Fault-Tolerant Metric Dimension of Wheel related Graphs

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Fault-Tolerant Metric Dimension of Gear, Anti-Web Gear and Anti-web Graphs

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Abstract. Concept of resolving set and metric basis has enjoyed a lot of success because of multi-purpose applications both in computer and mathematical sciences. A system in which failure of any single unit, another chain of units not containing the faulty unit can replace the originally used chain is called fault-tolerant self-stable system. Recent research reveal that the problem of finding metric dimension is NP-hard and problem of computing the exact values of fault tolerant metric dimension seems to be even harder although some bounds can be computed rather easily. In the present article we compute closed formulas for the fault-tolerant metric dimension of gear, anti-web gear and anti-web graphs. We conclude that out of these only anti-web graph has constant fault-tolerant metric dimension.

1. Introduction

Computer networks are graphs with vertices representing hosts, servers or hubs and edges as connecting medium between them. Vertex is actually a possible location to find fault or some damaged devices in a computer network. This idea somehow created urge in Slater and independently in Harary and Meletr in [7] to uniquely recognize each vertex of a graph in a network so that fault could be controlled in an efficient way. Thus basis for notion of locating sets and locating number of graphs came into existence. Since then, the resolving sets have been investigated a lot [1-5]. The resolving set contributes in various areas such as network discovery [9-10], connected joins in graphs, strategies for the mastermind games [23], applications of pattern recognition, combinatorial optimization, image processing[11], pharmaceutical chemistry and game theory [15]. A moving point in a graph may be located by finding the distance from the point to the collection of sonar stations which have been properly positioned in the graph see [10]. Thus finding a minimal sufficiently large set of labeled vertices such that robot can find its position is a problem known as robot navigation, already well studied in [9]. This sufficiently large set of labeled vertices is a resolving set of the graph space and the corresponding cardinality is the metric dimension. Similarly on another node, a real world problem is the study of networks whose structure has not been imposed by a central authority but arisen from local and distributed processes. It is very difficult and costly to obtain a map of all nodes and the links between them. A commonly used technique is to obtain local view of network from various locations and combine them to obtain a good approximation for the real network. Metric dimension also has some applications in this respect as well.

Consider a simple, connected graph G, and metric $d_G : V(G) \times V(G) \rightarrow \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers and $d_G(x, y)$ is the minimum number of edges in any path between x and y. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|W)$ of v with respect to W is the $k-$tuple $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If distinct vertices of G have distinct representation with respect to W, then W is called a resolving set of G, see [1]. Such a resolving set with minimum cardinality is a basis of G and metric dimension of G, denoted by $\beta(G)$ is its cardinality.

Buczkowski et. al. established metric dimension of wheel W_n to be $\lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$ [1], Caceres et. al. [3] found that the metric dimension of fan to be $\lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$ and Tomescu et. al. [16] determined dimension of Jahangir grphs J_{2n} to be $\lfloor \frac{2n}{3} \rfloor$ for all $n \geq 4$.

A particular metric-feature of the family of graph is independence of metric dimension on the particular element of the family. A connected graph has constant metric dimension if $\beta(G) = k$ where $k \in \mathbf{Z}^+$ is fixed. This feature has been presented in [26]. In [28], authors computed metric dimension of flower graph and some families of convex polytopes. In [1], Chartrand et al. proved that a graph has constant metric dimension 1 iff it is a path. Kasif et. al. computed partial results of metric dimension of Mobius ladder in [12] whereas Munir et.al. computed exact and complete results for metric dimension of Mobius Ladders in [22]. C. Poisson et.al. computed metric dimension of unicyclic graphs in [13].

