2H_x}D_{t11} = \underset{t\to 0^+}{O}\left(|\log(t)|\varepsilon_t^\eta\right).$$

Using Assumption (8),

$$\rho^{-\beta_{x_1-t_1}(s)-\beta_{x_2-t_2}(s)-1}\leq c\rho^{-2\beta_x(s)-1}\max(\rho^{2\varepsilon_t^\eta},\rho^{-2\varepsilon_t^\eta}).$$

Hence, due to Assumption (39),

$$\rho^{-\beta_{x_1-t_1}(s)-\beta_{x_2-t_2}(s)-1}\leq ct^{2\beta_x(s)+1}\max((t\rho)^{-2(\bar{\beta}+\epsilon)-1},(t\rho)^{-2(H-\epsilon)-1}),$$

for small t, and some $\epsilon<\min(H,1-\bar{\beta})$. Consequently,

$$D_{t12}\leq c\int_{\mathcal{S}}\left|\sqrt{\tau_{x_1-t_1}(s)}\sqrt{\tau_{x_2-t_2}(s)}-\delta_x(s)\right|t^{2\beta_x(s)}J_2(s)ds,$$

where the integrals

$$J_2(s)=\int_0^{+\infty}|e^{-i\rho\langle T_1e_i,s\rangle}-1||e^{i\rho\langle T_2e_i,s\rangle}-1|\max(\rho^{-2(H-\epsilon)-1},\rho^{-2(\bar{\beta}+\epsilon)-1})d\rho$$

are bounded independently of s by a sum of finite integrals of the form (51). Furthermore, for $s \in E_c^x$, $\delta_x(s) = 0$ and τ_x is bounded. On the other hand, for $x \in E_x$, $\delta_x(s) = \tau_x(s)$, and $\sqrt{\tau_{x_1-t_1}(s)}\sqrt{\tau_{x_2-t_2}(s)} - \delta_x(s)$ can be written as

$$\prod_{j=1}^{2}(\sqrt{\tau_{x_j-t_j}(s)} - \sqrt{\tau_x(s)}) + \sqrt{\tau_x(s)}\sum_{j=1}^{2}(\sqrt{\tau_{x_j-t_j}(s)} - \sqrt{\tau_x(s)}).$$

Therefore, using a Cauchy-Schwarz inequality, we obtain $t^{-2H_x} D_{t12} \leq c(\zeta_{t1} + \zeta_{t2})$, with

$$\zeta_{t1} = \int_{E_x^c} t^{2(\beta_x(s)-H_x)} ds,$$

and

$$\zeta_{t2} = \sum_{j=1}^{2} \sqrt{\int_{\mathcal{S}} \left(\sqrt{\tau_{x_j-t_j}(s)} - \sqrt{\tau_x(s)}\right)^2 ds} + \prod_{j=1}^{2} \sqrt{\int_{\mathcal{S}} \left(\sqrt{\tau_{x_j-t_j}(s)} - \sqrt{\tau_x(s)}\right)^2 ds}.$$

From Assumption (8), it follows that $\zeta_{t2} = \underset{t\to 0^+}{O}\left(\varepsilon_t^{\alpha}\right)$. Besides, let

$$F_{x,t} = \left\{ s \in \mathcal{S}, \beta_x(s) - H_x > |\log(t)|^{-\frac{1}{2}} \right\}.$$

Then,

$$\zeta_{t1} = \int_{F_{x,t}} t^{2(\beta_x(s)-H_x)} ds + \int_{E_x^c \setminus F_{x,t}} t^{2(\beta_x(s)-H_x)} ds \leq c\left(t^{2|\log(t)|^{-\frac{1}{2}}} + \mu(E_x^c \setminus F_{x,t}) \right),$$

where $\mu(E_x^c \setminus F_{x,t})$ is the measure of $E_x^c \setminus F_{x,t}$ on $\mathcal{S}$. This measure tends to 0 as t tends to 0, as well as the first term on the right-hand side of the inequality. As a consequence, there exists $(\zeta_t)_t$ converging to 0 as t tends to 0^+ such that $t^{-2H_x} D_{t12} = \underset{t\to 0^+}{O}(\max(\zeta_t, \varepsilon_t^{\alpha}))$, and

$$t^{-2H_x} D_{t1} = \underset{t\to 0^+}{O}(\max(\zeta_t, \varepsilon_t^{\alpha}, |\log(t)|\varepsilon_t^{\eta})). \tag{53}$$

Using Equations (45), (52), (53), we finally establish Equation (41) for some sequence $(\gamma_t)_t$ satisfying

$$\gamma_t = \underset{t\to 0^+}{O}\left(\max\left(\zeta_t, \varepsilon_t^{\alpha}, |\log(t)|\varepsilon_t^{\eta}, t^{2(1-H_x)}, t^{2(\min(\alpha,\eta)-H_x)}\right)\right).$$

This sequence converges to 0 as $t \to 0^+$, due to Assumption (39) and since $\bar{\beta} < \min(\alpha, \eta)$, and $H_x < \bar{\beta} < 1$.

