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Backbone colouring and algorithms for TDMA scheduling

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We investigate graph colouring models for the purpose of optimizing TDMA link scheduling in Wireless Networks. Inspired by the BPRN-colouring model recently introduced by Rocha and Sasaki, we introduce a new colouring model, namely the BMRN-colouring model, which can be used to model link scheduling problems where particular types of collisions must be avoided during the node transmissions.

In this paper, we initiate the study of the BMRN-colouring model by providing several bounds on the minimum number of colours needed to BMRN-colour digraphs, as well as several complexity results establishing the hardness of finding optimal colourings. We also give a special focus on these considerations for planar digraph topologies, for which we provide refined results. Some of these results extend to the BPRN-colouring model as well.

Keywords: Wireless networks, TDMA scheduling, Backbone colouring, Algorithmic complexity.

1 Introduction

1.1 Motivation

A *Radio Network* consists in a set of *nodes* distributed in space that communicate via broadcast radio waves, with all messages sent from a node transmitted to all nodes in its range. The *range* of a node is therefore the region in space within which it can communicate to others. Node a can transmit to node b if b is within the range of a, and we say there is a directional *link* ab. Typically, communication is done in a *multi-hop* fashion, with intermediate nodes forwarding data from a source to a destination that is distant. Since the channels are shared, transmissions are subject to collisions, that result in undesired effects such as loss of data and network efficiency. To avoid such collisions, medium access control techniques have been designed.

In the Time-Division Multiple Access (TDMA) method [10], time is divided into *frames* of fixed duration. Each frame is itself divided into a number of fixed duration *time slots*. A *link schedule* is then an assignment of time slots to the network transmissions. When the link ab is scheduled, it is required that b receives its message from a free of collision, although we do not require the same for other nodes receiving a message from a. The TDMA method is being used in standards such as IEEE 802.16 [18] and IEEE 802.11s [19], providing guaranteed Quality-of-Service (QoS). In particular, TDMA MAC protocols are being used for the increasingly popular Wireless Sensor and Mesh Networks [5]. Although there are particularities in the distinct network scenarios, an efficient use of TDMA methods is related to an increase in network throughput, and to a reduction of delays and packet losses.

Given a network and an interference model, the challenge is to find a link schedule that avoids conflicts and minimizes the number of used time slots [6, 9]. Minimizing the number of time slots is important, since it affects network throughput and multi-hop transmission delays. That is, if there are N_f different time slots in a frame, then, since a node or link is only active during its associated time slot, it becomes active only during a fraction $\frac{1}{N_f}$ of time. For this reason, finding a broadcast or link scheduling that minimizes the number of slots needed in a frame can lead to an improvement in the network efficiency.

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In a typical network, only a subset of the existing links become active. This is because the existence of a link ab does not necessarily imply that a ever communicates with b. As an example, consider a network formed by a Wi-Fi router that is connected to the internet by cable, and that is surrounded by a number of devices such as computers and smartphones, connected to the internet through it. The nearby devices may be in each other's ranges, but their transmissions are not intended to each other, so that their mutual links do not need to be scheduled. In the rest of this paper we refer to the subnetwork whose links are the active ones as the network *backbone*.

1.2 Modelling

Such problems on radio networks can be modelled as graph colouring problems, in the following way. A sensor network can be represented as a digraph D with vertex set $V(D)$ and arc set $A(D)$. The vertices of D correspond to the nodes, and there is an arc from a vertex u to a vertex v if the node corresponding to v is in the transmission range of the node corresponding to u. The backbone is represented by a spanning subdigraph B of D, whose arcs correspond exactly to the backbone links. The arcs in $A(D) \setminus A(B)$ correspond to the networks links that are not used by the network. The pair (D, B) is called a *backboned digraph* throughout. Now, we consider the time slots as colours to be assigned to the arcs of B. Hence, a link schedule of k time slots is a k-arc-colouring of B that must satisfy some constraints, depicting the conflicts which must be avoided during transmissions.

In practice, transmission collisions may occur for various reasons (physical constraints, device constraints, etc.), such as the following four ones, which shall be considered throughout this paper.

- **Type-1 constraint**: *During a time slot, a node cannot both transmit and receive messages along the backbone.*
  $\Rightarrow$ If uv and vw are two arcs of B, then they cannot be assigned the same colour.
- **Type-2 constraint**: *During a time slot, a node receiving a message along the backbone must not receive a message from another transmitting node.*
  $\Rightarrow$ If u_1v_1 is an arc of B, and e is either an arc u_2v_1 of B, or an arc u_2v_2 of B such that $u_2v_1 \in A(D) \setminus A(B)$, then uv and e cannot be assigned the same colour.
- **Type-3 constraint**: *During a time slot, a node transmits at most one message along the backbone.*
  $\Rightarrow$ If uv_1 and uv_2 are two arcs of B, then they cannot be assigned the same colour.
- **Type-3* constraint**: *A node transmits all its messages along the backbone during one time slot only.*
  $\Rightarrow$ If uv_1 and uv_2 are two arcs of B, then they must be assigned the same colour.

1.3 Colouring variants over the four types of constraints

1.3.1 BPRN-colouring

Arc-colourings of backboned digraphs fulfilling combinations of the constraints above were already considered in literature. In particular, an arc-colouring of a backboned digraph (D, B) verifying Type-1, Type-2 and Type-3 constraints is called a *Backbone-Packet-Radio-Network-colouring* (*BPRN-colouring* for short). Note that when B is an *out-branching* of D, i.e., a spanning oriented tree all arcs of which are oriented away from the root, BPRN-colourings model link schedules where, in the radio network, the root node wants to send a private message to all other nodes. The *BPRN-chromatic index* of (D, B), denoted by $\mathrm{BPRN}(D, B)$, is the minimum number of colours in a BPRN-colouring of (D, B).

The BPRN-colouring model is a generalization of the *PRN-colouring* model, first proposed by Arika [4], that appeared in other works about link scheduling [9, 13, 15]. It was introduced by Rocha and Sasaki in [14], who, motivated by applications in Wireless Sensor Networks, studied the case where the backbone is an in-branching, i.e., an oriented tree where all arcs are oriented towards the root. Among several results on BPRN-colourings, they exhibited bounds on the BPRN-chromatic index of backboned digraphs (D, B) where D is complete or a cycle. They also proved that, in general, determining the BPRN-chromatic index is $\mathcal{NP}$-hard, even when restricted to bipartite backboned digraphs.

1.3.2 BMRN-colouring

Let (D, B) be a backboned digraph. An arc-colouring of (D, B) verifying Type-1 and Type-2 constraints only is called a *Backbone-Multicast-Radio-Network-colouring* (*BMRN-colouring* for short). The *BMRN-*

chromatic index of (D, B), denoted by BMRN(D, B), is the minimum number of colours in a BMRN-colouring of (D, B). To the best of our knowledge, BMRN-colourings were not studied in literature. According to the definitions, BMRN-colourings are subject to less constraints than BPRN-colourings, and can thus be regarded as a simpler colouring notion. In particular, results on BPRN-colourings hold for BMRN-colourings. Moreover, the particular case when B is an out-branching models the case when a root wants to send the same message to all other vertices via a multicast tree.

1.3.3 BMRN*-colouring

For a backboned digraph (D, B), an arc-colouring verifying Type-1, Type-2, and Type-3* constraints is called a *Star-Backbone-Multicast-Radio-Network-colouring* or simply *BMRN*-colouring* of (D, B). The *BMRN*-chromatic index* of (D, B), denoted by BMRN$^*(D, B)$, is the minimum number of colours in a BMRN*-colouring of (D, B). A first main motivation for considering BMRN*-colourings is to derive results on the BMRN-colourings, since BMRN*-colourings are also BMRN-colourings. However, it should be pointed that BMRN*-colourings are also a model to *broadcast scheduling*, in which each node of the network is scheduled to a time slot, during which all of its neighbours must receive its transmission free of collisions. Other works such as [12, 13] study broadcast scheduling by considering that the network links are symmetric, the network being modelled by an undirected graph and the scheduling problem corresponding to finding a distance-2 colouring that uses the smallest number of colours. This approach can be seen as a simplification of our problem, since a typical network may have devices with different transmission ranges, resulting in a directed graph.

The BMRN*-colouring model can therefore be seen as a more realistic model for broadcast scheduling. This motivates the investigation of more results on this model and its corresponding algorithms. Other works such as [8] already point that distance-2 colouring of graphs is not the most accurate model to broadcast scheduling in real wireless networks, and consider the directed version. But to the best of our knowledge, we are the first to model the problem as an arc-colouring problem, and to relate it to link scheduling.

1.4 Organization of the paper

We start off with Section 2, in which we introduce some definitions and terminology that are useful for understanding our investigations.

It should be intuitive to the readers that the three colouring variants presented in the previous section are related. From the definitions, BPRN-colourings and BMRN*-colourings are also BMRN-colourings. Section 3 is devoted to understanding some of these relations further.

In Section 4, we exhibit upper bounds on the BMRN-chromatic, BPRN-chromatic and BMRN*-chromatic indices of a *spanned digraph*, that is a backboned digraph (D, T) where T is an out-branching in D. Firstly, in Subsection 4.1, we establish general bounds in terms of the maximum degree and maximum out-degree of the digraph and show that these bounds are tight up to a small constant factor. For example, we show that for every spanned digraph (D, T) we have BMRN$(D, T) \leq 2\Delta^+(D) + 1$ (where $\Delta^+(D)$ denotes the maximum out-degree of a vertex of D) and BMRN$(D, T) \leq 2\Delta(D) - 1$ (where $\Delta(D)$ denotes the maximum degree of the underlying graph of D) while there are spanned digraphs (D, T) for which BMRN$(D, T) \geq 2\Delta^+(D) - 1$ and BMRN$(D, T) \geq 2\Delta(D) - 3$. Secondly, in Subsection 4.2, we give more refined upper bounds in the realistic case where a planar topology is considered. We show that BMRN$(D, T) \leq$ BMRN$^*(D, T) \leq 8$ for every planar spanned digraph (D, T), and that BMRN$(D, T) \leq$ BMRN$^*(D, T) \leq 5$ for every outerplanar spanned digraph (D, T).

In Section 5, we show that determining any of the BMRN-, BPRN- and BMRN*-chromatic indices of a spanned digraph is $\mathcal{NP}$-hard in general, and that this remains true if one is allowed to choose the out-branching of the input spanned digraph. We also show that given a planar spanned digraph (D, T), deciding whether BMRN$(D, T) \leq 3$ is $\mathcal{NP}$-complete.

Drawing conventions: In every figure depicting a backboned digraph (D, B), the bold arcs are the arcs of B, the dashed arcs are the arcs not in B, and the undirected edges are pairs of opposite arcs. Moreover, if B is an out-branching, then the white vertex is its root.

2 Definitions and notation

The *square* G^2 of an undirected graph G is the graph with same vertex set as G and in which two vertices are adjacent if and only if they are at distance at most 2 in G.

The *underlying simple graph* $\tilde{U}(D)$ of a digraph D is the graph defined by $V(\tilde{U}(D)) = V(D)$ and $E(\tilde{U}(D)) = \{uv \mid uv \in A(D) \text{ or } vu \in A(D)\}$. We call D *symmetric* if $vu \in A(D)$ whenever $uv \in A(D)$. Observe that if D is symmetric, then $\tilde{U}(D)$ is obtained by replacing each directed 2-cycle by an edge. The digraph obtained from D by *contraction* of the arc $a = xy$ in $A(D)$ is the digraph D/a obtained from $D - \{x, y\}$ by adding a new vertex v_a and the arc $v_a w$ (resp. wv_a) for every vertex u in $V(D) \setminus \{x, y\}$ such that $xw \in A(D)$ or $yw \in A(D)$ (resp. $wx \in A(D)$ or $wy \in A(D)$).

