Compressed sensing applied to modeshapes reconstruction

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ABSTRACT

Modal analysis classicaly used signals that respect the *Shannon/Nyquist* theory. Compressive sampling (or Compressed Sampling, CS) is a recent development in digital signal processing that offers the potential of high resolution capture of physical signals from relatively few measurements, typically well below the number expected from the requirements of the *Shannon/Nyquist* sampling theorem. This technique combines two key ideas: sparse representation through an informed choice of linear basis for the class of signals under study; and incoherent (eg. pseudorandom) measurements of the signal to extract the maximum amount of information from the signal using a minimum amount of measurements. We propose one classical demonstration of CS in modal identification of a multi-harmonic impulse response function. Then one original application in modeshape reconstruction of a plate under vibration. Comparing classical ℓ_2 inversion and ℓ_1 optimization to recover sparse spatial data randomly localized sensors on the plate demonstrates the superiority of ℓ_1 reconstruction (RMSE).

1. Introduction

Compared to nondestructive testing (ultrasound techniques, analysis of magnetic fields, radiology, thermal methods) vibration-based analysis allows a mixed global/local analysis of the structure with the potential of being applied in situ (laser vibrometer, optical sensor, etc) [1,2]. Deciding on an optimal sensor placement and optimal frequency sampling is a common problem encountered in many engineering applications and is a critical issue in the construction and implementation of an effective Structural Health Monitoring system (SHM). As a first example we study the modeshape reconstruction from grid placement. It highlights the fact that mode shapes visualization is often biased due to spatial aliasing. On figure 1, we can see that 9 grid point measurement are not enough precise to reconstruct the (3,1) mode shape.

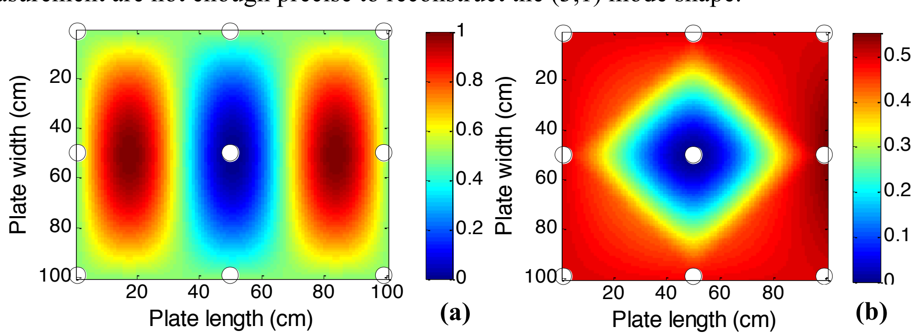


Fig. 1. : (3,1) mode of vibrating plate plus regular grid distribution of sensors in white circles (a) and The cubic interpolation which shows a spatial aliasing in mode shape reconstruction (b). A regular grid of 9 sensors permits only to reconstruct the (1,1).

In a first paper [3] we use Monte Carlo approach to demonstrate the ability of Kriging method at high spatial density to reconstruct mode shape with accuracy using regular or random grid. In this paper we are not analyzing a problem of Sensor Placement Optimization (SPO), which aims at identifying the sensor layout that will optimize one or more of the probabilistic performance measures. A detailed literature about SPO can be found in [2,4]. We prefer to have a "signal processing" approach. According to well known theorem sampling theorem, Stubbs and Park [5] introduced this theorem for spatial data for avoiding well know problem called "aliasing". Schulz et al. [6] address the issue of damage resolution as a function of spatial distribution of sensors. They show that damage can be located within a spatial resolution equal to the distance between sensors on a structure. Sazonov and Klinkhachorn have developed an optimal sampling theory [7]. This approach allows us to estimate high resolution modeshapes taking into account experimental noise and enables to evaluate even small damages. This method use the curvature mode shape properties to find relationship between optimal sampling data and Signal to Noise Ratio and have been adapted for wavelets approach by Morlier et al [8].
The first works combining acoustic measurement and parcimonious approach were presented last year at the French Congress of Acoustics [9,10]. The advanced mathematical techniques so-called Compressive Sensing (CS) benefit fields as diverse as sensors, signal processing, image compression etc ... CS is used to find some kind of underlying structure behind most of the analog signals on the condition that these signals are sparse [11,12,13]. It is then possible to acquire signal at lower sampling frequency, and therefore no longer verify the *Shannon/Nyquist* frequency.