Recent development in this context has paved way for a new related concept known as fault-tolerance in metric dimension. Suppose that, in a network, n processing units are interlinked, and of these units, forming a chain of maximal length, are used to solve some task. To have a fault-tolerant self-stable system, it is necessary that in case of failure of any single unit, another chain of units not containing the faulty unit can replace the originally used chain. So a fault-tolerant design enables a system to continue its intended operation, possibly at a reduced level, rather than failing completely. One uses graphs to represent the units and the links, see [2]. A resolving set $\acute{S}$ is considered as fault-tolerant if $\acute{S} \setminus \{v\}$ is also a resolving set, for each $v \in \acute{S}$, and the fault-tolerant metric dimension, $\beta'(G)$, is the minimum cardinality of such $\acute{S}$. A family $\mathcal{G}$ of connected graphs is said to have constant fault-tolerant metric dimension if it is independent of any choice of member of that family. Fault-tolerant designs are widely used in engineering and computer sciences [4]. Slater in [2] introduced the study of fault-tolerant locating-dominating sets. Hernando et.al. introduced the idea of a single fault-tolerant metric dimension in [30]. They discussed single fault-tolerant metric dimension of trees. They also proved that fault-tolerant metric dimension is bounded by a function of the metric dimension irrespective of the choice of graph given be $\beta(\acute{G}) \leq \beta(G)(1 + 2.5^{\beta(G)-1})$. Javed et. al. discussed fault-tolerance in resolvability [19] and computed fault tolerant metric dimension of some graphs [18]. It is easy to gather that, $\beta(\acute{G}) \geq \beta(G) + 1$, [18]. Shabbir et. al. discussed fault-tolerance in triangular lattice networks [5].

In the present article we discuss fault-tolerance metric dimension of some families of convex polytopes. A convex Polytope is the convex-hull of finite set in any Eucleadian space [29]. These are important families of graphs and have been consistently under serious discussion. For instance, Baca in [24] computed labelings of two important families of convex polytopes , and discussed magic labeling of convex polytopes in [25]. Nazeer et. al. computed the closed formulae for the center, eccentricity, periphery and average eccentricity for the convex polytopes in [21]. In [28], Imran, et. al. proved that the convex polytopes S_n and T_n have constant metric dimension and in [26] they proved that Q_n and D_n also have constant metric dimension. Recently in [31], authors computed bounds for the fault tolerant metric dimension of six families of convex polytopes. In this article we compute fault-tolerant metric dimension of gear graph and anti-web gear graphs. It is important to point out that we explicitly derive closed formulas and not the bounds as has been done in [31] for the case of convex polytopes.

2. Main Results

In this part we give our main results. At first we compute fault-tolerant metric dimension of gear graphs and anti-web gear graphs.

Definition 2.1. Gap in resolving set

If B' is a resolving set of a graph G which contains two or more vertices of G and we suppose that vertices of $B' = \{v_{j_1}, v_{j_2}, ..., v_{j_r}\}$ so that $j_1 < j_2 < ... < j_r$. We shall say that the vertices v_{j_a}, $v_{j_{a+1}}$ for $1 \leq a \leq r-1$ and v_{i_r}, v_{i_1} are neighboring vertices. Now the vertices lying between these neighboring vertices are referred to as gap of B' and the number of vertices in a gap will be referred to as size of gap. If $\mid B' \mid =$ t: then it means that we have t gaps, some of which may be empty.

2.1. Fault-tolerant metric dimension of Gear graph

Definition 2.2. The gear graph G_n is defined as follows: consider an even cycle $C_{2n} : v_1, v_2, \cdots, v_{2n}, v_1$, where $n \geq 2$ and a new vertex v_0 adjacent to

n vertices of C_{2n} : v_1, v_3, $\cdots$, v_{2n-1}. G_n has order $2n+1$ and size $3n$ and can be obtained from the wheel W_{2n} by alternately deleting n spokes.

The Following figure 1 is an instance of gear graph.

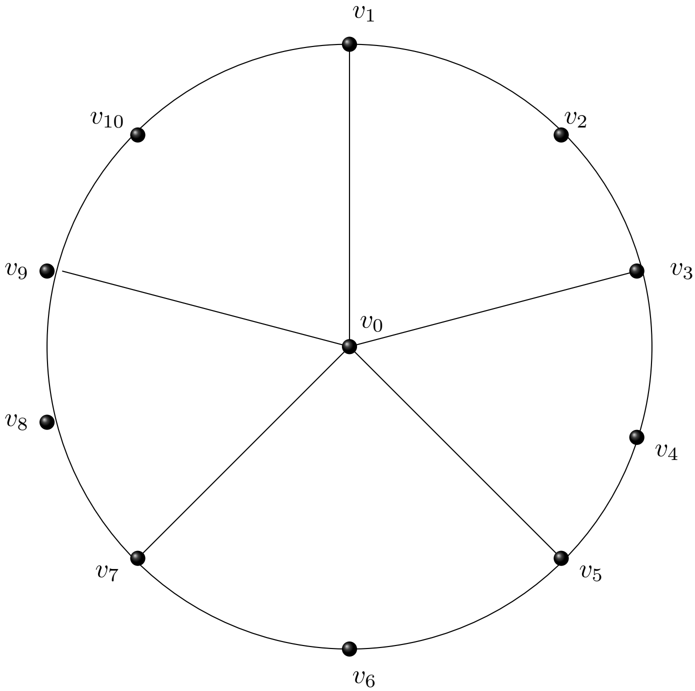

Figure 1: A graph of (G_5) gear graph.