A digraph D is *subcubic* if $\Delta(D) \leq 3$.

The notion of minor of a digraph that we use in this paper corresponds to the notion of minor in the underlying graph. That is, a digraph D' is a *minor* of D if it can be obtained from D by a succession of vertex-deletions, arc-deletions, and arc-contractions. A family $\mathcal{D}$ of digraphs is *minor-closed* if for every $D \in \mathcal{D}$, all minors of D are also in $\mathcal{D}$.

The *union* of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$. If D_1 and D_2 are disjoint (i.e., $V(D_1) \neq V(D_2)$), then it is the *disjoint union* of D_1 and D_2. A *matching* in a digraph D is a disjoint union of arcs. A matching M is *perfect* in D if every vertex of D is either the tail or the head of an arc of M. A *matched digraph* is a pair (D, M) where D is a digraph and M is a perfect matching of D.

In a digraph, we say that a vertex u *dominates* a vertex v if uv is an arc. An *out-star* is a digraph consisting in a vertex, called the *centre*, dominating all the others, called *spades*. A *galaxy* is a disjoint union of out-stars. For a digraph D, an *out-generator* of D is a vertex r such that D admits an out-branching rooted at r. In other words, there is a directed path from r to every other vertex of D.

If A is a directed path or cycle, and x, y are two vertices of A, then $A[x, y]$ denotes the directed subpath of A with origin x and terminus y.

3 Relations between BPRN-, BMRN- and BMRN*-indices

According to the definitions, for every backboned digraph (D, B) we have the following:

$$\text{BMRN}(D, B) \leq \text{BPRN}(D, B) \quad \text{and} \quad \text{BMRN}(D, B) \leq \text{BMRN}^*(D, B).$$

Conversely, one can easily see that $\text{BPRN}(D, B)$ cannot be bounded by a function of $\text{BMRN}(D, B)$. To see this, consider, for example, the directed out-star with k spades $\vec{S}_k$, for which we clearly have $\text{BMRN}(\vec{S}_k, \vec{S}_k) = 1$ and $\text{BPRN}(\vec{S}_k, \vec{S}_k) = k$.

On the other hand, we show that $\text{BMRN}^*(D, T)$ is bounded above by a function of $\text{BMRN}(D, T)$. The upper bound function we exhibit is actually best possible.

Theorem 3.1. *For every backboned digraph (D, B), we have $\text{BMRN}^*(D, B) \leq 2^{\text{BMRN}(D,B)} - 1$.*

Proof: Let ϕ be a BMRN-colouring of (D, B). For every vertex $v \in V(D)$, let

$$f(v) = \{\phi(vw) \mid w \in N^+(v)\}.$$

Now, let g be the arc-colouring of B defined by $g(xy) = f(x)$ for every arc $xy \in A(B)$. We claim that g is a BMRN*-colouring of (D, B):

- For every two arcs uv and vw of B, we have $\phi(uv) \notin f(v)$ because ϕ is a BMRN-colouring of (D, B). Hence $f(u) \neq f(v)$, and $g(uv) \neq g(vw)$. In particular, g is a proper arc-colouring. Thus Type-1 constraints are satisfied.
- For every two arcs u_1v_1 and u_2v_2 of B such that $u_1v_2 \in A(D)$, we have $\phi(u_2v_2) \notin f(u_1)$ because ϕ is a BMRN-colouring of (D, B). Hence $f(u_1) \neq f(u_2)$, and $g(u_1v_1) \neq g(u_2v_2)$.Thus Type-2 constraints are satisfied.
- By definition, any two arcs of B with the same tail are assigned the same colour by g. Hence Type-3* constraints are satisfied.

The conclusion is now obtained by noting that if ϕ takes values in a set S of k colours, then g takes values in a set of at most $2^k - 1$ colours, namely the non-empty subsets of S. □

We now prove that the upper bound in Theorem 3.1 is best possible.

Proposition 3.2. *For every $k \geq 1$, there exists a digraph D and a spanning galaxy B of D such that BMRN$(D, B) = k$ and BMRN$^*(D, B) = 2^k - 1$.*

Proof: Let X be a set of k vertices. Let us construct a galaxy B as follows: for each non-empty subset U of X, we create an out-star $S^+(U)$ with centre x_U and spades $y_U(u)$ for all $u \in U$. Let now D be the digraph obtained from B by adding arcs as follows. For every two elements $u \neq t$ of X, and every subset T such that $t \in T$, we add the arc $x_{\{u\}}y_T(t)$. This ensures that, in any BMRN*-colouring of (D, B), the arcs $x_{\{u\}}y_{\{u\}}(u)$ and $x_T y_T(t)$ get different colours. Then, for every pair $\{U, T\}$ of distinct subsets of X with $|U| \leq |T|$, choose a vertex t in $T \setminus U$, and add the arc $x_U y_T(t)$ (if not already present). Observe that this arc implies that, in any BMRN*-colouring of (D, B), the arcs of $S^+(U)$ and the arcs of $S^+(T)$ are assigned different colours, i.e., the two sets of colours are disjoint. Consequently, BMRN$^*(D, B) \geq 2^k - 1$.

Now consider the arc-colouring ϕ of B where every arc $x_U y_U(u)$ is assigned colour u. One easily checks that ϕ is a k-BMRN-colouring of (D, B). Indeed, the arcs of $A(D) \setminus A(B)$ are of the form $x_U y_T(t)$ with $t \in T \setminus U$. According to how ϕ was defined, this means that the arcs with tail x_U are assigned, by ϕ, a colour in U, while the arc $x_T y_T(t)$ is assigned colour t. Hence all Type-2 constraints are satisfied, and, consequently, BMRN$(D, B) \leq k$. Furthermore, we actually have equality, as every two arcs $x_{\{u\}}y_{\{u\}}(u)$, $x_{\{t\}}y_{\{t\}}(t)$ for $u \neq t$ must receive distinct colours, due to the arc $x_{\{u\}}y_{\{t\}}(t)$. So all k arcs of the form $x_{\{u\}}y_{\{u\}}(u)$ ($u \in U$) must be coloured differently.

From Theorem 3.1, we now get that BMRN$^*(D, B) = 2^k - 1$. □

The proof of Proposition 3.2 can be easily modified to hold for spanned digraphs (D, B).

Proposition 3.3. *For every $k \geq 1$, there exists a spanned digraph (D, T) such that BMRN$(D, T) = k$ and BMRN$^*(D, T) = 2^k - 1$.*

Proof: We start from the pair (D, B) constructed in the proof of Proposition 3.2. We construct a spanned digraph (D, T) as follows. We take an out-branching B^+ with vertex set $\{z_U \mid U \subseteq X, |U| \geq 1\}$ and root z_X. For every arc $a_U a_T$ of B^+, we add a directed path of length 3 with initial vertex a spade of $S^+(U)$ (i.e., $y_U(u)$ for some $u \in U$), terminal vertex x_T, and new (private) internal vertices. This forms our out-branching T. One then easily checks that BMRN$(D, T) =$ BMRN$(D, B) = k$ and BMRN$^*(D, T) =$ BMRN$^*(D, B) = 2^k - 1$. □

4 Bounds on BPRN(D, B), BMRN(D, B) and BMRN$^*(D, B)$

BPRN-, BMRN- and BMRN*-colourings of backboned digraphs can be viewed as particular cases of the classical colouring of graphs, where one aims at assigning colours to the vertices so that a *proper colouring* is attained, i.e., a colouring where no two adjacent vertices have the same colour. Let us now explain this in detail, for a backboned digraph (D, B).

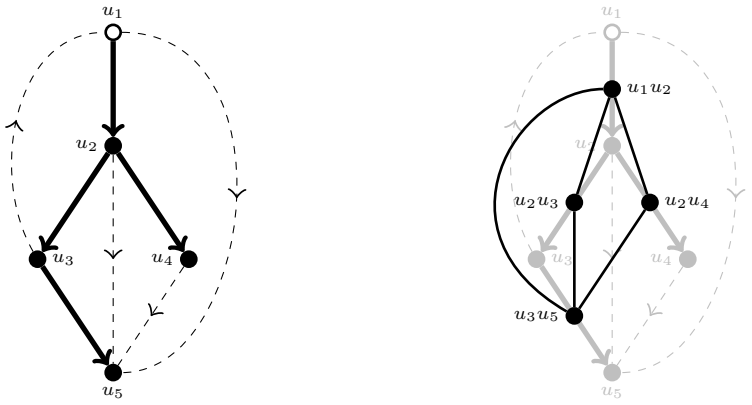


Figure 1: A spanned digraph (D, T) (left), and the BMRN-constraint graph $C_{\text{BMRN}}(D, T)$ (right).

The *BMRN-constraint graph* of (D, B) (see Figure 1 for an example) is the undirected graph $C_{\text{BMRN}}(D, B)$ defined as follows:

- $V(C_{\text{BMRN}}(D,B)) = A(B)$;
- There is an edge in $C_{\text{BMRN}}(D,B)$ between two vertices corresponding to arcs, say, u_1v_1 and u_2v_2 if either $u_1 = v_2$ or $u_2 = v_1$ (Type-1 constraints), or $u_1v_2 \in A(D)$ or $u_2v_1 \in A(D)$ (Type-2 constraints).

By construction, there is a trivial one-to-one correspondence between the BMRN-colourings of (D,B) and the proper colourings of $C_{\text{BMRN}}(D,B)$. In particular, we have $\text{BMRN}(D,B) = \chi(C_{\text{BMRN}}(D,B))$, where χ is the usual chromatic number of undirected graphs.

The *BPRN-constraint graph* of (D,B) is the graph $C_{\text{BPRN}}(D,B)$ defined as follows:

- $V(C_{\text{BPRN}}(D,B)) = A(B)$;
- There is an edge in $C_{\text{BPRN}}(D,B)$ between two vertices corresponding to arcs, say, u_1v_1 and u_2v_2 if either $u_1 = v_2$ or $u_2 = v_1$ (Type-1 constraints), or $u_1v_2 \in A(D)$ or $u_2v_1 \in A(D)$ (Type-2 constraints), or $u_1 = u_2$ (Type-3 constraints).

By construction, there is a trivial one-to-one correspondence between the BPRN-colourings of (D,B) and the proper colourings of $C_{\text{BPRN}}(D,B)$, and $\text{BPRN}(D,B) = \chi(C_{\text{BPRN}}(D,B))$.

The *BMRN*-constraint graph* $C_{\text{BMRN}^*}(D,B)$ of (D,B) is obtained from $C_{\text{BMRN}}(D,B)$ by identifying all vertices corresponding to arcs of B with same tail. It can also be defined as follows:

- $V(C_{\text{BMRN}^*}(D,B)) = V(D)$;
- u_1u_2 is an edge in $C_{\text{BMRN}^*}(D,B)$ if there are two arcs u_1v_1 and u_2v_2 of B such that either $u_1 = v_2$ or $u_2 = v_1$ (Type-1 constraints), or $u_1v_2 \in A(D)$ or $u_2v_1 \in A(D)$ (Type-2 constraints).

Again, by construction, there is a one-to-one correspondence between the BMRN*-colourings of (D,B) and the proper colourings of $C_{\text{BMRN}^*}(D,B)$. So $\text{BMRN}^*(D,B) = \chi(C_{\text{BMRN}^*}(D,B))$.

In this section, we give general bounds on $\text{BPRN}(D,B)$, $\text{BMRN}(D,B)$ and $\text{BMRN}^*(D,B)$, mainly by studying properties of $C_{\text{BPRN}}(D,B)$, $C_{\text{BMRN}}(D,B)$ and $C_{\text{BMRN}^*}(D,B)$.