2. Compressive sensing: learning by numerical examples

The least-squares solution to such problems is to minimize the ℓ_0 norm—that is, minimize the amount of energy in the system. This is usually simple mathematically (involving only a matrix multiplication by the pseudo-inverse of the basis sampled in). However, this leads to poor results for many practical applications, for which the unknown coefficients have nonzero energy. To enforce the sparsity constraint when solving for the underdetermined system of linear equations, one can minimize the number of nonzero components of the solution. The function counting the number of non-zero components of a vector was called the ℓ_0 norm by David Donoho. Candès. et. al. [11,12], proved that for many problems it is probable that the ℓ_1 norm is equivalent to the ℓ_0 norm, in a technical sense: This equivalence result allows one to solve the ℓ_1 problem, which is easier than the ℓ_0 problem. Finding the candidate with the smallest ℓ_1 norm can be expressed relatively easily as a linear program, for which efficient solution methods already exist [14].

The traditional approach to data acquisition is based on the *Shannon-Nyquist* theorem: to acquire a signal with a bandwidth of size W must be sampled at a higher frequency $2W$. The compressive sensing exploits the fact that many real signals can be expressed in a sparse way and the inconsistency between certain types of bases to reduce this number of samples. A vector S-sparse is a vector that has at most S nonzero components. Many natural signals, when expressed in a particular base, have a representation with many significant coefficients. Data compression exploits this fact by removing these low coefficients, which slightly reduces the signal quality. These real-world signals (e.g. sound, images, video) can be viewed as an n-dimensional vector. To acquire this signal, we consider a linear measurement model, in which we measure an m-dimensional vector $b = Ax \in R^m$ for some m × n measurement matrix A (thus we measure the inner products of x with the rows of A). For instance, if we are measuring a time series in the frequency domain, A would be some sort of *Fourier* matrix.

This leads to the following classical question in linear algebra :
How many measurements *m* do we need to make in order to recover the original signal *x* exactly from *b*?

The classical theory of linear algebra is as follows:

- If there are at least as many measurements as unknowns ($m \geq n$), and A has full rank, then the problem is determined or overdetermined, and one can easily solve $Ax = b$ uniquely (e.g. by gaussian elimination).
- If there are fewer measurements than unknowns ($m < n$), then the problem is underdetermined even when A has full rank. Knowledge of $Ax = b$ restricts x to an (affine) subspace of R^n, but does not determine x completely. However, if one has reason to believe that x is "small", one can use the least squares solution : $x = \mathrm{argmin}_{x:Ax=b} \|x\|_{\ell_2} = A^*(AA^*)^{-1}b$ as the "best guess" for x.

Compressed sensing is advantageous whenever signals are sparse in a known basis. So the advantage is obvious when measurements (or simulations) are expensive and mathematical inversion is cheap. Such situations can arise in imaging (e.g. the "single-pixel camera"), Sensor networks, MRI Astronomy etc…and modal analysis.
In fact, the above proof also shows how to reconstruct an *S-sparse* signal $x \in R^n$ from the measurements $b = Ax$.
x is the unique sparsest solution to $Ax = b$. In other words,

$$x = \operatorname{argmin}_{x:Ax=b} \|x\|_{\ell_0} \tag{1}$$

where $\|x\|_{\ell_0} := \sum_{i=1}^{n} |x_i|^0 = \#\{1 \le i \le n : xi \ne 0\}$ is the sparsity of x.

Unfortunately, in contrast to the ℓ_2 minimisation problem (least-squares), ℓ_0 minimisation is computationally intractable (in fact, it is an NP-hard problem). In part, this is because ℓ_0 minimisation is not a convex optimisation problem. A simple, yet surprisingly effective, way to do so is ℓ_1 minimisation (or basis pursuit); thus, our guess x* for the problem $Ax = b$ is given by the formula :

$$x^* = \operatorname{argmin}_{x:Ax=b} \|x\|_{\ell_1} \tag{2}$$

This is a convex optimisation problem, However, the ℓ_1 norm is not differentiable and this prevents from using classical optimization algorithm from differentiable optimization (such as SQP,...). Nonetheless, this non-differentiable convex optimization problem can be transformed in a linear programming optimization problem

$$x^* = \operatorname{argmin}_{x,y \in \Re^n, y>0 : Ax=b} \sum_{i=1}^{n} y_i$$

and then can be solved fairly quickly by linear programming methods. Note that compressed sensing has equivalent formulations in the statistical field of regression. Best subset regression methods for instence seek amongst all the predictors the best set that explain the best the outcome to be predicted. This equivalent to ℓ_0 minimization. Several heuristics exist to solve it through for instance best forward and best backward subset selection. The same way ℓ_1 minimization has equivalent in regression, these are for instance lasso or LAR (Least Angle Regression) techniques. A valuable reference on these specific regression techniques is [15]. The main assumption behind these statistical techniques, which is precisely the same underlying assumption for compressed sensing, is known in statistics as bet on sparsity.

