Unbiased Filtering for State and Unknown Input with Delay

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Abstract: In this paper, we consider linear network systems with unknown inputs. We present an unbiased recursive algorithm that simultaneously estimates states and inputs. We focus on delay-ℓ left invertible systems with intrinsic delay $\ell \geq 1$, where the input reconstruction is possible only by using outputs up to ℓ time steps later in the future. By showing an equivalence with a descriptor system, we state conditions under which the time-varying filter converges to a stationary stable filter, involving the solution of a discrete-time algebraic Riccati equation.

Keywords: Simultaneous input and state estimation, unknown input, delayed filter, unbiased estimator, network system, cyber-physical security.

1. INTRODUCTION

Monitoring a network system can require estimating not only the network state but also unknown inputs affecting it. For instance, in a social network with a fraction of the agents being leaders and the rest being followers, the influence of the leader on the dynamics of the follower can be thought of as an unknown input, so that unknown input estimation techniques can be used to understand how much influence a given leader has on the followers. Unknown inputs can also represent malicious attacks, such as denial-of-service attacks, false data injection attacks, or replay attacks (see Liang et al. (2016) and Slay and Miller (2007), to cite a few).

The effect of the unknown input on the outputs might not be immediate, that is, there might be a delay between the input injection and its measurable effect, due to the distance between the affected states and the sensors. Such delay needs to be considered in the estimation: at time k, when output measurement y_h is available for all $h \leq k$, one cannot estimate input u_h and state x_h for all $h \leq k$, but rather can estimate input for $h \leq k - \ell$ and state for $h \leq k - \ell + 1$.

Delay-ℓ unknown input estimation for stochastic linear systems, in particular for the case of $\ell = 1$, has a rich literature which can be divided in two categories. The first one considers systems for which there is no direct feedthrough between the input and output. An algorithm, which yields minimum-variance unbiased (MVU) estimates of state and unknown input, with delay 1, was given in Gillijns and De Moor (2007a) in a centralized way, while a distributed version was proposed in Esna-Ashari et al. (2012). In Chavan et al. (2014) the case of an arbitrary delay was considered, under some restrictive assumptions on the system matrices. The second category considers direct feedthrough between the unknown input and the output. For delay 1, an MVU estimator with a feedthrough matrix having full column rank was studied

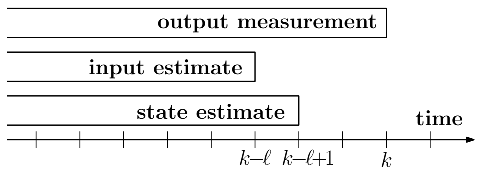

Fig. 1. Timeline of delay-ℓ estimation.

in Gillijns and De Moor (2007b). The full rank requirement was relaxed in Yong et al. (2016). For the more general case of $\ell > 1$, Kirtikar et al. (2011) and Yong et al. (2015) study the conditions for existence of a state and input estimator with delay ℓ, and the latter proposes an algorithm, but for the particular case of $\ell = 1$.

In the present paper, for systems with arbitrary direct feedthrough, we provide a recursive linear algorithm for estimating both states and inputs with delay ℓ: at time k, given output measurements up to y_k, an estimate of $u_{k-\ell}$ and $x_{k-\ell+1}$ is obtained (see Figure 1).

The paper is organized as follows. In Section 2 we state the problem being studied in this paper and summarize the preliminary material needed for developing the results, while we present our main result in Section 3. Section 4 deals with an appropriately chosen numerical example, while some concluding remarks are presented in Section 5.

Notation: As usual, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, while $\mathbb{E}$ denotes the expectation operator. $A = \text{diag}(A_1, A_2, \ldots, A_N)$ denotes a block-diagonal matrix whose blocks along the diagonal are $A_1, A_2, \ldots, A_N$. I_a denotes identity matrix of size a and $0_{a \times b}$ denotes an all-zero matrix with a rows and b columns; indexes denoting size will be often omitted, if clear from the context. $\ker A$ and $\operatorname{Im} A$ denote the kernel and the range of a matrix A, and $\dim V$ denotes the dimension of a vector space V.

2. PROBLEM FORMULATION

Consider a linear time-invariant system that is subject to unknown inputs, and whose dynamics are given as follows:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + w_k \\ y_k = Cx_k + Du_k + v_k \end{cases} \tag{1}$$

with state vector $x_k \in \mathbb{R}^n$, unknown input $u_k \in \mathbb{R}^p$ and output $y_k \in \mathbb{R}^m$; the matrices A, B, C, and D being of appropriate dimensions. Process noise w_k and measurement noise v_k are assumed to be white, zero mean, mutually uncorrelated with covariance matrices Q and R, respectively.

In what follows, we introduce various notions related to the joint input and state estimation problem.

