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Wentao Tang *, **Zhenhua Wang** *, **Yi Shen** *, **Mickael Rodrigues** **, **Didier Theilliol** ***

* *School of Astronautics, Harbin Institute of Technology, Harbin, 150001 P. R. China (e-mail: zhenhua.wang@hit.edu.cn).*
** *Automatic and Process Control Laboratory, University of Lyon, Lyon, F-69003, France (e-mail: mickael.rodrigues@univ-lyon1.fr)*
*** *Centre de Recherche en Automatique de Nancy, Université de Lorraine, CNRS UMR 7039, F-54506 Vandoeuvre-les-Nancy, France (e-mail: didier.theilliol@univ-lorraine.fr)*

Abstract: The design of a novel sensor fault detection scheme for discrete-time linear time-invariant systems with unknown but bounded uncertainties is proposed in the paper. To make the residual sensitive to fault and robust against the unknown disturbance and measurement noise, we propose a multi-objective fault detection observer method based on the criteria of H_- index and P-radius and further present an iterative LMI method to solve the observer design problem. In the design of residual evaluation, we propose a threshold computation method via the interval hull approximation of the residual reachable set. The merit of the proposed method is that its threshold computation exhibits less conservatism and higher computational efficiency than the zonotope-based method. Simulations are conducted to demonstrate the superiority of the proposed method.

Keywords: Fault detection, multi-objective observer, threshold computation.

1. INTRODUCTION

In the past decades, model-based fault diagnosis techniques have been extensively studied, see e.g. Ding (2008), Chen and Patton (2012) and the references therein. One of the challenges in model-based fault diagnosis is how to handle model uncertainties including the unknown disturbance and measurement noise. H_-/H_∞ robust fault detection observer which is sensitive to faults and robust to model uncertainties was first proposed by Hou and Patton (1996) and has attracted many researchers' attention see in Liu, Wang, and Yang (2005), Wang and Yang (2008) , Ding and Yang (2010) and Chadli, Abdo, and Ding (2013). In Wang, Rodrigues, Theilliol, and Shen (2015), H_∞ design is used for robust fault detection of descriptor systems. In H_∞ design, it is required that the energy of disturbance over entire time domain is bounded, which is seldom satisfied. Although H_∞ norm is widely used in control analysis and synthesis, it is not a suitable measure for residual evaluation (Wang et al., 2017b).

Different from H_∞ design, set-membership estimation, which assumes that the uncertainties are unknown but bounded, provides a natural way to compute threshold for fault detection. Recently, the zonotope-based method has received much attention due to its flexibility and simplicity. Some zonotope size criteria including P-radius (Le et al., 2013) and F-radius (Combastel, 2015b) have been used to design the set-membership observers robust against disturbance and noise. In recent years, zonotope-based set-membership estimation methods have been used for fault detection (Guerra et al., 2006; Combastel and Zhang, 2006; Puig, 2010). In Xu, Puig, Ocampo-Martinez, Stoican, and Olaru (2014), the zonotope-based method is also used for fault isolation. Combastel (2015a) combines Kalman filtering and zonotope-based method for robust fault detection.

Note that most of existing results on zonotopic fault detection only consider the robustness to disturbance and noise. Wang et al. (2017a) first combines the H_- analysis and the zonotopic analysis to achieve zonotopic fault detection with H_- performance. However, a reduction operator is required in the zonotopic fault detection and may cause conservatism.

The main contribution of this paper consists in two aspects. First, the fault detection observer design based on H_- and the P-radius is solved by iterative LMI method. The designed observer is sensitive to sensor faults and robust to disturbance and noise. Second, compared with the zonotope-based method, a more accurate threshold for residual evaluation is obtained by computing the interval hull of the residual reachable set. The proposed threshold computation method is inspired by Girard et al. (2006) and has high computational efficiency, which only involves simple calculations of low dimensional matrices and can get rid of the reduction operation.

