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On the long time convergence of potential MFG

Marco Masoero *

July 27, 2018

Abstract

We look at the long time behavior of potential Mean Field Games (briefly MFG) using some standard tools from weak KAM theory. We first show that the time-dependent minimization problem converges to an ergodic constant $-\lambda$, then we provide a class of examples where the value of the stationary MFG minimization problem is strictly greater than $-\lambda$. This will imply that the trajectories of the time-dependent MFG system do not converge to static equilibria.

Introduction

Mean Field Games were first introduced by Lasry and Lions [24, 25] and, simultaneously, by Huang, Caines and Malamhé [22]. This theory is a branch of the broader theory of Dynamic Games and it is devoted to those models where infinitely many players interact strategically with each other.

In many cases the Nash equilibria of those games can be analyzed through the solutions of the, so called, MFG system

$$\begin{cases} -\partial_t u - \triangle u + H(x, Du) = F(x, m) & \text{in } \mathbb{R}^d \times [0, T] \\ -\partial_t m + \triangle m + \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } \mathbb{R}^d \times [0, T] \\ m(t) = m_0,\ u(T, x) = u_T(x) & \text{in } \mathbb{R}^d. \end{cases}$$

with unknown the couple (u, m). We can think at $m(t)$ as the distribution of players at time t and $u(t, x)$ as the value function of any infinitesimal player starting from x at time t.

The aim of this paper is to shed some light on the long time behavior of potential MFG when monotonicity is not in place. The long time behavior and the existence of solutions which are periodic in time have been subject of several papers starting from [26] and the Mexican wave model in [21] to more recent results in [7, 10, 11, 20], but in these papers either monotonicity was assumed or the MFG was not of potential type. Cirant and Nurbekyan were the first to recently provide some results in the direction of periodic solutions for non monotone MFG. Cirant in [13] suggested the existence of non monotone configurations under which oscillatory behaviors were to be expected. Afterwards in [14], with Nurbekyan, they proved, through bifurcation methods, the existence of a path of branches which corresponds to a periodic trajectory. The main difference with our work is the choice of the class of solutions. In our case we look at paths which are energy minimizers whereas, in their, it might not be the case.

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Potential MFG are those games whose MFG system can be derived as optimality condition of the following minimization problem

$$\mathcal{U}(T, m_0) = \inf_{(m,w)} \int_0^T \int_{\mathbb{T}^d} H^* \left(x, -\frac{dw(t)}{dm(t)}(x) \right) dm(t) + \mathcal{F}(m(t))dt,$$

where (m, w) verifies the Fokker Plank equation $-\partial_t m + \triangle m + \mathrm{div} w = 0$ with $m(0) = m_0$ and the coupling function F in the MFG system is the derivative with respect to the measure of $\mathcal{F}$. These games have been largely studied (see Lasry and Lions [25] for existence results and, among others [9, 4, 28] for further properties) but, so far, not much is known regarding their long time behavior outside the assumption of monotonicity, where Cardaliaguet, Lasry, Lions and Porretta [10] proved the convergence to the ergodic system

$$\begin{cases} -\bar{\lambda} - \triangle u + H(x, Du) = F(x, m) & \text{in } \mathbb{T}^d \\ \triangle m + \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } \mathbb{T}^d. \end{cases}$$

We show that in general this is not the case, even in the very regular setting of non local coupling. We look at the problem from the point of view of weak KAM theory. The link between the two theories is not new and it was already proposed by Cardaliaguet [7] in the first order monotone case, even though in a different manner.

The paper is divided in three sections. In the first one we prove the convergence of $T^{-1}\mathcal{U}(T, \cdot)$ when T goes to infinity. The method we use is directly inspired by Lions, Papanicolaou, and Varadhan [27]: instead of looking directly at $\lim_T T^{-1}\mathcal{U}$ we define the infinite horizon, discounted problem

$$\mathcal{V}_\delta(m_0) = \inf_{(m,w)} \int_0^\infty e^{-\delta t} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w(t)}{m(t)} \right) dm(t) + \mathcal{F}(m(t))dt$$

and we prove that $\lim_\delta \delta \mathcal{V}_\delta(\cdot) = -\lambda$ when $\delta \to 0^+$and that this limit is uniform with respect to the initial distribution. A key assumption is the boundedness of the second derivative of $F(x, m)$ with respect to the state variable. This gives uniform semiconcavity estimates of the solutions of the MFG system associated to the discounted minimization problem. The existence of the limit $\lim_\delta \delta \mathcal{V}_\delta(\cdot)$ implies the existence of the limit $\lim_T T^{-1}\mathcal{U}$ and the two must coincide.

As byproduct, we have the existence of a corrector function χ on the space of measures which enjoys the following dynamic programming principle

$$\chi(m_0) = \inf_{(m,w)} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w(t)}{m(t)} \right) dm(s) + \mathcal{F}(m(s))ds + \chi(m(t_2)) \right) + \lambda(t_2 - t_1).$$

The second section is devoted to the study of the set of corrector functions. A corrector is any continuous function on the space of measures which verifies the dynamic programming principle above. Both the terminology and the techniques are borrowed from weak KAM theory, in particular we rely on Fathi's book [19], along with his seminal papers [16, 17, 18]. In principle, as in the standard weak KAM theory, the corrector functions verify an HJB equation in the space of probability measure. In this work nothing is said about this property which is the subject of a paper that is still in progress.