In a recent paper [16], the authors use CS on accelerometer signals of vibration of a bridge. The interest is obvious when engineers try to analyse vibrations during a year, using hundreds of sensors (MEMS and wireless network). We can note that the choice of the basis (wavelets, *Fourier*) is crucial with this type of approaches. In aeronautic we have the same interest in continuous monitoring of structures over a long period (flutter detection).

We try to illustrate a basic application of CS in the next example (using Matlab code ℓ_1-Magic of the University of Caltech [14]). Let's take the example of an analog signal (ie *Fs = 400Hz >> Nyquist* frequency) to 4 frequency components *30, 60, 100 and 130 Hz*. We will compare in figure 2 the reconstruction of the signal regularly sampled at *Fs=150 Hz* using linear reconstruction formula (Shannon) and compressive sensing with fewer points of observations (but random sample).

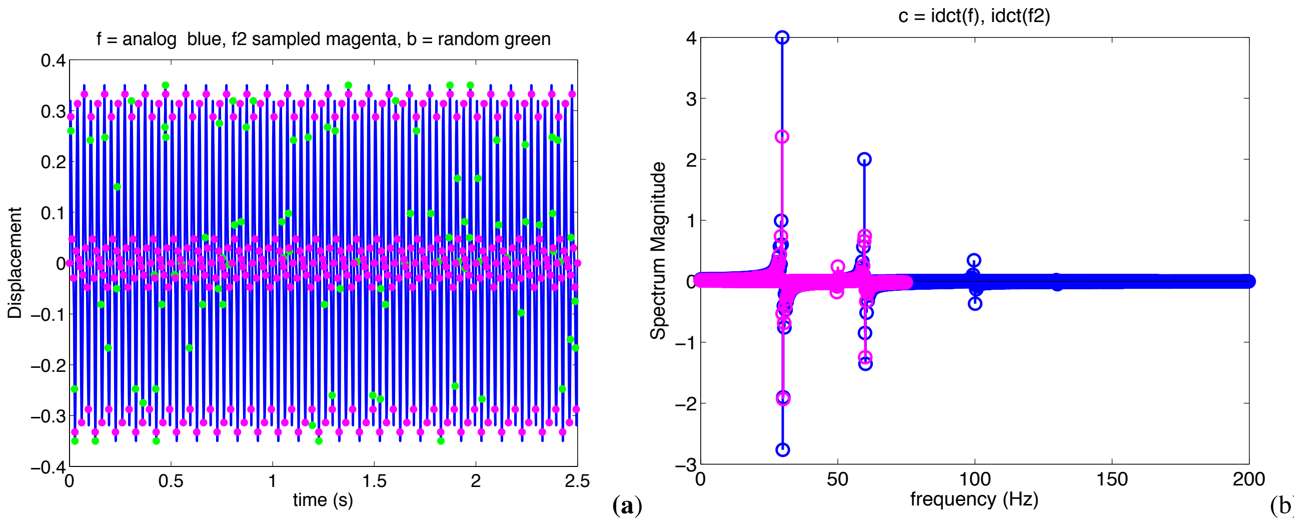


Fig. 2. Analog signal (in blue), discretized signal (magenta) respecting *Nyquist* frequency (N points) and randomized signals at low resolution (N/10) (a), and DCT spectrum comparison (b)

We see that the spectrum (DCT) has four resonances in the continuous signal, and 4 also in the digital signal but aliasing appears because *Fs* is too low: the time signal reconstruction will not be correct. When we solve this problem using Moore-Penrose pseudo inverse, we can note the appearance of noise in Figure 3b (whereas CS imposes zero coefficients).
It is then easy to compare the result of the spectrum reconstructed by ℓ_2 norm and the solution "magic" by using the ℓ_1 norm given in figure 3.