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2. PRELIMINARIES

A zonotope $\mathcal{Z} \subset \mathbb{R}^n$ is the affine image of an unit hypercube $\mathbf{B}^m = [-1, 1]^m$, $m \geq n$. Given the center vector $p \in \mathbb{R}^n$ and the generator matrix $H \in \mathbb{R}^{n\times m}$, the zonotope $\mathcal{Z}$ is defined as follows

$$\mathcal{Z} = \langle p, H\rangle = \{p + Hz, z \in \mathbf{B}^m\} \tag{1}$$

The Minkowski sum of two sets $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$\mathcal{X} \oplus \mathcal{Y} = \{x + y : x \in \mathcal{X},\, y \in \mathcal{Y}\} \tag{2}$$

And for sets $S_1, \ldots, S_n$,

$$\bigoplus_{i=1}^{n} S_i = S_1 \oplus \cdots \oplus S_n \tag{3}$$

which denotes the Minkowski sum of a group of sets.

The linear image of a set $\mathcal{X} \subset \mathbb{R}^n$ by a matrix $L \in \mathcal{R}^{m\times n}$ is defined as

$$L\mathcal{X} = \{Lx : x \in \mathcal{X}\} \tag{4}$$

For zonotopes, the following properties hold:

$$\begin{aligned} \langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle &= \langle p_1 + p_2, [H_1\ H_2]\rangle \\ L\langle p, H\rangle &= \langle Lp, LH\rangle \end{aligned} \tag{5}$$

For a vector $x \in \mathbb{R}^n$ and a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $\|x\|_P^2 = x^T P x$, where the superscript T denotes transposition, and $\|x\|^2 = x^T x$. In a symmetric block matrix, we use $*$ to represent a term that can be induced by symmetry. For a discrete-time signal z_k, its L_2 norm is defined as $\|z\|_2 = \sqrt{\sum_{k=0}^{\infty} z_k^T z_k}$.

3. PROBLEM FORMULATION

Consider the following system:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + D_w w_k \\ y_k = Cx_k + D_v v_k + Ef_k \end{cases} \tag{6}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^{n_y}$ denote the vectors of state, control input and measurement output, respectively. $w_k \in \mathbb{R}^{n_w}$ is the unknown disturbance, $v_k \in \mathbb{R}^{n_v}$ is the measurement noise and $f_k \in \mathbb{R}^{n_f}$ denotes the sensor fault. $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times n_u}$, $C \in \mathbb{R}^{n_y\times n_x}$, $D_w \in \mathbb{R}^{n_x\times n_w}$, $D_v \in \mathbb{R}^{n_y\times n_v}$ and $E \in \mathbb{R}^{n_y\times n_f}$ are known constant matrices.

Without loss of generality, we assume w_k, v_k and x_0 to be unknown but bounded as follows

$$w_k \in \mathcal{W} = \langle 0, I_{n_w}\rangle \tag{7}$$

$$v_k \in \mathcal{V} = \langle 0, I_{n_v}\rangle \tag{8}$$

$$x_0 \in \mathcal{X}_0 = \langle p_0, H_0\rangle \tag{9}$$

where I_{n_w} and I_{n_v} denote identity matrices, p_0 and H_0 are known constant vector and matrix.

For system (6), an observer-based residual generator is proposed as

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k) \\ r_k = M(y_k - C\hat{x}_k) \end{cases} \tag{10}$$

where $\hat{x}_k$ and r_k are the state estimation and residual, respectively. $L \in \mathbb{R}^{n_x\times n_y}$ is the observer gain and $M \in \mathbb{R}^{n_y\times n_y}$ denotes the weighting matrix to provide more design freedom. And the initial state estimation is set as

$$\hat{x}_0 = p_0 \tag{11}$$

The state estimation error is defined as

$$e_k = x_k - \hat{x}_k \tag{12}$$

Subtracting (10) from (6), the error dynamic system is obtained by

$$\begin{cases} e_{k+1} = (A - LC)e_k + D_w w_k - LD_v v_k - LEf_k \\ r_k = MCe_k + MD_v v_k + MEf_k \end{cases} \tag{13}$$

From (9) and (12), we have

$$e_0 \in \Omega_0 = \langle 0, H_0\rangle \tag{14}$$

In this paper, we first design the fault detection observer in (10) to generate the residual that is sensitive to the sensor fault and robust against the unknown disturbance and measurement noise. Following the fault detection observer design, we propose a residual evaluation method via computing the interval hull approximation.

