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Abstract

Methods for computing approximately bisimilar symbolic models for incrementally stable switched systems are often based on discretization of time and space, where the value of time and space sampling parameters must be carefully chosen in order to achieve a desired precision. These approaches can result in symbolic models that have a very large number of transitions, especially when the time sampling, and thus the space sampling parameters are small. In this paper, we present an approach to the computation of symbolic models for switched systems with dwell-time constraints using multirate time sampling, where the period of symbolic transitions is a multiple of the control (i.e. switching) period. We show that all the multirate symbolic models, resulting from the proposed construction, are approximately bisimilar to the original incrementally stable switched system with the precision depending on the sampling parameters, and the sampling factor between transition and control periods. The main contribution of the paper is the explicit determination of the optimal sampling factor, which minimizes the number of transitions in the class of proposed symbolic models for a prescribed precision. Interestingly, we prove that this optimal sampling factor is mainly determined by the state space dimension and the number of modes of the switched system. Finally, an illustration of the proposed approach is shown on an example, which shows the benefit of multirate symbolic models in reducing the computational cost of abstraction-based controller synthesis.

Key words: Approximate bisimulation; switched systems; symbolic control; multirate sampling; incremental stability.

1 Introduction

A switched system is a dynamical system consisting of a finite number of subsystems and a law that controls the switching among them [13,27,14]. The literature on switched systems principally focuses on the stability and stabilization problems. However, other objectives need also to be considered such as safety, reachability or more complex objectives such as those expressed in linear temporal logic. For this reason, over recent years, several studies focused on the use of discrete abstractions and symbolic control techniques. The area of symbolic control is concerned with the use of algorithmic discrete synthesis techniques for designing controllers for continuous and hybrid dynamical systems (see e.g. [29,25] and the references therein). A key concept in symbolic control is that of symbolic models, which consist in discrete abstractions of the continuous dynamics, and which are amenable to automata theoretic techniques for the synthesis of controllers enforcing a broad range of specifications [5,4]. Controllers for the original system, with strong formal guarantees, can then be obtained through dedicated refinement procedures [29,7,24]. This latter step requires the original system and the symbolic model to be related by some formal behavioral relationship such as simulation, bisimulation relations or their approximate and alternated versions [29,9].

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Numerous works have been dedicated to the computation of symbolic models for various classes of dynamical systems. Focusing on approximately bisimilar abstractions, existing approaches make it possible to deal with nonlinear systems [19,23], switched systems [10], time-delay systems [21,20], networked control systems [6,35], stochastic systems [32,34]... All these approaches are essentially based on discretization of time and space and require the considered system to satisfy some kind of incremental stability property [1]. However, incremental stability can be dropped if one seeks symbolic models related only by one-sided approximate simulation relations [28,36]. In most cases, symbolic models of arbitrary precision can be obtained by carefully choosing time and space sampling parameters. However, for a given precision, the choice of a small time sampling parameter imposes to choose a small space sampling parameter resulting in symbolic models with a prohibitively large number of transitions. This constitutes a limiting factor of the approach because the size of the symbolic models is crucial for computational efficiency of discrete controller synthesis algorithms. Several studies have been conducted in order to address this issue by enabling the computation of more parsimonious symbolic models with smaller numbers of transitions. Such approaches include compositional abstraction schemes where symbolic models of a system are built from symbolic models of its components [30,22,15]; multi-resolution or multi-scale symbolic models computed using non-uniform adaptive space discretizations [31,8]; symbolic models where the set of symbolic states is not given by a discretization of the state-space but by input sequences [12,33].

In this paper, we show how the size of symbolic models can be reduced using multirate sampling. Multirate sampling has been introduced in the area of sampled-data systems to face some of the sampling processes disadvantages such as the loss of relative degree and changes in the properties of the zero dynamics (see e.g. [17,11,18]). In this paper, we present an approach to the computation of multirate symbolic models for incrementally stable switched systems, where the period of symbolic transitions is a multiple of the control (i.e. switching) period. A similar approach has been explored in the symbolic control literature in the context of nonlinear digital control systems [16]. The first contribution of the paper is to extend this approach to the class of switched systems, with dwell-time constraints. We show that the obtained multirate symbolic models are approximately bisimilar to the original switched system. Then, the second and main contribution of the paper lies in the explicit determination of the optimal sampling factor between transition and control periods, which minimizes the number of transitions in the class of proposed symbolic models for a prescribed precision; this problem is not considered in [16]. Interestingly, we show that the optimal sampling factor is mainly determined by the state space dimension and the number of modes of the switched system.

This paper is organized as follows. In Section 2, we introduce the class of incrementally stable switched systems under study and we present the abstraction framework used in the paper. In Section 3, we present the construction of symbolic models for incrementally stable switched systems with dwell-time constraints, using multirate sampling. In Section 4, we establish the optimal sampling factor between control and transition periods which minimizes the number of transitions in the symbolic model. Finally, in Section 5, we illustrate our approach using an example taken from [10], which shows the benefits of the proposed multirate symbolic models.

A preliminary version of this work has been presented in the conference paper [26] where switched systems without dwell-time constraints are considered. The current paper extends the approach to consider dwell-time constraints; results of [26] being recovered as particular cases. We also provide novel numerical experiments.

Notations: $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^+$ denote the sets of integers, of non-negative integers and of positive integers, respectively. $\mathbb{R}$, $\mathbb{R}_0^+$ and $\mathbb{R}^+$ denote the sets of real numbers, of non-negative real numbers, and of positive real numbers, respectively. For $s \in \mathbb{R}_0^+$, $\lfloor s \rfloor$ denote its integer part, i.e. the largest nonnegative integer $r \in \mathbb{N}$ such that $r \leq s$. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm (i.e. the 2-norm) of x. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if γ is of class $\mathcal{K}$ and $\gamma(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for all fixed $t \in \mathbb{R}_0^+$, the map $\beta(\cdot, t)$ belongs to class $\mathcal{K}$, and for all fixed $s \in \mathbb{R}^+$, the map $\beta(s, \cdot)$ is strictly decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

2 Preliminaries

2.1 Incrementally stable switched systems

We introduce the class of switched systems:

Definition 1 *A switched system is a quadruple* $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$*, consisting of:*

- *a state space* $\mathbb{R}^n$*;*
- *a finite set of modes* $P = \{1, \ldots, m\}$*;*
- *a set of switching signals* $\mathcal{P} \subseteq \mathcal{S}(\mathbb{R}_0^+, P)$*, where* $\mathcal{S}(\mathbb{R}_0^+, P)$ *denotes the set of piecewise constant functions from* $\mathbb{R}_0^+$ *to* P*, continuous from the right and with a finite number of discontinuities on every bounded interval of* $\mathbb{R}_0^+$*;*
- *a collection of vector fields* $F = \{f_1, \ldots, f_m\}$*, indexed by* P*.*

The discontinuities $0 < t_1 < t_2 < \dots$ of a switching signal are called *switching times*; by definition of $\mathcal{S}(\mathbb{R}_0^+, P)$, there are only a finite number of switching times on every bounded interval of $\mathbb{R}_0^+$ and thus Zeno behaviors are avoided. A switching signal $\mathbf{p} \in \mathcal{S}(\mathbb{R}_0^+, P)$ has *dwell-time* $\tau_d \in \mathbb{R}^+$ if the sequence of switching times satisfies $t_{k+1} - t_k \geq \tau_d$, for all $k \geq 1$. The set of switching signals with dwell-time τ_d is denoted $\mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$.

