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To cite this version:
Yu Li. A class of cutoff functions for non-trivial Riemann zeros. 2019. hal-01860885v6

HAL Id: hal-01860885
https://hal.archives-ouvertes.fr/hal-01860885v6
Submitted on 15 Sep 2019

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A class of cutoff functions for non-trivial Riemann zeros

Yu Li

Abstract

Abstract: In this paper we construct a pseudo hermitian operator $\tilde{H}$ on the Hilbert space $L^2[\frac{1}{2}, 1]$ and show that the Hurwitz zeta functions over the interval $[\frac{1}{2}, 1]$ are eigenfunctions of $\tilde{H}$ for all complex numbers.

Keywords. Riemann hypothesis, non-trivial Riemann zero, Pseudo hermitian operator, Berry-Keating operator

1 Introduction

One of the most attracting problems in Mathematics is Riemann hypothesis and it states that all the non-trivial zeros of Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-t}}{e^t - 1} dt$$

have the real part $\frac{1}{2}$.

One proposed way of proving Riemann hypothesis, known as Hilbert-Pólya conjecture, is to find an unbounded self-adjoint operator such that the imaginary part of Riemann zeros corresponds to the eigenvalues of this operator. It has been suggested [5] that the possible approach to solve Hilbert-Pólya conjecture lies in a quantization of the classical Hamiltonian $H = XP$, where $P$ is the canonical momentum operator associated with position operator $X$. Inspired of the previous works, an operator $\tilde{H}$ similar to quantum analogy was proposed [4] and asymptotic analysis was performed [3]. However, the momentum operator $P$ in this case has no self-adjoint extension [2].

2 Hurwitz Zeta functions

The Hurwitz zeta function is originally defined for complex arguments $s$ with $\Re(s) > 1$ and $x > 0$ by

$$\zeta(s, x) = \sum_{n \geq 0} \frac{1}{(n + x)^s}$$
Hurwitz zeta function has an analytic continuation on \( \mathbb{C} \) with the only singularity \( s = 1 \) and it can be represented \([1, 10]\)

\[
\zeta(s, x) = \frac{-\Gamma(1 - s)}{2\pi i} \int_{\mathcal{C}} \frac{e^{\xi s - 1} e^{\xi x}}{e^{\xi} - 1} d\xi, \quad s \neq 1
\]

where \( \mathcal{C} \) is the contour around the negative real axis and it starts at \(-\infty\), encircles the origin once in the positive direction without enclosing any of the points \( \xi = \pm 2ni, \pm 4ni, \ldots \) and returns to \(-\infty\) \([10]\). The famous Riemann zeta function is the special case of Hurwitz zeta function with \( x = 1 \)

\[
\zeta(s, 1) = \zeta(s)
\]

and also with \( x = \frac{1}{2} \)

\[
\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)
\]

The partial derivative of Hurwitz zeta function with respect to \( x \)

\[
\frac{\partial \zeta(s, x)}{\partial x} = -s\zeta(s + 1, x), \quad s \neq 0, 1; x > 0
\]

Let \( s \) denote a point in critical strip \( 0 < \Re(s) < 1 \) and we define a cutoff function \( \zeta_s \) of Hurwitz function on interval \([\frac{1}{2}, 1]\)

\[
\zeta_s(x) := \zeta(s, x), \quad \frac{1}{2} \leq x \leq 1
\]

The functions \( \zeta_s \) is continuous and even smooth on the closed interval \([\frac{1}{2}, 1]\) and have finite norm and belong to the Hilbert space, \( \zeta_s \in L^2[\frac{1}{2}, 1] \). If a complex number \( z \) is a non-trivial Riemann zero,

\[
\zeta_z(1) = 0
\]

then the function \( \zeta_z \) vanishes at \( x = \frac{1}{2} \) from \( 2 \)

\[
\zeta_z\left(\frac{1}{2}\right) = 0
\]

## 3 Operators in Hilbert space \( L^2[\frac{1}{2}, 1] \)

We confine our discussion over the region \([\frac{1}{2}, 1]\). Let \( X \) be the position operator on \( L^2[\frac{1}{2}, 1] \) and \( P_0 \) the momentum operator \( P_0 = -i \frac{d}{dx} \) densely defined on \( L^2[\frac{1}{2}, 1] \) with domain

\[
D(P_0) = \left\{ f \in C^\infty[\frac{1}{2}, 1] : f\left(\frac{1}{2}\right) = f(1) \right\}
\]

\([6]\) and it has a unique self-adjoint extension \( P_0 = P \). The closure of the graph of \( P_0 \) is the graph of \( P \)

\[
\Gamma(P) = \Gamma(P_0) = L^2[\frac{1}{2}, 1] \times L^2[\frac{1}{2}, 1]
\]
From the Stone’s theorem on one-parameter unitary groups, \( \{ e^{itP} \} \) is a strongly continuous one-parameter group of unitary operators on \( L^2[\frac{1}{2}, 1] \). Thus, both operators \( e^{\pm itP} - I \) are bijective and bounded, and then the bounded inverses \( (e^{\pm itP} - I)^{-1} \) exist, from the bounded inverse theorem in functional analysis.