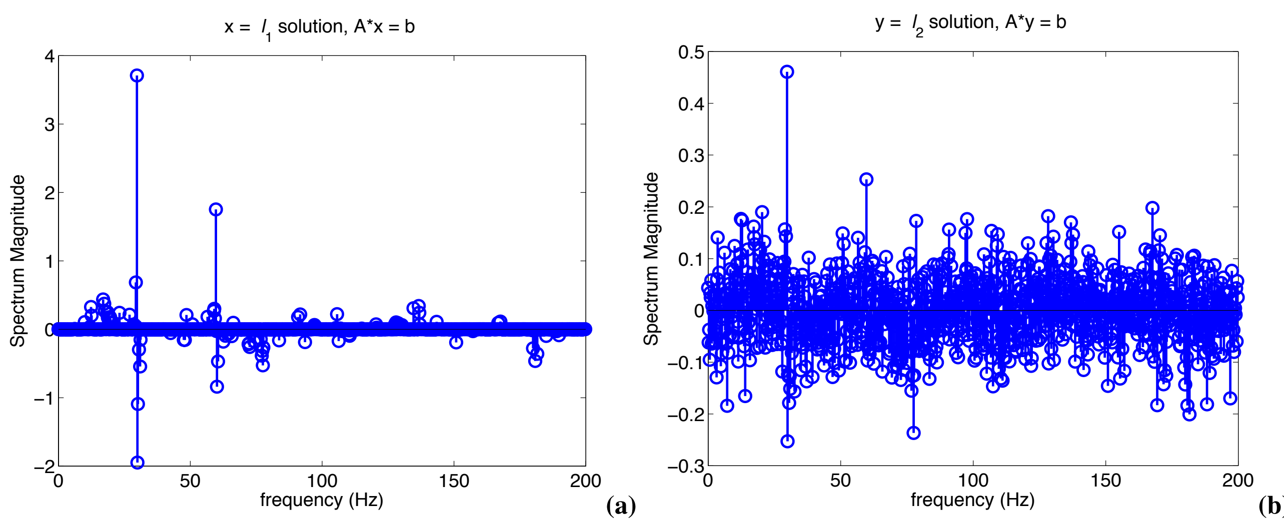


Fig. 3. Comparison of DCT spectrum of reconstructed signals by ℓ_1 inversion (a) and ℓ_2 inversion (b) of randomized signals. The ℓ_2 inversion is not capable of good reconstruction (noise)

We shall then compare the reconstruction on zoomed time signal (Figure 4) : N_{Random} varies from N/5 to N/10, where N is number of samples. We see that the ℓ_2 norm solutions and sampling the 'classic' is not correct with respect to the accuracy of ℓ_1 method.