We referred vertices of G_n having degree 3 with odd numbering and degree 2 with even numbering. In this part, we want to compute fault-tolerant metric dimension of gear graph G_n. In [16] the gear graph was denoted by J_{2n} and it was proved that $\beta(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ for $n \geq 4$. Now suppose that B' be any fault-tolerant metric basis of G_n. We make following observations.

Lemma 2.3. *If there is no empty gap then every gap of B' (fault-tolerant metric basis) contains exactly one vertex.*

Proof. Suppose on contrary, there exists a gap which contains two vertices. Then there exists vertices $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_j\}$ (where $v_j$is neighboring vertex of v_{i+3} and one of the neighboring gaps of v_j contain one vertex), such that v_i, v_{i+3}, v_j, $\in B'$. Then by $B' \setminus \{v_j\}$ we have $r(v_{i+2}/B') = r(v_{i+4}/B')$ $= (2, 1, 2, ..., 2)$ which is a contradiction. □

Lemma 2.4. *If two neighboring gaps of B' (fault-tolerant metric basis) are empty then each of their neighboring gaps contain two vertices.*

Proof. Suppose on contrary, two neighboring gaps of B' (fault-tolerant metric basis) are empty and at least one of their neighboring gaps contain one vertex. Then there exists vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7}$, such that $v_i, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+7} \in B'$. Then by $B' \setminus \{v_{i+5}\}$ we have $r(v_{i+6}/B') = r(v_{i+8}/B') = (2, 1, 2, ..., 2)$ which is a contradiction. □

Lemma 2.5. *The resolving set of G_n of the type $B' = \{v_1, v_3, v_5, ...v_{n-1}\}$ or $B' = \{v_1, v_4, v_5, v_6, v_9, v_{11}, v_{13}, ...v_{n-1}\}$ is fault-tolerant metric basis.*

Proof. We see from lemma 2.3 and 2.4 that B' can be a fault-tolerant resolving set because for each v in B', the set $B' \setminus \{v\}$ is a resolving set. Now we see that the set $B' \setminus \{v_i, v_j\}$ where $v_i, v_j \in B'$ is not a resolving set. Hence B' is a fault-tolerant metric basis. After the above observations we reach at the following main result of this part.

□

Theorem 2.6. *If $n \geq 4$, then we have $\beta'(G_n) = n$.*

Proof. In [16] it was proved that the central vertex v_0 of G_n is not included in B'. For a gear graph, denoted by (G_n), the fault-tolerant metric dimension for $n = 3$ is 4 and $B' = \{v_1, v_2, v_3, v_4\}$. Now we may write $n = \frac{k}{2}$, where $k \geq 8$ and k is even, then $\beta'(G_n) = n = \frac{k}{2}$. since $B' = \{V_{2i}, 1 \leq i \leq 2n\}$ is a fault-tolerant resolving set as it satisfies lemma 2.3.
It follows from above discussion that $\beta'(G_n) \leq n$.
Now we prove that $\beta'(G_n) \geq n$. Let B' be a fault-tolerant metric basis of G_n, so we have two cases
case 1: when every gap contains one vertex (lemma 2.3). Now if $\mid B' \mid = m$. Then it means that there are m vertices in fault-tolerant metric basis and hence there exists m gaps all of them contain one vertex. So the number of vertices belonging to the gaps of B' will be at most m. Hence $2n - m \leq m$. Therefore $\mid B' \mid = m \geq n$.
case 2: When two neighboring gaps are empty and each of their neighboring gaps contain two vertices (lemma 2.4). Now if $\mid B' \mid = m$. Then it means that there are m vertices in fault-tolerant metric basis and hence there exists m gaps and the number of vertices belonging to the gaps of B' will be at most m. Hence $2n - m \leq m$. Therefore $\mid B' \mid = m \geq n$. □

2.2. Fault-tolerant metric dimension of Anti web graph

Definition 2.7. The Anti web graph AW_n is defined as follows: consider an even cycle $C_{2n} : v_1, v_2, ... , v_{2n}, v_1$, where $n \geq 4$ and is even. Edges are obtained by joining the vertices of cycle and by v_i, v_{i+2} where $1 \leq i \leq n$. AW_n has order n and size $2n$.