Particular interest is given to the projected Mather set which is the set of probability measures contained in a calibrated curve. We say that $(\bar{m}, \bar{w})$ is a calibrated curve associated to a corrector function χ if, for any t_1, $t_2 \in \mathbb{R}$, $(\bar{m}, \bar{w})$ is optimal for the dynamic programming principle, that is

$$\chi(\bar{m}(t_1)) = \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{\bar{w}(t)}{\bar{m}(t)} \right) d\bar{m}(s) + \mathcal{F}(\bar{m}(s))ds + \chi(\bar{m}(t_2)) + \lambda(t_2 - t_1).$$

These curves play a fundamental role to understand the long time behavior of these MFG. They are indeed the attractors of the dynamics which minimize the discounted, infinite horizon MFG.

In the third section we focus on the relation between the limit value λ and the ergodic value $\bar{\lambda}$, associated to the stationary MFG, defined by

$$-\bar{\lambda} = \inf_{(m,w)} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w}{m}\right) dm + \mathcal{F}(m).$$

We propose two examples which highlight how much important it is the structure of the coupling function $F(x,\cdot)$ in the dynamic of potential MFG. In the first example we impose monotonicity and we recover part of the results in [10]. In this case the limit value and the ergodic one coincide.

On the other hand, in the second example, the minimization problems are no longer convex and we can prove that $\lambda > \bar{\lambda}$. This means that it is not possible that the MFG system converges to a stationary equilibrium. The fact that $\lambda > \bar{\lambda}$ implies that the energy of the finite horizon game goes below the energy of the stationary one. Looking at the projected Mather set we can say even more. As this set is compact and it can not contain any stationary curve, calibrated curves can not even approach any static configuration.

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Assumptions and definitions

We work on the $d-$dimensional flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ to avoid to deal with boundary conditions and to set the problem on a compact domain.

Notation: We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$. This is a compact, complete and separable set when endowed with the Monge-Kantorovich distance $\boldsymbol{d}(\cdot,\cdot)$. We define $\mathcal{M}(\mathbb{T}^d;\mathbb{R}^d)$ the set of Borel vector valued measures w with finite mass $|w|$. If m_t is a time dependent probability measure on $\mathbb{T}^d$, then $L^2_m([0,T]\times\mathbb{T}^d)$ is the set of m-measurable functions f such that the integral of $|f|^2 dm_t$ over $[0,T]\times\mathbb{T}^d$ is finite. Analogously for $L^2_m(\mathbb{T}^d)$ and $L^2_m(\mathbb{T}^d;\mathbb{R}^d)$, where in the latter case we consider vector valued functions.

We use throughout the paper the notion of derivative for functions defined on $\mathcal{P}(\mathbb{T}^d)$ introduced in [8]. We say that $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is C^1 if there exists a continuous function $\frac{\delta\Phi}{\delta m} : \mathcal{P}(\mathbb{T}^d)\times\mathbb{T}^d \to \mathbb{R}$ such that

$$\Phi(m_1) - \Phi(m_2) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta\Phi}{\delta m}((1-t)m_1 + tm_2, x)(m_2 - m_1)(dx)dt, \qquad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d).$$

As this derivative is defined up to an additive constant, we use the standard normalization

$$\int_{\mathbb{T}^d} \frac{\delta\Phi}{\delta m}(m,x)m(dx) = 0.$$

Assumptions: We impose the following assumptions on the hamiltonian H and the coupling function F so that we can derive uniform estimates on the solutions of the MFG system.

1. $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is of class C^2, $p \mapsto D_{pp}H(x,p)$ is Lipschitz continuous, uniformly with respect to x. Moreover there exists $\bar{C} > 0$ that verifies

$$\bar{C}^{-1} I_d \leq D_{pp}H(x,p) \leq \bar{C} I_d, \quad \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d \tag{1}$$

and $\theta \in (0,1)$, $C > 0$ such that the following conditions hold true

$$|D_{xx}H(x,p)| \leq C(1+|p|)^{1+\theta}, \quad |D_{x,p}H(x,p)| \leq C(1+|p|)^{\theta}, \quad \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{2}$$

2. $\mathcal{F} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is of class C^2. Its derivative $F : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is twice differentiable in x and $D^2_{xx}F$ is bounded. Examples of non monotone coupling functions which verify such conditions can be found in Lemma 4.2 and Lemma 4.3.