4. RESIDUAL GENERATION BY FAULT DETECTION OBSERVER

The task of fault detection observer design is to find an observer gain L such that the residual is sensitive to sensor faults and robust against model uncertainties simultaneously. The design conditions can be converted into solving a multi-objective optimization problem.

4.1 Fault sensitivity condition

Consider the following error system which is only affected by sensor faults.

$$\begin{cases} e_{k+1} = (A - LC)e_k - LEfk \\ r_k = MCe_k + MEf_k \end{cases} \tag{15}$$

where the initial error $e_0 = 0$.

The following theorem is proposed to design L such that the residual is sensitive to the sensor fault.

Theorem 1. For the system (15), given a scalar $\beta > 0$, if there exists a positive definite matrix Q such that the following inequality holds.

$$\begin{bmatrix} -Q & QA - QLC & -QLE \\ * & -Q - C^T YC & -C^T YE \\ * & * & -E^T YE + \beta^2 I_{n_f} \end{bmatrix} < 0, \tag{16}$$

where $Y = M^T M$. Then the following inequality holds such that r_k is sensitive to f_k.

$$\|r\|_2 > \beta\|f\|_2 \tag{17}$$

Proof. By using the Schur complement, (16) is equivalent

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix} < 0 \tag{18}$$

where

$$\begin{aligned} \Phi_{11} &= -C^T M^T MC + (A - LC)^T Q(A - LC) - Q \\ \Phi_{12} &= (A - LC)^T Q(-LE) - C^T M^T ME \\ \Phi_{22} &= \beta^2 I_{n_f} - E^T M^T ME + (LE)^T Q(LE) \end{aligned} \tag{19}$$

Define the following Lyapunov function

$$V_k = e_k^T Q e_k \tag{20}$$

where $Q \in \mathbb{R}^{n_x\times n_x}$ is a positive definite matrix.

The time difference of V_k is

$$\begin{aligned}\Delta V_k &= V_{k+1} - V_k \\ &= e_k^T[(A-LC)^T Q(A-LC) - Q]e_k \\ &\quad + e_k^T(A-LC)^T Q(-LE)f_k \\ &\quad + f_k^T(-LE)^T Q(A-LC)e_k \\ &\quad + f_k^T(LE)^T Q(LE)f_k\end{aligned} \tag{21}$$

Then, pre-multiply and post-multiply (18) with $\begin{bmatrix} e_k^T & f_k^T \end{bmatrix}^T$ and its transpose, we have

$$r_k^T r_k - \beta^2 f_k^T f_k - \Delta V_k > 0. \tag{22}$$

It follows that

$$\|r\|_2^2 - \beta^2\|f\|_2^2 - V_\infty + V_0 > 0 \tag{23}$$

Since $V_0 = 0$ and $V_\infty \geq 0$, we have

$$\|r\|_2^2 - \beta^2\|f\|_2^2 > 0 \tag{24}$$

which is equivalent to (17). □

4.2 Disturbance attenuation condition

To design the observer gain L such that the residual is robust against the system uncertainties, the error system only affected by disturbance and noise is considered as follows

$$\begin{cases} e_{k+1} = (A-LC)e_k + D_w w_k - LD_v v_k \\ r_k = MCe_k + MD_v v_k \end{cases} \tag{25}$$

For system (25), the reachable set of e_{k+1} can be obtained from those of e_k, w_k and v_k recursively by using the properties of zonotopes, (5). Since w_k, v_k and e_0 are bounded in centered zonotopes, e_k can be bounded in the zonotope denoted as $\Omega_k = \langle 0, H_k \rangle$, where $H_k \in \mathbb{R}^{n_x \times s}$.