The Following figure 2 is an instance of anti-web graph.

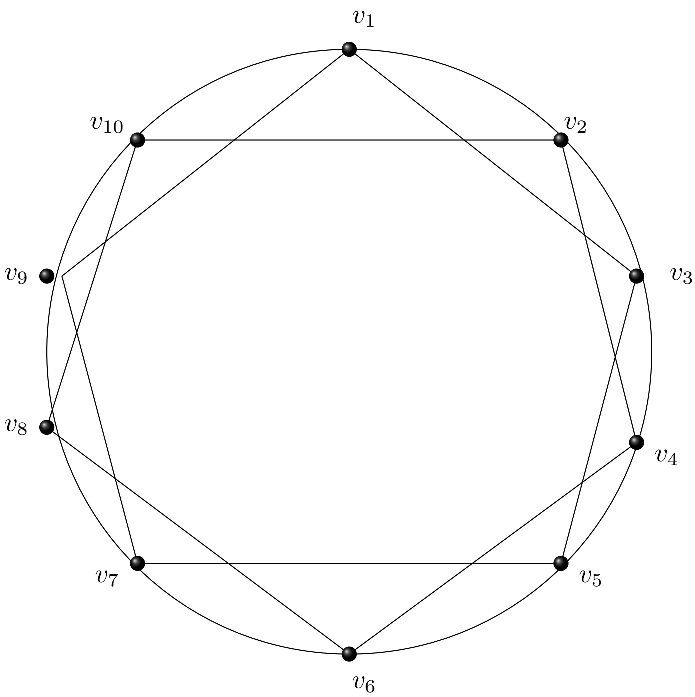

Figure 2: A graph of (AW_{10}) anti web graph.

In this section, we want to compute fault-tolerant metric dimension of anti web AW_n. Now suppose that B' be any fault-tolerant metric basis of AW_n. We observe the following key points

Lemma 2.8. *The resolving set of AW_n contain at least three vertices.*

Proof. One can easily check the following.

- Suppose on contrary, there exists a resolving set which contain one vertex v_i in AW_n, then there exists vertices v_j, v_k belong to AW_n such that we have $r(v_j/B') = r(v_k/B') = (1)$ or (2) or.....$(\lfloor \frac{n+2}{4} \rfloor)$ which is a contradiction.
- Now again suppose on contrary, there exists a resolving set which contain two vertices v_i and v_j in AW_n, we have $r(v_k/B') = r(v_r/B') = (1, 1)$ or $(2, 2)$ or.....$(\lfloor \frac{n+2}{4} \rfloor)$ which is a contradiction.

□

Theorem 2.9. *If $n \geq 4$, then we have $\beta'(AW_n) = 4$.*

Proof. Consider the resolving set of AW_n of the type $B' = \{v_2, v_4, \} \cup \{v_{i+2}, v_{i+4}, i = n$ (if n is even) or n-1 (if n is odd)$\}$. We see that B' is a fault-tolerant resolving set because for each v in B', the set $B' \setminus \{v\}$ is a resolving set. Now we see that the set $B' \setminus \{v_2, v_4\}$ or $B' \setminus \{v_{i+2}, v_{i+4}\}$ is not a resolving set, which is a contradiction. Now by lemma 2.8 any resolving set of AW_n contain at least three vertices. Hence B' is a fault-tolerant metric basis and its cardinality is 4. Hence $\beta'(AW_n) = 4$. $\square$

2.3. Fault-tolerant metric dimension of Anti web-gear graph

Definition 2.10. A anti web-gear graph denoted by AWG_n can be obtained as join of an anti web AW_n and a gear graph denoted by G_n. We have $V(AWG_n) = V(G_n)$ and $E(AWG_n) = E(G_n) \cup \{v_i v_{i+2} : 0 \leq i \leq n\}$, where the indices are taken modulo n. AWG_n has order $2n + 1$ and size $3n$.