A piecewise $\mathcal{C}^1$ function $\mathbf{x} : \mathbb{R}_0^+ \to \mathbb{R}^n$ is said to be a *trajectory* of Σ if it is continuous and there exists a switching signal $\mathbf{p} \in \mathcal{P}$ such that, at each $t \in \mathbb{R}_0^+$ where the function $\mathbf{p}$ is continuous, $\mathbf{x}$ is continuously differentiable and satisfies:

$$\dot{\mathbf{x}}(t) = f_{\mathbf{p}(t)}(\mathbf{x}(t)). \tag{1}$$

We make the assumption that the vector fields f_p, $p \in P$, are locally Lipschtiz and forward complete (see e.g. [2] for necessary and sufficient conditions), so that for all switching signals $\mathbf{p} \in \mathcal{P}$ and all initial states $x \in \mathbb{R}^n$, there exists a unique trajectory, solution to (1) with $\mathbf{x}(0) = x$, denoted $\mathbf{x}(., x, \mathbf{p})$. We will denote by ϕ_t^p the flow associated to the vector field f_p. Then, for a constant switching signal given by $\mathbf{p}(t) = p$, for all $t \in \mathbb{R}_0^+$, we have $\mathbf{x}(t, x, \mathbf{p}) = \phi_t^p(x)$, for all $t \in \mathbb{R}_0^+$.

In the following, we consider *incrementally globally uniformly asymptotically stable* (δ-GUAS) switched systems, see [10] for a formal definition. Intuitively, incremental stability means that all trajectories associated to the same switching signal converge to the same trajectory, independently of their initial conditions. Sufficient conditions for incremental stability are given in [10] in terms of existence of multiple Lyapunov functions.

Definition 2 *: Smooth functions $V_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$, $p \in P$, are multiple δ-GUAS Lyapunov functions for Σ if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, $\kappa \in \mathbb{R}^+$ and $\mu \geq 1$ such that for all $x, y \in \mathbb{R}^n$, and $p, p' \in P$,*

$$\underline{\alpha}(\|x - y\|) \leq V_p(x, y) \leq \overline{\alpha}(\|x - y\|); \tag{2}$$

$$\frac{\partial V_p}{\partial x}(x, y) f_p(x) + \frac{\partial V_p}{\partial y}(x, y) f_p(y) \leq -\kappa V_p(x, y); \tag{3}$$

$$V_p(x, y) \leq \mu V_{p'}(x, y). \tag{4}$$

In [10], it is proved that $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ is δ-GUAS if there exist multiple δ-GUAS Lyapunov functions for Σ and the set of switching signals $\mathcal{P} \subseteq \mathcal{S}_{\tau_d}(\mathbb{R}_0^+, P)$ with dwell-time $\tau_d > \frac{\ln(\mu)}{\kappa}$.

In this paper, we assume that the previous condition holds and in order to construct symbolic models for the switched systems, we shall make the supplementary assumption that there exists a $\mathcal{K}_\infty$ function γ such that for multiple δ-GUAS Lyapunov functions we have:

$$\forall x, y, z \in \mathbb{R}^n,\ p \in P,\ |V_p(x, y) - V_p(x, z)| \leq \gamma(\|y - z\|). \tag{5}$$

Remark 1 *In [10], it is shown that if we are interested in the dynamics of the switched system on a compact set $C \subseteq \mathbb{R}^n$ and V_p, $p \in P$, are $\mathcal{C}^1$ on C, then, (5) holds with the linear $\mathcal{K}_\infty$ function given by $\gamma(s) = c_\gamma s$ where*

$$c_\gamma = \max_{x,y \in C, p \in P} \left\| \frac{\partial V_p}{\partial y}(x, y) \right\|,$$

Remark 2 *For all $x \in \mathbb{R}^n$, (2) implies that for all $p \in P$, $V_p(x, x) = 0$, then for all $x, y \in \mathbb{R}^n$, $p \in P$ we have from (5) that:*

$$V_p(x, y) \leq |V_p(x, y) - V_p(x, x)| \leq \gamma(\|x - y\|).$$

Then, there is no loss of generality in assuming that the second inequality in (2) holds with $\overline{\alpha} = \gamma$.

Remark 3 *The smoothness assumption on functions V_p, $p \in P$, in Definition 2 can be relaxed, and condition (3) can be replaced by*

$$V_p(\phi_t^p(x), \phi_t^p(y)) \leq e^{-\kappa t} V_p(x, y) \tag{6}$$

for all $x, y \in \mathbb{R}^n, p \in P$ and $t \in \mathbb{R}_0^+$.

2.2 Transition systems

We present the notion of transition systems, which allows us to describe, in a unified framework, switched systems and their symbolic models.

Definition 3 *A transition system is a tuple $T = (X, U, Y, \Delta, X^0)$ consisting of:*

- *a set of states X;*
- *a set of inputs U;*
- *a set of outputs Y;*
- *a transition relation $\Delta \subseteq X \times U \times X \times Y$;*
- *a set of initial states $X^0 \subseteq X$.*

T is said to be metric if the set of outputs Y is equipped with a metric d, symbolic if X and U are finite or countable sets.

The transition $(x, u, x', y) \in \Delta$ will be denoted $(x', y) \in \Delta(x, u)$ and means that the system can evolve from state x to state x' under the action of input u, while producing output y. An input $u \in U$ belongs to the set of *enabled* inputs at state $x \in X$, denoted $\text{enab}_\Delta(x)$, if $\Delta(x, u) \neq \emptyset$. T is said to be *deterministic* if for all $x \in X$ and for all $u \in \text{enab}_\Delta(x)$, $\Delta(x, u)$ consists of a unique element. State $x \in X$ is said to be *blocking* if $\text{enab}_\Delta(x) = \emptyset$, otherwise it is said to be *non-blocking*.

A *trajectory* of the transition system is a finite or infinite sequence of transitions $\sigma = (x^0, u^0, y^0)(x^1, u^1, y^1)(x^2, u^2, y^2)\dots$ where $(x^{i+1}, y^i) \in \Delta(x^i, u^i)$, for $i \geq 0$.

