Bounds for the trace of small Salem numbers

V. Flammang *

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Abstract

In this work, we give a lower bound and an upper bound for the trace of a Salem number whose Mahler measure is less than 1.3. We use the method of explicit auxiliary functions combined with the recursive algorithm developed in [F3].

1 Introduction

A *Salem number* is an algebraic integer $\tau > 1$ of degree $n = 2d \geq 4$ whose other conjugates, except for τ^{-1} have modulus 1.

Now, the *Mahler measure* of an algebraic integer α with conjugates $\alpha_1 = \alpha, \dots, \alpha_n$ is $M(\alpha) = \prod_{j=1}^n \max(1, |\alpha_j|)$ and its trace is $\text{tr}(\alpha) = \sum_{j=1}^n \alpha_j$. Note that, if τ is a Salem number then $M(\tau) = \tau$.

If α is an algebraic integer and $M(\alpha) = 1$, then a classical theorem of Kronecker [K] tells us that α is a root of unity. It suggests the question: $\inf_{\alpha \text{ not a root of unity}} M(\alpha) > 1$? It is known as the *Lehmer's problem* and it is still open. Another formulation can be given as follows. Does there exist an absolute constant $c > 0$ such that: if $M(\alpha) > 1$ then $M(\alpha) > 1 + c$? The smallest known value for $M(\alpha)$ is due to Lehmer himself [L] and is $M(\lambda) = 1.176280\dots$ where λ is the positive root > 1 of the polynomial $L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$. Here is the link between *small Salem numbers* (with Mahler measure less than 1.3) and Lehmer's problem: λ is a Salem number and moreover is up to now the smallest known Salem number. Currently, there are 47 known small Salem numbers (see the table [M2]). D.W. Boyd compiled 43 of such numbers [B1], [B2] in the years 1977-1978. Later, in 1990, M. J. Mossinghoff discovered the last four ones [M1], including one of degree 46. In 1999, the author, M. Grandcolas and G. Rhin proved that this table was complete up to degree 40 [F2]. M.J. Mossinghoff, G. Rhin and Q. Wu extended the result up to the degree 44 [MRW]. For more details on Salem numbers, see [S1].

In this paper, we focus our attention on the trace of small Salem numbers and prove that the trace can only take a small number of values. More precisely, we get:

Theorem 1. *Let τ be a small Salem number of degree $n = 2d \geq 4$. Then, we have:*

$$-0.207188d \leq \text{tr}(\tau) \leq 0.193466d + 0.765083.$$

As the list of small Salem numbers is exhaustive up to the degree 44, this result is interesting from the degree 46. We have computed the bounds from this degree to the degree 30. We obtain:

*V. Flammang, IECL, France, valerie.flammang@univ-lorraine.fr

Corollary 1. *Let τ be a small Salem number of degree $n = 2d \geq 46$. Then, we have:*

$$\begin{array}{llll} d = 23 & -4 \leq \text{tr}(\tau) \leq 5 & d = 27 & -5 \leq \text{tr}(\tau) \leq 5 \\ d = 24 & -5 \leq \text{tr}(\tau) \leq 5 & d = 28 & -5 \leq \text{tr}(\tau) \leq 6 \\ d = 25 & -5 \leq \text{tr}(\tau) \leq 5 & d = 29 & -6 \leq \text{tr}(\tau) \leq 6 \\ d = 26 & -5 \leq \text{tr}(\tau) \leq 5 & d = 30 & -6 \leq \text{tr}(\tau) \leq 6 \end{array}$$

Obviously, we have tested our bounds on the 47 known small Salem numbers and the results are recorded in Table 2 at the end of the paper. The method used to obtain the bounds in Theorem 1 also allows us to obtain the following result, which is less efficient than the one on the trace but which nevertheless provides some information on small Salem numbers:

Theorem 2. *Let τ be a small Salem number of degree $n = 2d \geq 4$ with minimal polynomial P . Then, we have:*

$$|P(1)| \leq \exp(0.377596d - 0.153515).$$

Corollary 2. *Let τ be a small Salem number of degree $n = 2d \geq 46$ with minimal polynomial P . Then, we have:*

$$\begin{array}{llll} d = 23 & |P(1)| \leq 5070 & d = 27 & |P(1)| \leq 22961 \\ d = 24 & |P(1)| \leq 7396 & d = 28 & |P(1)| \leq 33495 \\ d = 25 & |P(1)| \leq 10790 & d = 29 & |P(1)| \leq 48862 \\ d = 26 & |P(1)| \leq 15740 & d = 30 & |P(1)| \leq 71279 \end{array}$$

Again, we have tested our bounds on the 47 known small Salem numbers and the results are recorded in Table 3 at the end of the paper.

2 Relations between Salem numbers and totally positive algebraic integers

Let τ be a Salem number with minimal polynomial $P(z)$ of degree $n = 2d$. Then $P(z)$ is a *reciprocal* polynomial i.e., $P(z) = z^n P(1/z)$. The change of variable $x = z + \frac{1}{z} + 2$ transforms the polynomial $P(z)$ into a monic integer polynomial $Q(x)$ of degree d with the following properties:

- To the roots τ and τ^{-1} of the polynomial P is associated a root $\alpha > 4$ of the polynomial Q . Moreover, we are interesting only in small Salem numbers. It means $\tau < 1.3$ and thus $\alpha < 4.069230$.
- To the roots of modulus 1 of the polynomial P are associated $d - 1$ roots in $(0, 4)$ of the polynomial Q . Thus, all the conjugates of α are positive real numbers so α is called a *totally positive algebraic integer*.
- $\text{tr}(\alpha) = \text{tr}(\tau) + 2d$.

In [F3], we proved that if α is a totally positive algebraic integer of degree d whose minimal polynomial is different from $x - 1$, $x^2 - 3x + 1$, $x^3 - 5x^2 + 6x - 1$, $x^4 - 7x^3 + 13x^2 - 7x + 1$ and $x^4 - 7x^3 + 14x^2 - 8x + 1$ then $\text{tr}(\alpha) \geq 1.792812d$. Hence, we immediately deduce the lower bound in Theorem 1: $\text{tr}(\tau) \geq 1.792812d - 2d$.

- $P(1)=Q(4)$.

From now, we will restrict our attention to positive algebraic integers α of degree d with $4 < \alpha < 4.069230$ and all of whose other conjugates lie in $(0, 4)$.

3 Proof of the Theorems

We detail the proof for Theorem 1 and we only indicate in the last below subsection what changes to get Theorem 2.

3.1 The principle of auxiliary function

It was introduced into number theory by C. J. Smyth in [S2]. The auxiliary function considered here is :

$$\text{for } x \in (0, 4.069230), \quad f(x) = -x - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \quad (1)$$

where the c_j 's are positive real numbers and the polynomials Q_j 's are nonzero polynomials in $\mathbb{Z}[x]$.

Let m_1 be the minimum of the function f on $(0, 4)$ and m_2 be the minimum of the function f on $(4, 4.069230)$. If Q does not divide any Q_j , then we have

$$\sum_{i=1}^d f(\alpha_i) \geq (d-1)m_1 + m_2$$

i.e.,

$$-\text{tr}(\alpha) \geq (d-1)m_1 + m_2 + \sum_{1 \leq j \leq J} c_j \log \left| \prod_{i=1}^d Q_j(\alpha_i) \right|.$$

Since Q does not divide any Q_j then $\prod_{i=1}^d Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of Q and Q_j .