We recall that, if μ, $\nu \in \mathcal{P}(\mathbb{T}^d)$, the 1-Wasserstein distance is defined by

$$\boldsymbol{d}(\mu,\nu) = \sup \left\{ \int_{\mathbb{T}^d} \phi(x)\, d(\mu-\nu)(x) \,\middle|\, \text{continuous } \phi : \mathbb{T}^d \to \mathbb{R},\ \mathrm{Lip}(\phi) \leq 1 \right\}.$$

Minimization Problems: Under the above assumptions, we can introduce two minimization problems. Each one of those will be proposed in two different but equivalent forms. The first one is

$$\mathcal{U}(T, m_0) = \inf_{(m,\alpha)} \int_0^T \int_{\mathbb{T}^d} H^*(x,\alpha) dm(t) + \mathcal{F}(m(t)) dt, \quad m_0 \in \mathcal{P}(\mathbb{T}^d)$$

where $m \in C^0([0,T], \mathcal{P}(\mathbb{T}^d))$, $\alpha \in L^2_m([0,T] \times \mathbb{T}^d, \mathbb{R}^d)$ and the following equation is verified in sense of distribution

$$\begin{cases} -\partial_t m + \triangle m + \mathrm{div}(m\alpha) = 0 & \text{in } [0,T] \times \mathbb{T}^d \\ m(0) = m_0 & \text{in } \mathbb{T}^d. \end{cases} \tag{3}$$

Equivalently (see [4] for more details),

$$\mathcal{U}(T, m_0) = \inf_{(m,w) \in \mathcal{E}_2^T(m_0)} \int_0^T \int_{\mathbb{T}^d} H^* \left(x, -\frac{dw(t)}{dm(t)}(x) \right) dm(t) + \mathcal{F}(m(t)) dt, \quad m_0 \in \mathcal{P}(\mathbb{T}^d)$$

where $\mathcal{E}_2^T(m_0)$ is the set of time dependent Borel measures $(m(t), w(t)) \in \mathcal{P}(\mathbb{T}^d) \times \mathcal{M}(\mathbb{T}^d; \mathbb{R}^d)$ such that $m \in C^0([0,T], \mathcal{P}(\mathbb{T}^d))$, w is absolutely continuous with respect to m, its density dw/dm belongs to $L^2_m([0,T] \times \mathbb{T}^d)$ and $-\partial_t m + \triangle m - \mathrm{div} w = 0$ is verified in sense of distributions with initial condition $m(0) = m_0$.

The second minimization problem reads

$$\mathcal{V}_\delta(m_0) = \inf_{(m,\alpha)} \int_0^\infty e^{-\delta t} \int_{\mathbb{T}^d} H^*(x,\alpha) dm(t) + \mathcal{F}(m(t)) dt, \quad m_0 \in \mathcal{P}(\mathbb{T}^d)$$

where $\delta > 0$, $m \in C^0([0,+\infty), \mathcal{P}(\mathbb{T}^d))$, $\alpha \in L^2_{m,\delta}([0,+\infty) \times \mathbb{T}^d, \mathbb{R}^d)$, that is L^2_m with weight $e^{-\delta t}$, and (m,α) verifies (3) in $[0,+\infty) \times \mathbb{T}^d$. Equivalently, $\mathcal{V}_\delta$ can be defined as

$$\mathcal{V}_\delta(m_0) = \inf_{(m,w) \in \mathcal{E}_2^\delta(m_0)} \int_0^{+\infty} e^{-\delta t} \int_{\mathbb{T}^d} H^* \left(x, -\frac{dw(t)}{dm(t)}(x) \right) dm(t) + \mathcal{F}(m(t)) dt, \quad m_0 \in \mathcal{P}(\mathbb{T}^d) \tag{4}$$

where $\mathcal{E}_2^\delta(m_0)$ is defined as $\mathcal{E}_2(m_0)$ with the only difference that we ask dw/dm to be L^2 in $[0,+\infty) \times \mathbb{T}^d$ with respect to $e^{-\delta t} m(t)$. For convenience we introduce the functional on $\mathcal{E}_2(m_0)$

$$J_\delta(m_0, m, w) = \int_0^{+\infty} e^{-\delta t} \int_{\mathbb{T}^d} H^* \left(x, -\frac{dw(t)}{dm(t)}(x) \right) dm(t) + \mathcal{F}(m(t)) dt,$$

so that $\mathcal{V}_\delta(m_0) = \inf_{(m,w)} J(m_0, m, w)$.

We also define the ergodic value $\bar{\lambda}$ as follows

$$-\bar{\lambda} = \inf_{m,\alpha} \int_{\mathbb{T}^d} H^*(x, \alpha) dm + \mathcal{F}(m) \tag{5}$$

where $(m, \alpha) \in \mathcal{P}(\mathbb{T}^d) \times L^2_m(\mathbb{T}^d; \mathbb{R}^d)$ verifies in sense of distribution $\triangle m + \operatorname{div}(m\alpha) = 0$ in $\mathbb{T}^d$.

Or, equivalently,

$$-\bar{\lambda} = \inf_{(m,w)\in\mathcal{E}} \int_{\mathbb{T}^d} H^* \left(x, -\frac{dw}{dm}(x) \right) dm(x) + \mathcal{F}(m) \tag{6}$$

where $\mathcal{E}$ is the set of $(m, w) \in \mathcal{P}(\mathbb{T}^d) \times \mathcal{M}(\mathbb{T}^d; \mathbb{R}^d)$ such that w is absolutely continuous with respect to m, its density dw/dm belongs to $L^2_m(\mathbb{T}^d)$ and $\triangle m - \operatorname{div} w = 0$ is verified in sense of distributions.

Throughout the paper we will use the constant $C > 0$ which may change from line to line.