It is *initialized* if $x^0 \in X^0$. A state $x \in X$ is reachable if there exists an initialized trajectory such that $x^i = x$, for some $i \geq 0$. The transition system is said to be *non-blocking* if all reachable states are non-blocking. The *output behavior* associated to the trajectory σ is the sequence of outputs $y^0 y^1 y^2 \dots$

In this paper, we consider the approximation relationship for transition systems based on the notion of approximate bisimulation [9], which requires that the distance between the output behaviors of two transition systems remains bounded by some specified precision. The following definition is taken from [8] and generalizes that of [9] to accommodate the encoding of the output map within the transition relation.

Definition 4 *Let $T_1 = (X_1, U, Y, \Delta_1, X_1^0)$ and $T_2 = (X_2, U, Y, \Delta_2, X_2^0)$ be two metric transition systems with the same input set U and the same output set Y equipped with a metric d. Let $\varepsilon \geq 0$ be a given precision. A relation $R \subseteq X_1 \times X_2$ is said to be an ε-approximate bisimulation relation between T_1 and T_2 if for all $(x_1, x_2) \in R$, $u \in U$:*

$$\forall (x_1', y_1) \in \Delta_1(x_1, u), \exists (x_2', y_2) \in \Delta_2(x_2, u), \quad d(y_1, y_2) \leq \varepsilon \text{ and } (x_1', x_2') \in R;$$
$$\forall (x_2', y_2) \in \Delta_2(x_2, u), \exists (x_1', y_1) \in \Delta_1(x_1, u), \quad d(y_1, y_2) \leq \varepsilon \text{ and } (x_1', x_2') \in R.$$

The transition systems T_1 and T_2 are said to be ε-approximately bisimilar, denoted $T_1 \sim_\varepsilon T_2$, if:

- $\forall x_1 \in X_1^0$, $\exists x_2 \in X_2^0$, *such that* $(x_1, x_2) \in R$;
- $\forall x_2 \in X_2^0$, $\exists x_1 \in X_1^0$, *such that* $(x_1, x_2) \in R$.

The approximate bisimulation relation guarantees that for each output behavior of T_1 (respectively of T_2), there exists an output behavior of T_2 (respectively of T_1) such that the distance between these output behaviors is uniformly bounded by ε (see [9]).

3 Symbolic models with multirate sampling

In this section, we extend the results of [26] to the case of switched systems with dwell-time constraints. Let us consider a switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, P, \mathcal{P}_{\tau_d}, F)$, in which the switching is periodically controlled with control period $\tau \in \mathbb{R}^+$ and in which a dwell-time $\tau_d \in \mathbb{R}^+$ is imposed on switching signals. For simplicity, we assume that $\tau = \tau_d / k$ where $k \in \mathbb{N}^+$.

The sampled dynamics of Σ_{τ_d} can then be described by the transition system $T_\tau(\Sigma_{\tau_d}) = (X, U, Y, \Delta_\tau, X^0)$ as follows:

- the set of states is $X = \mathbb{R}^n \times P$;
- the set of inputs is $U = P$;
- the set of outputs is $Y = \mathbb{R}^n \cup \mathbb{R}^{k \times n}$;
- the transition relation is given for $(x, p), (x', p') \in X$, $u \in U$, $y \in Y$, by $((x', p'), y) \in \Delta_\tau((x, p), u)$ if and only if

$$\begin{cases} x' = \phi_\tau^u(x), \ p' = u \\ y = x \end{cases} \quad \text{if } u = p$$
$$\begin{cases} x' = \phi_{k\tau}^u(x), \ p' = u, \\ y = (x, \phi_\tau^u(x), \dots, \phi_{(k-1)\tau}^u(x)) \end{cases} \quad \text{if } u \neq p$$

- the set of initial states $X^0 = \mathbb{R}^n \times P$.

We should emphasize that transitions in $T_\tau(\Sigma_{\tau_d})$ have either duration τ or $\tau_d = k\tau$. The state $(x, p) \in X$ indicates that the state of the switched system is $x \in \mathbb{R}^n$ and that the active mode is $p \in P$. Then, one can either go on with mode p, which corresponds to the first type of transitions of duration τ; or switch to another mode $p' \neq p$, which corresponds to the second type of transitions where the new mode p' is held for duration τ_d. It is easy to see that the dwell-time constraint is fulfilled by construction. It is noteworthy that this construction differs from, and is more compact than, that of [10].

$T_\tau(\Sigma_{\tau_d})$ is non-blocking ($\text{enab}_{\Delta_\tau}((x, p)) = U$ for all $(x, p) \in X$), deterministic, and metric when the set of outputs Y is equipped with the metric given by $d_Y(y, y') = +\infty$ if y, y' do not have the same dimension and

$$d_Y(y, y') = \|y - y'\| \quad \text{if } y, y' \in \mathbb{R}^n$$
$$d_Y(y, y') = \max_{j=1}^{k} \|y_j - y_j'\| \quad \text{if } y, y' \in \mathbb{R}^{k \times n}$$

with $y = (y_1, \dots, y_k)$, $y' = (y_1', \dots, y_k')$.

3.1 Multirate sampling of switched systems

In the previous transition system, one transition coincides with the control period (of duration τ or $k\tau$ if the transition system keeps the same mode or changes it, respectively). In this paper, we consider multirate sampling where a transition corresponds to a sequence of r control periods, where the *sampling factor* $r \in \mathbb{N}^+$. Thus, multirate transition systems are obtained by concatenating r successive transitions of $T_\tau(\Sigma_{\tau_d})$.

Let us define $T_\tau^r(\Sigma_{\tau_d}) = (X, U^r, Y^r, \Delta_\tau^r, X^0)$ where:

- the set of states is $X = \mathbb{R}^n \times P$;
- the set of inputs is $U^r = P^r$;
- the set of outputs $Y^r = (\mathbb{R}^n \cup \mathbb{R}^{k \times n})^r$;
- the transition relation is given for $(x, p), (x', p') \in X$, $u \in U^r$, with $u = (u_1, \dots, u_r)$, and $y \in Y^r$, with

$y = (y_1, \dots, y_r)$ by $((x', p'), y) \in \Delta_\tau^r((x, p), u)$ if and only if

$$(x, p) = (x_1, p_1),\ (x', p') = (x_{r+1}, p_{r+1}),\ \text{with}\ ((x_{i+1}, p_{i+1}), y_i) \in \Delta_\tau((x_i, p_i), u_i),\ i = 1, \dots, r$$

- the set of initial states $X^0 = \mathbb{R}^n \times P$.