Hence, if α is not a root of Q_j , we have

$$-(d-1)m_1 - m_2 \geq \text{tr}(\alpha).$$

3.2 Link between the integer transfinite diameter and the auxiliary functions

Let K be a compact subset of $\mathbb{C}$ and φ be a positive function defined on K . The φ integer transfinite diameter of K is defined by

$$t_{\mathbb{Z}, \varphi}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{H \in \mathbb{Z}[X] \\ \deg(H) = n}} \sup_{z \in K} \left(|H(z)|^{\frac{1}{n}} \varphi(z) \right).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A].

In the auxiliary function (1), we replace the coefficients c_j by rational numbers a_j/q where q is a positive integer such that $q \cdot c_j$ is an integer for all $1 \leq j \leq J$. Then we can write:

$$\text{for } x \in (0, 4.069230), \quad f(x) = -x - \frac{t}{r} \log |H(x)| \geq (d-1)m_1 + m_2 \quad (2)$$

where $H = \prod_{j=1}^J Q_j^{a_j} \in \mathbb{Z}[X]$ is of degree $r = \sum_{j=1}^J a_j \deg Q_j$ and $t = \sum_{j=1}^J c_j \deg Q_j$ (this formulation was introduced by J. P. Serre). Thus we seek a polynomial $H \in \mathbb{Z}[X]$ such that

$$\sup_{x \in (0, 4.069230)} |H(x)|^{t/r} e^x \leq e^{-(d-1)m_1 - m_2}.$$

If we suppose that t is fixed, it is equivalent to find an effective upper bound for the weighted integer transfinite diameter over the interval $(0, 4.069230)$ with the weight $\varphi(x) = e^x$:

$$t_{\mathbb{Z}, \varphi}((0, 4.069230)) = \liminf_{\substack{r \geq 1 \\ r \rightarrow \infty}} \inf_{\substack{H \in \mathbb{Z}[X] \\ \deg(H) = r}} \sup_{x \in (0, 4.069230)} \left(|H(x)|^{t/r} \varphi(x) \right)$$

3.3 Construction of an explicit auxiliary function

The main point is to find a set of “good ” polynomials Q_j , i.e., which gives the best possible value for m_1 and m_2 . Until 2003, the polynomials were found heuristically. In 2003, Q. Wu [Wu] developed an algorithm that allows a systematic search of “good ” polynomials. In 2009 [F1], we made two improvements to this previous algorithm in the use of the LLL algorithm. The idea is to get good polynomials Q_j by induction. Thus, we call this algorithm the *recursive algorithm*. For more details, see [F3].

Suppose that we have some polynomials $Q_1, Q_2, \dots, Q_J$. Then we use the semi-infinite linear programming (introduced into number theory by C. Smyth [S2]) to optimize f for this set of polynomials (i.e., to get the greatest possible m_1 and m_2). We obtain the numbers $c_1, c_2, \dots, c_J$

and f in the form (2) with $t = \sum_{i=1}^J c_i \deg(Q_i)$. We seek a polynomial $R \in \mathbb{Z}[x]$ of degree k ($k = 3$ for instance) such that

$$\sup_{x \in I} |H(x)R(x)|^{t/(r+k)} e^x \leq e^{-(d-1)m_1 - m_2}$$

where $H = \prod_{j=1}^J Q_j$ and I is an interval composed with points uniformly distributed on $(0, 4.069230)$.

We want the quantity

$$\sup_{x \in I} |H(x)R(x)| \exp\left(\frac{x(r+k)}{t}\right)$$

to be as small as possible. We apply the LLL algorithm to the linear forms

$$Q(x_i)R(x_i) \exp\left(\frac{x_i(r+k)}{t}\right).$$

The x_i are control points which are points uniformly distributed on the interval $I = (0, 4.069230)$ to which we have added points where f has local minima. Thus we find a polynomial R whose irreducible factors R_j are good candidates to enlarge the set $\{Q_1, \dots, Q_J\}$. We only keep the factors R_j that have a nonzero coefficient in the newly optimized auxiliary function f . After optimization, some previous polynomials Q_j may have a zero exponent and so are removed. To obtain the upper bound in Theorem 1, k has varying from 3 to 7. The polynomials and their coefficient can be read off from Table 4.