The Following figure 1 is an instance of anti-web gear graph.

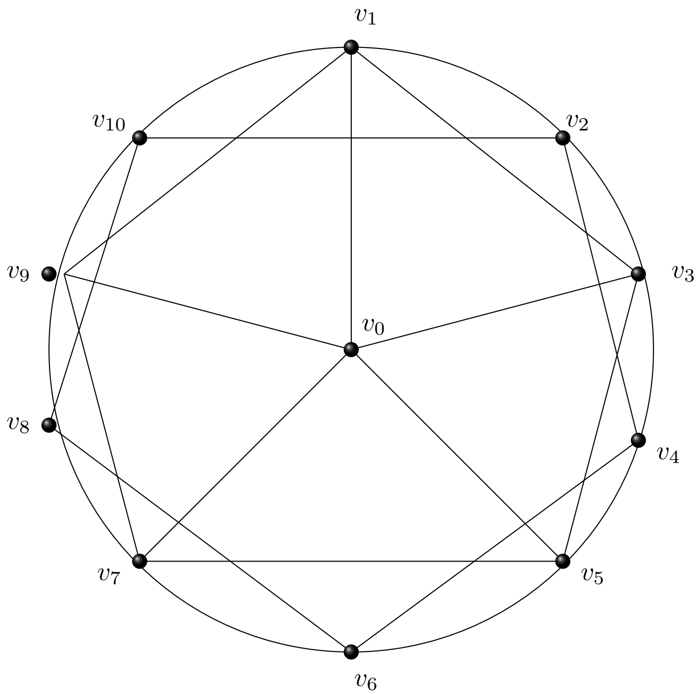

Figure 3: A graph of (AWG_5) anti web-gear graph.

we referred vertices of AWG_n having degree 5 with odd numbering and degree 4 with even numbering. So we have four types of gaps 5-5, 4-4, 5-4 and 4-5. In this section, we computed the fault-tolerant metric dimension of anti web-gear graphs. In [32] anti web-gear graph was denoted by AWJ_{2n} and it was proved that $\beta(AWJ_{2n}) = \lceil \frac{n+1}{3} \rceil$ for $n \geq 15$. Now suppose that B' be any fault-tolerant metric basis of G_n. We observe that if AWG_n for $n \geq 3$, then the central vertex v_0 does not belong to any fault-tolerant basis. Since $diam(AWG_n) = 4$, if v_0 belongs to any fault-tolerant metric basis, say B', then there must exist two distinct vertices v_i and v_j, for $1 \leq i \neq j \leq n$ such that $r(v_i/B') = r(v_j/B')$. Consequently, the fault-tolerant basis vertices belong to the rim vertices of AWG_n only. We also obtain the following

Lemma 2.11. *If B' is a fault-tolerant metric basis of AWG_n and $n \geq 8$, then every gap of B' contain three vertices and is of the type 5-5 or 4-4, except one gap which contains one vertex when n is odd.*

Proof.
- Suppose on contrary, there exists a gap of the type 5-4. Then there exists vertices v_i, v_{i+1}, v_{i+2}, $v_{i+3}, v_{i+4}, v_{i+5}, v_j$ (we take gap between v_{i+5} and v_j at least 3 in order to keep cardinality minimum) where $v_i, v_{i+5}, v_j \in B'$. Now $r(v_{i+4}/B')$ $= r(v_{i+6}/B')$ by $B' \setminus \{v_j\}$ which is a contradiction.
- Suppose on contrary, there exists a gap of the type 4-5. Then there exists vertices $v_j, v_i, v_{i+1}, v_{i+2}, v_{i+3}$, v_{i+4}, v_{i+5} (we take gap between v_{i+5} and v_j at least 3 in order to keep cardinality minimum) where $v_i, v_{i+5}, v_j \in B'$. Now $r(v_{i-1}/B') = r(v_{i+1}/B')$ by $B' \setminus \{v_j\}$ which is a contradiction.