According to (25), the generator matrix of $\Omega_{k+1} = \langle 0, H_{k+1} \rangle$ can be obtained by

$$H_{k+1} = [(A-LC)H_k \quad D_w \quad -LD_v] \tag{26}$$

To minimize the system uncertainties effects on the estimation error and the residual, the observer gain L is designed such that the size of Ω_k is minimized. A zonotope size criterion named P-radius is proposed in Le et al. (2013). The P-radius of $\langle 0, H_k \rangle$ is defined as

$$l_k = \max_{z \in \mathbf{B}^s} \|H_k z\|_P^2 \tag{27}$$

If there exists a scalar $\gamma \in (0, 1)$ such that $l_{k+1} < \gamma l_k$, the size of Ω_k(i.e. l_k) is decreasing. Due to the disturbance and noise, this condition is hard to verify. A relaxation of this condition can be

$$l_{k+1} < \gamma l_k + \epsilon \tag{28}$$

where ϵ is a positive constant that represents the max influence of unknown disturbance and measurement noise as follows

$$\epsilon = \max_{s_1 \in \mathbf{B}^{n_w}} \|D_w s_1\|^2 + \max_{s_2 \in \mathbf{B}^{n_v}} \|D_v s_2\|^2 \tag{29}$$

Under the condition (28), l_k is bounded. If γ is smaller or P is bigger, the size of Ω_k is smaller.

Theorem 2. Given a scalar $\gamma \in (0, 1)$, (28) holds if there exists a positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ such that the following inequality holds

$$\begin{bmatrix} -\gamma P & 0 & 0 & (A-LC)^T P^T \\ * & -D_w^T D_w & 0 & D_w^T P^T \\ * & * & -D_v^T D_v & (-LD_v)^T P^T \\ * & * & * & -P \end{bmatrix} < 0 \tag{30}$$

Proof. Using (27) and (29), (28) can be rewritten as

$$\begin{aligned}\max_{\hat{z} \in \mathbf{B}^{s+n_w+n_v}} \|H_{k+1}\hat{z}\|_P^2 \leq \max_{z \in \mathbf{B}^{n_x}} \|H_k z\|_P^2 + \max_{s_1 \in \mathbf{B}^{n_w}} \|D_w s_1\|^2 \\ + \max_{s_2 \in \mathbf{B}^{n_v}} \|D_v s_2\|^2\end{aligned} \tag{31}$$

A sufficient condition of (31) is that $\forall z_1, z, s_1, s_2$, the following inequality holds.

$$\|H_{k+1}z_1\|_P^2 - \gamma\|H_k z\|_P^2 - \|D_w s_1\|^2 - \|D_v s_2\|^2 < 0 \tag{32}$$

where $z_1 = \begin{bmatrix} z^T & s_1^T & s_2^T \end{bmatrix}^T$. The explicit form of the previous inequality is

$$\begin{aligned}&\begin{bmatrix} z \\ s_1 \\ s_2 \end{bmatrix}^T H_{k+1}^T P H_{k+1} \begin{bmatrix} z \\ s_1 \\ s_2 \end{bmatrix} - z^T H_k^T \gamma P H_k z \\ &-s_1^T D_w^T D_w s_1 - s_2^T D_v^T D_v s_2 < 0\end{aligned} \tag{33}$$

Denoting $\theta = H_k z$ and substituting (26) into (33), the inequality above can be written as

$$\begin{aligned}&\begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix}^T [A-LC \quad D_w \quad -LD_v]^T P [A-LC \quad D_w \quad -LD_v] \\ &\begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix} - \begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix}^T \begin{bmatrix} \gamma P & 0 & 0 \\ * & D_w^T D_w & 0 \\ * & * & D_v^T D_v \end{bmatrix} \begin{bmatrix} \theta \\ s_1 \\ s_2 \end{bmatrix} < 0\end{aligned} \tag{34}$$

Based on the definition of the negative definite matrix and Schur complement, (34) is equivalent to (30). □

4.3 Fault detection observer design using iterative LMI

The fault detection observer is required to be sensitive to the sensor fault and robust against the uncertainties, which is a multi-objective optimization problem. Since there exist coupling terms PL and QL, (16) and (30) are not LMIs.