4.1 General upper bounds

4.1.1 BMRN*-colouring

Observe that if u_1u_2 is an edge of $C_{\text{BMRN}^*}(D,B)$, then u_1 and u_2 are at distance at most 2 in D. This means that $C_{\text{BMRN}^*}(D,B)$ is a subgraph of $\tilde{U}(D)^2$. Hence

$$\text{BMRN}^*(D,B) \leq \chi(\tilde{U}(D)^2) \leq \Delta(\tilde{U}(D)^2) + 1 \leq \Delta(\tilde{U}(D))^2 + 1 \leq \Delta(D)^2 + 1,$$

which provides a first upper bound on $\text{BMRN}^*(D,B)$ (and, hence, on $\text{BMRN}(D,B)$) in terms of $\Delta(D)$.

We now prove a better upper bound on $\text{BMRN}^*(D,T)$ for spanned digraphs (D,T). It is obtained from a particular orientation of $C_{\text{BMRN}^*}(D,T)$ and the following well-known lemma, whose short proof is given for sake of completeness.

Lemma 4.1. *For every digraph D, we have $\chi(D) \leq 2\Delta^+(D) + 1$.*

Proof: For every subdigraph H of D, we have

$$\sum_{v \in V(H)} d_H(v) = 2 \cdot |A(H)| = 2 \sum_{v \in V(H)} d_H^+(v) \leq 2 \cdot |V(H)| \cdot \Delta^+(H) \leq 2 \cdot |V(H)| \cdot \Delta^+(D).$$

Hence every subgraph H of D has a vertex of degree at most $2\Delta^+(D)$. In other words, D is $2\Delta^+(D)$-degenerate, and thus $\chi(D) \leq 2\Delta^+(D) + 1$. □

Theorem 4.2. *For every spanned digraph (D,T), we have*

$$\textit{BMRN}^*(D,T) \leq 2\Delta^+(D) + 3 \quad \textit{and} \quad \textit{BMRN}^*(D,T) \leq 2\Delta(D) + 1.$$

Proof: Let $\vec{C}$ be the orientation of $C_{\text{BMRN}^*}(D,T)$ defined by

- $V(\vec{C}) = V(D)$, and
- u_1 dominates u_2 in $\vec{C}$ if either
  - $u_2u_1 \in A(T)$ (Type-1 constraints), or
  - there exists v_1 such that $u_2v_1 \in A(T)$ and $u_1v_1 \in A(D)$ (Type-2 constraints).

For every $u \in V(D)$, we have $d^+_{\vec{C}}(u) \leq d^-_T(u) + \sum_{v \in N^+_D(u)} d^-_T(v)$. Now since T is an out-branching, we have $d^-_T(x) \leq 1$ for every vertex x, so

$$d^+_{\vec{C}}(u) \leq d^-_T(u) + d^+_D(u) \leq d^+_D(u) + 1 \leq \Delta^+(D) + 1,$$

so $\Delta^+(\vec{C}) \leq \Delta^+(D) + 1$. Thus, by Lemma 4.1, we get

$$\text{BMRN}^*(D,T) = \chi(\vec{C}) \leq 2\Delta^+(\vec{C}) + 1 \leq 2\Delta^+(D) + 3.$$

Similarly $d^+_{\vec{C}}(u) \leq d^-_T(u) + d^+_D(u) - 1 \leq \Delta(D) - 1$, and so we get $\text{BMRN}^*(D,T) \leq 2\Delta(D) - 1$. □

4.1.2 BPRN-colouring

As in the previous section, using an orientation of the BPRN-constraint graph $C_{\text{BPRN}}(D,T)$ (for some spanned digraph (D,T)), we now establish upper bounds on $\text{BPRN}(D,T)$ in terms of $\Delta^+(D)$ and $\Delta(D)$.

Theorem 4.3. *For every spanned digraph (D,T), we have*

$$\textit{BPRN}(D,T) \leq 2\Delta^+(D) + 1 \qquad \textit{and} \qquad \textit{BPRN}(D,T) \leq 2\Delta(D) - 1.$$

Proof: Let $\vec{C}$ be the orientation of $C_{\text{BPRN}}(D,T)$ defined by

- $V(\vec{C}) = A(T)$, and
- u_1v_1 dominates u_2v_2 in $\vec{C}$ if either
  - $v_2 = u_1$ (Type-1 constraints),
  - $u_1v_2 \in A(D) \setminus A(T)$ (Type-2 constraints), or
  - $u_1 = u_2$ that is $u_1v_2 \in A(T)$ (Type-3 constraints).

For every arc $uv \in A(T)$, we have $d^+_{\vec{C}}(uv) \leq d^-_T(u) + \sum_{w \in N^+_D(u) \setminus \{v\}} d^-_T(w)$. Since T is an out-branching, we have

$$d^+_{\vec{C}}(uv) \leq d^-_T(u) + d^+_D(u) - 1 \leq d^+_D(u) \leq \Delta^+(D),$$

so $\Delta^+(\vec{C}) \leq \Delta^+(D)$. Thus, by Lemma 4.1, we get

$$\text{BPRN}(D,T) = \chi(\vec{C}) \leq 2\Delta^+(\vec{C}) + 1 \leq 2\Delta^+(D) + 1.$$

Similarly $d^+_{\vec{C}}(uv) \leq d^-_T(u) + d^+_D(u) - 1 \leq \Delta(D) - 1$, and so we get $\text{BPRN}(D,T) \leq 2\Delta(D) - 1$. □

4.1.3 BMRN-colouring

Since $\text{BMRN}(D,B) \leq \text{BPRN}(D,B)$ holds for every backboned digraph (D,B), we directly get the following from Theorem 4.3.

Corollary 4.4. *For every spanned digraph (D,T), we have*

$$\textit{BMRN}(D,T) \leq 2\Delta^+(D) + 1 \qquad \textit{and} \qquad \textit{BMRN}(D,T) \leq 2\Delta(D) - 1.$$

4.1.4 Tightness of the bounds

The bounds of Theorems 4.2 and 4.3 and Corollary 4.4 are tight up to a small additive factor, as shown in the following proposition.

Proposition 4.5. *For every $k \geq 2$, there exists a spanned digraph (D_k, T_k) with $\Delta^+(D_k) = k$, $\Delta(D_k) = k+1$, and BMRN$(D_k, T_k) = 2k-1$.*

Proof: Let (D_k, T_k) be the spanned digraph defined as follows (see Figure 2):

- $V(D_k) = V(T_k) = \bigcup_{i=1}^{2k-1} \{t_i, u_i, v_i\}$,
- $A(T_k) = \bigcup_{i=1}^{2k-2} \{t_i t_{i+1}\} \cup \bigcup_{i=1}^{2k-1} \{t_i u_i, u_i v_i\}$, and
- $A(D_k) = A(T_k) \cup \bigcup_{i=1}^{2k-1} \{u_i v_{i+j} \mid 1 \leq j \leq k-1\}$ *(indices are modulo $2k-1$).*

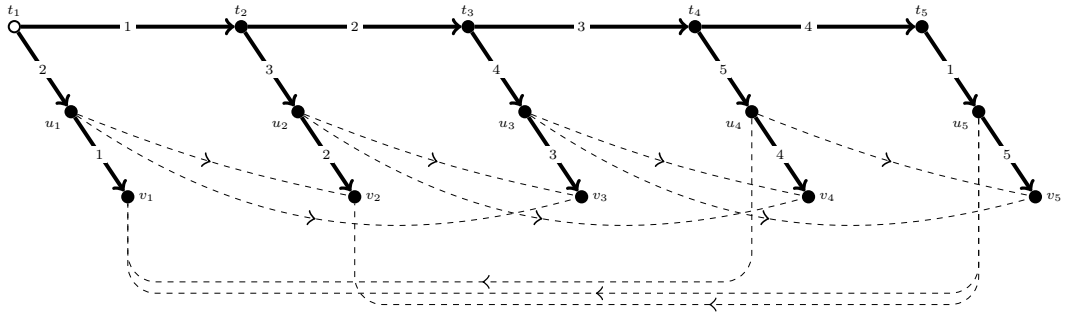


Figure 2: The spanned digraph (D_3, T_3) described in the proof of Proposition 4.5.

Clearly $\Delta^+(D_k) = k$, and $\Delta(D_k) = k+1$. Now, in every BMRN-colouring of (D_k, T_k), every two of the $u_i v_i$'s must receive distinct colours because they are involved in a Type-2 constraint. Consequently, BMRN$(D_k, T_k) \geq 2k-1$. It is easy to check that $(2k-1)$-BMRN-colourings of (D_k, T_k) actually exist (by, e.g., generalizing the colouring scheme of (D_3, T_3) depicted in Figure 2), so BMRN$(D_k, T_k) = 2k-1$. □

Remark 4.6. Note that in the construction described in the proof of Proposition 4.5, we have $\Delta^+(T_k) = 2$. Therefore, Theorems 4.2 and 4.3 and Corollary 4.4 are tight in the sense that there is no pair (ϵ, f) where ϵ is a positive real and f a function such that BMRN$(D,T) \leq (2-\varepsilon)\Delta^+(D) + f(\Delta^+(T))$ for any spanned digraph (D,T).

The bounds of Theorems 4.2 and 4.3 and Corollary 4.4 are almost tight (as shown in Proposition 4.5), but not tight. If $\Delta(D) = 1$, then D and T have two vertices and one arc, so BMRN$(D,T) =$ BMRN$^*(D,T) =$ BPRN$(D,T) = 1$. If $\Delta(D)^+ = 1$, then D is a directed path or cycle and T is a directed path, in which case BMRN$(D,T) =$ BMRN$^*(D,T) =$ BPRN$(D,T) = 2$ (if $A(T) \geq 2$). If $\Delta(D) = 2$, then D is an oriented path or cycle and T is an oriented path, and one easily sees that BMRN(D,T), BMRN$^*(D,T)$, BPRN$(D,T) \leq 3$ Moreover the upper bound 3 is attained by the spanned digraph (D,T) with $V(D) = V(T) = \{v_1, v_2, v_3, v_4\}$, $A(T) = \{v_1v_2, v_2v_3, v_3v_4\}$ and $A(D) = A(T) \cup \{v_1v_4\}$.

For larger values of $\Delta(D)$ and $\Delta^+(D)$, one could wonder whether the bounds of Theorems 4.2 and 4.3 and Corollary 4.4 can be improved. We thus address the following questions:

Question 4.7.

- What is the maximum value $M_{\Delta^+}(k)$ (resp. $M_{\Delta}(k)$) of BMRN(D,T) over all spanned digraphs (D,T) with $\Delta^+(D) \leq k$ (resp. $\Delta(D) \leq k$)?
- What is the maximum value $M^*_{\Delta^+}(k)$ (resp. $M^*_{\Delta}(k)$) of BMRN$^*(D,T)$ over all spanned digraphs (D,T) with $\Delta^+(D) \leq k$ (resp. $\Delta(D) \leq k$)?

- What is the maximum value $P_{\Delta^+}(k)$ (resp. $P_{\Delta}(k)$) of BPRN(D,T) over all spanned digraphs (D,T) with $\Delta^+(D) \leq k$ (resp. $\Delta(D) \leq k$)?

In the rest of this section, we make a first step towards these questions by studying $M_\Delta(3)$ and $M^*_\Delta(3)$. Corollary 4.4 and Proposition 4.5 yield $3 \leq M_\Delta(3) \leq 5$. We show that $M_\Delta(3) = M^*_\Delta(3) = 4$.