Let us now turn to the general case when v is a kernel of order $K \geq 0$. In this case, the characteristic polynomial $Q_v(z) = \sum_{k\in[[0,L]]} v[k] z_1^{k_1} \cdots z_d^{k_d}$ of the kernel v admits $(1, \cdots, 1)$ as a root. Consequently, for some $i \in [[1, d]]$, there exists a polynomial $Q_{\tilde{v}}$, such that $Q_v(z) = (1 - z_i)Q_{\tilde{v}}(z)$. By identification, one can check that the polynomial $Q_{\tilde{v}} = \sum_{k\in[[0,L-e_i]]^d} \tilde{v}[k] z_1^{k_1} \cdots z_d^{k_d}$, where $v[k] = \tilde{v}[k] - \tilde{v}[k - e_i]$ for all $k \in [[e_i, L - e_i]]^d$, $v[0] = \tilde{v}[0]$, and $v[L] = -\tilde{v}[L - e_i]$. As a consequence, we may write

$$V_{tj}(y) = \sum_{k\in[[0,L-e_i]]^d} \tilde{v}[k]\, V_{tj}^0(y - tT_j k),$$

where $V_{tj}^0(y) = \sum_{m\in[[0,e_i]]^d} v_0[m] Z(y - tT_j m)$ is an increment of order 0 defined with the kernel v_0 having terms $v_0[0] = 1$, $v_0[e_i] = -1$. Moreover,

$$\hat{v}(w) = Q_v(e^{iw_1}, \cdots, e^{iw_d}) = (1 - e^{i\langle e_i, w\rangle}) Q_{\tilde{v}}(e^{iw_1}, \cdots, e^{iw_d}),$$

so that $I(y) = \sum_{k,l\in[[0,L-e_i]]^d} \tilde{v}[k]\tilde{v}[l]I^0(y - T_1 k + T_2 l)$, with

$$I^0(u) = \int_{\mathbb{R}^d} (1 - e^{-i\langle e_i, T_1' w\rangle})(1 - e^{i\langle e_i, T_2' w\rangle})e^{i\langle u,w\rangle}\delta_x(w)|w|^{-2H_x - d}\,dw.$$

So, in this case, the quantity D_t defined in Equation (42) may be expressed as

$$D_t = \sum_{k,l\in[[0,L-e_i]]^d} \tilde{v}[k]\tilde{v}[l]D_t^0[k,l],$$

with

$$D_t^0[k,l] = t^{-2H_x}\mathbb{E}(V_{t1}^0(y_1 - tT_1 k)V_{t2}^0(y_2 - tT_2 l)) - I^0(t^{-1}(y_1 - y_2 - tT_1 k + tT_2 l)).$$

This implies that $|D_t| \le c \sum_{k,l\in[[0,L-e_i]]^d} |D_t^0[k,l]|$, for some $c > 0$. But, from the first part, each $D_t^0[k,l]$ satisfies Equation (41) for a sequence $(\gamma_t)_t$. So, does D_t.

Proof (Lemma 1) Let $\tilde{S}_{a,x}^N$ be as given by Equation (29). Using moment properties of Gaussian variables, we obtain

$$\begin{aligned}
\mathrm{Cov}(\tilde{S}_{a,x}^N, \tilde{S}_{b,y}^N) &= \frac{1}{n_\epsilon} \sum_{m,n\in\mathcal{V}_N} \left(\mathrm{Cov}((\tilde{X}_{a,x}^N[m])^2, (\tilde{X}_{b,y}^N[n])^2)\right)\\
&= \frac{1}{(2N_\epsilon + 1)^d} \sum_{m,n\in[[-N_\epsilon,N_\epsilon]]^d} 2\left(\mathbb{E}(\tilde{X}_{a,x}^N[m]\tilde{X}_{b,y}^N[n])\right)^2 - 1\\
&= \sum_{k\in[[-2N_\epsilon+1,2N_\epsilon+1]]^d} \prod_{i=1}^d \left(1 - \frac{|k_i|}{2N_\epsilon + 1}\right)|r[k + p_x^N - p_y^N]|^2 - 1,
\end{aligned}$$