1 Ergodic limit value

1.1 Minimizers and dynamic programming principle for $\mathcal{V}_\delta$

We start proving that the minimization problem (4) admits a minimizer and we also give a characterization of such a minimizer in terms of solutions of the associated MFG system.

Proposition 1.1. *For any $\delta > 0$ and any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, $\mathcal{V}_\delta(m_0)$ admits a minimizer (m, w). Moreover there exists $u \in C^{1,2}([0, +\infty) \times \mathbb{T}^d)$ and $m \in C^0([0, +\infty) \times \mathcal{P}(\mathbb{T}^d))$ solutions of*

$$\begin{cases} -\partial_t u - \triangle u + \delta u + H(x, Du) = \frac{\delta \mathcal{F}}{\delta m}(x, m) := F(x, m) & \text{in } \mathbb{T}^d \times [0, +\infty) \\ -\partial_t m + \triangle m + \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } \mathbb{T}^d \times [0, +\infty) \\ m(0) = m_0,\ u \in L^\infty([0, +\infty) \times \mathbb{T}^d) \end{cases} \tag{7}$$

such that $w = -m D_p H(x, Du)$.

Proof. First of all we show that $\mathcal{V}_\delta(m_0)$ is finite and that it is bounded by a constant K_δ independent of m_0. We can always use as competitor for $\mathcal{V}_\delta(m_0)$ the couple $(\mu, 0)$ where μ is the solution of

$$\begin{cases} -\partial_t \mu + \triangle \mu = 0 & \text{in } \mathbb{T}^d \times [0, +\infty) \\ \mu(0) = m_0 & \text{in } \mathbb{T}^d. \end{cases}$$

Given that $\mathcal{F}$ is bounded, if we use $(\mu, 0)$ as a competitor, we get

$$\mathcal{V}_\delta(m_0) \le \int_0^{+\infty} e^{-\delta t} H^*(x, 0) + \sup_m \mathcal{F}(m) dt := K_\delta.$$

We fix a minimizing sequence (m_n, w_n). If we use (1), we have that

$$\int_0^{+\infty} \int_{\mathbb{T}^d} e^{-\delta t} |w_n(t, x)| dx dt \le M_\delta$$

where M_δ is a fixed constant that does not depend on m_0 and n. Hence, for any fixed $k > 0$

$$\int_0^k \int_{\mathbb{T}^d} |w_n(t, x)| dx dt \le M_\delta e^{\delta k}. \tag{8}$$

Following Lemma 3.1 in [9], we get that for any $t, s \in [0,k]$

$$\boldsymbol{d}(m_n(t), m_n(s)) \leq C_\delta^k |t-s| + C|t-s|^{1/2}, \tag{9}$$

where C_δ^k depends only on δ and k. Inequality (9) tells us that $\{m_n\}_n$ is uniformly bounded in $C^{1/2}([0,k), \mathcal{P}(\mathbb{T}^d))$. We have then that m_n converges uniformly on any compact set to a limit $\bar{m} \in C^0([0,+\infty), \mathcal{P}(\mathbb{T}^d))$. Thanks to the bounds (8) we also know that w_n converges in $\mathcal{M}(I \times \mathbb{T}^d; \mathbb{R}^d)$ to a certain $\bar{w}$ on any bounded interval $I \subset [0,+\infty)$.

As the couple $(\bar{m}, \bar{w})$ belongs to $\mathcal{E}_2^\delta(m_0)$ we have that $J_\delta(m_0, \bar{m}, \bar{w}) < +\infty$. This means that

$$J_\delta(m(0), \bar{m}, \bar{w}) = \lim_{k \to +\infty} \int_0^k e^{-\delta t} \int_{\mathbb{T}^d} H^*\left(x, -\frac{d\bar{w}(t)}{d\bar{m}(t)}(x)\right) d\bar{m}(t) + \mathcal{F}(\bar{m}(t))dt$$

Note also that the functional is bounded from below, so there exists a constant C_δ such that, for any $(m,w) \in \mathcal{E}_2^\delta$,

$$\int_k^{+\infty} e^{-\delta t} \int_{\mathbb{T}^d} H^*\left(x, -\frac{dw(t)}{dm(t)}(x)\right) dm(t) + \mathcal{F}(m(t))dt$$

$$\geq e^{-\delta k} \int_0^{+\infty} e^{-\delta t} \inf_{(x,p) \in \mathbb{T}^d \times \mathbb{R}^d} H^*(x,p) + \inf_m \mathcal{F}(m)dt = e^{-\delta k} C_\delta.$$

Therefore,

$$J_\delta(m_0, m_n, w_n) \geq \int_0^k e^{-\delta t} \int_{\mathbb{T}^d} H^*\left(x, -\frac{dw_n(t)}{dm_n(t)}(x)\right) dm_n(t) + \mathcal{F}(m_n(t))dt + e^{-\delta k} C_\delta.$$

Thanks to the convergence of (m_n, w_n) on compact sets, we can pass to the limit in n and we get

$$\mathcal{V}_\delta(m_0) \geq \int_0^k e^{-\delta t} \int_{\mathbb{T}^d} H^*\left(x, -\frac{d\bar{w}(t)}{d\bar{m}(t)}(x)\right) d\bar{m}(t) + \mathcal{F}(\bar{m}(t))dt + C_\delta e^{-\delta k}.$$

Taking the limit on k we finally get that $\mathcal{V}_\delta(m_0) \geq J(m_0, \bar{m}, \bar{w})$.