Figure 3 shows a spanned digraph (D,T) such that $\Delta(D) = 3$ and BMRN$(D,T) = 4$ (one easily checks that its BMRN-constraint graph is K_4, the complete graph on four vertices). Hence $M^*_\Delta(3) \geq M_\Delta(3) \geq 4$.

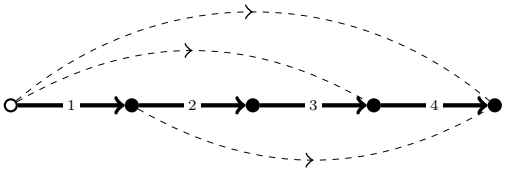


Figure 3: A spanned digraph (D,T) with $\Delta(D) = 3$ and BMRN$(D,T) = 4$.

We now prove that $M_\Delta(3) \leq M^*_\Delta(3) \leq 4$.

Theorem 4.8. *For every subcubic spanned digraph (D,T), we have BMRN$^*(D,T) \leq 4$.*

Proof: Let D be a subcubic digraph and T an out-branching of D. Let $D' = D \setminus A(T)$. We partition $V(D) = V(T)$ into four sets according to their in- and out-degrees in T. Recall that the *root* r is the unique vertex such that $d^-_T(r) = 0$, and that the *leaves* are the vertices with out-degree 0 in T. A vertex v is *flat* if $d^+_T(v) = d^-_T(v) = 1$ and it is *branching* if $d^-_T(v) = 1$ and $d^+_T(v) = 2$. If u is a flat vertex, then u^+ denotes its out-neighbour in T.

To prove that BMRN$^*(D,T) \leq 4$, we shall prove that $C^* = C_{\text{BMRN}^*}(D,B)$ is 3-degenerate, and thus 4-colourable.

Suppose for a contradiction that C^* is not 3-degenerate. Then it has a subgraph H such that $\delta(H) \geq 4$. The graph H contains no leaves of T because they are isolated vertices in C^*.

Consider a flat vertex u in $V(H)$. The only possibility for it to have degree 4 in C^* is that there exist three distinct vertices v, u_1, u_2 in $V(H)$ such that there exists w such that $vw \in A(T)$ and $uw \in A(D')$, $u_1u^+ \in A(D')$, and either $u_2u^+ \in A(D')$ or $u_2 = u^+$. In the latter case, note that u_2 has degree at most 3 in C^*; so let us suppose the first situation occurs. Note that u_1 and u_2 are either flat vertices or the root. Therefore if u is a flat vertex in H, then u^+ is a leaf which is the tail of two arcs u_1u^+ and u_2u^+ of $A(D')$ (with u_1, u_2 either flat vertices or the root).

Let U be the set of flat vertices in H and let U^+ be the set of leaves dominated by a vertex of U in T. We have $|U| = |U^+|$. Moreover, in D' each vertex of U^+ has two in-neighbours in $U \cup \{r\}$, while every vertex of U has at most one out-neighbour of U^+ and r has at most two out-neighbours in U^+. Hence $2|U| = 2|U^+| \leq |U| + 2$, so $|U| \leq 2$. Furthermore, if $|U| = 2$, say $U = \{u_1, u_2\}$, then $u_1u_2^+, u_2u_1^+ \in A(D')$ and so u_1 (and u_2 as well) has degree 3 in C^*, a contradiction; and $U = \{u\}$ is not possible because u^+ must have an in-neighbour in $U \setminus \{u\}$. Therefore $U = \emptyset$.

Now $V(H)$ contains only branching vertices and possibly the root r. Thus all arcs in D' originate from r. Let v be a vertex in $V(H)$ such that its out-neighbours in T are not in H; recall that a such vertex exists since all leaves of T are not in H. Then v can only be adjacent in H to r, and its in-neighbour in T, a contradiction to the fact that $\delta(H) \geq 4$. □

We believe that a similar result holds for BPRN-colourings of spanned digraphs with maximum degree 3. Note in particular that the spanned digraph (D,T) in Figure 3 verifies BPRN$(D,T) \leq 4$.

Question 4.9. Does every subcubic spanned digraph (D,T) satisfy BPRN$(D,T) \leq 4$?

Let us end up this section with discussing the notion of proper edge-colourings, which seem legitimate to consider in our context as they are perhaps the most investigated type of edge-colourings. Recall that a *proper edge-colouring* of an undirected graph is an edge-colouring where no two adjacent edges receive the same colour. We note that a BPRN-colouring is always a proper edge-colouring because of Type-1 and Type-3 constraints. A BMRN*-colouring is not a proper edge-colouring as soon as the backbone has vertices with out-degree at least 2, because of Type-3* constraints. A BMRN-colouring can be a proper

edge-colouring, although this is not always the case as vertices are allowed to have several out-going arcs with the same colour.

A well-known result of Vizing [17] states that every undirected graph G with maximum degree Δ has *chromatic index* $\chi'(G)$ (smallest number of colours in a proper edge-colouring) Δ or $\Delta + 1$. In light of the previous arguments, we wonder about a possible connection between $\text{BPRN}(D, T)$ and $\chi'(\tilde{U}(D)$ for a spanned digraph (D, T), for instance of the following form:

Question 4.10. Is there a function f, such that $\text{BPRN}(D, T) \leq f(\chi'(\tilde{U}(D))$ for every spanned digraph (D, T)?

In case such a function f were to exist, it would also be interesting investigating the existence of such a function for BMRN-colourings and BMRN*-colourings as well.

4.2 Upper bounds for some families of digraphs

4.2.1 Minor-closed families of digraphs

We start off by pointing out the following obvious result for BMRN*-colouring.

Lemma 4.11. *Let D be a digraph and B_1 and B_2 be two subdigraphs of D. Then*

$$\textit{BMRN}^*(D, B_1 \cup B_2) \leq \textit{BMRN}^*(D, B_1) + \textit{BMRN}^*(D, B_2).$$

In view of Lemma 4.11, to get upper bounds on $\text{BMRN}^*(D, T)$ for spanned digraphs (D, T), it might be interesting to get upper bounds on $\text{BMRN}^*(D, S)$ when S is a galaxy. We now use this approach for minor-closed families of backboned digraphs. For any class $\mathcal{F}$ of digraphs, we define $\chi(\mathcal{F}) = \max\{\chi(D) \mid D \in \mathcal{F}\}$.

Theorem 4.12. *Let $\mathcal{F}$ be a minor-closed family of digraphs. If $D \in \mathcal{F}$ and S is a galaxy in D, then $\textit{BMRN}^*(D, S) \leq \chi(\mathcal{F})$.*

Proof: Consider the BMRN*-constraint graph $C_{\text{BMRN}^*}(D, S)$. One easily sees that it is the minor of D obtained by contracting the arcs of S. Hence $\text{BMRN}^*(D, S) = \chi(C_{\text{BMRN}^*}(D, S)) \leq \chi(\mathcal{F})$. □

Theorem 4.12 and Lemma 4.11 immediately imply the following.

Corollary 4.13. *Let $\mathcal{F}$ be a minor-closed family of digraphs. If (D, B) is a backboned digraph with $D \in \mathcal{F}$, then $\textit{BMRN}^*(D, B) \leq \chi(B)\chi(\mathcal{F})$. In particular, for every spanned digraph (D, T) where $D \in \mathcal{F}$, we have $\textit{BMRN}^*(D, T) \leq 2\chi(\mathcal{F})$.*

4.3 Upper bounds for planar spanned digraphs

Corollary 4.13 and the Four-Colour Theorem [1, 2, 3] yield the following.

Corollary 4.14. *For every planar spanned digraph (D, T), we have*

$$\textit{BMRN}(D, T) \leq \textit{BMRN}^*(D, T) \leq 8.$$

There exist planar spanned digraphs (D, T) verifying $\text{BMRN}(D, T) = 7$. One such example is given in Figure 4. One easily checks that the BMRN-constraint graph of this example is K_7, the complete graph on seven vertices.

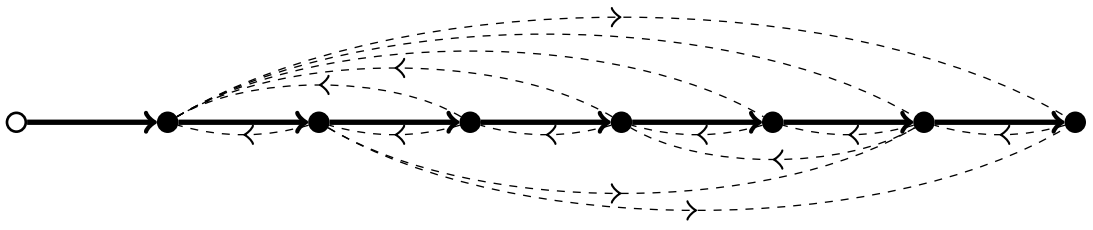


Figure 4: A planar spanned digraph (D, T) with $\text{BMRN}(D, T) = 7$.

However, we still do not know whether all planar spanned digraphs have BMRN- or BMRN*-chromatic index at most 7.

Question 4.15. Is it true that, for every planar spanned digraph (D,T), we have:

(a) $\text{BMRN}(D,T) \leq 7$?

(b) $\text{BMRN}^*(D,T) \leq 7$?

In the next section, we manage to answer such questions for outerplanar spanned digraphs.

4.4 Upper bounds for outerplanar spanned digraphs

Since outerplanar graphs have chromatic number at most 3, Corollary 4.13 yields that $\text{BMRN}^*(D,T) \leq 6$ for every outerplanar spanned digraph (D,T). The aim of this section is to improve this bound and show the following theorem.

Theorem 4.16. *For every outerplanar spanned digraph (D,T), we have*

$$\textit{BMRN}(D,T) \leq \textit{BMRN}^*(D,T) \leq 5.$$

The bound 5 of Theorem 4.16 is best possible as shown by the example depicted in Figure 5. One easily sees that, for this spanned digraph (D,T), we have $\text{BMRN}(D,T) \geq 5$ because its BMRN-constraint graph contains a K_5.

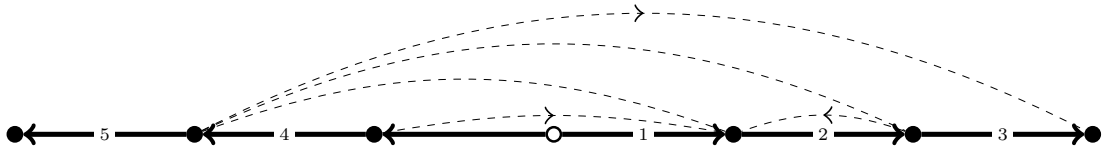


Figure 5: An outerplanar spanned digraph (D,T) with $\text{BMRN}(D,T) \geq 5$.

A digraph D is *outerplanar-maximal* if it is outerplanar and adding any new arc to D results in a non-outerplanar digraph. An outerplanar-maximal digraph is symmetric and its underlying simple graph $G = \tilde{U}(D)$ of D is 2-connected and *inner triangulated* (i.e., all faces except the outer one are 3-faces).

Let (D,T) be an outerplanar spanned digraph. A *chord* is an arc of D which is not incident to the outer face, and a *T-chord* is a chord in $A(T)$. The proof of Theorem 4.16 is by induction on the number of T-chords. We first prove the following which corresponds to the basis of the induction, that is the case when there is no T-chord.

Lemma 4.17. *Let (D,T) be a 2-connected outerplanar spanned digraph such that all arcs of T are on the outer face of D. Then $\textit{BMRN}^*(D,T) \leq 5$.*

Proof: By considering a minimum counterexample (i.e., with the minimum number of vertices). Free to add arcs, we may assume that D is outerplanar-maximal. Since D is a symmetric digraph, in what follows we sometimes regard it as an undirected graph G, in which every pair of arcs $\{uv, vu\}$ is replaced with an edge uv.