□

Lemma 2.12. *If n is even, then resolving set of AWG_n of the type $B' = \{v_{4i+1}, 0 \leq i \leq \frac{n-2}{2}\}$ or $B' = \{V_{4i+2}, 0 \leq i \leq \frac{n-2}{2}\}$ is fault-tolerant metric basis.*

Proof. We see from lemma 2.11 that B' can be a fault-tolerant resolving set because for each v in B', the set $B' \setminus \{v\}$ is a resolving set. Now we see that the set $B' \setminus \{v_i, v_j\}$ where $v_i, v_j \in B'$ is not a resolving set. Hence B' is a fault-tolerant metric basis.

□

Theorem 2.13. *If $n \geq 8$, then we have $\beta'(AWG_n) = \lceil \frac{n}{2} \rceil$.*

Proof. For an Anti web-gear graph, denoted by (AWG_n),the fault-tolerant metric dimension for $2 \leq n < 8$ are $\beta'(AWG_n) = 4$.
case 1: when n is even, then we may write $2n = k$, (where $n \geq 8$ and is even) then $\beta'(AWG_n) = \frac{n}{2} = \frac{k}{4}$. since $B' = \{v_{4i+1}, 0 \leq i \leq \frac{n-2}{2}\}$ or $B' = \{V_{4i+2}, 0 \leq i \leq \frac{n-2}{2}\}$. It is a fault-tolerant resolving set as it satisfies lemma 2.11 and 2.12.
case 2: when n is odd, then we may write $2n = k$, (where $n \geq 8$ and is odd) and $\beta'(AWG_n) = \lceil \frac{n}{2} \rceil = \lceil \frac{k}{4} \rceil$. since $B' = \{v_1, v_3, v_{4i+3}, 1 \leq i \leq \frac{n-3}{2}\}$ or $B' = \{v_2, v_4, v_{4i}, 2 \leq i \leq \frac{n-1}{2}\}$. It is a fault-tolerant resolving set as it satisfies lemma 2.11 and 2.12.
It follows from above discussion that $\beta'(AWG_n) \leq \lceil \frac{n}{2} \rceil$.
Now we prove that $\beta'(AWG_n) \geq \lceil \frac{n}{2} \rceil$. Let B' be a fault-tolerant metric basis of AWG_n.
If $\mid B' \mid = m$. Then it means that there are m vertices in fault-tolerant metric basis and hence there exists m gaps all of them contain three vertices except one gap which possibly contain one vertex. So the number of vertices belonging to the gaps of B' will be at most $3m$. Hence $2n - m \leq 3m$. Therefore $\mid B' \mid = m \geq \lceil \frac{n}{2} \rceil$.

□

3. Conclusions

In this article we computed fault-tolerant metric dimension of graphs of three families of wheel related graphs. We conclude that anti-web graph has constant fault-tolerant metric dimension. If we keep all the notations of the second section intact, then we arrive at the main main results

Theorem 3.1. *If* $n \geq 4$, *then we have* $\beta'(G_n) = n$.

Theorem 3.2. *If* $n \geq 4$, *then we have* $\beta'(AW_n) = 4$.

Theorem 3.3. *If* $n \geq 8$, *then we have* $\beta'(AWG_n) = \lceil \frac{n}{2} \rceil$.

Author Contributions

Formal analysis is done by Mobeen Munir, investigation is done by Zafar hussain, Imtiaz Ali. Manuscript writing is carried out by Imtiaz Ali and Ashfaq Ahmed and Jia Bao Liu.

Data Availability Statement

Data sharing is not applicable to this article as no data-set was generated or analyzed during the current study.

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Conflict of Interests

"The authors declare no conflict of interest."

References

[1] P. S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, *On k-dimensional graphs and their bases*, Pariodica Math. Hung, 46(1)(2003), 9-15.

[2] Slater, P.J., *Fault-Tolerant Locating-Dominating Sets*, Discrete Math., 249 (2002), pp. 179-189.

[3] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, *On the metric dimension of some families of graphs*, Electronic Notes in Disc. Math., 22(2005), 129-133.

[4] J. P. Hayes: *A graph model for Fault-tolerant computer systems*, IEEE Trans. Comput., 34 (1985).

[5] A. Shabbir, T.Zumferertscu, *Fault-tolerant designs in triangular lattice networks*, Appl. Anal. Discrete Math. 10 (2016), 447456.

[6] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, *Resovability in graphs and metric dimension of a graph*, Disc. Appl. Math., 105(2000), 99-113.