where

$$r[k] = \frac{\int_{[0,2\pi]^d} e^{i\langle k,w\rangle}\tilde{h}_{a,x;b,y}(\omega)d\omega.}{\tilde{\rho}_a^{H_x}\sqrt{C_x(\varphi_a)}\tilde{\rho}_b^{H_y}\sqrt{C_y(\varphi_b)}}, \tag{54}$$

with notations of Equations (23) and (26). The term $r[k]$ is a Fourier coefficient of a function which is in $L^2([0,2\pi]^d)$ for $K > d/4$ (see [34]). Using the Bessel Parseval equality, we thus have

$$\sum_{k\in\mathbb{Z}^d} |r[k]|^2 = \frac{1}{(2\pi)^d} \frac{\int_{[0,2\pi]^d} |\tilde{h}_{a,x;b,y}(\omega)|^2 d\omega.}{\tilde{\rho}_a^{2H_x}C_x(\varphi_a)\tilde{\rho}_b^{2H_y}C_y(\varphi_b)} = \Sigma_{a,x;b,y}.$$

Hence, to establish the proposition, it suffices to show that

$$S^N = \sum_{k\in[[-2N_\epsilon+1,2N_\epsilon+1]]^d} \left(|r[k + p_x^N - p_y^N]|^2 - |r[k]|^2\right)$$

converges to 0 as N tends to $+\infty$. If $x = y$, this sum vanishes and the convergence is trivial. So, let us assume that $x \ne y$. Since $|r[k]|$ tends to 0 as $|k|$ tend to $+\infty$, we have

$$|S^N| \le c \sum_{k\in[[-2N_\epsilon+1,2N_\epsilon+1]]^d} s^N[k], \tag{55}$$

with $s^N[k] = |r[k + p_x^N - p_y^N] - r[k]|$ and $c > 0$. Moreover,

$$\begin{aligned}
s^N[k] &\le c\left|\int_{[0,2\pi]^d} (e^{i\langle k+p_x^N - p_y^N,\omega\rangle} - e^{i\langle k,\omega\rangle})\tilde{h}_{a,x;b,y}(\omega)d\omega\right|\\
&\le c\left|\int_{[0,2\pi]^d} e^{-i\langle \frac{p_y^N - p_x^N}{2},\omega\rangle} e^{i\langle k,\omega\rangle} \sin(\langle p_x^N - p_y^N, \omega\rangle/2)\tilde{h}_{a,x;b,y}(\omega)d\omega\right|,
\end{aligned}$$

for some $c > 0$. Therefore,

$$s^N[k] \leq c \left| \int_{\mathbb{R}^d} e^{-i\langle \frac{p_y^N - p_x^N}{2} - k, \omega\rangle} b(\omega) d\omega \right|, \tag{56}$$

where $b(\omega) = |\sin(\langle p_x^N - p_y^N, \omega\rangle/2)|\,|\omega|^{-H_x - H_y - d}$.

Let us now study the integral $B(u) = \int_{\mathbb{R}^d} e^{-i\langle u, \omega\rangle} b(\omega) d\omega$. Assume that all components of u are non-zero. Using integration by parts on the first component of w, we get that $B(u)$ is equal to

$$\int\limits_{\mathbb{R}^{d-1}} \prod_{m\neq 1} e^{i\langle u_m, \omega_m\rangle} \left(\left[\frac{e^{i\langle u_1, \omega_1\rangle} - 1}{iu_1} b(\omega) \right]_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} \frac{(e^{i\langle u_1, \omega_1\rangle} - 1)}{iu_1} \frac{\partial b}{\partial \omega_1}(\omega) d\omega_1 \right) \prod_{m\neq 1} d\omega_m.$$

Since $\lim\limits_{|w|\to+\infty} b(w) = 0$, it follows that

$$B(u) = -\frac{1}{iu_1} \int\limits_{\mathbb{R}^d} \prod_{m\neq 1} e^{i\langle u_m, \omega_m\rangle} (e^{i\langle u_1, \omega_1\rangle} - 1) \frac{\partial b}{\partial \omega_1}(\omega) d\omega.$$