Definition 1. (Definition 2.5 Sundaram (2012)). Let ℓ be a non-negative integer. The system $\{A, B, C, D\}$ is delay-ℓ left invertible if the unknown input u_0 is uniquely determined by the initial state x_0 and the output sequence $\{y_0, y_1, \ldots, y_\ell\}$ (in the absence of noise). The smallest ℓ for which this condition is satisfied is called the inherent delay of the system. ■

We define $\Gamma_\ell \in \mathbb{R}^{(\ell+1)m\times(\ell+1)p}$ (known as delay-ℓ left-invertibility matrix) and $N_\ell \in \mathbb{R}^{(\ell(m+n)+m)\times(\ell(p+n)+p)}$ as:

$$\Gamma_\ell = \begin{bmatrix} D & 0 & \ldots & 0 \\ CB & D & \ddots & 0 \\ CAB & CB & \ddots & \ddots & . \\ . & . & \ddots & \ddots & . \\ CA^{\ell-1}B & CA^{\ell-2}B & \ldots. & CB & D \end{bmatrix}$$

and

$$N_\ell = \begin{bmatrix} D & 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\ B & -I & \ldots & \ldots & & \ldots & \ldots & 0 \\ 0 & C & D & \ldots & \ldots & & \ldots & 0 \\ 0 & A & B & -I & \ldots & & \ldots & 0 \\ . & . & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ . & . & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots C & D \end{bmatrix} = \begin{bmatrix} E & 0 \\ F_\ell & N_{\ell-1} \end{bmatrix},$$

where $E = \begin{bmatrix} D & 0 \\ B & -I_n \end{bmatrix}$, $F = \begin{bmatrix} 0 & -C \\ 0 & -A \end{bmatrix}$, $F_\ell = \begin{bmatrix} -F \\ 0 \end{bmatrix}$.

By a suitable re-writing of (1), over ℓ consecutive time-steps, the following system of equations is readily obtained:

$$N_\ell \begin{bmatrix} u_{k-\ell} \\ x_{k-\ell+1} \\ u_{k-\ell+1} \\ x_{k-\ell+2} \\ \vdots \\ u_{k-1} \\ x_k \\ u_k \end{bmatrix} = \begin{bmatrix} y_{k-\ell} - Cx_{k-\ell} \\ -Ax_{k-\ell} \\ y_{k-\ell+1} \\ 0 \\ \vdots \\ y_{k-1} \\ 0 \\ y_k \end{bmatrix} - \begin{bmatrix} v_{k-\ell} \\ w_{k-\ell} \\ v_{k-\ell+1} \\ w_{k-\ell+1} \\ \vdots \\ v_{k-1} \\ w_{k-1} \\ v_k \end{bmatrix} \tag{2}$$

The following algebraic characterizations for delay-ℓ left invertibility can be provided in terms of Γ_ℓ and N_ℓ.

Proposition 2. The following statements are equivalent:

1) The system $\{A, B, C, D\}$ is delay-ℓ left invertible;
2) $\text{rank}(\Gamma_\ell) = p + \text{rank}(\Gamma_{\ell-1})$;
3) $\text{rank}(N_\ell) = p + n + \text{rank}(N_{\ell-1})$. ■

Proof: Equivalence of items 1) and 2) is stated in Massey and Sain (1968) (Theorem 4). The proof of equivalence of items 1) and 3) is based on the same idea, as detailed below (see Garin (2017) for the case $\ell = 1$). Notice that u_0 is uniquely determined by $x_0, y_0, y_1, \ldots, y_\ell$ if and only if both u_0 and x_1 are uniquely determined by x_0, $y_0, y_1, \ldots, y_\ell$, since x_1 is completely determined by u_0 and x_0. From (2), setting noise to zero, the following is immediate:

$$\begin{bmatrix} E & 0 \\ F_\ell & N_{\ell-1} \end{bmatrix} \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \vdots \\ u_{\ell-1} \\ x_\ell \\ u_\ell \end{bmatrix} = \begin{bmatrix} y_0 - Cx_0 \\ -Ax_0 \\ y_1 \\ 0 \\ \vdots \\ y_{\ell-1} \\ 0 \\ y_\ell \end{bmatrix}. \tag{3}$$

The solution for the first part u_0, x_1 of the unknown vector is unique if and only if $\text{rank}(N_\ell) = p + n + \text{rank}(N_{\ell-1})$. □

In the noise-free case, it was shown in Sundaram and Hadjicostis (2007) that delay-ℓ left invertibility and strong detectability, i.e., rank $\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = n + p, \forall z \in \mathbb{C}$ s.t. $|z| \geq 1$, are necessary and sufficient to ensure the existence of an observer with delay ℓ[1]. In the next section, we will take noise into consideration. Assuming the system is delay-ℓ left-invertible, we will construct an unbiased linear estimator for the input and the state. Further assuming that the system is strongly detectable and satisfies a suitable reachability condition, we will use results from the analysis of descriptor systems to ensure the convergence to stationary stable error dynamics, involving the unique solution of a discrete-time algebraic Riccati equation.

3. MAIN RESULT

3.1 Construction of a recursive filter with delay ℓ

We consider a filter structure where, at time k, we estimate $u_{k-\ell}$ and $x_{k-\ell+1}$ as linear functions of the measurements $y_{k-\ell+1}, \ldots, y_k$ and of the previous state estimate $\hat{x}_{k-\ell}$, assumed to be unbiased. Precisely, we look for an estimate of the following form:

$$\begin{bmatrix} \hat{u}_{k-\ell} \\ \hat{x}_{k-\ell+1} \end{bmatrix} = M_k \tilde{y}_{k-\ell,\ell}\,, \tag{4}$$

where the innovation $\tilde{y}_{k-\ell,\ell}$ is defined as

$$\tilde{y}_{k-\ell,\ell} = \bar{y}_{k-\ell,\ell} - F_{\ell+1} \begin{bmatrix} 0 \\ I_n \end{bmatrix} \hat{x}_{k-\ell} \tag{5}$$

with $\bar{y}^T_{k-\ell,\ell} = \left[y^T_{k-\ell}, 0, y^T_{k-\ell+1}, 0, \ldots, y^T_{k-1}, 0, y^T_k\right]$.