The proof of the second statement relies again on classic tools for potential MFG (see for instance [9] or [4]). Using the convexity of H^* and the regularity of $\mathcal{F}$, we can easily show that if $(\bar{m}, \bar{w})$ is a minimizer for $\mathcal{V}_\delta(m_0)$ then it must be a minimizer for $\bar{J}_\delta : \mathcal{E}_2^\delta(m_0) \to \mathbb{R}$, which reads

$$\bar{J}_\delta(m,w) = \int_0^{+\infty} e^{-\delta t} \int_{\mathbb{T}^d} H^*\left(x, -\frac{dw(t)}{dm(t)}(x)\right) dm(t) + F(x, \bar{m}(t))dm(t)dt.$$

As $\bar{J}_\delta$ is convex we can define its dual problem (in the sense of Fenchel-Rockafellar)

$$\inf_{u \in C_b^2([0,+\infty) \times \mathbb{T}^d)} \left\{ -\int_{\mathbb{T}^d} u(0,x)dm_0(x) \text{ where } -\partial_t u - \triangle u + \delta u + H(x, Du) \leq F(x, \bar{m}) \right\}.$$

Thanks to the comparison principle, the minimizer $\bar{u}$ is the unique solution of $-\partial_t \bar{u} - \triangle \bar{u} + \delta \bar{u} + H(x, D\bar{u}) = F(x, \bar{m})$ in $C_b^2([0,+\infty) \times \mathbb{T}^d)$.

If we sum the two problems we get

$$\int_0^{+\infty} e^{-\delta t} \int_{\mathbb{T}^d} H^*\left(x, -\frac{d\bar{w}(t)}{d\bar{m}(t)}(x)\right) d\bar{m}(t) + F(x, \bar{m}(t))d\bar{m}(t)dt - \int_{\mathbb{T}^d} \phi(0,x)dm_0(x) = 0$$

Using the differential constraints of $(\bar{m}, \bar{w})$ and $\bar{u}$ we get that $\bar{w} = -\bar{m} D_p H(x, D\bar{u})$ $\bar{m}$-a.e.. This means that $\bar{w}$ is bounded that in turn implies that $\bar{m} > 0$ so that $\bar{w} = -\bar{m} D_p H(x, D\bar{u})$ is verified everywhere.

□

We now state without proof the dynamic programming principle for $\mathcal{V}_\delta$. The proof relies on standard arguments in optimal control theory (see for instance [5]).

Lemma 1.2. *The function $\mathcal{V}_\delta$ verifies the dynamic programming principle, which reads*

$$\mathcal{V}_\delta(m_0) = \inf_{(m,\alpha)\in\mathcal{E}_2^\delta(m_0)} \left(\int_0^t e^{-\delta s} \int_{\mathbb{T}^d} H^*(x,\alpha) dm(s) + \mathcal{F}(m(s)) ds + e^{-\delta t} \mathcal{V}_\delta(m(t)) \right).$$

1.2 Existence of a corrector

The main result of this section is Theorem 1.5 where show that the function $\mathcal{V}_\delta(\cdot)$ is uniformly Lipschitz with respect to δ. As a consequence, we have Proposition 1.6 which claims on one side that the limit $\lim_{\delta\to 0} \delta \mathcal{V}_\delta(m_0)$ is well defined and it is uniform in m_0 and, on the other, that, up to subsequence, also $\mathcal{V}_\delta(\cdot) - \mathcal{V}_\delta(m_0)$ converges to a continuous function χ. In Lemma 1.7 we prove that χ enjoys the dynamic programming principle and, therefore, we have the existence of a corrector function.

The idea behind the proof of Theorem 1.5 is the following: we want to prove that there exists a constant $\bar{K} > 0$ independent of δ, such that

$$|\mathcal{V}_\delta(m_2^0) - \mathcal{V}_\delta(m_1^0)| \le \bar{K} \boldsymbol{d}(m_1^0, m_2^0).$$

We fix an horizon $T > 0$, to be chosen later, and we take $(m_1(\cdot), \alpha_1(\cdot))$ a minimizer for $\mathcal{V}_\delta(m_1^0)$. We consider any couple (m_2, α_2) such that (3) is verified in $[0, T]$, $m_2(0) = m_2^0$, $m_2(T) = m_1(T)$ and $m_2 \equiv m_1$, $\alpha_2 \equiv \alpha_1$ in $[T, \infty)$. The couple (m_2, α_2) is admissible and

$$\mathcal{V}_\delta(m_2^0) \le \int_0^T e^{-\delta s} \int_{\mathbb{T}^d} H^*(x,\alpha_2) dm_2(s) + \mathcal{F}(m_2(s)) ds + e^{-\delta T} V_\delta(m_1(T)). \tag{10}$$

Therefore,

$$\mathcal{V}_\delta(m_2^0) - \mathcal{V}_\delta(m_1^0) \le \int_0^T e^{-\delta t} \int_{\mathbb{T}^d} H^*(x,\alpha_2) dm_2(t) - H^*(x,\alpha_1) dm_1(t) + \mathcal{F}(m_2(t)) - \mathcal{F}(m_1(t)) dt.$$

In order to prove the continuity of $\mathcal{V}_\delta$ with respect to the initial data we need to introduce some standard estimates on the solutions of the MFG system (7).