In Wang et al. (2017a), by letting $W = PL$ and $Q = \alpha P$, (30) and (16) are reformulated as LMIs. Then the observer gain L can be obtained by solving the following optimization problem:

$$\min_{\beta, P, W, Y} -tr(P) - \beta^2 \tag{35a}$$

subject to

$$\begin{bmatrix} -\gamma P & 0 & 0 & A^T P - C^T W^T \\ * & -D_w^T D_w & 0 & D_w^T P^T \\ * & * & -D_v^T D_v & -D_v^T W^T \\ * & * & * & -P \end{bmatrix} < 0 \tag{35b}$$

$$\begin{bmatrix} -\alpha P & \alpha PA - \alpha WC & -\alpha WE \\ * & -\alpha P - C^T YC & -C^T YE \\ * & * & -E^T YE + \beta^2 I_{n_f} \end{bmatrix} < 0 \tag{35c}$$

Nevertheless, this method may cause some conservatism due to the linearization of PL and QL. To deal with the coupling terms, P and Q are obtained by solving LMIs as L given. Next, using the P and Q obtained in the last

step, L is also obtained by solving LMIs. The two steps continues until the termination condition is satisfied.

The iterative method can achieve less conservatism. The overall process of the proposed method can be denoted as algorithm 1.

Algorithm 1

Give the parameters of system (6):A, D_w, C, D_v, E and set a termination constant ε.

1. Given $\gamma \in (0,1)$, solve the optimization problem (35) and obtain the matrix L.

2. Fix L as obtained at last step, solve the following optimization problem and denote the solution of β^2 as β_1.

$$\begin{aligned} &\min_{P,Q,\beta,Y} -tr(P) - \beta^2 \\ &s.t. \quad (30),\ (16) \end{aligned}$$

3. Fix P and Q as obtained at last step, solve the following optimization problem and denote the solution of β^2 as β_2.

$$\begin{aligned} &\min_{P,Q,\beta,Y} -\beta^2 \\ &s.t. \quad (30),\ (16) \end{aligned}$$

If $|\beta_2 - \beta_1| < \varepsilon$ is satisfied, terminate the algorithm and output L. If not, go back to the step 2.

5. THRESHOLD COMPUTATION FOR RESIDUAL EVALUATION

In this section, the threshold for residual evaluation is obtained by a recursive algorithm to compute the interval hull approximation of the residual reachable set.

Denote the reachable sets of e_k and r_k as Ω_k and R_k. According to (25), Ω_{k+1} and R_k in fault-free case can be obtained by the following equations.

$$\begin{cases} \Omega_{k+1} = (A - LC)\Omega_k \oplus D_w\mathcal{W} \oplus (-LD_v)\mathcal{V} \\ R_k = MC\Omega_k \oplus MD_v\mathcal{V} \end{cases} \tag{36}$$

According to (36), Ω_{k+1} can be obtained by

$$\begin{aligned} \Omega_{k+1} = (A - LC)^k\Omega_0 &\oplus \bigoplus_{i=0}^{k-1}(A - LC)^i D_w\mathcal{W} \\ &\oplus \bigoplus_{i=0}^{k-1}(A - LC)^i(-LD_v)\mathcal{V} \end{aligned} \tag{37}$$

Then the reachable set of r_{k+1} can be obtained by

$$\begin{aligned} R_{k+1} = MC(A - LC)^k\Omega_0 &\oplus \bigoplus_{i=0}^{k-1} MC(A - LC)^i D_w\mathcal{W} \\ &\oplus \bigoplus_{i=0}^{k-1} MC(A - LC)^i(-LD_v)\mathcal{V} \oplus MD_v\mathcal{V} \end{aligned} \tag{38}$$

The equation (38) indicates that R_{k+1} can be separated into four parts. The boundaries of R_{k+1} can be easily obtained by a recursive algorithm.

