

An identity for the infinite sum $\sum_{n=0}^{\infty} \frac{1}{(n!)^3}$
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An identity for the infinite sum

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^3}$$

Hassan Jolany

Abstract. In this short note, we give an identity for the alpha function

$$\alpha(x, s) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^s},$$

where $s \in \mathbb{N}$, $x \in \mathbb{R}$, in the case $s = 3$.

1. INTRODUCTION We know that when $s = 1$, then the alpha function is the exponential function $\alpha(x, 1) = e^x$. The question is that what about when $s \geq 2$?. We have the following nice differential equation for the alpha function [2]. In fact, it is known that the alpha function is the solution of the following differential equation.

$$\sum_{k=1}^s \sigma_s^k x^{k-1} y^{(k)} - y = 0$$

where σ_s^k is the Stirling numbers of the second kind which satisfies in the following generating function,

$$\frac{1}{k!} (e^x - 1)^k = \sum_{s=k}^{\infty} \sigma_s^k \frac{x^n}{n!}$$

Hence by using the language of generalized hypergeometric function, we get [3],

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^3} = {}_0F_2(; 1, 1; 1) \approx 2.1297$$

To deal with this question, we first give the following theorem which is the direct consequence of Bessel-Parseval identity[1].

Theorem 1. *Let we have the two following entire series, with real the coefficients,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_n x^n$$

with respectively non-zero radius R and R' . Take

$$h(x) = \sum_{n=0}^{\infty} a_n b_n x^n$$

, then if $-R < u < R$ and $-R' < v < R'$, then we have

$$h(uv) = \frac{1}{2\pi} \int_0^{2\pi} f(ue^{it})g(ve^{-it})dt$$

In the Theorem 1, if we take $f(x) = g(x) = e^x$, and $u = x, v = 1$, then we get the following amazing identity,

$$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{(x+1)\cos t} \cos((x-1)\sin t)dt$$

Hence by taking $x = 1$ we get

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{2\cos t} dt$$

But we know,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a \cos x + b \sin x)} dx = I_0(\sqrt{a^2 + b^2}), \quad (1)$$

where I_0 is the modified Bessel function of the first kind.

2. THE MAIN RESULT In this section we give an integral identity for the alpha function when $s = 3$. We take $f(x) = e^x, g(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{(x+1)\cos t} \cos((x-1)\sin t)dt$, and $u = x, v = 1$. Hence, from Theorem 1, we have

$$A = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^3} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{xe^{is}} e^{(e^{-is}+1)\cos t} \cos((e^{-is}-1)\sin t) ds dt$$

But, we have

$$e^{xe^{is}} e^{(e^{-is}+1)\cos t} = (e^{x \cos s + \cos s \cos t + \cos t}) (\cos(x \sin s - \sin s \cos t) + i \sin(x \sin s - \sin s \cos t))$$

and also,

$$\cos((e^{-is}-1)\sin t) = \cos(\cos s \sin t - \sin t) \cosh(\sin s \sin t) + i \sin(\cos s \sin t - \sin t) \sinh(\sin s \sin t)$$

Hence

$$\begin{aligned}
A &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left((e^{x \cos s + \cos s \cos t + \cos t}) (\cos(x \sin s - \sin s \cos t)) \right) \cos(\cos s \sin t - \sin t) \cosh(\sin s \sin t) ds dt \\
&\quad - \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left(e^{x \cos s + \cos s \cos t + \cos t} \right) \sin(x \sin s - \sin s \cos t) \sin(\cos s \sin t - \sin t) \sinh(\sin s \sin t) ds dt
\end{aligned}$$

and this gives an identity for the alpha function in the case $s = 3$.

REFERENCES

1. Knopp, Konrad, *Theory and Application of Infinite Series*. Dover Publications.(1990)
2. Gerard Eguether, *Nombres de Stirling et Nombres de Bell*.
<http://www.iecl.univ-lorraine.fr/~Gerard.Eguether/zARTICLE/BF.pdf>
3. Wolfram Alpha,
<http://www.wolframalpha.com/>