3.4 Proof of Theorem 2

The only change is the type of the auxiliary function. In the proof of this theorem, the function involved is:

$$\text{for } x \in (0, 4.069230), \quad f(x) = -\log(x+4) - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)|$$

where, as previously, the c'_j s are positive real numbers and the polynomials Q'_j s are nonzero polynomials in $\mathbb{Z}[x]$. Then, the rest of the proof follows that of Theorem 1. The polynomials and their coefficients can be read off from Table 5.

4 Remarks on the computations

It is obvious that to be able to do the computations we had to set a value of d . As the table is complete up to degree 44, we started with $d = 23$. This gave us values for m_1 and m_2 and the corresponding upper bound for $\text{tr}(\tau)$. With these values of m_1 and m_2 , we then calculated the different other bounds for $d = 24$ to $d = 30$ and this gave the results recorded in Corollary 1. Moreover, to be sure of having optimum results, we have redone each calculation entirely for each value of d , from 24 to 30. As the bounds obtained are not better than the previous ones, we preferred to present the results under the form of Theorem 1, rather than only in the form of Corollary 1.

Table 1: The 47 known small Salem numbers

N°	Polynomial P
1	$z^8 - z^5 - z^4 - z^3 + 1$
2	$z^{10} - z^6 - z^5 - z^4 + 1$
3	$z^{10} - z^7 - z^5 - z^3 + 1$
4	$z^{10} - z^8 - z^5 - z^2 + 1$
5	$z^{10} - z^8 - z^7 + z^5 - z^3 - z^2 + 1$
6	$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$
7	$z^{12} - z^{11} + z^{10} - z^9 - z^6 - z^3 + z^2 - z + 1$
8	$z^{14} - z^{12} - z^7 - z^2 + 1$
9	$z^{14} - z^{13} - z^8 + z^7 - z^6 - z + 1$
10	$z^{14} - z^{11} - z^{10} + z^7 - z^4 - z^3 + 1$
11	$z^{14} - z^{12} - z^{11} + z^9 - z^7 + z^5 - z^3 - z^2 + 1$
12	$z^{16} - z^{15} - z^8 - z + 1$
13	$z^{18} - z^{17} - z^{10} + z^9 - z^8 - z + 1$
14	$z^{18} - z^{17} - z^{14} + z^{13} - z^9 + z^5 - z^4 - z + 1$
15	$z^{18} - z^{12} - z^{11} - z^{10} - z^9 - z^8 - z^7 - z^6 + 1$
16	$z^{18} - z^{14} - z^{12} - z^{11} - z^9 - z^7 - z^6 - z^4 + 1$
17	$z^{18} - z^{17} - z^{14} + z^{13} - z^{12} + z^{10} - z^9 + z^8 - z^6 + z^5 - z^4 - z + 1$
18	$z^{18} - z^{17} + z^{16} - z^{15} - z^{12} + z^{11} - z^{10} + z^9 - z^8 + z^7 - z^6 - z^3 + z^2 - z + 1$
19	$z^{18} - 2z^{17} + 2z^{16} - 2z^{15} + 2z^{14} - 2z^{13} + 2z^{12} - 3z^{11} + 3z^{10} - 3z^9 + 3z^8 - 3z^7 + 2z^6 - 2z^5 + 2z^4 - 2z^3 + 2z^2 - 2z + 1$
20	$z^{20} - z^{18} - z^{15} - z^5 - z^2 + 1$
21	$z^{20} - z^{18} - z^{15} - z^{12} + z^{10} - z^8 - z^5 - z^2 + 1$
22	$z^{20} - z^{19} - z^{15} + z^{14} - z^{11} + z^{10} - z^9 + z^6 - z^5 - z + 1$
23	$z^{20} - 2z^{19} + 2z^{18} - 2z^{17} + 2z^{16} - 2z^{15} + z^{14} - z^{12} + z^{11} - z^{10} + z^9 - z^8 + z^6 - 2z^5 + 2z^4 - 2z^3 + 2z^2 - 2z + 1$
24	$z^{22} - z^{21} - z^{17} + z^{11} - z^5 - z + 1$
25	$z^{22} - z^{21} - z^{20} + z^{19} - z^{13} + z^{11} - z^9 + z^3 - z^2 - z + 1$