The matrix M_k will be constructed so that the estimates are unbiased with minimum error covariance, and then we will propose an approximation leading to sub-optimal covariance but simpler implementation.

For this purpose, we set the linear model linking the variables to be estimated with the available information. In-

[1] A delayed observer is an observer capable to reconstruct the state despite the presence of the unknown input. In our paper, an observer with delay ℓ reconstructs $x_{k-\ell+1}$ from outputs up to y_k; Sundaram and Hadjicostis (2007) uses a different convention and denotes the same delay as $\ell - 1$.

troducing the notation $\tilde{\epsilon}_{k-\ell,\ell} = F_{\ell+1} \begin{bmatrix} 0 \\ I_n \end{bmatrix} (x_{k-\ell} - \hat{x}_{k-\ell}) + \bar{\epsilon}_{k-\ell,\ell}$, with
$\bar{\epsilon}^T_{k-\ell,\ell} = \left[v^T_{k-\ell}, w^T_{k-\ell}, v^T_{k-\ell+1}, w^T_{k-\ell+1}, \ldots, v^T_{k-1}, w^T_{k-1}, v^T_k\right]$,
(2) can be re-written as

$$\begin{bmatrix} E \\ F_\ell \end{bmatrix} \begin{bmatrix} u_{k-\ell} \\ x_{k-\ell+1} \end{bmatrix} + \begin{bmatrix} 0 \\ N_{\ell-1} \end{bmatrix} \begin{bmatrix} u_{k-\ell+1} \\ x_{k-\ell+2} \\ \vdots \\ u_{k-1} \\ x_k \\ u_k \end{bmatrix} = \tilde{y}_{k-\ell,\ell} - \tilde{\epsilon}_{k-\ell,\ell} \,. \quad (6)$$

Assuming that $\hat{x}_{k-\ell}$ is unbiased, then $\tilde{\epsilon}_{k-\ell,\ell}$ has zero mean and covariance $\Sigma_{k-\ell} = \mathbb{E}(\tilde{\epsilon}_{k-\ell,\ell}\tilde{\epsilon}^T_{k-\ell,\ell})$.

Lemma 3. Assuming that $\hat{x}_{k-\ell}$ is unbiased, the linear estimator (4) is unbiased if and only if

$$M_k \begin{bmatrix} E \\ F_\ell \end{bmatrix} = I \text{ and } M_k \begin{bmatrix} 0 \\ N_{\ell-1} \end{bmatrix} = 0 \,. \quad (7)$$

■

Proof: Pre-multiplying (6) by M_k, taking expectation, and recalling that $\mathbb{E}\tilde{\epsilon}_{k-\ell} = 0$, we obtain

$$M_k \begin{bmatrix} E \\ F_\ell \end{bmatrix} \begin{bmatrix} u_{k-\ell} \\ \mathbb{E}x_{k-\ell+1} \end{bmatrix} + M_k \begin{bmatrix} 0 \\ N_{\ell-1} \end{bmatrix} \begin{bmatrix} u_{k-\ell+1} \\ \mathbb{E}x_{k-\ell+2} \\ \vdots \\ u_{k-1} \\ \mathbb{E}x_k \\ u_k \end{bmatrix} = \mathbb{E}(M_k \tilde{y}_{k-\ell,\ell}).$$

Recall that $M_k\tilde{y}_{k-\ell,\ell} = \left[\begin{smallmatrix} \hat{u}_{k-\ell} \\ \hat{x}_{k-\ell+1} \end{smallmatrix}\right]$, and that we want this estimate to be unbiased for all input and state sequence (we do not allow M_k to be state-dependent). This is true if and only if the two conditions in (7) are fulfilled. □

When the system is delay-ℓ left invertible, it is always possible to find a matrix M_k fulfilling the two conditions (7) [2]. Such a matrix can be written as a product $M_k = G_k H$, with $H \left[\begin{smallmatrix} 0 \\ N_{\ell-1} \end{smallmatrix}\right] = 0$, H having full row-rank, and a number of rows equal to $n+m+\dim\ker N^T_{\ell-1}$, i.e., rows of H form a basis of $\ker\left[0, N^T_{\ell-1}\right]$. Pre-multiplying (6) by H, we obtain

$$H \begin{bmatrix} E \\ F_\ell \end{bmatrix} \begin{bmatrix} u_{k-\ell} \\ x_{k-\ell+1} \end{bmatrix} = H\tilde{y}_{k-\ell,\ell} - H\tilde{\epsilon}_{k-\ell,\ell}. \quad (8)$$

The covariance of $H\tilde{\epsilon}_{k-\ell,\ell}$ is $H\Sigma_{k-\ell}H^T$, which is a positive definite matrix since H has full row rank. From (8), by Gauss-Markov theorem, the BLUE estimate is given by

$$\begin{bmatrix} \hat{u}_{k-\ell} \\ \hat{x}_{k-\ell+1} \end{bmatrix} = P_{k-\ell+1} \left[E^T, F^T_\ell\right] H^T \left(H\Sigma_{k-\ell}H^T\right)^{-1} H\tilde{y}_{k-\ell,\ell}, \quad (9)$$

where

$$P_{k-\ell+1} = \left(\left[E^T, F^T_\ell\right] H^T \left(H\Sigma_{k-\ell}H^T\right)^{-1} H \left[\begin{smallmatrix} E \\ F_\ell \end{smallmatrix}\right]\right)^{-1} \quad (10)$$

is its error covariance matrix.