Let us number the vertices of D by $v_1, \ldots, v_n$ so that v_iv_{i+1} or $v_{i+1}v_i$ is an arc of T for all $1 \leq i \leq n-1$. For $1 \leq i \leq n-1$, let a_i be the arc of T between v_i and v_{i+1}. In other words, $\{a_i\} = \{v_iv_{i+1}, v_{i+1}v_i\} \cap A(T)$.

The *span* of an arc v_iv_j of D or an edge v_iv_j of G is $|j-i|$. In particular, all arcs of T have span 1. $T[i,j]$ denotes the subdipath of T induced by $\{v_i, \ldots, v_j\}$. Let i_0 be the index of the root r of T (i.e., $r = v_{i_0}$).

The *edge-tree* of (D,T) is the out-tree whose vertices are the edges of G and such that every edge v_iv_j (with $i<j$) of span at least 2 dominates the two edges v_iv_k and v_kv_j such that $i<k<j$ and $v_iv_kv_jv_i$ is a 3-cycle in G. Observe that the root of the edge-tree of (D,T) is v_1v_n, and its leaves are the edges of span 1. Furthermore, the span of every edge that is not a leaf is the sum of the spans of the two edges it dominates.

If v_iv_j is an edge of G and $T[i,j]$ is a directed path, then the *v_iv_j-reduced spanned digraph* is the spanned digraph $(D_{i,j}, T_{i,j})$ defined as follows:

- $D_{i,j}$ is obtained from $D - \{v_{i+1}, \ldots, v_{j-1}\}$ by adding a vertex $x_{i,j}$ and the arcs of the two directed cycles $(v_i, x_{i,j}, v_i)$ and $(v_j, x_{i,j}, v_j)$.

- If $T[i,j]$ is a directed path from v_i to v_j (resp. from v_j to v_i), then $T_{i,j}$ is obtained from $T - \{v_{i+1}, \dots, v_{j-1}\}$ by adding a new vertex $x_{i,j}$ and the arcs $v_i x_{i,j}$ and $x_{i,j} v_j$ (resp. $v_j x_{i,j}$ and $x_{i,j} v_i$).

In this case, $b = b_{i,j}$ (resp. $b' = b'_{i,j}$) denotes the arc of $T_{i,j}$ between v_i (resp. v_j) and $x_{i,j}$.

If $v_i v_j$ is an edge of G and $T[i,j]$ is not a directed path, then the *$v_i v_j$-reduced spanned digraph* is the spanned digraph $(D_{i,j}, T_{i,j})$ defined as follows:

- $D_{i,j}$ is obtained from $D - \{v_{i+1}, \dots, v_{j-1}\}$ by adding two vertices $x_{i,j}$ and $y_{i,j}$ and the arcs of the three directed cycles $(v_i, x_{i,j}, v_i)$, $(x_{i,j}, y_{i,j}, x_{i,j})$, and $(v_j, y_{i,j}, v_j)$.
- $T_{i,j}$ is obtained from $T - \{v_{i+1}, \dots, v_{j-1}\}$ by adding the two vertices $x_{i,j}$ and $y_{i,j}$ and the arcs $x_{i,j} v_i$, $x_{i,j} y_{i,j}$, and $y_{i,j} v_j$.

In that case, $b = b_{i,j}$ (resp. $b' = b'_{i,j}$) denotes the arc $x_{i,j} v_i$ (resp. $y_{i,j} v_j$).

Observe moreover that if $v_i v_j$ has span at least 3 or span 2 and $T[i,j]$ is a directed path, then the $v_i v_j$-reduced spanned digraph has smaller order than (D,T). Therefore by minimality, it admits a 5-BMRN*-colouring ϕ. Moreover, the colours assigned to a_{i-1}, a_j, b and b' are all distinct, except possibly $\phi(a_{i-1}) = \phi(a_j)$ when $a_{i-1} = v_i v_{i-1}$ and $a_j = v_j v_{j+1}$.

The general idea of the proof is to show that there is an edge $v_i v_j$ such that any 5-BMRN*-colouring of the $v_i v_j$-reduced spanned digraph can be modified to get a 5-BMRN*-colouring of (D,T), which is a contradiction.

Claim 4.18. If $v_i v_{i+3}$ is an edge of span 3, then $i < i_0 < i+3$.

Proof of claim. Assume for a contradiction that G has an edge $v_i v_{i+3}$ of span 3 with $i_0 \leq i$ or $i \geq i+3$. By symmetry, we may assume that $i_0 \leq i$. Hence $T[i, i+3]$ is a directed path from v_i to v_{i+3}.

By minimality of (D,T), the $v_i v_{i+3}$-reduced spanned digraph $(D_{i,i+3}, T_{i,i+3})$ has a 5-BMRN*-colouring ϕ. Without loss of generality, $\phi(b) = 2$, $\phi(b') = 3$, $\phi(a_{i-1}) \in \{1,2\}$ (if a_{i-1} exists), and $\phi(a_{i+3}) \in \{1,4\}$ (if a_{i+3} exists). Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 5$, and $\phi(a_{i+2}) = 3$. One easily checks that this yields a 5-BMRN*-colouring of (D,T), a contradiction. ◊

Claim 4.19. If $v_i v_{i+4}$ is an edge of span 4 and $v_i v_{i+2}$ and $v_{i+2} v_{i+4}$ are edges, then $i_0 < i$ or $i_0 > i+4$.

Proof of claim. Assume for a contradiction that $v_i v_{i+4}$, $v_i v_{i+2}$ and $v_{i+2} v_{i+4}$ are edges and $i \leq i_0 \leq i+4$. By symmetry, we may assume $i_0 \in \{i, i+1, i+2\}$. By minimality of (D,T), the $v_i v_{i+4}$-reduced spanned digraph has a 5-BMRN*-colouring ϕ. Without loss of generality, $\phi(a_1) = 1$ (if a_{i-1} exists), $\phi(b) = 2$ if $i_0 \neq i$ and $\phi(b) = 1$ if $i_0 = i$, $\phi(b') = 3$, and $\phi(a_{i+3}) \in \{1,4\}$ (if a_{i+3} exists).

Set $\phi(a_i) = 1$ if $i_0 = i$ and $\phi(a_i) = 2$ if $i_0 \in \{i+1, i+2\}$, $\phi(a_{i+1}) = 2$ if $i_0 \in \{i, i+1\}$ and $\phi(a_{i+1}) = 5$ if $i_0 = i+2$, $\phi(a_{i+2}) = 5$ and $\phi(a_{i+3}) = 3$. This gives a 5-BMRN*-colouring of (D,T), a contradiction. ◊

Consider a *deepest* edge e of span 2 in the edge-tree of (D,T), that is an edge with longest distance from $v_1 v_n$ in the edge-tree. Set $e_1 = v_i v_{i+2}$. Let e_2 be the edge that dominates e_1 in the edge-tree, and let e'_1 be the second edge that is dominated by e_2. Since e_1 is a deepest edge of span 2, necessarily e'_1 is an edge of span 1 or 2, for otherwise the branch of the edge-tree spanned at e'_1 would contain an edge of span 2 deeper than e_1. Therefore the span of e_2, which is the sum of the spans of e_1 and e'_1, is either 3 or 4. Without loss of generality, either $e_2 = v_i v_{i+3}$ or $e_2 = v_i v_{i+4}$ and $v_{i+2} v_{i+4}$ is an edge.

Case 1: $e_2 = v_i v_{i+3}$.

Then $i_0 \in \{i+1, i+2\}$ by Claim 4.18. Let $e_3 = v_k v_l$ be the edge that dominates e_2 in the edge-tree, and let e'_2 be the second edge dominated by e_3. Since e_1 was the deepest edge of span 2, e'_2 has span at most 4. If e'_2 has span 4, then swapping the names of e_2 and e'_2 we are in Case 2. If e'_2 has span 3, then e'_2 contradicts Claim 4.18. Hence e'_2 has span 1 or 2. Henceforth, we must be in one of the subcases below. For each of them, we take a 5-BMRN*-colouring ϕ of the e_3-reduced spanned digraph, which exists by minimality of (D,T). Without loss of generality, we may assume that $\phi(a_{k-1}) = 1$ (if a_{k-1} exists), $\phi(b) = 2$, $\phi(b') = 3$, and $\phi(a_l) \in \{1,4\}$ (if a_l exists). We now show for each subcase how to derive a 5-BMRN*-colouring of (D,T), which is a contradiction.

- $e_3 = v_i v_{i+4}$.
  Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 2$ if $i_0 = i+1$ and $\phi(a_{i+1}) = 5$ if $i_0 = i+2$, $\phi(a_{i+2}) = 5$, and $\phi(a_{i+3}) = 3$.

- $e_3 = v_{i-1} v_{i+3}$.
  Set $\phi(a_{i-1}) = 2$, $\phi(a_i) = 5$, $\phi(a_{i+1}) = 5$ if $i_0 = i+1$ and $\phi(a_{i+1}) = 3$ if $i_0 = i+2$, and $\phi(a_{i+2}) = 3$.

- $e_3 = v_i v_{i+5}$.
  Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 2$ if $i_0 = i+1$ and $\phi(a_{i+1}) = 5$ if $i_0 = i+2$, $\phi(a_{i+2}) = 5$, $\phi(a_{i+3})$ is a colour of $\{1,4\} \setminus \{\phi(a_{i+5})\}$, and $\phi(a_{i+4}) = 3$.

- $e_3 = v_{i-2} v_{i+4}$.
  Set $\phi(a_{i-2}) = 2$, $\phi(a_{i-1}) = 4$, $\phi(a_i) = 5$, $\phi(a_{i+1}) = 5$ if $i_0 = i+1$ and $\phi(a_{i+2}) = 3$ if $i_0 = i+2$, and $\phi(a_{i+3}) = 3$.

Case 2: $e_2 = v_i v_{i+4}$ and $v_{i+2} v_{i+4}$ is an edge.

By Claim 4.19, and by symmetry, we may assume that $i_0 < i$. Let $e_3 = v_k v_l$ be the edge that dominates e_2 in the edge-tree, and let e'_2 be the second edge dominated by e_3. Since e_1 was the deepest edge of span 2, e'_2 has span at most 4. Furthermore if e'_2 has span 4, then it dominates two edges of span 2.

One of the subcases below must thus occur. For each of them, we take a 5-BMRN*-colouring ϕ of the e_3-reduced spanned digraph, which exists by minimality of (D,T). Without loss of generality, we may assume that $\phi(a_{k-1}) \in \{1,2\}$ (if a_{k-1} exists), $\phi(b) = 2$, $\phi(b') = 3$, and $\phi(a_l) \in \{1,4\}$ (if a_l exists). Moreover $\phi(a_{k-1}) = 2$ only if $i_0 = k$ and $\phi(a_l)$ only if $k \leq i_0 \leq l$.

We now show for each subcase how to derive a 5-BMRN*-colouring of (D,T), which is a contradiction. We first consider the subcases when $k = i$. In those subcases, $\phi(a_l) = 4$ because $i_0 < i$.

(a) $e_3 = v_i v_{i+5}$.
  Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 4$, $\phi(a_{i+2}) = 5$, $\phi(a_{i+3}) = 1$, and $\phi(a_{i+4}) = 3$.

(b) $e_3 = v_i v_{i+6}$.
  Set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 3$, $\phi(a_{i+2}) = 4$, $\phi(a_{i+3}) = 1$, $\phi(a_{i+4}) = 5$, and $\phi(a_{i+5}) = 3$.

(c) $e_3 = v_i v_{i+7}$.
  In this subcase, $e'_2 = v_{i+4} v_{i+7}$ which contradicts Claim 4.18.