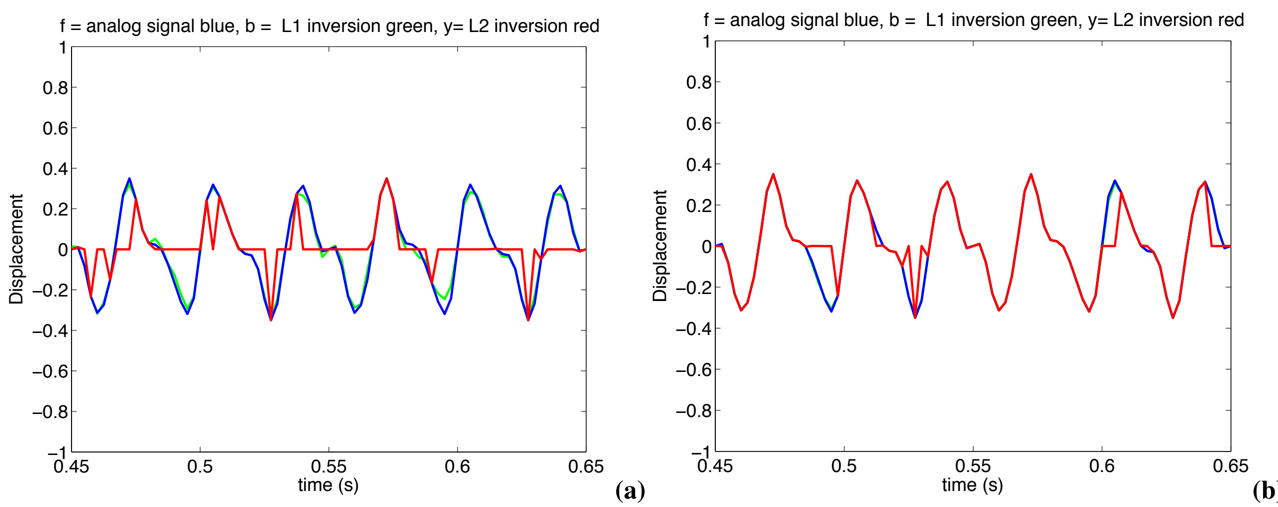


Fig. 4. Comparison of reconstructed signals by ℓ_1 inversion (green) for different sampling N/10 (a) and N/5: (b) of randomized signals. From the time domain (zoom) The ℓ_2 inversion (red) is not capable of good reconstruction of the continuous signal (blue) whereas ℓ_1 optimization (green) is reliable even for low sampling.

Of course reconstructions ℓ_2 and ℓ_1 inversion become better with increasing the number of observations. The current trend is to develop sampling acquisition cards for CS, but the requirements go beyond the technological limits of today's sensors.

3. Modeshapes reconstruction

When dealing with modeshapes reconstruction the principle is classicaly to make a regular grid of sensors. CS principles will permit to make sensor placement random. In numerical computation, surface interpolation functions create a continuous (or prediction) surface from sampled point values. The continuous surface representation of a raster dataset represents height, concentration, or magnitude. In *Matlab*, *zi=griddata(x,y,z,xi,yi)* fits a surface of the form *z=f(x,y)* to the data in the (usually) non uniformly-spaced vectors *(x,y,z)*. Here we study empirically the feasibility of using. *Shannon*'s sampling theorem is well known in 1D (time domain) but it also exists in 2D as demonstrate in the figure 5. Generally, the estimation accuracy will increase together with the sampling density.

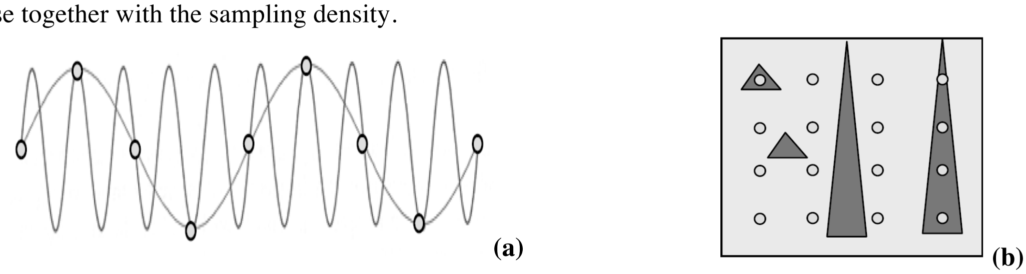
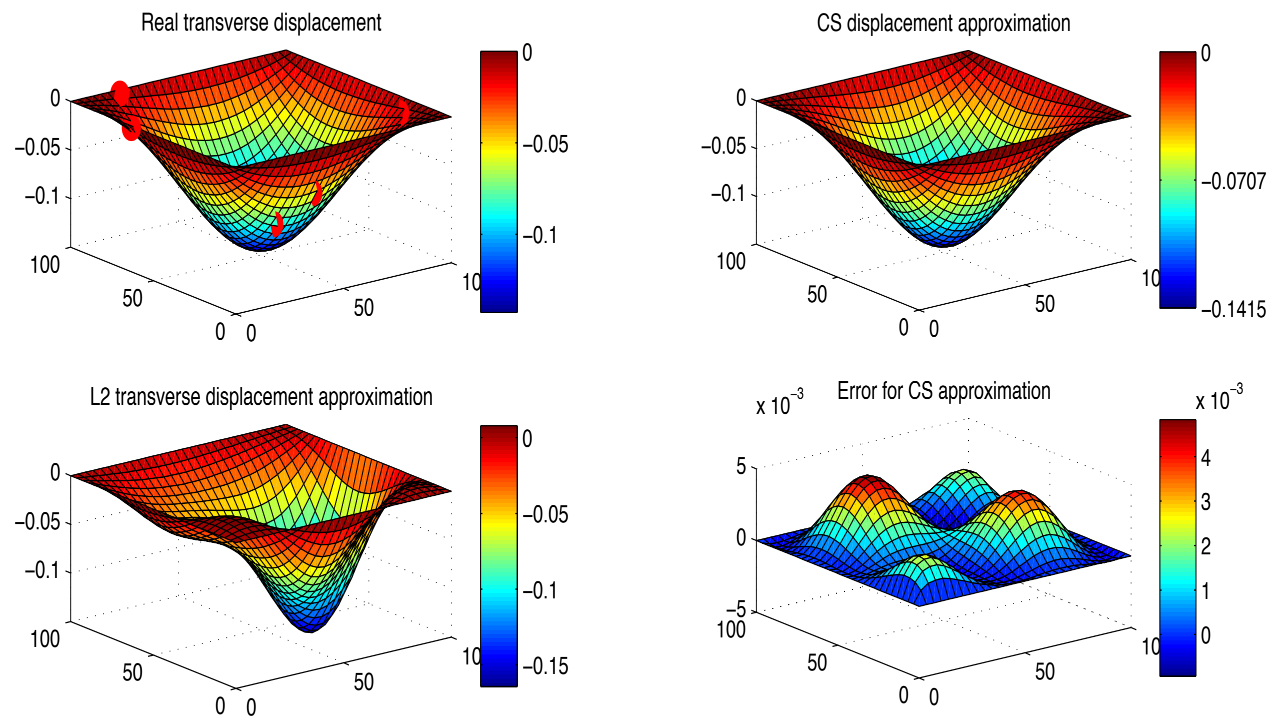