Repeating the process on other components of w, we obtain

$$B(u) = i^d \prod_{m\in[[1,d]]} \frac{1}{u_m} \int\limits_{\mathbb{R}^d} \prod_{m\in[[1,d]]} (e^{i\langle u_l, \omega_l\rangle} - 1) \frac{\partial^d b}{\partial \omega_1 \cdots \partial \omega_d}(\omega) d\omega,$$

so that, for some $c > 0$,

$$|B(u)| \leq c \prod_{m\in[[1,d]]} \frac{1}{|u_m|}.$$

Hence, using Equation (56), we eventually get

$$s^N[k] \leq c \prod_{m\notin J} \frac{1}{|p_{x,m}^N - p_{y,m}^N|} \prod_{m\in J} \frac{1}{|k_m|},$$

where $J = \{m \in [[1,d]], x_m - y_m = 0\}$ is a non-empty set of indices.

Using Condition (17), it follows that

$$s^N[k] = \underset{N\to+\infty}{O}(N^{-(d-|J|)}) \prod_{m\in J} \frac{1}{|k_m|},$$

where $|J|$ stands for the cardinal of J. Moreover, we have

$$\sum_{k_m\in[[-2N_\epsilon+1, 2N_\epsilon+1]]} |k_m|^{-1} \leq 1 + 2\int_1^{2N_\epsilon} \frac{1}{x} dx \leq 1 + 2\log(2N_\epsilon).$$

Therefore, from Equation (55), we obtain

$$|S^N| \leq (1 + 2\log(2N_\epsilon))^{|J|} N_\epsilon^{d-|J|} \underset{N\to+\infty}{O}(N^{-(d-|J|)}).$$

Due to Condition (19), $N_\epsilon = \underset{N\to+\infty}{O}(N\epsilon_N)$. Thus,

$$|S^N| = \underset{N\to+\infty}{O}\Big((1 + \log(N) + \log(\epsilon_N))^{|J|} \epsilon_N^{d-|J|} \Big).$$

If $|J| = 0$ then $|S^N| = \underset{N\to+\infty}{O}(\epsilon_N^d)$. If $|J| > 0$, then

$$|S^N| = \underset{N\to+\infty}{O}\Big((1 + \log(N) + \log(\epsilon_N)) \epsilon_N^{d/|J|-1})^{|J|} \Big).$$

Hence, in both cases, $\lim\limits_{N\to+\infty} |S^N| = 0$ due to Conditions (20).

Proof (Lemma 2) Since Z is the sum of two independent fields Z_L and Z_H, $\mathbb{E}((V_{a,x}^N[m])^2)$ can be decomposed as a sum of $\mathbb{E}((V_{L,a,x}^N[m])^2)$ and $\mathbb{E}((V_{H,a,x}^N[m])^2)$ where $V_{L,a,x}^N[m]$ and $V_{H,a,x}^N[m]$ are increment fields of Z_L and Z_H, respectively. Since the increment is of order $K \geq M$, we have

$$\mathbb{E}((V_{L,a,x}^N[m])^2) = \int_{\mathbb{R}^d} \left|\hat{v}\left(\frac{T_a' w}{N}\right)\right|^2 g(w)dw.$$

We split the integral above into the sum of integrals on two domains, $\{w, |w| > A\}$ and $\{w, |w| < A\}$. Since v is of order K, we have $|\hat{v}(u)|^2 = \underset{u \to 0^+}{O}(|u|^{2K+2})$. Thus, for N large enough,

$$E_1^N = \int_{|w|<A} \left|\hat{v}\left(\frac{T_a' w}{N}\right)\right|^2 g(w)dw \leq N^{-2K-2} \int_{|w|<A} |w|^{2K+2} g(w)dw.$$

Since Z_L is an intrinsic random field of order M and $K \geq M$, the integral on the right-hand side is finite and $E_1^N = \underset{N \to +\infty}{O}(N^{-2K-2})$. Using Assumption (11), we also have

$$E_2^N \leq c \int_{|w|>A} \left|\hat{v}\left(\frac{T_a' w}{N}\right)\right|^2 |w|^{-2\bar{H}-d} dw.$$

After a variable change $\zeta = w/N$, we obtain

$$E_2^N = \underset{N \to +\infty}{O}(N^{-2\bar{H}}) \int_{\mathbb{R}^d} |\hat{v}(\zeta)|^2 |\zeta|^{-2\bar{H}-d} d\zeta.$$

Since the integral on the right-hand side is finite, we get $E_2^N = \underset{N \to +\infty}{O}(N^{-2\bar{H}})$. Therefore,

$$N^{2H_x} \mathbb{E}((V_{L,a,x}^N[m])^2) = \underset{N \to +\infty}{O}(N^{2(H_x - \bar{H})}) \tag{57}$$

and converges to 0 as N tends to $+\infty$ since $\bar{H} > \tilde{\beta} \geq H_x$.