Remark: The BLUE estimate (9)-(10) has an expression which involves the matrix H. However, having fixed one matrix H such that $H \left[\begin{smallmatrix} 0 \\ N_{\ell-1} \end{smallmatrix}\right] = 0$, having full row-rank, and a number of rows equal to $n + m + \dim\ker N^T_{\ell-1}$, any other matrix $\tilde{H}$ satisfying the same properties can be obtained as $\tilde{H} = JH$, for some J square and invertible matrix (a change of basis of the row space). Then, looking at (9)-(10), it is easy to see that replacing H with JH gives exactly the same estimate, since J cancels out. The estimate being the same irrespective of the choice of H, we can use any construction of H; we will use the following one. We construct a matrix U_2 whose columns form an orthonormal basis of $\ker(N^T_{\ell-1})$, as follows. We consider the (full size) singular value decomposition (svd) $N_{\ell-1} = USV^T$, and the partition $U = [U_1, U_2]$, where U_1 has r columns and U_2 has $d = (\ell-1)(m+n)+m-r$ columns, $r = \operatorname{rank} N_{\ell-1}$. Using U_2, we define $H = \begin{bmatrix} I_{n+m} & 0 \\ 0 & U_2^T \end{bmatrix}$ □.

The difficulty in implementing a filter based on the BLUE estimate (9)-(10) lies in the error covariance matrix $\Sigma_{k-\ell}$. Below, we propose a simpler albeit suboptimal filter, where we approximate $\Sigma_{k-\ell}$ by the block-diagonal matrix $\operatorname{diag}(\hat{\Sigma}_{k-\ell}, \bar{\Sigma}_\ell)$, where $\bar{\Sigma}_\ell = \operatorname{diag}(R, Q, \cdots, R, Q, R)$ and $\hat{\Sigma}_{k-\ell} = \begin{bmatrix} R + CP^{xx}_{k-\ell}C^T & CP^{xx}_{k-\ell}A^T \\ AP^{xx}_{k-\ell}C^T & Q + AP^{xx}_{k-\ell}A^T \end{bmatrix}$. This amounts at disregarding cross-correlations between $\hat{x}_{k-\ell}$ and $y_{k-\ell}$, …, y_{k-1}. With this approximation, together with the above construction of H, we can exploit the block-diagonal structure of these matrices, to obtain a simpler version of (9)-(10), reminiscent of a Kalman filter. Notice that

$$\left(H \begin{bmatrix} \hat{\Sigma}_{k-\ell} & 0 \\ 0 & \bar{\Sigma}_\ell \end{bmatrix} H^T\right)^{-1} = \begin{bmatrix} \hat{\Sigma}^{-1}_{k-\ell} & 0 \\ 0 & (U_2^T\bar{\Sigma}_\ell U_2)^{-1} \end{bmatrix}.$$

We define $\Psi_\ell = F^T_\ell U_2(U_2^T\bar{\Sigma}_\ell U_2)^{-1}U_2^T$ and $\Omega_\ell = \Psi_\ell F_\ell$. With these definitions, and with the above-mentioned approximation for $\Sigma_{k-\ell}$, from (10) we get

$$P_{k-\ell+1} = \left(E^T\hat{\Sigma}^{-1}_{k-\ell}E + \Omega_\ell\right)^{-1} \quad (11)$$

and (9) becomes

$$\begin{bmatrix} \hat{u}_{k-\ell} \\ \hat{x}_{k-\ell+1} \end{bmatrix} = P_{k-\ell+1} \left[E^T\hat{\Sigma}^{-1}_{k-\ell}, \Psi_\ell\right] \tilde{y}_{k-\ell,\ell}.$$

Recalling the definition of $\tilde{y}_{k-\ell,\ell}$ (see (5)), one can see that $\tilde{y}^T_{k-\ell,\ell} = \left[(y_{k-\ell} - CA\hat{x}_{k-\ell})^T, -A\hat{x}^T_{k-\ell}, \bar{y}^T_{k-\ell+1,\ell-1}\right]$. Hence, partitioning columns of $E^T\hat{\Sigma}^{-1}_{k-\ell}$ in two blocks of size m and n denoted as $E^T\hat{\Sigma}^{-1}_{k-\ell} = [K^{(1)}_{k-\ell}, -K^{(2)}_{k-\ell}]$, we obtain

$$\begin{bmatrix} \hat{u}_{k-\ell} \\ \hat{x}_{k-\ell+1} \end{bmatrix} = P_{k-\ell+1} \Big(K^{(1)}_{k-\ell}(y_{k-\ell} - C\hat{x}_{k-\ell}) + K^{(2)}_{k-\ell}A\hat{x}_{k-\ell} + \Psi_\ell\,\bar{y}_{k-\ell+1,\ell-1} \Big). \quad (12)$$

The proposed recursive filter is summarized in the following algorithm.

Algorithm: Delay-$\boldsymbol{\ell}$ unbiased recursive estimator for state and unknown input.