26	$z^{22} - z^{20} - z^{19} + z^{15} + z^{14} - z^{12} - z^{11} - z^{10} + z^8 + z^7 - z^3 - z^2 + 1$
27	$z^{22} - z^{21} - z^{19} + z^{18} - z^{14} + z^{13} - z^{12} + z^{11} - z^{10} + z^9 - z^8 + z^4 - z^3 - z + 1$
28	$z^{24} - z^{23} - z^{18} - z^6 - z + 1$
29	$z^{24} - z^{23} - z^{20} + z^{19} - z^{17} + z^{16} - z^{15} + z^{13} - z^{12} + z^{11} - z^9 + z^8 - z^7 + z^5 - z^4 - z + 1$
30	$z^{26} - z^{25} - z^{20} + z^{13} - z^6 - z + 1$
31	$z^{26} - z^{24} - z^{21} - z^{18} + z^{16} + z^{13} + z^{10} - z^8 - z^5 - z^2 + 1$
32	$z^{26} - z^{24} - z^{23} + z^{19} - z^{17} - z^{16} + z^{14} + z^{13} + z^{12} - z^{10} - z^9 + z^7 - z^3 - z^2 + 1$
33	$z^{26} - z^{20} - z^{19} - z^{18} - z^{17} - z^{16} - z^{15} - z^{14} - z^{13} - z^{12} - z^{11} - z^{10} - z^9 - z^8 - z^7 - z^6 + 1$
34	$z^{26} - z^{25} - z^{22} + z^{21} - z^{20} + z^{18} - z^{17} + z^{16} - z^{14} + z^{13} - z^{12} + z^{10} - z^9 + z^8 - z^6 + z^5 - z^4 - z + 1$
35	$z^{26} - 2z^{25} + z^{24} + z^{23} - 2z^{22} + z^{21} - z^{18} + z^{17} - z^{15} + z^{14} - z^{13} + z^{12} - z^{11} + z^9 - z^8 + z^5 - 2z^4 + z^3 + z^2 - 2z + 1$
36	$z^{26} - z^{25} - z^{24} + 2z^{22} - 2z^{20} - z^{19} + 2z^{18} + 2z^{17} - 2z^{16} - 2z^{15} + 3z^{13} - 2z^{11} - 2z^{10} + 2z^9 + 2z^8 - z^7 - 2z^6 + 2z^4 - z^2 - z + 1$
37	$z^{28} - z^{24} - z^{23} - z^{22} - z^{21} - z^{20} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} - z^8 - z^7 - z^6 - z^5 - z^4 + 1$
38	$z^{30} - z^{29} - z^{22} - z^{18} - z^{15} - z^{12} - z^8 - z + 1$
39	$z^{30} - z^{28} - z^{25} - z^{24} + z^{20} + z^{17} - z^{15} + z^{13} + z^{10} - z^6 - z^5 - z^2 + 1$
40	$z^{30} - z^{25} - z^{24} - z^{23} - z^{22} - z^{21} - z^{20} + z^{15} - z^{10} - z^9 - z^8 - z^7 - z^6 - z^5 + 1$
41	$z^{30} - 2z^{29} + 2z^{28} - 2z^{27} + z^{26} - z^{24} + 2z^{23} - 2z^{22} + z^{21} - z^{19} + z^{18} - z^{17} + z^{16} - z^{15} + z^{14} - z^{13} + z^{12} - z^{11} + z^9 - 2z^8 + 2z^7 - z^6 + z^4 - 2z^3 + 2z^2 - 2z + 1$
42	$z^{34} - z^{33} - z^{30} + z^{29} - z^{28} + z^{26} - z^{25} + z^{24} - z^{22} + z^{21} - z^{20} + z^{18} - z^{17} + z^{16} - z^{14} + z^{13} - z^{12} + z^{10} - z^9 + z^8 - z^6 + z^5 - z^4 - z + 1$
43	$z^{34} - z^{33} - z^{31} + z^{29} + z^{27} - 2z^{26} + z^{23} + z^{22} - z^{21} - z^{20} - z^{19} + z^{18} + z^{17} + z^{16} - z^{15} - z^{14} - z^{13} + z^{12} + z^{11} - 2z^8 + z^7 + z^5 - z^3 - z + 1$
44	$z^{36} + z^{35} - z^{33} - 2z^{32} - 2z^{31} - z^{30} + z^{28} + z^{27} - z^{25} - z^{24} + z^{22} + z^{21} - z^{19} - z^{18} - z^{17} + z^{15} + z^{14} - z^{12} - z^{11} + z^9 + z^8 - z^6 - 2z^5 - 2z^4 - z^3 + z + 1$
45	$z^{40} - z^{37} - z^{35} - z^{33} - z^{31} - z^{29} + z^{26} + z^{24} + z^{22} + z^{20} + z^{18} + z^{16} + z^{14} - z^{11} - z^9 - z^7 - z^5 - z^3 + 1$
46	$z^{44} - z^{43} - z^{37} - z^{33} + z^{25} + z^{22} + z^{19} - z^{11} - z^7 - z + 1$
47	$z^{46} - z^{42} - z^{41} - z^{40} - z^{39} + z^{25} + z^{24} + z^{23} + z^{22} + z^{21} - z^7 - z^6 - z^5 - z^4 + 1$