$T_\tau^r(\Sigma_{\tau_d})$ is non-blocking ($\text{enab}_{\Delta_\tau^r}((x, p)) = U^r$ for all $(x, p) \in X$), deterministic, and metric when the set of outputs Y^r is equipped with the following metric d_{Y^r}:

$$\forall y = (y_1, \dots, y_r), y' = (y'_1, \dots, y'_r) \in Y_r,\quad d_{Y^r}(y, y') = \max_{i=1}^{r} d_Y(y_i, y'_i). \tag{7}$$

Let us remark that for $r = 1$, $T_\tau^r(\Sigma_{\tau_d})$ coincides with $T_\tau(\Sigma_{\tau_d})$. When $r \neq 1$, the following result shows that $T_\tau(\Sigma_{\tau_d})$ and $T_\tau^r(\Sigma_{\tau_d})$ produce equivalent infinite output behaviors. This result is straightforward and is stated without proof.

Claim 1 *For any infinite output behavior $(y^0, y^1, y^2, \dots)$ of $T_\tau(\Sigma_{\tau_d})$, there exists an infinite output behavior $(z^0, z^1, z^2, \dots)$ of $T_\tau^r(\Sigma_{\tau_d})$ with $z^i = (z_1^i, \dots, z_r^i)$ such that*

$$\forall i \in \mathbb{N}, j = 1, \dots, r,\ z_j^i = y^{ir+j-1}. \tag{8}$$

Conversely, for any infinite output behavior $(z^0, z^1, z^2, \dots)$ of $T_\tau^r(\Sigma_{\tau_d})$ with $z^i = (z_1^i, \dots, z_r^i)$, there exists an infinite output behavior $(y^0, y^1, y^2, \dots)$ of $T_\tau(\Sigma_{\tau_d})$ such that (8) holds.

Remark 4 *Using $T_\tau(\Sigma_{\tau_d})$ or $T_\tau^r(\Sigma_{\tau_d})$ for the purpose of synthesis provides identical guarantees on the sampled behavior of the switched system, since the infinite output behaviors of both transition systems are equivalent. However, it leads to different implementations of switching controllers. For controllers synthesized using $T_\tau(\Sigma_{\tau_d})$, the sensing and actuation periods (of duration τ or $k\tau$) are equal; while for controllers synthesized using $T_\tau^r(\Sigma_{\tau_d})$, the sensing period consists of r actuation periods. In the latter case, at sensing instants, the controller selects a sequence of r modes, each of which is actuated for a duration τ or $k\tau$.*

3.2 Construction of symbolic models

For an incrementally stable switched system Σ with multiple δ-GUAS Lyapunov functions, a construction of symbolic models that are approximately bisimilar to $T_\tau(\Sigma_{\tau_d})$ has been proposed in [10], based on a discretization of the state-space $\mathbb{R}^n$. Theorem 4.2 in that paper, shows that symbolic models of arbitrary precision can be computed by using a sufficiently fine discretization of the state-space. However, this usually results in symbolic models that have a very large number of transitions, especially when the control period τ is small.

In this section, we establish a similar result for the multirate transition system $T_\tau^r(\Sigma_{\tau_d})$. This idea is inspired by the work presented in [16], in which symbolic models are computed for digital control systems using multirate sampling. Our results can be seen as an extension to the class of switched systems with dwell-time. In addition, in the following sections, we will provide a theoretical analysis allowing us to choose the optimal sampling factor r, minimizing the number of transitions in the symbolic model, which is not available in [16].

Let $\eta \in \mathbb{R}^+$ be a space sampling parameter, the set of states $\mathbb{R}^n$ is approximated by the lattice:

$$[\mathbb{R}^n]_\eta = \left\{ q \in \mathbb{R}^n | q_i = k_i \frac{2\eta}{\sqrt{n}},\ k_i \in \mathbb{Z},\ i = 1, \dots, n \right\}.$$

We associate a quantizer $Q_\eta : \mathbb{R}^n \longrightarrow [\mathbb{R}^n]_\eta$ given by $Q_\eta(x) = q$ if and only if

$$\forall i = 1, \dots, n,\ q_i - \frac{\eta}{\sqrt{n}} \leq x_i < q_i + \frac{\eta}{\sqrt{n}}.$$

where x_i and q_i denote the i-th coordinates, $i = 1, \dots, n$ of x and q, respectively. We can easily show that for all $x \in \mathbb{R}^n$, $\|Q_\eta(x) - x\| \leq \eta$.

Let us define the transition system $T_{\tau,\eta}^r(\Sigma_{\tau_d}) = (X_\eta, U^r, Y^r, \Delta_{\tau,\eta}^r, X_\eta^0)$ as follows:

- the set of states is $X_\eta = [\mathbb{R}^n]_\eta \times P$;
- the set of inputs is $U^r = P^r$;
- the set of outputs $Y^r = (\mathbb{R}^n \cup \mathbb{R}^{k \times n})^r$;
- the transition relation is given for $(q, p), (q', p') \in X_\eta$, $u \in U^r$, $y \in Y^r$, by $((q', p'), y) \in \Delta_{\tau,\eta}^r((q, p), u)$ if and only if

$$q' = Q_\eta(x')\ \text{and}\ ((x', p'), y) \in \Delta_\tau^r((q, p), u);$$

- the set of initial states is $X_\eta^0 = [\mathbb{R}^n]_\eta \times P$.

$T_{\tau,\eta}^r(\Sigma_{\tau_d})$ is symbolic, non-blocking ($\text{enab}_{\Delta_{\tau,\eta}^r}((q, p)) = U^r$ for all $(q, p) \in X_\eta$), deterministic and metric when the set of outputs Y^r is equipped with the metric d_{Y^r} given by (7).

Theorem 1 *Consider a switched system Σ_{τ_d}, and let us assume that there exist multiple δ-GUAS Lyapunov functions V_p, $p \in P$, for Σ_{τ_d} such that (5) holds for some $\mathcal{K}_\infty$ function γ, let the dwell-time $\tau_d > \frac{\ln(\mu)}{\kappa}$. Let time and space sampling parameters $\tau, \eta \in \mathbb{R}^+$, sampling factor $r \in \mathbb{N}^+$ and precision $\varepsilon \in \mathbb{R}^+$ satisfy:*

$$\eta \leq \gamma^{-1}\left(\tfrac{1}{\mu}(1 - \lambda(\tau)^r)\underline{\alpha}(\varepsilon)\right) \tag{9}$$

where $\lambda(\tau) = \max(e^{-\kappa\tau}, \mu e^{-\kappa\tau_d})$, then, the transition systems $T_\tau^r(\Sigma_{\tau_d})$ and $T_{\tau,\eta}^r(\Sigma_{\tau_d})$ are ε-approximately bisimilar.