Pre-processing: Given system matrices A, B, C, D and noise covariance matrices R, Q, build:

- $E = \begin{bmatrix} D & 0 \\ B & -I_n \end{bmatrix}$, $F = \begin{bmatrix} 0 & -C \\ 0 & -A \end{bmatrix}$, $F_\ell = \begin{bmatrix} -F \\ 0_{(\ell-2)(m+n)+m,\, p+n} \end{bmatrix}$,
- $N_{\ell-1} = \begin{bmatrix} D & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ B & -I & \ldots & \ldots & & \ldots & \ldots & 0 \\ 0 & C & D & \ldots & \ldots & & \ldots & 0 \\ 0 & A & B & -I & \ldots & & \ldots & 0 \\ . & . & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ . & . & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots C & D \end{bmatrix}$, where D appears ℓ times,

[2] This means that also in the case $D = 0$ we consider a larger family of systems than Chavan et al. (2014), where the proposed filter was unbiased only under further assumptions.

- $\bar{\Sigma}_\ell = \mathrm{diag}(R, Q, \cdots, R, Q, R)$, where R appears ℓ times,

Compute:

- $[U, S, V] = \mathrm{svd}(N_{\ell-1})$ and build $U_2 = U \left[\begin{smallmatrix} 0 \\ I_d \end{smallmatrix}\right]$, $d = \ell m + (\ell-1)n - \mathrm{rank}\, N_{\ell-1}$
- $\Psi_\ell = F_\ell^T U_2 (U_2^T \bar{\Sigma}_\ell U_2)^{-1} U_2^T$, $\Omega_\ell = \Psi_\ell F_\ell$.

Initialization: $\hat{x}_0$, $P_0^{xx} > 0$.

Filter iterations: for $k \geq \ell$, use $\hat{x}_{k-\ell}$, $P_{k-\ell}^{xx}$, and measurement $y_{k-\ell}, \ldots, y_k$ to compute $\hat{u}_{k-\ell}$, $\hat{x}_{k-\ell+1}$ and $P_{k-\ell+1}^{xx}$, as follows.

Extended measurements vector:
$\bar{y}_{k-\ell+1,\ell-1} = \left[y_{k-\ell+1}^T, 0, y_{k-\ell+2}^T, 0, \ldots, y_{k-1}^T, 0, y_k^T\right]^T$.

Approximate covariances and gains:

- $\hat{\Sigma}_{k-\ell} = \begin{bmatrix} R + CP_{k-\ell}^{xx}C^T & CP_{k-\ell}^{xx}A^T \\ AP_{k-\ell}^{xx}C^T & Q + AP_{k-\ell}^{xx}A^T \end{bmatrix}$,
- compute $\hat{\Sigma}_{k-\ell}^{-1}$,
- $P_{k-\ell+1} = \left(E^T \hat{\Sigma}_{k-\ell}^{-1} E + \Omega_\ell\right)^{-1}$,
- $K_{k-\ell}^{(1)} = E^T \hat{\Sigma}_{k-\ell}^{-1} \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]$, $\quad K_{k-\ell}^{(2)} = -E^T \hat{\Sigma}_{k-\ell}^{-1} \left[\begin{smallmatrix} 0 \\ I_n \end{smallmatrix}\right]$.

Estimates:

$$\begin{bmatrix} \hat{u}_{k-\ell} \\ \hat{x}_{k-\ell+1} \end{bmatrix} = P_{k-\ell+1}\Big(K_{k-\ell}^{(1)}(y_{k-\ell} - C\hat{x}_{k-\ell}) + K_{k-\ell}^{(2)} A\hat{x}_{k-\ell} + \Psi_\ell\, \bar{y}_{k-\ell+1,\ell-1}\Big).$$

Approximate state error covariance:

$$P_{k-\ell+1}^{xx} = [0, I_n]\, P_{k-\ell+1} \left[\begin{smallmatrix} 0 \\ I_n \end{smallmatrix}\right].$$

3.2 Performance analysis

In order to analyze the performance of the proposed filter, we first show its equivalence with a recursive minimum variance estimator designed within the linear descriptor systems framework. We introduce the notation $z_k^T = \left[u_{k-\ell-1}^T, x_{k-\ell}^T\right]$, $\bar{y}_k^T = \left[y_{k-\ell}^T, 0\right]$, $\epsilon_k^T = -\left[v_{k-\ell}^T, w_{k-\ell}^T\right]$, and $\chi_{k,\ell} = U_2^T \bar{y}_{k-\ell,\ell-1}$ and we refer to Sect. 3.1 for other notations.

Proposition 4. The system (1) can be written in the following descriptor form

$$\begin{cases} Ez_{k+1} = Fz_k + \bar{y}_k + \epsilon_k \\ \chi_{k,\ell} = U_2^T F_\ell z_k + U_2^T \bar{\epsilon}_{k-\ell,\ell-1}\,. \end{cases} \tag{13}$$