The boundaries of a set $S \in \mathbb{R}^n$ can be formulated as an interval hull as follows

$$\mathrm{Box}(S) = ([a_1, b_1], \ldots, [a_i, b_i], \ldots, [a_n, b_n]) \tag{39}$$

where $[a_i, b_i]$ is an interval, a_i and b_i are the lower and upper boundary of the i-th element of any vector variable $s \in S$. For simplicity, define $a = [a_1, \ldots, a_n]$ and $b = [b_1, \ldots, b_n]$, then the interval hull of S can be denoted as $\mathrm{Box}(S) = [a, b]$.

For a zonotope $\mathcal{Z} = \langle p, H\rangle \in \mathbb{R}^n$, where $H \in \mathbb{R}^{n\times s}$, the components of $\mathrm{Box}(\mathcal{Z}) = [a, b]$ can be obtained by

$$\begin{cases} a_i = p_i - \sum_{j=0}^{s} |H_{i,j}|, \quad i = 1, \ldots, n \\ b_i = p_i + \sum_{j=0}^{s} |H_{i,j}|, \quad i = 1, \ldots, n \end{cases} \tag{40}$$

For interval hulls $\mathrm{Box}(S_1) = [a, b]$, $\mathrm{Box}(S_2) = [c, d]$, the following property holds.

$$\begin{aligned} &\mathrm{Box}(S_1 \oplus S_2) = \mathrm{Box}(S_1) \oplus \mathrm{Box}(S_2) \\ &= [a, b] \oplus [c, d] = [a + c, b + d] \end{aligned} \tag{41}$$

Using (38) and (41), $\mathrm{Box}(R_{k+1})$ can be obtained by

$$\begin{aligned} &\mathrm{Box}(R_{k+1}) = \mathrm{Box}\Big(MC(A - LC)^k\Omega_0\Big) \\ &\oplus \bigoplus_{i=0}^{k-1} \mathrm{Box}\Big(MC(A - LC)^i D_w\mathcal{W}\Big) \\ &\oplus \bigoplus_{i=0}^{k-1} \mathrm{Box}\Big(MC(A - LC)^i(-LD_v)\mathcal{V}\Big) \oplus \mathrm{Box}\Big(MD_v\mathcal{V}\Big) \end{aligned} \tag{42}$$

According to the structures of the components of (42), $\mathrm{Box}(R_{k+1})$ can be obtained by the recursive algorithm as follows.

Algorithm 2

Give the parameters of system (13):A, D_w, C, D_v, L, Ω_0, $\mathcal{W}$, $\mathcal{V}$

1. Initialize the follow sets with the known sets:
$X_0 \leftarrow \Omega_0$
$M_0 \leftarrow D_w\mathcal{W}$
$D_0 \leftarrow \{0\}$
$S_0 \leftarrow LD_v\mathcal{V}$
$N_0 \leftarrow \{0\}$

2. The interval hull of R_{k+1} is obtained by the following iteration process:
$X_{k+1} = (A - LC)X_k$
$D_{k+1} = D_k \oplus \mathrm{Box}(MCM_k)$
$N_{k+1} = N_k \oplus \mathrm{Box}(MCS_k)$
$M_{k+1} = (A - LC)M_k$
$S_{k+1} = (A - LC)S_k$
$\mathrm{Box}(R_{k+1}) = \mathrm{Box}(MCX_{k+1}) \oplus D_{k+1} \oplus N_{k+1} \oplus \mathrm{Box}(MD_v\mathcal{V})$

Remark 1. A zonotopic fault detection procedure by testing if the origin of coordinate is in the residual zonotope is proposed in Wang et al. (2017a). However, this test suffers a large computational burden. It is more practical to calculate the boundaries of the components of the residual as the threshold, which can be obtained by calculating the interval hull of the residual zonotope. The recursive algorithm proposed in this section can obtain more accurate thresholds than those by the zonotope-based method and has higher computational efficiency.