Table 2: The bounds of Theorem 1 for the 47 known small Salem numbers

N^*	$\deg(P)$	Lower bound	$\text{tr}(P)$	Upper bound
1	8	-0.82875200	0	1.5389470
2	10	-1.0359400	0	1.7324130
3	10	-1.0359400	0	1.7324130
4	10	-1.0359400	0	1.7324130
5	10	-1.0359400	0	1.7324130
6	10	-1.0359400	-1	1.7324130
7	12	-1.2431280	1	1.9258790
8	14	-1.4503160	0	2.1193450
9	14	-1.4503160	1	2.1193450
10	14	-1.4503160	0	2.1193450
11	14	-1.4503160	0	2.1193450
12	16	-1.6575040	1	2.3128110
13	18	-1.8646920	1	2.5062770
14	18	-1.8646920	1	2.5062770
15	18	-1.8646920	0	2.5062770
16	18	-1.8646920	0	2.5062770
17	18	-1.8646920	1	2.5062770
18	18	-1.8646920	1	2.5062770
19	18	-1.8646920	2	2.5062770
20	20	-2.0718800	0	2.6997430
21	20	-2.0718800	0	2.6997430
22	20	-2.0718800	1	2.6997430
23	20	-2.0718800	2	2.6997430
24	22	-2.2790680	1	2.8932090
25	22	-2.2790680	1	2.8932090
26	22	-2.2790680	0	2.8932090
27	22	-2.2790680	1	2.8932090
28	24	-2.4862560	1	3.0866750
29	24	-2.4862560	1	3.0866750
30	26	-2.6934440	1	3.2801410
31	26	-2.6934440	0	3.2801410
32	26	-2.6934440	0	3.2801410
33	26	-2.6934440	0	3.2801410
34	26	-2.6934440	1	3.2801410
35	26	-2.6934440	2	3.2801410
36	26	-2.6934440	1	3.2801410
37	28	-2.9006320	0	3.4736070
38	30	-3.1078200	1	3.6670730
39	30	-3.1078200	0	3.6670730
40	30	-3.1078200	0	3.6670730
41	30	-3.1078200	2	3.6670730
42	34	-3.5221960	1	4.0540050
43	34	-3.5221960	1	4.0540050
44	36	-3.7293840	-1	4.2474710
45	40	-4.1437600	0	4.6344030
46	44	-4.5581360	1	5.0213350
47	46	-4.7653240	0	5.2148010