Proof: Similarly to (6), from (2) we obtain

$$\left[\begin{smallmatrix} E \\ F_\ell \end{smallmatrix}\right] \left[\begin{smallmatrix} u_{k-\ell} \\ x_{k-\ell+1} \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ N_{\ell-1} \end{smallmatrix}\right] \begin{bmatrix} u_{k-\ell+1} \\ x_{k-\ell+2} \\ \cdots \\ \cdots \\ x_k \\ u_k \end{bmatrix} = \left[\begin{smallmatrix} \bar{y}_k \\ \bar{y}_{k-\ell+1,\ell-1} \end{smallmatrix}\right] - \left[\begin{smallmatrix} -F \\ 0_{(\ell-1)(m+n).(p+n)} \end{smallmatrix}\right] \left[\begin{smallmatrix} u_{k-\ell-1} \\ x_{k-\ell} \end{smallmatrix}\right] - \left[\begin{smallmatrix} -\epsilon_k \\ \bar{\epsilon}_{k-\ell+1,\ell-1} \end{smallmatrix}\right].$$

This can be rewritten as

$$Ez_{k+1} = Fz_k + \bar{y}_k + \epsilon_k \tag{14}$$

$$F_\ell z_{k+1} + N_{\ell-1} \begin{bmatrix} u_{k-\ell+1} \\ x_{k-\ell+2} \\ \cdots \\ \cdots \\ x_k \\ u_k \end{bmatrix} = \bar{y}_{k-\ell+1,\ell-1} - \bar{\epsilon}_{k-\ell+1,\ell-1}. \tag{15}$$

Equation (14) is the state equation. Pre-multiplying (15) with U_2^T, noting that $U_2^T N_{\ell-1} = 0$, and replacing $k+1$ with k, we get $U_2^T F_\ell z_k = U_2^T \bar{y}_{k-\ell,\ell-1} - U_2^T \bar{\epsilon}_{k-\ell,\ell-1}$, which gives the measurement equation. □

Notice that (13) is a linear descriptor system, with state z_k, measurements $\chi_{k,\ell}$ and known input $\bar{y}_k$. Its process noise ϵ_k has covariance $\Sigma = \mathrm{diag}(R, Q)$, while the measurement noise $U_2^T \bar{\epsilon}_{k-\ell,\ell-1}$ has covariance $U_2^T \bar{\Sigma}_\ell U_2$.

By applying (Darouach et al., 1993b, Thm. 3) and (Darouach et al., 1993a, Thm. 4) to the descriptor system (13), we obtain the following filter and its stability and convergence properties. We introduce the required notation and then give the result in Prop. 5 below.

Assuming $\mathrm{rank} \left[\begin{smallmatrix} E \\ U_2^T F_\ell \end{smallmatrix}\right] = n + p$, let E_1 be a non-singular upper triangular matrix, of size $n + p$, obtained as

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} = T \begin{bmatrix} E \\ U_2^T F_\ell \end{bmatrix},$$

with T an orthogonal matrix; this decomposition can be obtained using QR factorization. Then, use T to obtain

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = T \begin{bmatrix} F \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} W_1 & S_1 \\ S_1^T & W_2 \end{bmatrix} = T\, \mathrm{diag}(\Sigma, U_2^T \bar{\Sigma}_\ell U_2) T^T.$$

Let $Q_s^{1/2}$ denote any square root of

$$Q_s = E_1^{-1}\left(W_1 - S_1 W_2^{-1} S_1^T\right) E_1^{-T}$$

and let $F_s = E_1^{-1}\left(F_1 - S_1 W_2^{-1} F_2\right)$.

We also introduce the following discrete-time algebraic Riccati equation (DARE)

$$P = F_s P F_s^T - F_s P F_2^T \left(F_2 P F_2^T + W_2\right)^{-1} F_2 P F_s^T + Q_s. \tag{16}$$

With this notation in place, we have the following result.

Proposition 5. (Darouach et al. (1993b) Thm. 3 and Darouach et al. (1993a) Thm. 4). If

i) $\mathrm{rank} \begin{bmatrix} E \\ U_2^T F_\ell \end{bmatrix} = n + p$;

ii) $\mathrm{rank} \begin{bmatrix} zE - F \\ U_2^T F_\ell \end{bmatrix} = n + p$, $\forall z \in \mathbb{C}$ s.t. $|z| \geq 1$,

then there exists a recursive estimator $\hat{z}_{k-\ell+1}$ given by

$$\bar{P}_{k+1}^{-1} = E^T(\Sigma + F\bar{P}_k F^T)^{-1} E + F_\ell^T U_2 (U_2^T \bar{\Sigma}_\ell U_2)^{-1} U_2^T F_\ell, \tag{17}$$

$$\hat{z}_{k+1} = \bar{P}_{k+1} E^T (\Sigma + F\bar{P}_k F^T)^{-1} (F\hat{z}_k + \bar{y}_k) + \bar{P}_{k+1} F_\ell^T U_2 (U_2^T \bar{\Sigma}_\ell U_2)^{-1} \chi_{k+1,\ell}. \tag{18}$$

Furthermore, if

iii) the pair $(F_s, Q_s^{1/2})$ has no unreachable mode on the unit circle, i.e., $\mathrm{rank}\left[\begin{smallmatrix} zI - F_s & Q_s^{1/2} \end{smallmatrix}\right] = p + n$, $\forall z \in \mathbb{C}$ s.t. $|z| = 1$,

then the DARE (16) has a unique solution P, $\bar{P}_k$ converges exponentially fast to P, and the corresponding steady-state filter is stable. ■

We will now show that the estimator obtained in Sect. 3.1 is equivalent to the filter in Prop. 5, and hence inherits its convergence and stability.