To establish the convergence of $N^{2H_x} \mathbb{E}((V_{H,a,x}^N[m])^2)$, we use Lemma 3.

Indeed, set $t = 1/N$ and $\varepsilon_t = \epsilon_N$. Then, due to Conditions (20) (i) and (ii), ε_t fulfills Condition (39). Setting $y_1 = y_2 = \frac{p_x^N + m}{N}$ for an arbitrary $m \in \mathcal{V}_N$, we observe that, for $j = 1, 2$, $V_{tj}(y_j) = V_{H,a,x}^N[m]$ when $T_j = T_a$. We further notice that $I(0) = \tilde{\rho}_a^{2H_x} C_x(\varphi_a)$. Hence, Lemma 3 implies that, for any $a \in \mathcal{F}$, $m \in \mathcal{V}_N$,

$$|N^{2H_x} \mathbb{E}((V_{H,a,x}^N[m])^2) - \tilde{\rho}_a^{2H_x} C_x(\varphi_a)| \leq \gamma_{\frac{1}{N}}, \tag{58}$$

for some sequence $(\gamma_t)_t$ converging to 0 as t tends to 0^+. Equation (32) follows from Equation (57) and (58) for $\zeta_N = \gamma_{\frac{1}{N}} + \underset{N \to +\infty}{O}(N^{2(H_x - \bar{H})})$. Equation (33) can be established in a same way using the following lemma.

B Proof of Theorem 1

Proof Let $x \in \mathbb{R}^d$. We aim at proving that the random field Z is Locally asymptotically self-similar of order H_x at x, and admits the AFBF $\tilde{Z}_x$ of Equation (10) as a tangent field. For $t > 0$, let $V_{x,t}$ be the Gaussian field defined, for any $u \in \mathbb{R}^d$, by

$$V_{x,t}(u) = \frac{Z(x + tu) - Z(x)}{t^{H_x}}. \tag{59}$$

We shall show that, for any $u \in \mathbb{R}^d$, $\{V_{x,t}(u)\}_{u \in \mathbb{R}^d}$ converges to $\{\tilde{Z}_x(u)\}_{u \in \mathbb{R}^d}$, as $t \to 0$, in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets [24].

For that, we use a convergence criterion which is stated in Theorem 2.1.10 of [23]. Due to this criterion, we only have to show

1. that $V_{x,t}$ converges to $\tilde{Z}_x$ in terms of finite dimensional distributions, as $t \to 0$,
2. and that the sequence of random fields $(V_{x,t_n})_{n\in\mathbb{N}^*}$, defined with $t_n = 1/n \in [0,1]$, satisfies a tightness property

$$\forall I > 0, \forall u, v \in [-I, I], \sup_{n>0} \mathbb{E}\left(|V_{x,t_n}(u) - V_{x,t_n}(v)|^{\alpha_1}\right) \le C_0(I)|u-v|^{d+\alpha_2}, \tag{60}$$

for some positive constant $C_0(I)$ which may depend on I, and positive constants α_1, α_2.

Convergence in finite dimensional distribution. Due to the Lévy theorem, the convergence of finite distributions of a field can be studied using characteristic functions of field variables. In the case when the field is Gaussian, it can be further established through the field autocovariance which determines characteristic functions. Here, since fields are Gaussian, it suffices to show that, for any $u, v \in \mathbb{R}^d$,

$$\lim_{t\to 0} \mathbb{E}(V_{x,t}(u)V_{x,t}(v)) = \mathbb{E}(\tilde{Z}_x(u)\tilde{Z}_x(v)). \tag{61}$$

For that, we apply Lemma 3 of Appendix A to variations $V_{t1}(x)$ and $V_{t2}(x)$ defined with a kernel v having terms $v[0] = 1, v[e_1] = -1$, $v[k] = 0$, otherwise, and transforms

$$T_1 = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix},$$

respectively. Let us notice that $V_{t1} = t^{H_x} V_{x,t}(u)$ and $V_{t2} = t^{H_x} V_{x,t}(v)$. Hence, setting $\epsilon_t = t \max(|u|_\infty, |v|_\infty)$, we get

$$\lim_{t\to 0} \mathbb{E}(V_{x,t}(u)V_{x,t}(v)) = I(0) = \int_{\mathbb{R}^d} \hat{v}(T_1' w)\overline{\hat{v}(T_2' w)}\delta_x(w)|w|^{-2H_x-d} dw, \tag{62}$$

the later integral corresponding to the term $\mathbb{E}(\tilde{Z}_x(u)\tilde{Z}_x(v))$ of the autocovariance of $\tilde{Z}_x$.