6. SIMULATIONS

In this section, a numerical example adapted from the subsection 7.4 of Chen and Patton (1999) is used to illustrate the effectiveness of the proposed method. The system has the form of (6) with

$$A = \begin{bmatrix} 0.5 & -0.7 & 0.7 & 0 \\ 0 & 0.8 & 0.09 & 0 \\ -1.0 & 0 & 0 & 0.1 \\ 0 & 0 & -0.15 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_w = 0.01 I_4,$$
$$D_v = 0.01 I_3, \quad E = C$$

In the simulation study, the initial state x_0, p_0 and H_0 are set as

$$x_0 = [0.09\ 0.07\ 0.08\ 0.06]^T, \quad p_0 = [0\ 0\ 0\ 0]^T,$$
$$H_0 = 0.01 I_4$$

Given $\alpha = 10$ and $\gamma = 0.33$, using the method in Wang et al. (2017a) gives the solutions of L and M as follows.

$$L_1 = \begin{bmatrix} 0.2208 & -0.6049 & -0.9884 \\ 0.0589 & 0.6541 & -0.0445 \\ 0.1363 & -0.0308 & 1.3849 \\ -0.0163 & 0.0073 & 0.4723 \end{bmatrix}$$
$$M_1 = \begin{bmatrix} 3.1761 & 0.0249 & 0.2025 \\ 0.0249 & 4.8651 & -1.5145 \\ 0.2025 & -1.5145 & 3.3654 \end{bmatrix}$$

The sensor faults are formulated as

$$f_k = \begin{cases} [0\ 0\ 0]^T, & k < 40, k > 60 \\ [0.05\ -0.03\ 0.02]^T, & 40 \le k \le 60 \end{cases} \tag{43}$$

The simulation results are shown in Figure 1-3. The thresholds obtained by algorithm 2 are less conservative than those by the zonotope-based method. But both the two thresholds fail to detect the faults, which is due to the conservatism introduced by the linearization of PL and QL in (16) and (30), respectively.

Given the same α and γ, the solutions of L and M obtained by algorithm 1 are as follows. And the sensor faults are still set as (43).

$$L_2 = \begin{bmatrix} 0.3481 & -0.4923 & -0.0509 \\ 0.0697 & 0.6446 & 0.0001 \\ 0.2424 & 0.0622 & 0.1088 \\ -0.0745 & -0.0415 & 0.3224 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} 3.1587 & 0.2973 & -0.0395 \\ 0.2973 & 6.0840 & -0.0423 \\ -0.0395 & -0.0423 & 1.6793 \end{bmatrix}$$

The simulation results are depicted by Figure 4-6. These figures show that the threshold by the proposed method has higher fault detection rate than that by Wang et al. (2017a). Moreover, the method in Wang et al. (2017a) has heavier computation burden since it involves operations on high dimensional matrices and needs reduction operator. Algorithm 2 only involves simple calculations of low dimensional matrices and thus has higher computational efficiency.

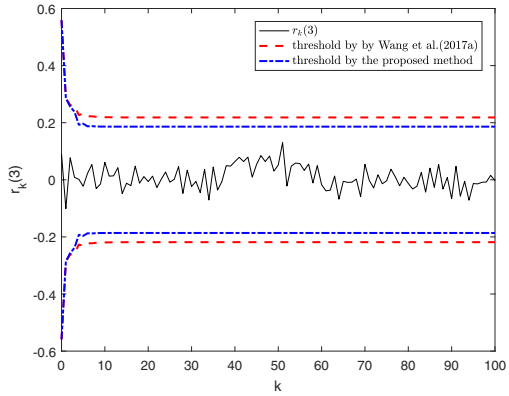

Fig. 1. $r_k(1)$ and its thresholds with L_1 and M_1.