Proposition 6. Given $\hat{x}_{k-\ell}$ and $P_{k-\ell}^{xx}$, use an arbitrary $\hat{u}_{k-\ell-1}^T$ to set $\hat{z}_k^T = \left[\hat{u}_{k-\ell-1}^T, \hat{x}_{k-\ell}^T\right]$, and let $\bar{P}_k$ be such that $[0, I_n]\, \bar{P}_k \left[\begin{smallmatrix} 0 \\ I_n \end{smallmatrix}\right] = P_{k-\ell}^{xx}$, other entries of $\bar{P}_k$ being arbitrarily chosen, provided $\bar{P}_k > 0$. Then, the estimates and covariance matrices from (12)-(11) and those from (17)-(18) are related as follows:

$$\begin{bmatrix} \hat{u}_{k-\ell} \\ \hat{x}_{k-\ell+1} \end{bmatrix} = \hat{z}_{k+1} \quad \text{and} \quad P_{k-\ell+1} = \bar{P}_{k+1}. \blacksquare$$

Proof: The proof is immediate by noting that $\hat{\Sigma}_{k-\ell} = \Sigma + FP_{k-\ell}F^T$ and $F\hat{z}_k + \bar{y}_k = \begin{bmatrix} y_{k-\ell} - C\hat{x}_{k-\ell} \\ -A\hat{x}_{k-\ell} \end{bmatrix}$. $\square$

Prop. 6 means that the algorithm described in Sect. 3.1 and the filter for the descriptor system (13) (running for $k \geq \ell$) give exactly the same estimates, provided they are consistently initialized. Hence, they also share the same convergence properties. The two lemmas below show the relation between properties of the system (1) (delay-ℓ left-invertibility and strong detectability) and the conditions in Prop. 5 concerning the descriptor system (13).

Lemma 7. If the system is delay-ℓ left invertible, then $\text{rank} \begin{bmatrix} E \\ U_2^T F_\ell \end{bmatrix} = n + p$. $\blacksquare$

Proof: To prove that $\text{rank} \begin{bmatrix} E \\ U_2^T F_\ell \end{bmatrix} = n + p$, we will prove that $\begin{bmatrix} E \\ U_2^T F_\ell \end{bmatrix} w = 0$ implies $w = 0$. The delay-ℓ left invertibility condition $\text{rank}(N_\ell) = p + n + \text{rank}(N_{\ell-1})$ can be equivalently rephrased as: $\text{rank} \begin{bmatrix} E \\ F_\ell \end{bmatrix} = p + n$ and $\text{Im} \begin{bmatrix} E \\ F_\ell \end{bmatrix} \cap \text{Im} \begin{bmatrix} 0 \\ N_{\ell-1} \end{bmatrix} = \{0\}$. Notice that $\begin{bmatrix} E \\ U_2^T F_\ell \end{bmatrix} = H \begin{bmatrix} E \\ F_\ell \end{bmatrix}$, with $H = \begin{bmatrix} I_{n+m} & 0 \\ 0 & U_2^T \end{bmatrix}$. By definition of U_2, $\text{Im}(U_2) = \ker(N_{\ell-1}^T)$, which implies that $\text{Im}(H) = \ker \left[0, N_{\ell-1}^T\right]$ and hence $\ker(H) = \text{Im} \begin{bmatrix} 0 \\ N_{\ell-1} \end{bmatrix}$. If $\begin{bmatrix} E \\ U_2^T F_\ell \end{bmatrix} w = 0$, then $Hv = 0$, with $v = \begin{bmatrix} E \\ F_\ell \end{bmatrix} w$. This implies $v \in \text{Im} \begin{bmatrix} E \\ F_\ell \end{bmatrix} \cap \ker H$, and hence $v = 0$. Then also $w = 0$, since $\text{rank} \begin{bmatrix} E \\ F_\ell \end{bmatrix} = n + p$. $\square$

Lemma 8. If $\text{rank} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = n+p, \forall z \in \mathbb{C}$ s.t. $|z| \geq 1$, then $\text{rank} \begin{bmatrix} zE - F \\ U_2^T F_\ell \end{bmatrix} = n + p$, $\forall z \in \mathbb{C}$ s.t. $|z| \geq 1$.

Proof: Notice that $[zE - F] = \begin{bmatrix} zD & C \\ zB & A-zI \end{bmatrix}$, so that $\text{rank}\,[zE - F] = \text{rank} \begin{bmatrix} A-zI & zB \\ C & zD \end{bmatrix}$. For every $z \neq 0$, the latter is equal to $\text{rank} \begin{bmatrix} A-zI & B \\ C & D \end{bmatrix}$, and hence is $n+p$ for all $|z| \geq 1$. As a consequence, $\text{rank} \begin{bmatrix} zE - F \\ U_2^T F_\ell \end{bmatrix} = n+p$, $\forall z \in \mathbb{C}$ s.t. $|z| \geq 1$. $\square$

Propositions 5 and 6 and Lemmas 7 and 8, together, give the following result.