Lemma 1.3. *There exists $C > 0$ independent of m_0, T, δ such that, if (u, m) is a classical solution of (7), then*

- $\|Du\|_{L^\infty([0,+\infty)\times\mathbb{T}^d)} \le C$
- $|D^2 u(s,\cdot)| \le C$ *for any* $s \in [0, +\infty)$
- $\boldsymbol{d}(m(s), m(l)) \le C|l - s|^{1/2}$ *for any* $l, s \in [0, +\infty)$

Consequently, we also have that $|\partial_t u(s.\cdot)| \le C$ for any $s \in [0, +\infty)$.

Proof. The proof follows closely the one proposed in [12] and it relies on semiconcavity estimates for the value function u. We recall that if $\phi \in C^\infty(\mathbb{T}^d)$ then

$$\|D\phi\|_{L^\infty(\mathbb{T}^d)} \leq d^{\frac{1}{2}} \sup_{x\in\mathbb{T}^d,\, |\xi|\leq 1} D^2\phi(x)\xi\cdot\xi. \tag{11}$$

We first prove the result for $u^T : [0,T]\times\mathbb{T}^d \to \mathbb{R}$, solution of

$$\begin{cases} -\partial_t u - \triangle u + \delta u + H(x,Du) = F(x,m) & \text{in } \mathbb{T}^d\times[0,T] \\ -\partial_t m + \triangle m + \operatorname{div}(mD_pH(x,Du)) = 0 & \text{in } \mathbb{T}^d\times[0,T] \\ m(t) = m_0,\ u(T,x) = 0 & \text{in } \mathbb{T}^d \end{cases}$$

We consider $\xi\in\mathbb{R}^d$, $|\xi|\leq 1$, that maximizes $\sup_{t,x} D^2u^T(t,x)\xi\cdot\xi = M$ and we look at the equation solved by $w(t,x) = D^2u^T(t,x)\xi\cdot\xi$ deriving twice in space the HJB equation in (7):

$$-\partial_t w - \triangle w + \delta w + D_{\xi\xi}H(x,Du) + 2D_{\xi p}H(x,Du)\cdot D^2u\xi$$

$$+D_{pp}H(x,Du)D^2u\xi\cdot D^2u\xi + D_pH(x,Du)\cdot Dw = D^2_{\xi\xi}F(x,m).$$

The maximum of w can be achieved either at $t=T$, but using the terminal condition of u^T we get $M=0$, or at a point (s,x) in the interior. In this case, if we use hypothesis (1) and (2) on H and the boundedness of $D^2_{xx}F$, then, at the maximum (s,x), we get the following inequality

$$\delta M - C(1+|Du|)^{1+\theta} - 2C(1+|Du|)^\theta|D^2u\xi| + \bar{C}^{-1}|D^2u\xi|^2 \leq C.$$

By Cauchy-Schwarz inequality we have that $M = w(s,x) \leq |D^2u(s,x)\xi|$. If we also plug (11) we get, for a possible different constant C

$$-C(1+M)^{1+\theta} - 2C(1+M)^{2\theta} + \bar{C}^{-1}M^2 \leq C.$$

Given that $\theta<1$ the above inequality ensures that M is bounded by a constant that does not depend on m_0, δ and T. The bound on $\|Du^T\|_{L^\infty([0,T]\times\mathbb{T}^d)}$ follows from (11). Now that we proved that Du^T is bounded so that Theorem V 5.4 in [23] gives us the boundedness of D^2u^T. Note that the estimates on Du^T and D^2u^T imply directly from the HJB equation that $\partial_t u^T$ is bounded as well. As all the estimates are independent of T, if we look at the sequence of u^T we have that, on any compact set, u^T is uniformly bounded and continuous. This means that u^T converges to u solution of the HJB equation on $[0,+\infty)$ and the same estimates hold true for u. Furthermore, it implies that also $D_pH(x,Du)$, the drift of the Fokker Planck equation, is uniformly bounded. Standard results on SDEs (for instance Lemma 3.4 in [6]) ensure the Holder continuity of $s\mapsto m(s)$ uniformly with respect to m_0 and δ. □