We construct an operator \( \tilde{H} \) on \( L^2[\frac{1}{2}, 1] \)

\[
\tilde{H} = (e^{itP} - I)^{-1} (XP + PX) (e^{itP} - I)
\]

It holds obviously for \( \frac{1}{2} \leq x \leq 1 \) and \( 0 < R(s) < 1 \)

\[
(XP + PX)x^{-s} = i(2s - 1)x^{-s}
\]

(6)

The operator \( \tilde{H} \) is pseudo hermitian [7, 8, 9], i.e. there exists an invertible and self-adjoint operator \( \eta \) such that the adjoint operator \( \tilde{H}^\dagger \) can be represented

\[
\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}
\]

**Proposition 3.1.** \( \tilde{H} \) is pseudo hermitian and all eigenvalues are real.

**Proof.** Let \( \eta = (e^{itP} - I)^\dagger (e^{itP} - I) \) and it is obviously self-adjoint, then

\[
\tilde{H}^\dagger = \eta \tilde{H} \eta^{-1}
\]

The operator \( \eta \) is positive definite and self-adjoint by construction, the operator \( \tilde{H} \) is hermitian in the new inner product with respect to \( \eta \)

\[
(\tilde{H} \phi, \varphi)_\eta = (\tilde{H} \phi, \eta \varphi) = (\phi, \eta \tilde{H} \eta^{-1} \eta \varphi) = (\phi, \tilde{H} \varphi)_\eta, \quad \phi, \varphi \in L^2[\frac{1}{2}, 1]
\]

Then all eigenvalues of \( \tilde{H} \) are real.

**Proposition 3.2.** For any complex number \( z \), the function \( \zeta_z(x) - \zeta_z(\frac{1}{2}) \) is an eigenfunction of \( \tilde{H} \) with eigenvalue \( i(2z - 1) \)

\[
\tilde{H} \left( \zeta_z(x) - \zeta_z(\frac{1}{2}) \right) = i(2z - 1) \left( \zeta_z(x) - \zeta_z(\frac{1}{2}) \right)
\]

**Proof.** We consider for non-trivial Riemann zeros \( z \) the term

\[
\frac{1}{iP} \left( \frac{iP}{e^{itP} - I} \right) x^{-z}, \quad \frac{1}{2} \leq x \leq 1
\]

where \( \frac{1}{iP} \) is defined as an integral operator on \( L^2[\frac{1}{2}, 1] \) with boundary \( x = \frac{1}{2} \)

\[
\frac{1}{iP} f(x) := \int_{\frac{1}{2}}^{x} f(\xi) d\xi, \quad f \in L^2[\frac{1}{2}, 1]
\]

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and \( \frac{iP}{e^{iP} - I} f \) is defined as a Bernoulli generating function

\[
\frac{iP}{e^{iP} - I} f(x) := \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n f(x), \quad f \in L^2[\frac{1}{2}, 1]
\]

Notice

\[
(iP)^n x^{-z} = \frac{\Gamma(-z + 1)}{\Gamma(-z + 1 - n)} x^{-z-n}
\]

We use Hankel loop contour integral formula [10, 11]

\[
\frac{1}{\Gamma(-z + 1 - n)} = \frac{1}{2\pi i} \int_C \xi^{-(1-z-n)} e^\xi d\xi
\]

where the integration is done along the Hankel-Bromwich contour and it begins at \(-\infty\), circles the origin once in the positive direction, and returns to \(-\infty\)

\[
(iP)^n x^{-z} = \Gamma(1-z) \left( \frac{1}{2\pi i} \int_C \xi^{-(1-z-n)} e^\xi d\xi \right) x^{-z-n}
\]

and then

\[
\frac{iP}{e^{iP} - I} x^{-z} = \sum_{n \geq 0} \frac{B_n}{n!} (iP)^n x^{-z}
\]

\[
= \frac{\Gamma(1-z)}{2\pi i} \sum_{n \geq 0} \int_C \frac{B_n}{n!} \left( \frac{\xi}{x} \right)^{z+n-1} e^{\xi} d\xi
\]

Let \( u = \frac{\xi}{x} \) and \( C' = \frac{\xi}{x} \) is still a loop around the negative real axis. Uniform convergence on closed interval justifies the interchange of integral and sum. Then Hurwitz zeta function can be reproduced (1)

\[
\frac{iP}{e^{iP} - I} x^{-z} = \frac{\Gamma(1-z)}{2\pi i} \int_{C'} \frac{u^z}{e^u - 1} e^{uz} du = -z \zeta(z+1, x)
\]

and the inverse

\[
\frac{iP}{e^{iP} - I} x^{-z} = -z \zeta(z+1, x)
\]

Then from (3)

\[
\frac{1}{iP} \left( \frac{iP}{e^{iP} - I} \right) x^{-z} = \int_{\frac{1}{2}}^x -z \zeta(z+1, \xi) d\xi
\]

\[
= \zeta(z, x) - \zeta(z, \frac{1}{2})
\]  

\[\square\]
References