[7] F. Harary, R. A. Melter, *On the metric dimension of a graph*, ARS Combinatoria, 2(1976), 191-195.

[8] I. Javaid, M. T. Rahim, K. Ali, *Families of regular graphs with constant metric dimension*, Utilitas Math., 75(2008), 21-33.

[9] S. Khuller, B. Raghavachari, A. Rosenfeld, *Location in graphs*, Technical Report CS-TR-3326, University of Maryland at Colleg Park, 1994.

[10] S. Khuller, B. Raghavachari and A. Rosenfeld, *Landmarks in graphs*, Disc. Appl. Math. 70 (1996), 217-229.

[11] R. A. Melter, I. Tomescu, *Metric bases in digital geometry*, Computer Vision, Graphics, and Image Processing, 25(1984), 113-121.

[12] Murtaza Ali, Gohar Ali, Muhammad Imran, A. Q. Baig, and Muhammad Kashif Shafiq, *On the metric dimension of Möbius ladders*, ARS Combinatoria, 105(2012), 403-410.

[13] C. Poisson and P. Zhang, *The metric dimension of unicyclic graphs*, J. Comb. Math Comb. Comput. 40 (2002), 17-32.

[14] P. J. Slater, *Leaves of trees*, Congress. Number., 14(1975), 549-559.

[15] P. J. Slater, *Dominating and reference sets in graphs*, J. Math. Phys., 22(1998), 445-455.

[16] I. Tomescu, I. Javaid, *On the metric dimension of the Jahangir graph*, Bull. Math. Soc. Math. Roumanie, 50(98), 4(2007), 371-376.

[17] I. Tomescu, M. Imran, *On metric and partition dimensions of some infinite regular graphs*, Bull. Math. Soc. Math. Roumanie, 52(100)(2009), 461-472.

[18] M. A. Chaudhry, I. Javaid and M. Salman, *Fault-Tolerant metric and partition dimension of graphs*, Util. Math. 83(2010) 187-199.

[19] I. Javaid, M. Salman, M. A. Chaudhry and S. Shokat,*Fault-Tolerance in Resolvability*, Util. Math. 80(2009) 263-275.

[20] G.Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, *Resolvibility in graphs and the metric dimension of a grap*, Disc. Appl. Math., 105(2000), 99-133.

[21] W. Nazeer, S. M. Kang, S. Nazeer, M. Munir, I. Kousar, A. Sehar and Y. C. Kwun, *On Center, Periphery and Average Eccentricity for the Convex Polytopes*, Symmetry, 8, (2016), 145; doi:10.3390/sym8120145

[22] M. Munir, A. R. Nizami and H. saeed, On the metric dimension of Möbius Ladder, 135 (2017), 249-256, ARS Combinatoria.

[23] Vasek Chvatal. Mastermind. Combinatorica, 3(3-4):325-329 1983.

[24] M. Baca,*Labellings of two classes of convex polytopes*, Utilitas Mathematica, 34, (1988), 24-31.

[25] M. Baca,*On magic Labellings of convex polytopes*, Annals Disc. Math, 88, (2012), 43-57.

[26] M. Imran, A.Q. Baig, A. Ahmed, *Families of plane Grapghs with constant metric dimension*, Utilitas Mathematica, 88, (2012), 43-57.

[27] I. Tomescu and M. Imran, *On metric and partition dimensions of some infinite regular graphs*, Bull. Math. Soc. Sci. Math. Roumanie(100),(2009) 461-472.

[28] M. Imran, F. Bashir, A.Q. Baig, S. Bokhary, A. Riasat *On metric dimension of flower graphs $f_{n,m}$, and convex polytopes*, Utilitas Mathematica, 92,(2013), 389-409.

[29] B. Grnbaum, G. M. Ziegler, *Convex Polytopes*, Springer, GTM, 978-0387404097, 2003.

[30] C. Hernando, M. Mora, P. Slater, D. Wood *Fault-tolerant metric dimension of graphs*, Convexity in discrete structures 5, (2008), 81-85

[31] H. Raza, S. Hayat, X. Pan, *On the fault-tolerant metric dimension of convex polytopes*, Applied Mathematics and Computation 339, (2018) 172-185.

[32] H. M. Afzal, J. Imran, *Computing the metric dimension of Wheel related graph*, applied mathematics and computation, 242 (2014) 624-632.

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