We now fix m_1^0, $m_2^0\in\mathcal{P}(\mathbb{T}^d)$. According to (10) we have

$$\mathcal{V}_\delta(m_2^0) - \mathcal{V}_\delta(m_1^0) \leq$$

$$\int_0^T e^{-\delta s}\int_{\mathbb{T}^d} H^*(x,\alpha_2)dm_2(s) - H^*(x,DH(x,Du_1))dm_1(s) + \mathcal{F}(m_2(s)) - \mathcal{F}(m_1(s))ds, \tag{12}$$

where (u_1,m_1) is a solution of (7) related to a minimizer (m_1,w_1) of $\mathcal{V}_\delta(m_1^0)$ that we found in Proposition 1.1. The couple (m_2,α_2) is such that (3) is verified in $[0,h+\tau]$ with $m_2(0)=m_2^0$

and $m_2(T) = m_1(T)$. The key point to prove the continuity of $\mathcal{V}_\delta$ is to construct a suitable (m_2, α_2). We first consider $\tilde{m}_2$ solution of

$$\begin{cases} -\partial_t \tilde{m}_2 + \triangle \tilde{m}_2 + \operatorname{div}(\tilde{m}_2 D_p H(x, Du_1)) = 0 \quad \text{in } [0,T] \times \mathbb{T}^d \\ \tilde{m}_2(0) = m_2^0 \end{cases} \tag{13}$$

and then we set

$$m_2(s,x) = \begin{cases} \tilde{m}_2(s,x), & \text{if } s \in (0,h] \\ \frac{\tau+h-s}{\tau}\tilde{m}_2(s,x) + \frac{s-h}{\tau} m_1(s,x), & \text{if } s \in [h, h+\tau] \\ m_1(s,x) & \text{if } s \in [\tau+h, T], \end{cases} \tag{14}$$

where h and τ will be chosen later. Note that, thanks to the boundedness of their drifts, Corollary 6.3.2 in [3] ensures that $\tilde{m}_2$ and m_1 have a density for any $s > 0$. What we still need is to define α_2 in $[h, h+\tau]$. We compute the equation verified by m_2 in $[h, h+\tau]$ and, using (7) and (13), we get

$$\partial_t m_2 - \triangle m_2 = \frac{\tau + h - s}{\tau} \operatorname{div}\left(\tilde{m}_2 D_p H(x, Du_1)\right) + \frac{s-h}{\tau} \operatorname{div}\left(m_1 D_p H(x, Du_1)\right) + \frac{m_1 - \tilde{m}_2}{\tau}$$

that is, by linearity,

$$\partial_t m_2 - \triangle m_2 = \operatorname{div}(m_2 D_p H(x, Du_1)) + \frac{m_1 - \tilde{m}_2}{\tau}.$$

Let $\zeta : [0, h+\tau] \times \mathbb{T}^d \to \mathbb{R}$ be the solution to

$$\begin{cases} \triangle \zeta = m_1 - \tilde{m}_2, \quad \text{in } [0, h+\tau] \times \mathbb{T}^d \\ \int_{\mathbb{T}^d} \zeta(s,x) = 0. \end{cases} \tag{15}$$

We can now define the drift α_2 as follows: $\alpha_2 = D_p H(x, Du_1) + \frac{D\zeta}{m_2 \tau}$ in $[h, h+\tau]$ and $\alpha_2 = D_p H(x, Du_1)$ elsewhere. As (m_2, α_2) verifies the Fokker Plank equation by construction with $m_2(0) = m_2^0$, it is admissible. For the continuity of $\mathcal{V}_\delta$ we still need estimates on the drift α_2. We prove those estimates in the next lemma using the regularity of the solutions of the adjoint of the Fokker-Plank equation.

Lemma 1.4. *For any time $s < h + \tau$, there exists a constant $K_s > 0$, bounded for $s > \varepsilon > 0$, such that*

$$\|D\zeta(s,\cdot)\|_{L^2(\mathbb{T}^d)} \leq K_s \boldsymbol{d}(m_1^0, m_2^0).$$

The constant K_s is independent of m_1^0, m_2^0.

Proof. We first note that, if we multiply (15) by ζ and we use Cauchy-Schwarz inequality on the right hand side, we get

$$\|D\zeta(s)\|^2_{L^2(\mathbb{T}^d)} \leq \|m_1(s) - \tilde{m}_2(s)\|_{L^2(\mathbb{T}^d)} \|\zeta(s)\|_{L^2(\mathbb{T}^d)}.$$

Now Pointcaré-Wirtinger inequality gives us

$$\|D\zeta(s)\|_{L^2(\mathbb{T}^d)} \leq C \|m_1(s) - \tilde{m}_2(s)\|_{L^2(\mathbb{T}^d)}. \tag{16}$$

If we define $\mu(s) = m_1(s) - \tilde{m}_2(s)$ then μ verifies the following equation

$$\begin{cases} -\partial_t \mu + \triangle \mu + \operatorname{div}(\mu D_p H(x, Du_1)) = 0 \quad \text{in } [0,s] \times \mathbb{T}^d \\ \mu(0) = m_1^0 - m_2^0 \qquad\qquad\qquad\qquad\quad \text{in } \mathbb{T}^d. \end{cases} \tag{17}$$

We now fix a $\bar{\phi} \in L^2(\mathbb{T}^d)$ and we consider the adjoint backward equation

$$\begin{cases} -\partial_t \phi - \triangle \phi + D_p H(x, Du_1) D\phi = 0 & \text{in } [0,s] \times \mathbb{T}^d \\ \phi(s,x) = \bar{\phi}(x) & \text{in } \mathbb{T}^d. \end{cases} \tag{18}$$

