An identity for the infinite sum $\sum\_n = 0^{\infty}\frac{1}{(n!)^3}$

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Abstract. In this short note, we give an identity for the alpha function

$$\alpha(x,s) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^s},$$

where $s \in \mathbb{N},\ x \in \mathbb{R}$, in the case $s = 3$.

1. INTRODUCTION We know that when $s = 1$, then the alpha function is the exponential function $\alpha(x,1) = e^x$. The question is that what about when $s \geq 2$?. We have the following nice differential equation for the alpha function [**2**]. In fact, it is known that the alpha function is the solution of the following differential equation.

$$\sum_{k=1}^{s} \sigma_s^k x^{k-1} y^{(k)} - y = 0$$

where σ_s^k is the Stirling numbers of the second kind which satisfies in the following generating function,

$$\frac{1}{k!}(e^x - 1)^k = \sum_{s=k}^{\infty} \sigma_s^k \frac{x^n}{n!}$$

Hence by using the language of generalized hypergeometric function, we get [**3**],

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^3} =_0 F_2(;1,1;1) =\sim 2.1297$$

To deal with this question, we first give the following theorem which is the direct consequence of Bessel-Parseval identity[**1**].

Theorem 1. *Let we have the two following entire series, with real the coefficients,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_n x^n$$

with respectively non-zero radius R and R'. Take

$$h(x) = \sum_{n=0}^{\infty} a_n b_n x^n$$

, then if $-R < u < R$ and $-R' < v < R'$, then we have

$$h(uv) = \frac{1}{2\pi} \int_0^{2\pi} f(ue^{it}) g(ve^{-it}) dt$$

In the Theorem 1, if we take $f(x) = g(x) = e^x$, and $u = x, v = 1$, then we get the following amazing identity,

$$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{(x+1)\cos t} \cos((x-1)\sin t) dt$$

Hence by taking $x = 1$ we get

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{2\cos t} dt$$

But we know,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a\cos x + b\sin x)} dx = I_0\left(\sqrt{a^2+b^2}\right), \tag{1}$$

where I_0 is the modified Bessel function of the first kind.

2. THE MAIN RESULT In this section we give an integral identity for the alpha function when $s = 3$. We take $f(x) = e^x$, $g(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{(x+1)\cos t} \cos((x-1)\sin t) dt$, and $u = x, v = 1$. Hence, from Theorem 1, we have

$$A = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^3} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{xe^{is}} e^{(e^{-is}+1)\cos t} \cos\left((e^{-is}-1)\sin t\right) ds dt$$

But, we have

$$e^{xe^{is}} e^{(e^{-is}+1)\cos t} = \left(e^{x\cos s + \cos s\cos t + \cos t}\right)\left(\cos(x\sin s - \sin s\cos t) + i\sin(x\sin s - \sin s\cos t)\right)$$

and also,

$$\cos\left((e^{-is}-1)\sin t\right) = \cos(\cos s\sin t - \sin t)\cosh(\sin s\sin t) + i\sin(\cos s\sin t - \sin t)\sinh(\sin s\sin t)$$

Hence

$$A = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left((e^{x\cos s+\cos s\cos t+\cos t})(\cos(x\sin s-\sin s\cos t))\right)\cos(\cos s\sin t-\sin t)\cosh(\sin s\sin t)dsdt$$
$$-\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(e^{x\cos s+\cos s\cos t+\cos t}\right)\sin(x\sin s-\sin s\cos t)\sin(\cos s\sin t-\sin t)\sinh(\sin s\sin t)dsdt$$

and this gives an identity for the alpha function in the case $s = 3$.

REFERENCES

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