(d) $e_3 = v_i v_{i+8}$.
  In this subcase, $v_{i+4} v_{i+6}$, $v_{i+6} v_{i+8}$, and $v_{i+4} v_{i+8}$ are edges. Then, set $\phi(a_i) = 2$, $\phi(a_{i+1}) = 3$, $\phi(a_{i+2}) = 5$, $\phi(a_{i+3}) = 1$, $\phi(a_{i+4}) = 4$, $\phi(a_{i+5}) = 5$, $\phi(a_{i+6}) = 2$ and $\phi(a_{i+7}) = 3$.

We now consider the subcases when $l = i+4$.

(e) $e_3 = v_{i-1} v_{i+4}$.
  Set $\phi(a_{i-1}) = 2$, $\phi(a_i) = 4$, $\phi(a_{i+1}) = 5$, $\phi(a_{i+2}) = 1$, and $\phi(a_{i+3}) = 3$.

(f) $e_3 = v_{i-2} v_{i+4}$.
  On the one hand, if $i_0 \leq i-2$, then set $\phi(a_{i-2}) = 2$, $\phi(a_{i-1}) = 5$, $\phi(a_i) = \phi(a_{i+4})$, $\{\phi(a_{i+1})\} = \{1,4\} \setminus \{\phi(a_i)\}$, $\phi(a_{i+2}) = 2$, and $\phi(a_{i+3}) = 3$. On the other hand, if $i_0 = i-1$, then set $\phi(a_{i-2}) = 2$, $\phi(a_{i-1}) = 2$, $\phi(a_i) = \phi(a_{i+4})$, $\{\phi(a_{i+1})\} = \{1,4\} \setminus \{\phi(a_i)\}$, $\phi(a_{i+2}) = 5$, and $\phi(a_{i+3}) = 3$.

(g) $e_3 = v_{i-3} v_{i+4}$.
  In that subcase, by Claim 4.18, $i_0 \in \{i-2, i-1\}$. Then, set $\phi(a_{i-3}) = 2$, $\phi(a_{i-2}) = 2$ if $i_0 = i-2$ and $\phi(a_{i-2}) = 3$ if $i_0 = i-1$, $\phi(a_{i-1}) = 3$, $\phi(a_i) = 1$, $\phi(a_{i+1}) = 2$, $\phi(a_{i+2}) = 5$, and $\phi(a_{i+3}) = 3$.

(h) $e_3 = v_i v_{i+8}$.
  We are in Subcase 2 (d), with e_2 and e'_2 swapped.

This concludes the proof. □

Proof of Theorem 4.16: By induction on the number of T-chords and next on the order of D. Without loss of generality, we may assume that D is outerplanar-maximal. Let C be the oriented cycle around the outer face.

If there is no T-chord, then we have the result by Lemma 4.17. Assume now that there is a T-chord uv. This chord divides D into two outerplanar digraphs, D_1 with outer face $C_1 = C[u, v] \cup \{uv\}$ and D_2 with outer face $C_2 = C[v, u] \cup \{uv\}$. Without loss of generality, we may assume that the root r of T is in D_1. For $i = 1, 2$, let D_i^* be the symmetric outerplanar graph obtained from D_i by adding the arc vu (if necessary), and let $T_i = T \cap D_i$. Observe that T_i is an out-branching of D_i^* and that the number of T_i-chords in (D_i^*, T_i) is less than the number of T-chords in (D, T).

We distinguish two major cases, each consisting of two subcases.

Case 1: There is an arc vw_1 in T_1.

We distinguish two subcases depending on whether u is the root of T or not.

Subcase 1.1: u is the root of T.

By the induction hypothesis, there is a 5-BMRN*-colouring ϕ_i of each (D_i^*, T_i). Free to permute the colours, we may assume that $\phi_1(uv) = \phi_2(uv)$ and that the arcs of T_2 with tail v (if some exist) are coloured (by ϕ_2) with $\phi_1(vw_1)$. One can easily check that the colouring ϕ of $A(T)$, defined by $\phi(a) = \phi_1(a)$ if $a \in A(T_1)$ and $\phi(a) = \phi_2(a)$ if $a \in A(T_2)$, is a 5-BMRN*-colouring of (D, T).

Subcase 1.2: u is not the root of T.

In that subcase, u has a unique in-neighbour t in T, which must be in T_1. Let D_2^+ be the subdigraph of D induced by $V(D_2) \cup \{t\}$, and T_2^+ be the out-branching of D_2^+ obtained from T by adding t and the arc tu. Observe that the number of T_2^+-chords in (D_2^+, T_2^+) is not greater than the number of T-chords in (D, T) and $|V(D_2^+)| < |V(D)|$. By the induction hypothesis, there are 5-BMRN*-colourings ϕ_1 of (D_1, T_1) and ϕ_2^+ of (D_2, T_2). Free to permute the colours, we may assume that $\phi_1(uv) = \phi_2^+(uv)$, $\phi_1(tu) = \phi_2^+(tu)$ and that the arcs of T_2 with tail v (if some exist) are coloured (by ϕ_2^+) with $\phi_1(vw_1)$. Note that this is possible because, in both colourings, tu and the arcs with tail v receive distinct colours because vu is an arc. One can easily check that the colouring ϕ of $A(T)$, defined by $\phi(a) = \phi_1(a)$ if $a \in A(T_1)$ and $\phi(a) = \phi_2^+(a)$ if $a \in A(T_2^+)$, is a 5-BMRN*-colouring of (D, T).

Case 2: There is no arc with tail v in T_1.

Since uv is a T-chord, $|V(D_1)| \geq 3$ and so u has a unique in-neighbour t in T_1, which is also its unique in-neighbour in T.

Subcase 2.1: $|V(D_1)| \geq 4$.

We get a 5-BMRN*-colouring of (D, T) exactly as in Subcase 1.2.

Subcase 2.2: $|V(D_1)| \geq 3$.

We may assume that uv is the unique T-chord, for otherwise Case 1 or Subcase 2.1 would apply, and we would be done. Let v' be the neighbour of u in C distinct from t. Let D' be the digraph obtained from D by replacing the arcs tv and vt by the arcs tv' and $v't$. Observe that (D, T) and (D', T) have the same BMRN*-chromatic index because those four arcs do not create any new constraint. If $uv' \notin A(T)$, then, by Lemma 4.17, (D', T) has a 5-BMRN*-colouring which is also a 5-BMRN*-colouring of (D, T). Henceforth, we may assume that $uv' \in A(T)$. In particular, $T - t$ is the union of two directed paths, P with first arc uv and P' with first arc uv'.

Assume u has at most five neighbours in D_2^*. Recall that D is symmetric, so every neighbour is both an in- and an out-neighbour. Two of these neighbours are v and v'. By Lemma 4.17, (D_2^*, T_2) admits a 5-BMRN*-colouring ϕ_2. Now at most four colours are forbidden for tu, namely $\phi_2(uv) = \phi_2(uv')$ and the colours assigned to arcs with tail a neighbour of u in D_2 distinct from v and v'. Hence one can extend ϕ_2 to (D, T) by assigning to tu a non-forbidden colour. Henceforth, we may assume that u has at least five neighbours in D_2^*.

Free to consider (D', T) instead of (D, T), we may assume that u has at least three neighbours in P. Let z be the last neighbour of u along P. Let z^- be the in-neighbour of z in P and let z^+ be the out-neighbour of z in P if it has one and $z^+ = z$ otherwise, and let z^* be the terminal vertex of P.

Let D_3 be the subdigraph induced by the vertices of $V(P[u, z^+]) \cup \{t\}$. By Lemma 4.17, (D_3, T_3) admits a 5-BMRN*-colouring ϕ_3. Observe moreover that the colours assigned to tu, uv, z^-z and zz^+ (if $z \neq z^+$) are all distinct.

Let D_4 be the digraph obtained from the subdigraph induced by the vertices of $V(P[z^-, z^*]) \cup V(P')$ by adding the 2-cycle (u, z^-, u). Let $T_4 = \{tu\} \cup P' \cup \{uz^-\} \cup P[z^-, z^*]$. The number of T_4-chords in (D_4, T_4) is equal to the number of T-chords in (D, T) and $|V(D_4)| < |V(D)|$, because there are at least three neighbours of u in $P[v, z]$. Thus, by minimality of (D, T), (D_4, T_4) admits a 5-BMRN*-colouring. In addition, the colours assigned to tu, uv, z^-z and zz^+ (if $z \neq z^+$) are all distinct. Hence, free to permute the colours, we may assume that ϕ_3 and ϕ_4 agree on those four arcs. Now one easily checks that the colouring ϕ of $A(T)$, defined by $\phi(a) = \phi_3(a)$ if $a \in A(T_3)$ and $\phi(a) = \phi_4(a)$ if $a \in A(T_4)$, is a 5-BMRN*-colouring of (D, T).

This concludes the proof. □

5 Complexity

5.1 Determining the exact value of an index

In this section, we prove several results showing that the problems of determining BMRN(D, T), BPRN(D, T) and BMRN$^*(D, T)$ are $\mathcal{NP}$-hard, even when restricted to particular spanned digraphs (D, T). We define these decision problems in the usual way:

k-BMRN-COLOURING
Input: A spanned digraph (D, T).
Question: Do we have BMRN$(D, T) \leq k$?

k-BPRN-COLOURING
Input: A spanned digraph (D, T).
Question: Do we have BPRN$(D, T) \leq k$?

k-BMRN*-COLOURING
Input: A spanned digraph (D, T).
Question: Do we have BMRN$^*(D, T) \leq k$?

Recall that finding a k-BMRN-colouring (resp., k-BPRN-colouring, k-BMRN*-colouring) of (D, T) is equivalent to finding a k-colouring of $C_{\text{BMRN}}(D, T)$ (resp., $C_{\text{BPRN}}(D, T)$, $C_{\text{BMRN}^*}(D, T)$). Furthermore, the usual k-COLOURING problem is well-known to be polynomial-time solvable when $k = 2$ and $\mathcal{NP}$-complete for all $k \geq 3$. Since the constraint graphs of (D, T) can clearly be constructed in polynomial time, we directly get that 2-BMRN-COLOURING, 2-BPRN-COLOURING, and 2-BMRN*-COLOURING are polynomial-time solvable.

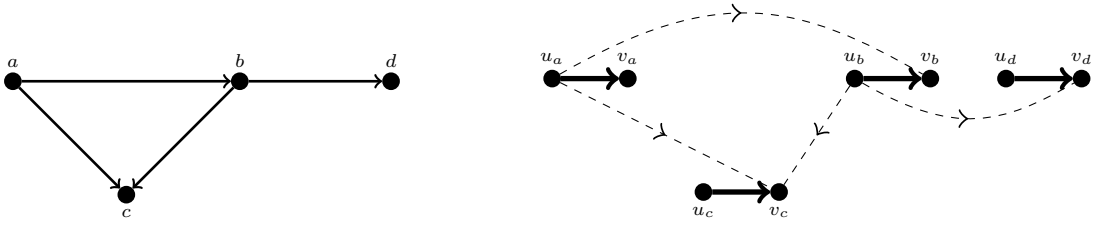


Figure 6: An oriented graph $\vec{G}$ (left), and the matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ (right).

Still from the previous colouring equivalence, we now establish the hardness of the three problems above for all $k \geq 3$, using the following construction (see Figure 6 for an example). Let $\vec{G}$ be an oriented graph.