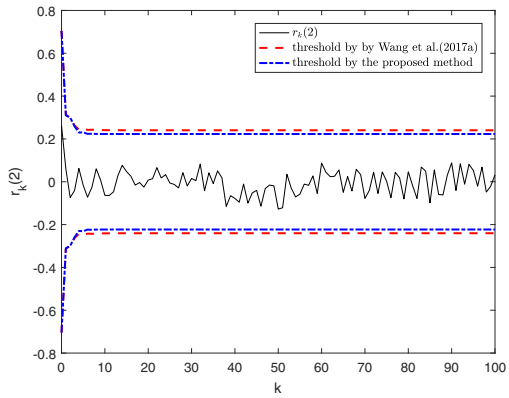

Fig. 2. $r_k(2)$ and its thresholds with L_1 and M_1.

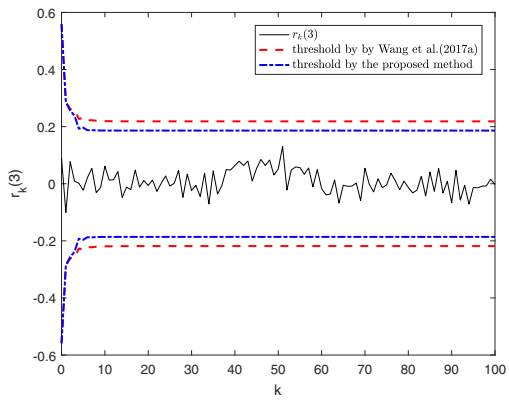

Fig. 3. $r_k(3)$ and its thresholds with L_1 and M_1.

7. CONCLUSION

In this paper, a fault detection observer is designed by combining the H_- index and the P-radius. The multi-objective optimization problem is solved by iterative LMI method. The designed observer is sensitive to the sensor fault and robust to the unknown disturbance and measurement noise. The threshold obtained by computing the interval hull approximation of the residual reachable set has less conservatism than that by the zonotope-based method.

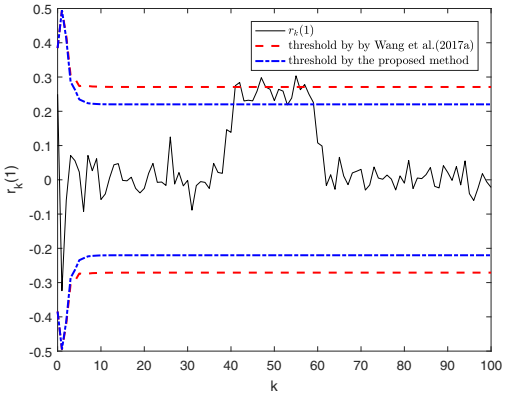

Fig. 4. $r_k(1)$ and its thresholds with L_2 and M_2.

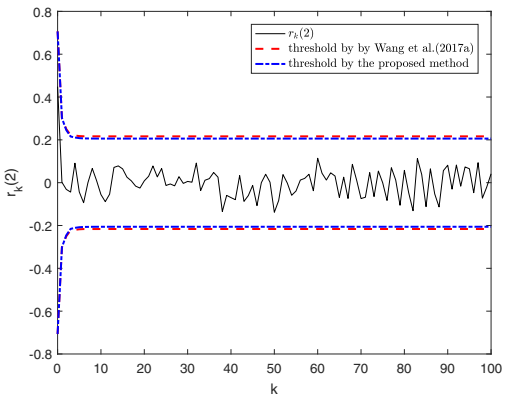

Fig. 5. $r_k(2)$ and its thresholds with L_2 and M_2.

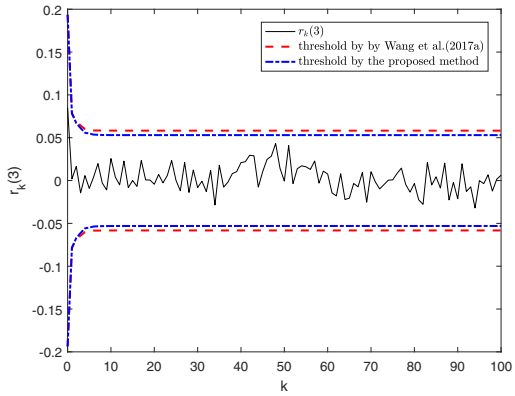

Fig. 6. $r_k(3)$ and its thresholds with L_2 and M_2.

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