Fig. 5.: In general artefacts are due to under-sampling or poor reconstruction: Temporal aliasing (Shannon's theorem) (a), Spatial aliasing (b) due to limited spatial resolution and induce loss of details.

We propose to compare least square ℓ_2 inversion with CS method using randomly chosen sensors on the vibrating structures. A FEM modal analysis of a (Simply Supported) SSSS plate was done using following geometrical and material properties (*length=width= 0.8m; height= 0.01m;E = 210e9 Pa; nu = 0.33; rho = 7700*). We choose to illustrate the CS principle using few sensors and a well-chosen (physical) dictionary basis (Fourier Basis). In these numerical experiments, we used 8 sensors and chose a natural basis of the first 25 eigenmodes of simply supported rectangular plate. The ℓ_2 norm minimization is computed through Moore-Penrose pseudo-inverse of the observation matrix A, while the ℓ_1 norm minimization is performed through simplex method.



(a) (b)

Fig. 6. First modeshape reconstruction of a SSSS plate. The sensors are in red (randomly chosen) on the 'continuous' modeshape (a) one can see the ℓ_2 inversion results on bottom. (b) We demonstrate the CS ability to reconstruct the modeshape, on bottom one can see the error versus continuous modeshape (maximum error of 5E-3).

It should be noted whenever the dimension of x remains low, there is no substantial difference in terms computational burden between the two methods. However, for larger dimension (ranging from 100 for instance), the dimension of linear program, which is the double of the dimension of x, can lead to a prohibitive execution time. For such large problem decomposition-aggregation methods can help to save computational time. Indeed, the linear problem has a specific structure (often denote block-angular) that can be exploited through classical decomposition algorithm of large-scale linear programming (such as Benders decomposition or Dantwig-Wolfe decomposition).
Experimentaly modeshapes are commonly estimated from the residues obtained by curve fitting algorithm from set of FRFs [17]. This numerical study can be compared to experimental test where Laser Doppler Vibrometer can be moved automatically and so control the succession of acquisition for each point of the grid (regular or random). One can notice there are a variety of ways to derive a prediction for each location; each method is referred to as a model. With each model, there are different assumptions made of the data, and certain models are more applicable for specific data (for example, one model may account for local variation better than another). The existing methods of interpolation method can be found in [18,19]: Inverse Distance, Polynomial Regression, Kriging, Nearest Neighbour, Minimum Curvature, Radial Basis Function etc. Error due to sensor placement uncertainties can be defined by thrust regions (different radius depending of each sensor accuracy). This is not taken into account in this study but clearly affect the modeshape estimation.

4. Conclusion

We present an interesting approach for modal analysis using signals (1D, 2D) that do not respect the *Shannon*'s theorem. A simple multi harmonic vibration signal is proposed to illustrate the CS principles (ℓ_1 minimization). A more realistic example based on modeshape reconstruction has been done and the reconstruction appears to be enhanced by the use of CS method comparing to classical ℓ_2 inversion. We also exhibit on the plate example (modal analysis) the crucial choice (physical) of dictionary basis (*Fourier* Basis). Future works will test the CS ability on more complex structures (thin-walled structure) using 'optimized' dictionary basis adapted for experimental modal analysis using random sensors placement.

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