The *matched digraph associated to* $\vec{G}$ is the matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ defined by:

$$\bullet\ V(D_{\vec{G}}) = V(M_{\vec{G}}) = \bigcup_{x \in V(\vec{G})} \{u_x, v_x\},$$
$$\bullet\ A(M_{\vec{G}}) = \{u_x v_x \mid x \in V(\vec{G})\}, \text{ and}$$
$$\bullet\ A(D_{\vec{G}}) = A(M_{\vec{G}}) \cup \{u_x v_y \mid xy \in A(\vec{G})\}.$$

One can easily check that $\vec{G}$ is nothing but the BMRN-constraint graph $C_{\text{BMRN}}(D_{\vec{G}}, M_{\vec{G}})$. Observe moreover that $\Delta^+(D_{\vec{G}}) = \Delta^+(\vec{G}) + 1$.

Theorem 5.1. *For every $k \geq 3$, the problems k-BMRN-COLOURING, k-BPRN-COLOURING and k-BMRN*-COLOURING are $\mathcal{NP}$-complete, even when restricted to spanned digraphs (D, T) where T is a directed path.*

Proof: Fix $k \geq 3$. The problems are clearly in $\mathcal{NP}$. Let us now prove that they are $\mathcal{NP}$-hard. Observe that when T is a directed path, the Type-3 and Type-3* constraints are vacuously fulfilled by any arc-colouring. Therefore, in that case, any BMRN-colouring of (D, T) is also a BPRN-colouring and a BMRN*-colouring. It thus suffices to prove the $\mathcal{NP}$-completeness of k-BMRN-COLOURING for such restricted instances.

The reduction is from k-COLOURING. Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$. Consider any orientation $\vec{G}$ of G. Let (D, T) be the spanned digraph obtained from the associated matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ by adding the arc $v_{x_i} u_{x_{i+1}}$ for all $1 \leq i \leq n-1$, i.e., $D = D_{\vec{G}} \cup \{v_{x_i} u_{x_{i+1}} \mid 1 \leq i \leq n-1\}$ and $T = M_{\vec{G}} \cup \{v_{x_i} u_{x_{i+1}} \mid 1 \leq i \leq n-1\}$. By construction, T is a directed path. One easily sees that each $v_{x_i} u_{x_{i+1}}$ is subject to at most two colouring constraints, or, in other words, that the associated vertex has degree 2 in the BMRN-constraint graph $C_{\text{BMRN}}(D, T)$. Hence, since $k \geq 3$, the graph $C_{\text{BMRN}}(D, T)$ is k-colourable if and only if G is k-colourable. In other words, (D, T) is k-BMRN-colourable if and only if G is k-colourable. □

We proved, in Theorem 4.8, that subcubic spanned digraphs have BMRN-chromatic index at most 4. As already pointed out, by looking at the BMRN-constraint graph, it can be decided in polynomial time whether the BMRN-chromatic index of a given subcubic spanned digraph is at most 2. It is not clear, however, whether it can easily be decided whether this index is 3 or 4; so we leave the following question open:

Question 5.2. Is 3-BMRN-COLOURING $\mathcal{NP}$-complete when restricted to subcubic spanned digraphs?

5.2 Finding a particular out-branching

We now consider problems where, for a given digraph D, one aims at finding an out-branching T (rooted at a given vertex r or not), such that BMRN(D, T) (resp. BMRN(D, T) and BPRN$^*(D, T)$) is small. More precisely, we consider the following decision problems (which we define for BMRN-colouring only, but they can be derived for BPRN-colouring and BMRN*-colouring in an obvious way):

k-BMRN-ROOT
Input: A digraph D and an out-generator r of D.
Question: Is there an out-branching T of D spanned at r such that BMRN$(D, T) \leq k$?

k-BMRN-BRANCHING
Input: A digraph D.
Question: Is there an out-branching T of D such that BMRN$(D, T) \leq k$?

We show in Theorem 5.3 that all arising problems are $\mathcal{NP}$-complete for all $k \geq 3$. We then prove the $\mathcal{NP}$-completeness of 2-BMRN-ROOT, 2-BMRN-BRANCHING, 2-BMRN*-ROOT and 2-BMRN*-BRANCHING in Theorems 5.4 and 5.5. The same result for 2-BPRN-ROOT and 2-BPRN-BRANCHING is then proved in Theorem 5.6.

Theorem 5.3. *For every $k \geq 3$, k-BMRN-ROOT, k-BMRN-BRANCHING, k-BMRN*-ROOT, k-BMRN*-BRANCHING, k-BPRN-ROOT and k-BPRN-BRANCHING are $\mathcal{NP}$-complete.*

Proof: The problems are clearly in $\mathcal{NP}$. We now prove that they are $\mathcal{NP}$-hard, by focusing, again, on BMRN-colourings (for the same reasons as in the proof of Theorem 5.1). Fix $k \geq 3$. The reduction is from k-COLOURING. Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$. Let $\vec{G}$ be the orientation of G such that if $x_i x_j$ is an arc, then $i > j$. Again consider, as in the proof of Theorem 5.1, the associated matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ of $\vec{G}$. Let D be the digraph obtained from $D_{\vec{G}}$ by adding a vertex r and the arcs ru_{x_1} and $v_{x_i}u_{x_{i+1}}$ for all $1 \leq i \leq n-1$. One easily sees that the unique out-branching of D is the directed path $T = (r, u_{x_1}, v_{x_1}, u_{x_2}, v_{x_2}, \ldots, u_{x_n}, v_{x_n})$. Moreover, all $u_{x_i}v_{x_{i+1}}$'s and ru_{x_1} have degree at most 2 in the BMRN-constraint graph $C_{\text{BMRN}}(D,T)$ of (D,T). Hence, since $k \geq 3$, we have that (D,T) is k-BMRN-colourable if and only if G is k-colourable. □

Theorem 5.4. 2-BMRN-ROOT *and* 2-BMRN-BRANCHING *are $\mathcal{NP}$-complete.*

Proof: The proof of the $\mathcal{NP}$-hardness is by reduction from MONOTONE NOT-ALL-EQUAL 3-SAT. In this variant, the boolean formula $\mathcal{F}$ consists of clauses all of whose literals are non-negated variables. We want to decide whether $\mathcal{F}$ admits an *NAE-assignment*, that is a truth assignment such that every clause has a true literal and a false literal. MONOTONE NOT-ALL-EQUAL 3-SAT was shown $\mathcal{NP}$-complete by Schaefer [16].

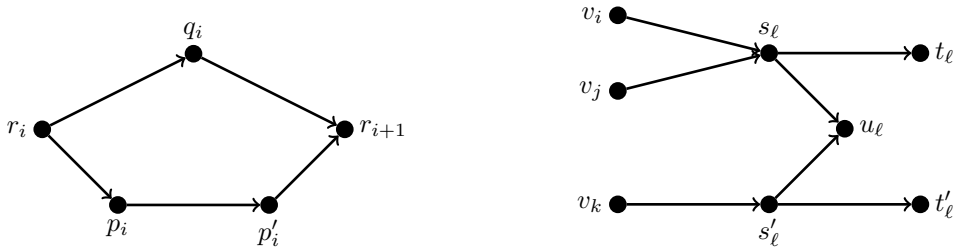


Figure 7: The two gadgets used in the proof of Theorem 5.4: the $\{r_i, r_{i+1}\}$-connector gadget (left), and the clause gadget for a clause $C_\ell = x_i \vee x_j \vee x_k$ (right).

Let $\mathcal{F}$ be an instance of MONOTONE NOT-ALL-EQUAL 3-SAT with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$. Let D be the digraph constructed as follows. For $1 \leq i \leq n$, add the vertices r_i, v_i and the arc $r_i v_i$. For every $1 \leq i \leq n-1$, we add an $\{r_i, r_{i+1}\}$-*connector gadget* (see the left of Figure 7), which is a digraph with vertex set $\{r_i, p_i, p'_i, q_i, r_{i+1}\}$ (where r_i and r_{i+1} are existing vertices) and arc set $\{r_i q_i, q_i r_{i+1}, r_i p_i, p_i p'_i, p'_i r_{i+1}\}$. For every clause $C_\ell = x_i \vee x_j \vee x_j$, we add a *clause gadget* (see the right of Figure 7) with vertex set $\{v_i, v_j, v_k, s_\ell, s'_\ell, u_\ell, t_\ell, t'_\ell\}$ (where v_i, v_j, v_k already exist) and arc set $\{v_i s_\ell, v_j s_\ell, s_\ell t_\ell, s_\ell u_\ell, v_k s'_\ell, s'_\ell t'_\ell, s'_\ell u_\ell\}$.

Observe that r_1 is the unique out-generator of D. Furthermore, every out-branching T of D must contain every arc $r_i v_i$, $1 \leq i \leq n$, because each v_i has in-degree 1. Let us show that D has an out-branching T (with root r_1) such that BMRN$(D,T) \leq 2$ if and only if $\mathcal{F}$ admits an NAE-assignment.

Assume first that there is an out-branching T of D such that BMRN$(D,T) \leq 2$. Let ϕ be a 2-BMRN-colouring of (D,T). Let ψ be the truth assignment defined by $\psi(x_i) = true$ if and only if $\phi(r_i v_i) = 1$. This is well-defined because all $r_i v_i$'s are in $A(T)$. We claim that ψ is an NAE-assignment of $\mathcal{F}$. Suppose for a contradiction that there is a clause $C_\ell = x_i \vee x_j \vee x_k$ such that all literals have the same value. Without loss of generality, $\psi(x_i) = \psi(x_j) = \psi(x_k) = true$. Then, by definition, $\phi(r_i v_i) = \phi(r_j v_j) = \phi(r_k v_k) = 1$. Now $v_k s'_\ell$ and $s'_\ell t'_\ell$ must be in T (because s'_ℓ and t'_ℓ have in-degree 1), and we have $\phi(v_k s'_\ell) = 2$ and $\phi(s'_\ell t'_\ell) = 1$. Furthermore, T contains an arc $a \in \{v_i s_\ell, v_j s_\ell\}$ and $s_\ell t_\ell$, and $\phi(a) = 2$ and $\phi(s_\ell t_\ell) = 1$. Finally, one arc a' of $s_\ell u_\ell$ and $s'_\ell u_\ell$ is in $A(T)$. Because of Type-1 constraints, we must have $\phi(a') = 1$, which creates Type-2 constraints since $\phi(s_\ell t_\ell) = \phi(s'_\ell t'_\ell) = 1$. This is a contradiction.

Reciprocally, assume that there exists an NAE-assignment ψ of $\mathcal{F}$. We shall construct an out-branching T of D and a 2-BMRN-colouring ϕ of (D,T). For $1 \leq i \leq n$, add $r_i v_i$ to $A(T)$ and set $\phi(r_i v_i) = 1$ if $\psi(x_i) = true$, and $\phi(r_i v_i) = 2$ if $\psi(x_i) = false$. For every clause $C_\ell = x_i \vee x_j \vee x_k$, either $\psi(x_i) \neq \psi(x_k)$ or $\psi(x_j) \neq \psi(x_k)$ because ψ is an NAE-assignment. Let a be $v_i s_\ell$ if $\psi(x_i) \neq \psi(x_k)$ and let a be $v_j s_\ell$ otherwise. Add a, $v_k s'_\ell$, $s_\ell t_\ell$, $s'_\ell t'_\ell$, and $s_\ell u_\ell$ to $A(T)$, and let $\phi(a) = \phi(s'_\ell t'_\ell) = \phi(r_k v_k)$ and $\phi(s_\ell t_\ell) = \phi(s_\ell u_\ell) = \phi(v_k s'_\ell) = 3 - \phi(a)$. For each $1 \leq i \leq n-1$, do the following:

- If $\phi(r_i v_i) = \phi(r_{i+1} v_{i+1})$, then add $r_i q_i$, $r_i p_i$, $p_i p'_i$ and $q_i r_{i+1}$ to $A(T)$ and set $\phi(r_i q_i) = \phi(r_i p_i) = \phi(r_i v_i)$ and $\phi(p_i p'_i) = \phi(q_i r_{i+1}) = 3 - \phi(r_i v_i)$;

- If $\phi(r_iv_i) \neq \phi(r_{i+1}v_{i+1})$, then add r_iq_i, r_ip_i, $p_ip'_i$ and p'_ir_{i+1} to $A(T)$ and set $\phi(r_iq_i) = \phi(r_ip_i) = \phi(p'_ir_{i+1}) = \phi(r_iv_i)$ and $\phi(p_ip'_i) = 3 - \phi(r_iv_i)$.