Tightness property. Let us consider with $t = \frac{1}{n} \in [0,1]$

$$D_t = \mathbb{E}\left((V_{x,t}(u) - V_{x,t}(v))^2\right) = \frac{\mathbb{E}\left((Z(x+tu) - Z(x+tv))^2\right)}{|t|^{2H_x}}. \tag{63}$$

Notice that

$$Z(x+tu) - Z(x+tv) = \sum_k v[k] Z(x + tu - tTk),$$

for the filter v having terms $v[0] = 1, v[1, 0, \cdots, 0] = -1$, $v[k] = 0$ otherwise, and the transform

$$T = \begin{pmatrix} u_1 - v_1 & -u_2 + v_2 \\ u_2 - v_2 & u_1 - v_1 \end{pmatrix}.$$

Hence, setting $V_{t1}(x+tu) = V_{t2}(x+tu) = Z(x+tu) - Z(x+tv)$ in Equation (42), we obtain

$$D_t = t^{-2H_x} \int_{|\omega|\ge A} |R_t(\omega)|^2 d\omega, \tag{64}$$

where $R_t(\omega) = (e^{i\langle x+tu,\omega\rangle} - 1)\sqrt{f_{x+tu}(\omega)} - (e^{-i\langle x+tv,\omega\rangle} - 1)\sqrt{f_{x+tv}(\omega)}$. Furthermore, $R_t(\omega) = \sum_{k=1}^3 R_{tk}(\omega)$, where terms $R_{tk}(\omega)$ are defined as in Equations (43). Therefore,

$$D_t = t^{-2H_x} \sum_{k,l=1}^{3} \int_{|w|>A} R_{tk}(w)\, \overline{R_{tl}(w)} dw.$$

Consequently, due to the Cauchy-Schwarz inequality, $D_t \le t^{-2H_x} \sum_{k,l=1}^3 \sqrt{\Delta_{tk}}\sqrt{\Delta_{tl}}$, where Δ_{tk} are as in Equation (46). Hence, Lemma 5 yields

$$\begin{aligned} D_t \le\; & c \max\left(|T|^{2\bar{\beta}}, |T|^{2H}\right) + (|T|^{2\eta} + |T|^{2\alpha}) \underset{t\to 0^+}{O}(t^{2(\min(\alpha,\eta)-H_x)}) \\ & + \max\left(|T|^{\bar{\beta}+\alpha} + |T|^{\bar{\beta}+\eta}, |T|^{H+\alpha} + |T|^{H+\eta}\right) \underset{t\to 0^+}{O}(t^{\min(\alpha,\eta)-H_x}) \\ & + |T|^{\alpha+\eta} \underset{t\to 0^+}{O}(t^{\alpha+\eta-2H_x}). \end{aligned}$$

Noticing that $|T| = |u-v|$ and using assumptions that $H < \bar{\beta} < \alpha < 1$ and $H < \bar{\beta} < \eta < 1$, we further have

$$D_t \leq \; c \max\left(|u-v|^{2\bar{\beta}}, |u-v|^{2H}\right)\left(1 + \max\left(|u-v|^{\min(\alpha,\eta)-H}, |u-v|^{2(\max(\alpha,\eta)-\bar{\beta})}\right)\right),$$

the second term reaching a bound $C_2(I)$ for any (u,v) on the compact set $[-I,I]^2 \times [-I,I]^2$. Therefore,

$$\sup_{t>0} \mathbb{E}\left((V_{x,t}(u) - V_{x,t}(v))^2\right) \leq C_2(I) \max\left(|u-v|^{2\bar{\beta}}, |u-v|^{2H}\right). \tag{65}$$

Consequently, choosing $p \in \mathbb{N}$ so that $pH > d$, we obtain

$$\mathbb{E}\left((V_{x,t}(u) - V_{x,t}(v))^{2p}\right) \leq C_p(I) \max\left(|u-v|^{2p\bar{\beta}}, |u-v|^{2pH}\right). \tag{66}$$

In other words, the tightness property (60) is shown for $\alpha_1 = 2p$ and $\alpha_2 = 2pH - d$.