Theorem 9. If the system is

i) delay-ℓ left-invertible,
ii) strongly detectable, and
iii) the pair $(F_s, Q_s^{1/2})$ has no unreachable mode on the unit circle,

then, for the filter described in Section 3.1, the matrix P_k converges exponentially fast to the unique solution P of the DARE (16), and the corresponding steady-state filter is stable. $\blacksquare$

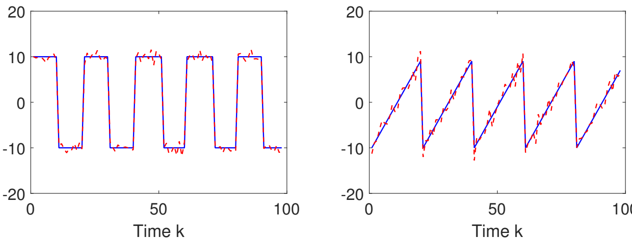

Fig. 2. Input signals estimation: True signals (blue solid lines) vs estimated signals (red dashed lines)

4. NUMERICAL EXAMPLE

To illustrate the performance of the proposed algorithm, simulation results are given in this section. The considered system is defined by the following matrices:

$$A = \begin{bmatrix} 1 & -1/2 & -1/2 & -1/2 \\ 0 & 1/2 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Since $D \neq 0$, this example cannot be treated with methods from Chavan et al. (2014). This system has inherent delay 2 and is strongly observable (see Sundaram and Hadjicostis (2007)). Moreover, after obtaining Q_s and F_s, one can see that the reachability condition on the unit circle is also satisfied.

The system is affected by a square wave and a sawtooth wave, assumed to be unknown, while the initial state, $x_0 = [8; 4; 6; 7]$, is also unknown. In addition, the process noise and the measurement noise are independent identically distributed zero-mean Gaussian processes with covariance $R = Q = \sigma^2 I$, with $\sigma = 0.35$; they are mutually uncorrelated.

The purpose of the proposed algorithm is then to estimate both the four states of the system and the two inputs with a delay $\ell = 2$. It is initialized with $\hat{x}_0 = 0$ and $P_0 = 10^3 I$. Performance with respect to input estimation is depicted in Figure 2, while performance with respect to the four states is shown in Figure 3. These figures show that both inputs and states are very well reconstructed.

The convergence of the algorithm is illustrated in Figure 4, which depicts the time evolution of the trace of the sample error covariance matrix and the trace of P_k (i.e., the approximate covariance matrix computed by the algorithm). After 25 iterations, trace(P_k) is equal to 3.846, the same as trace(P), P being the unique strong solution of the DARE (16); this is consistent with the result in Thm. 9. The sample covariance is computed over 1000 runs, with same initial condition and the same noise distributions as described above. The initial sample covariance trace is small, due to all runs having a same initial condition, and then the evolution shows an almost stationary behavior, around a value near the one computed by the algorithm.

5. CONCLUSION

In this paper we have presented a recursive algorithm providing unbiased estimates of state and unknown input, with delay; this algorithm is a simpler approximation of

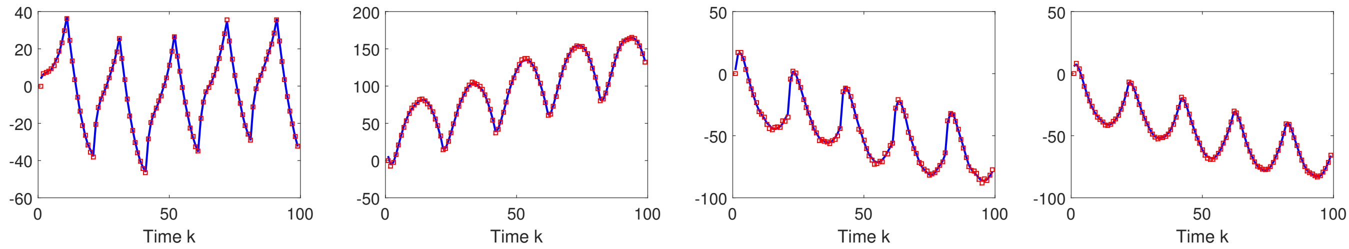

Fig. 3. State estimation: True states (blue solid lines) vs estimated states (red square lines)

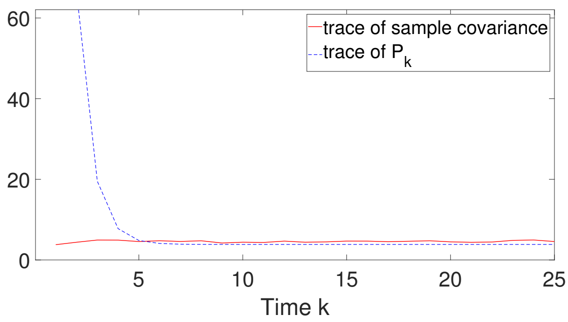

Fig. 4. In solid red, trace of the sample covariance of the error $(u_k^T - \hat{u}_k^T, x_k - \hat{x}_k^T)^T$; in dashed blue trace of the approximate covariance matrix P_k, computed by the algorithm.

the minimum-variance unbiased estimator. We have studied stability and convergence of this filter, and we have illustrated its performance in an example. Future work will explore theoretical guarantees on the quality of the proposed covariance approximation. Also, it will be interesting to look for simple ways to implement the optimal estimator itself. Given the importance of scalability for large network systems, another line of future work is the study of distributed algorithms for delay-ℓ estimation.

This work concerns the case where the external input is completely unknown. A broad area for future works includes problems where some partial knowledge about the input is available, and can be exploited to improve estimation. For example, some knowledge could be available on the input statistics or about its dynamics, or some physical limitations could be known, that impose constraints on the possible inputs, such as maximum intensity or maximum number of non-zero entries.

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