Now, it is simple matter to check that T is an out-branching of D and ϕ a 2-BMRN-colouring of (D, T). □

The exact same proof yields the following.

Theorem 5.5. 2-BMRN*-ROOT *and* 2-BMRN*-BRANCHING *are $\mathcal{NP}$-complete.*

In the next result, we prove the similar result for 2-BPRN-ROOT and 2-BPRN-BRANCHING:

Theorem 5.6. 2-BPRN-ROOT *and* 2-BPRN-BRANCHING *are $\mathcal{NP}$-complete.*

Proof: The proof of the $\mathcal{NP}$-hardness of the two problems is by reduction from a restriction of the DIRECTED HAMILTONIAN CYCLE problem, which asks whether a given digraph has a directed Hamiltonian cycle. In [11], Plesńik proved that DIRECTED HAMILTONIAN CYCLE remains $\mathcal{NP}$-complete when restricted to *small-degree digraphs*, that are digraphs in which the in-degree and out-degree of each vertex are either 1 or 2.

A *track* is a vertex with in-degree 1 and out-degree 1, an *out-switch* is a vertex with in-degree 1 and out-degree 2, and an *in-switch* is a vertex with in-degree 2 and out-degree 1. A digraph is *nice* if every vertex is either a track, an out-switch or an in-switch.

Claim 5.7. DIRECTED HAMILTONIAN CYCLE remains $\mathcal{NP}$-complete for nice digraphs.

Proof of claim. Reduction from DIRECTED HAMILTONIAN CYCLE restricted to small-degree digraphs. Consider a small-degree digraph D. Let D' be the digraph obtained by "exploding" each vertex v to an arc v^-v^+. Formally,

$$\bullet\ V(D') = \bigcup_{v \in V(D)} \left\{v^-, v^+\right\}, \text{ and}$$
$$\bullet\ A(D') = \left\{v^-v^+ \mid v \in V(D)\right\} \cup \left\{u^+v^- \mid uv \in A(D)\right\}.$$

Clearly D' is nice and it has a directed Hamiltonian cycle if and only if D has one. ◊

We now give a reduction from DIRECTED HAMILTONIAN CYCLE restricted to nice digraphs to 2-BPRN-ROOT and 2-BPRN-BRANCHING. Let D be a nice digraph. Choose a vertex v of D with in-degree 1. Let u be its in-neighbour. Observe that every directed Hamiltonian cycle of D must contain the arc uv. Let D' be the digraph obtained from $D - uv$ by adding two vertices r and s and the arcs rv and us. Note that a directed Hamiltonian path in D' necessarily starts in r and ends in s, so D' has a directed Hamiltonian path if and only if D has a directed Hamiltonian cycle.

We now construct a digraph H such that H admits an out-branching T such that BPRN$(H, T) = 2$ if and only if D' has a directed Hamiltonian path, and so if and only if D has a directed Hamiltonian cycle. To that end, we first associate a gadget in H to each vertex of D', in the following way:

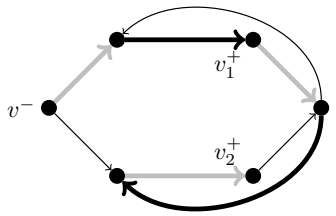


Figure 8: The out-gadget O_v. A directed Hamiltonian path P of O_v is displayed with bold arcs. The black and gray bold arcs form a 2-BPRN-colouring of (O_v, P).

- For each of r and s, we add to H two gadgets that are actually exactly the vertices r and s. For the vertex gadget corresponding to r (resp. s), we call r (resp. s) its *exit* (resp. *entry*).

- Consider a track v of D'. In H, we associate a *track gadget* T_v which is a directed path of length 3. The origin v^- of T_v is its *entry*, and the terminus v^+ is its *exit*.
- Consider an out-switch v of D'. In H, we associate an *out-gadget* O_v depicted in Figure 8. This gadget has one *entry* v^- and two *exits* v_1^+, v_2^+.
- Consider an in-switch v of D'. In H, we associate an *in-gadget* I_v, which is just an arc v^-v^+. We call v^- the *entry* of I_v, and v^+ its *exit*.

Note that all gadgets have at most one entry. Furthermore, only out-gadgets have two exits. To finish the construction of H, we now connect the gadgets as follows. Consider every arc uv of D'; then:

- If u is an out-switch, then we choose a degree-2 exit of the gadget associated to u in H, and we add an arc from that exit to the unique entry of the gadget associated to v. In other words, for the gadgets with two exits, each exit is added exactly one out-going arc.
- Otherwise, we add an arc from the unique exit of the gadget associated to u in H to the unique entry of the gadget associated to v.

Note that an out-gadget has only two directed Hamiltonian paths: they both start in its entry and they end in different exits. Moreover once a directed Hamiltonian path of H enters a gadget (track, out-gadget or in-gadget), it has to go through all the vertices of the gadget at once, due to the number of entries and exits. Therefore the digraph H has a directed Hamiltonian path if and only if D' has one. In particular, recall that all directed Hamiltonian paths start in r and end in s.

We claim that we have the desired equivalence between D' and H. Assume first that there is an out-branching T of H such that (H, T) has a 2-BPRN-colouring. Because of Type-1 and Type-3 constraints, note that T must be a directed Hamiltonian path of H.
Reciprocally, assume H has a directed Hamiltonian path P. We claim that colouring its arcs alternately with colours 1 and 2 (starting, say, with 1) yields a 2-BPRN-colouring of (H, P). Indeed, Type-1 constraints are satisfied by definition, and, since P is a path, Type-3 constraints are trivially satisfied. To be convinced that no Type-2 constraints can arise, consider the following arguments. First of all, all arcs of H not in P either belong to an out-gadget (which raise no Type-2 constraints, see Figure 8), or go from an exit of an out-gadget to the entry of an in-gadget. Also, it can be checked that all arcs of P entering and exiting any gadget necessarily have colour 1. Under this assumption, for every arc uv from an exit of an out-gadget to the entry of an in-gadget, the only arc of P out-going from u has colour 2, while the only arc of P in-coming to v has colour 1. Hence no Type-2 constraints arise.

Hence there is an out-branching T of H such that (H, T) has a 2-BPRN-colouring if and only if H has a directed Hamiltonian path, so if and only if D' has a directed Hamiltonian path. □

5.3 Restriction to planar spanned digraphs

We here consider PLANAR k-BMRN-COLOURING, which is the restriction of k-BMRN-COLOURING to planar spanned digraphs. For $k = 2$, the problem is polynomial-time solvable because k-BMRN-COLOURING is polynomial-time solvable. For $k \geq 8$, PLANAR k-BMRN-COLOURING is trivial as the answer is always 'Yes' by Corollary 4.14.

Theorem 5.8. PLANAR 3-BMRN-COLOURING *is $\mathcal{NP}$-complete.*

Proof: The proof is by reduction from PLANAR 3-COLOURING which consists in deciding whether a given planar graph is 3-colourable. This problem was proved $\mathcal{NP}$-complete in [7].

From a planar graph G, we shall construct, in polynomial time, a spanned planar digraph (D, T) such that $\chi(G) \leq 3$ if and only if $\text{BMRN}(D, T) \leq 3$.

Since G is planar, it admits a planar straight-line embedding $\tilde{G}$ in the plane. Moreover, free to move slightly some vertices, we may assume that no two vertices of $\tilde{G}$ lie on a same horizontal or vertical line. Let $\vec{G}$ be the orientation of G obtained by orienting every edge towards its higher vertex in $\tilde{G}$: hence if $uv \in A(\vec{G})$, then vertex v lies above u.

Let us consider the matched digraph $(D_{\vec{G}}, M_{\vec{G}})$ associated to $\vec{G}$. As observed earlier, G is actually the BMRN-constraint graph $C_{\text{BMRN}}(D_{\vec{G}}, M_{\vec{G}})$. So G is 3-colourable if and only if $(D_{\vec{G}}, M_{\vec{G}})$ is 3-BMRN-colourable.

Our goal is now to extend $(D_{\vec{G}}, M_{\vec{G}})$ into a planar spanned digraph (D, T), preserving the colouring equivalence with G. From $\tilde{G}$, one can easily derive a plane straight-line embedding of $D_{\vec{G}}$ in which all arcs $u_x v_x$, for $x \in V(\vec{G})$, are drawn horizontally and from left to right, of tiny length ϵ and whose middle is x. Towards getting (D, T), we first add, to $(D_{\vec{G}}, M_{\vec{G}})$, a vertex r below all other vertices (and thus on the outer face), which shall be the root of T. For every inner face F of $\tilde{G}$, let $m(F)$ be the lowest vertex of F (for the particular case of the outer face F, let $m(F) = r$). For every face F of $(D_{\vec{G}}, M_{\vec{G}})$, let $M(F)$ be the set of vertices x such that u_x is inside F. Observe that the vertices of $M(F)$ are incident to F, and that $m(F) \notin M(F)$. For each face F, we add a directed path P_x (both in D and T) of length 3 from $u_{m(F)}$ to u_x for each vertex $x \in M(F)$. Of course, we do this in such a way that the added paths do not cross. This results in the planar spanned digraph (D, T) where $D = D_{\vec{G}} \cup \bigcup_{x \in V(G)} P_x$ and $T = M_{\vec{G}} \cup \bigcup_{x \in V(G)} P_x$.

Let us now check that $\text{BMRN}(D_{\vec{G}}, M_{\vec{G}}) \leq 3$ if and only if $\text{BMRN}(D, T) \leq 3$. Since $(D_{\vec{G}}, M_{\vec{G}})$ is the restriction of (D, T) to $V(D_{\vec{G}})$, every 3-BMRN-colouring of (D, T) induces a 3-BMRN-colouring of $(D_{\vec{G}}, M_{\vec{G}})$. Conversely, every 3-BMRN-colouring ϕ of $(D_{\vec{G}}, M_{\vec{G}})$ can be easily extended in a 3-BMRN-colouring of (D, T), for the following reasons. First, the added paths P_x do not create any new constraint between the arcs of $M_{\vec{G}}$. Moreover, for every face F and every $x \in M(F)$, we can assign the colour $\phi(u_{m(F)}, v_{m(F)})$ to the first arc of P_x, and then extend the colouring to the two other arcs, which are subject to at most two constraints in (D, T).

Thus, (D, T) is a planar spanned digraph that is 3-BMRN-colourable if and only if G is 3-colourable. □

PLANAR k-COLOURING is trivial for all $k \geq 4$, as the answer is always 'Yes' according to the Four-Colour Theorem. Henceforth, the above proof cannot be generalized to show that PLANAR k-BMRN-COLOURING is $\mathcal{NP}$-complete for k larger than 3.

Question 5.9. For every $k \in \{4, 5, 6, 7\}$, what is the complexity of PLANAR k-BMRN-COLOURING?

Similarly, one can compute the chromatic number of an outerplanar graph in polynomial time. So we cannot establish the hardness of the restriction of k-BMRN-COLOURING to outerplanar digraphs via a proof similar to that of Theorem 5.8. We thus leave the following question open:

Question 5.10. For any outerplanar spanned digraph (D, T), can $\text{BMRN}(D, T)$ and $\text{BMRN}^*(D, T)$ be determined in polynomial time?

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