PROOF. Let us prove that the relation R defined by:

$$R = \left\{ ((x,p^1),(q,p^2)) \in X \times X_\eta \;\middle|\; \begin{array}{c} p^1 = p^2 = p \\ V_p(x,q) \le \frac{1}{\mu}\underline{\alpha}(\varepsilon) \end{array} \right\}$$

is an ε-approximate bisimulation relation between $T^r_\tau(\Sigma_{\tau_d})$ and $T^r_{\tau,\eta}(\Sigma_{\tau_d})$.

Let $((x,p^1),(q,p^2)) \in R$, then we have $p^1 = p^2 = p$ and $V_p(x,q) \le \frac{1}{\mu}\underline{\alpha}(\varepsilon)$. Let $u = (u_1,\dots,u_r) \in U^r$ and $((x',p'),y) \in \Delta^r_\tau((x,p),u)$, where $y = (y_1,\dots,y_r) \in Y^r$, then by definition of Δ^r_τ:

$$\begin{gathered} (x,p) = (x_1,p_1),\ (x',p') = (x_{r+1},p_{r+1}),\ \text{with} \\ ((x_{i+1},p_{i+1}),y_i) \in \Delta_\tau((x_i,p_i),u_i),\ i = 1,\dots,r. \end{gathered}$$

Similarly, let $((q',p'),z) \in \Delta^r_{\tau,\eta}((q,p),u)$, where $z = (z_1,\dots,z_r) \in Y^r$, then by definition of $\Delta^r_{\tau,\eta}$:

$$\begin{gathered} (q,p) = (q_1,p_1),\ (q',p') = (Q_\eta(q_{r+1}),p_{r+1}),\ \text{with} \\ ((q_{i+1},p_{i+1}),y_i) \in \Delta_\tau((q_i,p_i),u_i),\ i = 1,\dots,r. \end{gathered}$$

By the definition of Δ_τ, we have for all $i = 1,\dots,r$, $p_{i+1} = u_i$, and

$$\begin{aligned} V_{p_{i+1}}(x_{i+1},q_{i+1}) &\le \mathrm{e}^{-\kappa\tau}\, V_{p_i}(x_i,q_i), && \text{if } p_i = p_{i+1}, \\ V_{p_{i+1}}(x_{i+1},q_{i+1}) &\le \mu\, \mathrm{e}^{-\kappa\tau_d}\, V_{p_i}(x_i,q_i), && \text{if } p_i \ne p_{i+1}. \end{aligned}$$

where the two inequalities are obtained by (3) and (4). Then, it follows that for all $i = 1,\dots,r+1$,

$$V_{p_i}(x_i,q_i) \le \lambda(\tau)^{i-1} V_{p_1}(x_1,q_1) \le \lambda(\tau)^{i-1}\tfrac{1}{\mu}\underline{\alpha}(\varepsilon). \quad (10)$$

Then, from (5), (10) and (9), we have

$$\begin{aligned} V_{p'}(x',q') &= V_{p_{r+1}}(x_{r+1},Q_\eta(q_{r+1})) \\ &\le V_{p_{r+1}}(x_{r+1},q_{r+1}) + \gamma(\eta) \\ &\le \lambda(\tau)^r \tfrac{1}{\mu}\underline{\alpha}(\varepsilon) + \gamma(\eta) \le \tfrac{1}{\mu}\underline{\alpha}(\varepsilon). \end{aligned}$$

Thus, $((x',p'),(q',p')) \in R$.

Let $i = 1,\dots,r$, if $u_i = p_i$, we have $y_i = x_i$, $z_i = q_i$, then from (2), (10) and since $\lambda(\tau) \le 1$ and $\frac{1}{\mu} \le 1$,

$$d_Y(y_i,z_i) = \|x_i - q_i\| \le \underline{\alpha}^{-1}(V_{p_i}(x_i,q_i)) \le \varepsilon. \quad (11)$$

If $u_i \ne p_i$, we have $y_i = (y_{i,1},\dots,y_{i,k})$, $z_i = (z_{i,1},\dots,z_{i,k})$ where $y_{i,j} = \phi^{u_i}_{(j-1)\tau}(x_i)$ and $z_{i,j} = \phi^{u_i}_{(j-1)\tau}(q_i)$, $j = 1,\dots,k$. Then, from (3), (4), (10) and since $\lambda(\tau) \le 1$, we have for all $j = 1,\dots,k$,

$$V_{u_i}(y_{i,j},z_{i,j}) \le V_{u_i}(x_i,q_i) \le \mu V_{p_i}(x_i,q_i) \le \underline{\alpha}(\varepsilon).$$

Then, by (2), we have for all $j = 1,\dots,k$,

$$\|y_{i,j} - z_{i,j}\| \le \underline{\alpha}^{-1}(V_{u_i}(y_{i,j},z_{i,j})) \le \varepsilon.$$

Hence,

$$d_Y(y_i,z_i) = \max_{j=1}^{k} \|y_{i,j} - z_{i,j}\| \le \varepsilon. \quad (12)$$

It then follows from (11), (12) that

$$d_{Y^r}(y,z) = \max_{i=1}^{r} d_Y(y_i,z_i) \le \varepsilon.$$

Hence, the first condition of Definition 4 holds.

In a similar way, we prove that for all $((q',p'),z) \in \Delta^r_{\tau,\eta}((q,p),u)$ there exists $((x',p'),y) \in \Delta^r_\tau((x,p),u)$ such that $((x',p'),(q',p')) \in R$ and $d_{Y^r}(y,z) \le \varepsilon$. Hence, R is an ε-approximate bisimulation relation between $T^r_\tau(\Sigma_{\tau_d})$ and $T^r_{\tau,\eta}(\Sigma_{\tau_d})$.

Now, let $(x,p) \in X^0 = \mathbb{R}^n \times P$, and $(q,p) \in X^0_\eta = [\mathbb{R}^n]_\eta \times P$, given by $q = Q_\eta(x)$, then $\|x - q\| \le \eta$. Following Remark 2, we have that the second inequality of (2) holds with $\overline{\alpha} = \gamma$. It follows that

$$V_p(x,q) \le \gamma(\|x-q\|) \le \gamma(\eta) \le \tfrac{1}{\mu}\underline{\alpha}(\varepsilon)$$

where the last inequality comes from (9). Hence $((x,p),(q,p)) \in R$. Conversely, for all $(q,p) \in X^0_\eta = [\mathbb{R}^n]_\eta \times P$, let $(x,p) \in X^0 = \mathbb{R}^n \times P$, given by $x = q$, then $V_p(x,q) = 0$ and $((x,p),(q,p)) \in R$. Hence, $T^r_\tau(\Sigma_{\tau_d})$ and $T^r_{\tau,\eta}(\Sigma_{\tau_d})$ are ε-approximately bisimilar. □

Some remarks regarding the size of the symbolic models are in order. It appears from (9) that, for a given precision $\varepsilon \in \mathbb{R}^+$ and control period $\tau \in \mathbb{R}^+$, using larger sampling factor $r \in \mathbb{N}^+$ allows us to use larger values of $\eta \in \mathbb{R}^+$ and thus coarser discretizations of the state space. This results in symbolic models with fewer symbolic states. However, the number of transitions initiating from a symbolic state is m^r and thus grows exponentially with the sampling factor. Hence, the advantage of using multirate symbolic models in terms of number of transitions in the symbolic model is still unclear. This issue is addressed in the following section, where we determine the optimal value of the sampling factor.