Table 3: The bound of Theorem 2 for the 47 known small Salem numbers

N°	$\deg(P)$	$ P(1) $	Upper bound
1	8	1	3.8840134
2	10	1	5.6658957
3	10	1	5.6658957
4	10	1	5.6658957
5	10	1	5.6658957
6	10	1	5.6658957
7	12	1	8.2652584
8	14	1	12.057140
9	14	1	12.057140
10	14	1	12.057140
11	14	1	12.057140
12	16	1	17.588636
13	18	1	25.657835
14	18	1	25.657835
15	18	5	25.657835
16	18	5	25.657835
17	18	1	25.657835
18	18	1	25.657835
19	18	1	25.657835
20	20	2	37.428970
21	20	3	37.428970
22	20	1	37.428970
23	20	1	37.428970
24	22	1	54.600389
25	22	1	54.600389
26	22	1	54.600389
27	22	1	54.600389
28	24	2	79.649599
29	24	1	79.649599
30	26	1	116.19072
31	26	1	116.19072
32	26	1	116.19072
33	26	13	116.19072
34	26	1	116.19072
35	26	1	116.19072
36	26	1	116.19072
37	28	3	169.49594
38	30	5	247.25619
39	30	1	247.25619
40	30	9	247.25619
41	30	1	247.25619
42	34	1	526.16613
43	34	1	526.16613
44	36	7	767.55720
45	40	1	1633.3771
46	44	1	3475.8593
47	46	1	5070.4913

Table 4: Polynomials and their coefficients used for the upper bound of Theorem 1

Polynomials	Coefficients
x	0.0019679665
$x - 2$	0.1898490761
$x - 3$	0.5308931151
$x - 4$	0.5815137153
$x^2 - 4x + 1$	0.0725582571
$x^2 - 4x + 2$	0.0334017400
$x^2 - 5x + 5$	0.1931207414
$x^3 - 6x^2 + 9x - 1$	0.0288203880
$x^3 - 6x^2 + 9x - 3$	0.0185605107
$x^3 - 7x^2 + 14x - 7$	0.1095768403
$x^4 - 9x^3 + 26x^2 - 24x + 1$	0.0514099247
$x^5 - 11x^4 + 43x^3 - 69x^2 + 36x + 1$	0.0551475719
$x^5 - 11x^4 + 44x^3 - 77x^2 + 55x - 11$	0.0324052775
$x^6 - 14x^5 + 76x^4 - 199x^3 + 252x^2 - 132x + 17$	0.0030224114
$2x^7 - 34x^6 + 236x^5 - 856x^4 + 1719x^3 - 1854x^2 + 945x - 163$	0.0058282451

Table 5: Polynomials and their coefficients used for the upper bound of Theorem 2

Polynomials	Coefficients
x	0.3620883142
$x - 1$	0.5053687655
$x - 2$	0.3014609143
$x - 3$	0.1287326889
$x^2 - 3x + 1$	0.2940495893
$x^2 - 4x + 1$	0.0027131692
$x^2 - 4x + 2$	0.1046760710
$x^2 - 5x + 5$	0.0413670035
$x^3 - 5x^2 + 6x - 1$	0.1989461843
$x^3 - 6x^2 + 9x - 1$	0.1138658462
$x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1$	0.1023209293
$x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1$	0.0544779065

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