Remark 5 *The results of [26] on multirate symbolic models for incrementally stable switched systems (without dwell-time) can be recovered easily from the particular case of the present study when $\mu = 1$ (i.e. there exists common δ-GUAS Lyapunov function), and $k = 1$ (i.e. the dwell-time coincides with the control period, $\tau_d = \tau$).*

4 Optimal sampling factor

In the following, we consider multirate symbolic models $T^r_{\tau,\eta}(\Sigma_{\tau_d})$ computed using the approach described above, where we restrict the set of states to some compact set $C \subseteq \mathbb{R}^n$ with nonempty interior. The number of symbolic states in $X_\eta \cap (C \times P)$ can be accurately estimated by $\frac{v_C}{\eta^n} \times m$, where $v_C \in \mathbb{R}^+$ is a positive constant proportional to the volume of C. Then the number of symbolic transitions initiating from states in $X_\eta \cap (C \times P)$ is $v_C \frac{m^{r+1}}{\eta^n}$. We assume that the number of modes $m \geq 2$.

4.1 Problem formulation

In this section, given a desired precision $\varepsilon \in \mathbb{R}^+$, and a control period $\tau \in \mathbb{R}^+$, we establish the optimal values $r^* \in \mathbb{N}^+$ and $\eta^* \in \mathbb{R}^+$, which characterizes the multirate symbolic model $T^r_{\tau,\eta}(\Sigma_{\tau_d})$ of precision ε (as guaranteed by Theorem 1) with the minimal number of symbolic transitions initiating from states in $X_\eta \cap (C \times P)$.

Since, C is a compact set, following Remark 1, we assume that (5) holds for a linear $\mathcal{K}_\infty$ function γ given by $\gamma(s) = c_\gamma s$ where $c_\gamma \in \mathbb{R}^+$. Thus, we aim at solving the following mixed integer nonlinear program:

$$\begin{array}{ll} \text{Minimize} & v_C \frac{m^{r+1}}{\eta^n} \\ \text{over} & r \in \mathbb{N}^+,\ \eta \in \mathbb{R}^+ \\ \text{under} & \eta \leq (1-\lambda(\tau)^r)\frac{\underline{\alpha}(\varepsilon)}{\mu c_\gamma} \end{array} \tag{13}$$

where the inequality constraints comes from (9) to guarantee a precision ε.

Let us first remark that for a given $r \in \mathbb{N}^+$, the optimal value $\eta \in \mathbb{R}^+$ is obviously obtained as $\eta = (1 - \lambda(\tau)^r)\frac{\underline{\alpha}(\varepsilon)}{\mu c_\gamma}$. It follows that (13) is equivalent to the following integer program:

$$\begin{array}{ll} \text{Minimize} & v_C \frac{(\mu c_\gamma)^n}{(\underline{\alpha}(\varepsilon))^n} \frac{m^{r+1}}{(1-\lambda(\tau)^r)^n} \\ \text{over} & r \in \mathbb{N}^+ \end{array} \tag{14}$$

The value $v_C \frac{(\mu c_\gamma)^n}{(\underline{\alpha}(\varepsilon))^n} \in \mathbb{R}^+$ does not depend on r and thus does not affect the solution of (14), which can finally be equivalently formulated as:

$$\begin{array}{ll} \text{Minimize} & g(r) = \frac{m^{r+1}}{(1-\lambda(\tau)^r)^n} \\ \text{over} & r \in \mathbb{N}^+ \end{array} \tag{15}$$

A first interesting information that can be inferred from (15) is that the optimal sampling factor only depends on the control period $\tau \in \mathbb{R}^+$, the dimension of the state-space $n \in \mathbb{N}^+$, the number of modes $m \in \mathbb{N}^+$, the dwell time τ_d, the constant $\mu \geq 1$ given in (4) and the decay rate $\kappa \in \mathbb{R}^+$ of the multiple δ-GUAS Lyapunov functions. In particular, it is noteworthy that it is independent of the desired precision $\varepsilon \in \mathbb{R}^+$ and of the compact set C.

4.2 Explicit solution

In this section, we show that the previous optimization problems can be solved explicitly. We first consider the relaxation of the integer program (15) over the positive real numbers:

Lemma 1 *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be given as in (15). Then, g has a unique minimizer $\tilde{r}^* \in \mathbb{R}^+$ given by*

$$\tilde{r}^* = \frac{1}{-\ln(\lambda(\tau))} \ln\left(1 - \frac{n\ln(\lambda(\tau))}{\ln(m)}\right) \tag{16}$$

with $\lambda(\tau) = \max(e^{-\kappa\tau}, \mu e^{-\kappa\tau_d})$.

Moreover, g is strictly decreasing on $(0, \tilde{r}^*]$ and strictly increasing on $[\tilde{r}^*, +\infty)$.

PROOF. Let us compute the first order derivative of g:

$$\begin{aligned} g'(r) =& \frac{m}{(1-\lambda(\tau)^r)^{2n}} \Big(\ln(m) m^r (1-\lambda(\tau)^r)^n \\ &+ m^r n \ln(\lambda(\tau))\lambda(\tau)^r (1-\lambda(\tau)^r)^{n-1} \Big) \\ =& \frac{m^{r+1}}{(1-\lambda(\tau)^r)^{n+1}} \Big(\ln(m)(1-\lambda(\tau)^r) \\ &+ n\ln(\lambda(\tau))\lambda(\tau)^r \Big) \\ =& \frac{\ln(m) m^{r+1}}{(1-\lambda(\tau)^r)^{n+1}} \left(1 - \lambda(\tau)^r \left(1 - \frac{n \ln(\lambda(\tau))}{\ln(m)} \right) \right). \end{aligned}$$

By remarking that $\frac{\ln(m)m^{r+1}}{(1-\lambda(\tau)^r)^{n+1}} > 0$ for all $r \in \mathbb{R}^+$, it is easy to see that $g'(r)$ has the same sign as $\left(1 - \lambda(\tau)^r\left(1 - \frac{n\ln(\lambda(\tau))}{\ln(m)}\right)\right)$, and that it is negative on $(0, \tilde{r}^*)$, zero at $\tilde{r}^*$ and positive on $(\tilde{r}^*, +\infty)$. The result stated in Lemma 1 follows immediately. □

We can now state the main result of the section:

Theorem 2 *For any desired precision $\varepsilon \in \mathbb{R}^+$, and any control period $\tau \in \mathbb{R}^+$, the optimal parameters $r^* \in \mathbb{N}^+$ and $\eta^* \in \mathbb{R}^+$, solutions of (13), which minimize the number of symbolic transitions of $T^r_{\tau,\eta}(\Sigma_{\tau_d})$, initiating from states in $X_\eta \cap (C \times P)$, while satisfying (9), are given by*

$$r^* = \lfloor \tilde{r}^* \rfloor \text{ or } r^* = \lfloor \tilde{r}^* \rfloor + 1 \tag{17}$$

$$\text{and } \eta^* = (1-\lambda(\tau)^{r^*})\frac{\underline{\alpha}(\varepsilon)}{\mu c_\gamma} \tag{18}$$

where $\tilde{r}^*$ is given by (16).

PROOF. From Lemma 1, it follows that

$$\forall r \in \mathbb{N}^+,\ \text{with } r < \lfloor \tilde{r}^* \rfloor,\ g(r) > g(\lfloor \tilde{r}^* \rfloor)$$

and

$$\forall r \in \mathbb{N}^+,\ \text{with } r > \lfloor \tilde{r}^* \rfloor + 1,\ g(r) > g(\lfloor \tilde{r}^* \rfloor + 1).$$

Then, it follows that the minimal value of g over $\mathbb{N}^+$ is obtained for $r^* = \lfloor \tilde{r}^* \rfloor$ or $r^* = \lfloor \tilde{r}^* \rfloor + 1$. Then, from the discussions in Section 4.1, it follows that the solution of (13) is given by r^* and $\eta^* = (1 - \lambda(\tau)^{r^*})\frac{\underline{\alpha}(\varepsilon)}{\mu c_\gamma}$. □

In practice, we compute the optimal parameters of the multirate symbolic models by evaluating the function g at $\lfloor \tilde{r}^* \rfloor$ and $\lfloor \tilde{r}^* \rfloor + 1$. We then pick the one, out of two possible values of r^*, which minimizes g and compute η^* using (18).

We would like to point out that the previous result can be applied to either linear or nonlinear switched systems. The only requirement is that we restrict the analysis to a compact subset of $\mathbb{R}^n$. Finally, it is interesting to remark that for small values of the control period $\tau \in \mathbb{R}^+$, the optimal sampling factor r^* is mainly determined by the state space dimension and the number of modes.

Corollary 1 *There exists $\overline{\tau} \in \mathbb{R}^+$, such that for any desired precision $\varepsilon \in \mathbb{R}^+$, and any control period $\tau \in (0, \overline{\tau}]$, the optimal parameters $r^* \in \mathbb{N}^+$ and $\eta^* \in \mathbb{R}^+$, solutions of (13), which minimize the number of symbolic transitions of $T^r_{\tau,\eta}(\Sigma_{\tau_d})$, initiating from states in $X_\eta \cap (C \times P)$, while satisfying (9), are given by*

$$r^* = \left\lfloor \frac{n}{\ln(m)} \right\rfloor \ \ \textit{or}\ r^* = \left\lfloor \frac{n}{\ln(m)} \right\rfloor + 1$$
$$\textit{and}\ \eta^* = (1 - e^{-r^*\kappa\tau})\frac{\underline{\alpha}(\varepsilon)}{\mu c_\gamma}.$$

PROOF. First let us remark that for $\tau \leq \tau_d - \frac{\ln(\mu)}{\kappa}$, we have $\lambda(\tau) = e^{-\kappa\tau}$. Then, let $\overline{\tau}$ be given by

$$\overline{\tau} = \min\left(\tau_d - \frac{\ln(\mu)}{\kappa}, \frac{2\ln(m)}{n\kappa}\left(1 - \frac{\left\lfloor \frac{n}{\ln(m)} \right\rfloor}{\frac{n}{\ln(m)}}\right)\right). \tag{19}$$

From Theorem 2.2 in [3], we have that for all $n, m \in \mathbb{N}^+$ with $m \geq 2$, $\frac{n}{\ln(m)} \in \mathbb{R}^+ \setminus \mathbb{N}^+$. Then, it follows that $\lfloor \frac{n}{\ln(m)} \rfloor < \frac{n}{\ln(m)}$ and that $\frac{2\ln(m)}{n\kappa}\left(1 - \frac{\lfloor \frac{n}{\ln(m)} \rfloor}{\frac{n}{\ln(m)}}\right) > 0$. Moreover $\tau_d - \frac{\ln(\mu)}{\kappa} > 0$. Hence, $\overline{\tau} > 0$.

Now, let us remark that for all $\theta \in \mathbb{R}^+$, we have that $\theta(1 - \frac{\theta}{2}) \leq \ln(1+\theta) \leq \theta$. Let $\tilde{r}^*$ be given by (16), then it follows from the previous inequalities that for all $\tau \in \mathbb{R}^+$.

$$\frac{n}{\ln(m)}\left(1 - \frac{n\kappa\tau}{2\ln(m)}\right) \leq \tilde{r}^* \leq \frac{n}{\ln(m)}.$$

Then, using (19), it follows that for all $\tau \in (0, \overline{\tau}]$,

$$\left\lfloor \frac{n}{\ln(m)} \right\rfloor \leq \tilde{r}^* \leq \frac{n}{\ln(m)}$$

which implies that $\lfloor \tilde{r}^* \rfloor = \lfloor \frac{n}{\ln(m)} \rfloor$. The stated result is then a consequence of Theorem 2. □

5 Illustrating example

In this section, we illustrate our main results and demonstrate the benefits of the proposed approach by considering the same example as in [10]. We consider a two-dimensional switched affine system with two modes (i.e. $n = 2$, $m = 2$) and given by

$$\dot{\mathbf{x}}(t) = A_{\mathbf{p}(t)}\mathbf{x}(t) + b_{\mathbf{p}(t)}$$

with $b_1 = [-0.25\ \ -2]^T$, $b_2 = [0.25\ \ 1]^T$ and

$$A_1 = \begin{bmatrix} -0.25 & 1 \\ -2 & -0.25 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.25 & 2 \\ -1 & -0.25 \end{bmatrix}.$$

The system does not have a common δ-GUAS Lyapunov function but admits multiple δ-GUAS Lyapunov functions of the form $V_p(x,y) = \sqrt{(x-y)^T M_p (x-y)}$, with

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let us first remark that even if the Lyapunov function $V_i, i \in \{1,2\}$ is not smooth, it satisfies condition (6) with $\kappa = 0.25$. The equations (2), (4) and (5) hold with $\underline{\alpha}(s) = s$, $\overline{\alpha}(s) = \sqrt{2}s$, $\mu = \sqrt{2}$ and $\gamma(s) = \sqrt{2}s$. Imposing a dwell-time $\tau_d = 2 > \frac{\ln(\mu)}{\kappa}$, the switched system is incrementally stable.

We compute multirate symbolic models using the approach described in Section 5. We set the control period $\tau = 0.2$ (i.e. $k = 10$) and the desired precision $\varepsilon = 0.25 \times \sqrt{2}$. We restrict the dynamics to a compact subset of $\mathbb{R}^2$ given by $C = [-6, 6] \times [-4, 4]$. We compute the symbolic models for several sampling factors $r = 1, \ldots, 9$, the space sampling parameter is then chosen as $\eta = (1 - \lambda(\tau)^r)\frac{\underline{\alpha}(\varepsilon)}{\mu c_\gamma}$. Figure 1 shows the number of symbolic transitions as a function of r and we can see that this number is minimal for $r = 3$.

Using (19), we compute $\overline{\tau} = 0.61$. Thus, $\tau \in (0, \overline{\tau}]$ and the assumptions of Corollary 1 hold. In particular, since

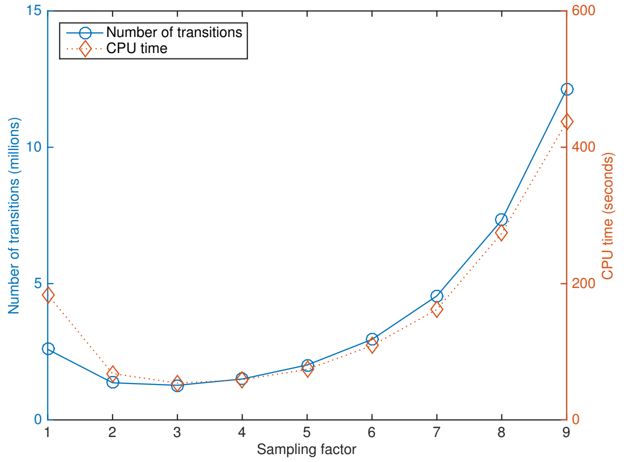

Fig. 1. Number of symbolic transitions in the multirate symbolic models $T^r_{\tau,\eta}(\Sigma_{\tau_d})$ of the switched system with dwell–time for different values of the sampling factor r; Computation times for generating symbolic models and synthesizing safety controllers (MATLAB implementation, processor 2.2 GHz Intel Core i7, memory 16 GB 1600 MHz DDR3).

$\frac{n}{\ln(m)} = 2.89$, the optimal sampling factor is either 2 or 3. We can then check numerically that the optimal sampling factor is $r^* = 3$, which is consistent with the experimental data.

We now synthesize safety controllers (see e.g. [29]), which keep the output of the symbolic models inside the compact region C while avoiding $C' = [-1.5, 1.5] \times [-1, 1]$. Figure 1 reports the computation times for generating symbolic models and synthesizing controllers for $r = 1, \ldots, 9$. The time for generating the symbolic model is linear with respect to the number of transitions and thus perfectly correlated with the number of transitions. However, the time for synthesizing the controller depends on the fixed point algorithm (see [29]) for which the worst case complexity is polynomial in the number of transitions which explains why the number of transitions and the CPU time are not perfectly correlated (a higher number of iterations is needed to reach the fixed point for $r = 1$ and $r = 2$ than for the other values). We can check that using the optimal sampling factor $r = 3$ allows us to reduce, for that example the computation times by about 70% in comparison to the approach corresponding to $r = 1$. For $r = 3$, Figure 2 shows a trajectory of the switched system and the associated switching signal controlled with the symbolic controller for the initial state $x^0 = [0 \;\; 3]^T$.

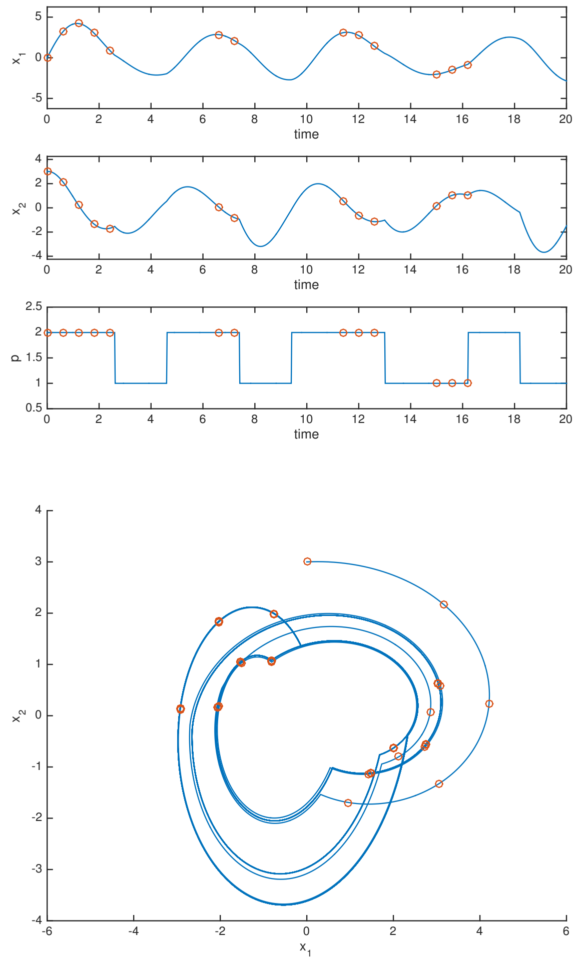

Fig. 2. Top: Trajectory of the switched system with dwell–time and the associated switching signal controlled with the symbolic controller for the initial state $x^0 = [0 \;\; 3]^T$. The control period is $\tau = 0.2$ while the sampling factor $r = 3$ (instants of transitions are indicated with circles). Bottom: Same trajectory in the state-space.

6 Conclusion

In this paper, we have proposed the use of multirate sampling for the computation of symbolic models for incrementally stable switched systems, with dwell-time constraints. We have demonstrated that our technique makes it possible to use more compact abstractions (i.e. with fewer transitions) than the standard existing approach presented in [10]. Moreover, the optimal sampling factor has been determined theoretically and we provided a simple expression depending solely on the number of modes and on the dimension of the state space, which makes it possible to use this result as a rule of thumb when computing symbolic models of switched systems. Our approach has been validated experimentally on a numerical example, which showed that multirate symbolic models indeed enable controller synthesis at a reduced computational cost. We are confident that similar results can be established for other classes of incrementally stable dynamical systems.

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