Given that $D_p H(x, Du)$ is bounded, if ϕ is the solution of (18), then there exists a constant K_s (Theorem 11.1 in [23]), such that

$$\|\phi(0)\|_{C^{1+\alpha}(\mathbb{T}^d)} \leq K_s \|\bar{\phi}\|_{L^2(\mathbb{T}^d)}. \tag{19}$$

As the equation (18) is the adjoint of (17),

$$\int_{\mathbb{T}^d} \phi(s)\mu(s)dx = \int_{\mathbb{T}^d} \phi(0)\mu(0)dx.$$

We now plug in the initial and terminal conditions and we estimate the righthand side as follows

$$\int_{\mathbb{T}^d} \bar{\phi}(x)(m_1(s)(dx) - \bar{m}_2(s)(dx)) = \int_{\mathbb{T}^d} \phi(0,x)(m_1^0(dx) - m_2^0(dx)) \leq \|D\phi(0)\|_{L^\infty} \boldsymbol{d}(m_1^0, m_2^0).$$

If we use the interior estimate (19) on the righthand side and we take the supremum over $\|\bar{\phi}\|_{L^2} \leq 1$, we finally end up with

$$\|m_1(s) - \tilde{m}_2(s)\|_{L^2} \leq K_s \boldsymbol{d}(m_1^0, m_2^0).$$

If we plug the last inequality into (16), we get the result. □

Theorem 1.5. *The family of functions $\{\mathcal{V}_\delta(\cdot)\}_\delta$ is uniformly Lipschitz continuous.*

Proof. Let $\alpha_1(t,x) = D_p H(x, Du_1)$ for any $t \in [0, h+\tau]$. We consider the same (m_2, α_2) that we defined earlier: m_2 is defined in (14), $\alpha_2 = \alpha_1 + \frac{D\zeta}{m_2 \tau}$ in $[h, h+\tau]$ and $\alpha_2 = \alpha_1$ elsewhere, where ζ solves (15). According to (12) we have

$$\mathcal{V}_\delta(m_2^0) - \mathcal{V}_\delta(m_1^0) \leq \int_0^h e^{-\delta s} \int_{\mathbb{T}^d} H^*(x, \alpha_1) d(m_2 - m_1) + \tag{20}$$

$$+ \int_h^{h+\tau} e^{-\delta s} \int_{\mathbb{T}^d} H^*(x, \alpha_2) dm_2 - H^*(x, \alpha_1) dm_1 + \int_0^{h+\tau} \mathcal{F}(m_2(s)) - \mathcal{F}(m_1(s)) ds.$$

Using the convexity of H, we can estimate the term $H^*(x, \alpha_2(s))$ for any time $s \in [h, h+\tau]$ as follows

$$\int_{\mathbb{T}^d} H^*(x, \alpha_2(s)) dm_2(s) \leq \int_{\mathbb{T}^d} H^*(x, \alpha_1(s)) dm_2(s) + D_p H^*(x, \alpha_2(s)) \cdot \frac{D\zeta(s)}{\tau} dx \tag{21}$$

$$\leq \int_{\mathbb{T}^d} H^*(x, \alpha_1(s)) dm_2(s) + \frac{1}{\tau} |\alpha_1(s)||D\zeta(s)| + \bar{C} \frac{|D\zeta(s)|^2}{\tau^2 m_2(s)} dx,$$

where in the last inequality we add and subtract $\alpha_1(s) \cdot D\zeta/\tau$ and we used the growth condition (1) on $D_{pp} H$.

We recall that, as the drift $D_p H(x, Du_1)$ is continuous and bounded, according to Theorem 2.2.1 [2], the measure m_1 has a density $m_1(s,x)$ for any $s > 0$, then, using Theorem 2.5.1 in [2], for any $l \in (0,s)$, we have

$$m_1(s,x) > m_1(l, x_0) e^{-Q(1 + \frac{1}{s-l} + \frac{1}{l})},$$

where Q does not depend on m_1^0, l and s. As $\mathbb{T}^d$ is bounded, for any $l > 0$, there exists a x_0 such that $m_1(l, x_0) > 1/2$. Given that the same holds true for $\tilde{m}_2$ then, for any $s \in [h, h+\tau]$, the definition of m_2 in (14) implies that

$$m_2(s,x) > \frac{1}{2} e^{-Q(1+\frac{1}{s-l}+\frac{1}{l})} \qquad \forall l \in (0,h).$$

For $l = h/2$ we obtain that the infimum, with respect to s, in the righthand side is achieved when $s = h$. Thus,

$$m_2(s,x) > \frac{1}{2} e^{-Q(1+\frac{4}{h})}. \tag{22}$$

We can now plug (21) and (22) into (20), which becomes

$$\mathcal{V}_\delta(m_2^0) - \mathcal{V}_\delta(m_1^0) \leq \int_0^{h+\tau} e^{-\delta s} \int_{\mathbb{T}^d} H^*(x, \alpha_1) d(m_2 - m_1) + \mathcal{F}(m_2(s)) - \mathcal{F}(m_1(s)) ds$$

$$+ \int_h^{h+\tau} e^{-\delta s} \int_{\mathbb{T}^d} \frac{C_2}{\tau} |D\zeta| + 2\frac{C}{\tau^2} |D\zeta|^2 e^{Q(1+\frac{4}{h})} dx ds.$$

Using the bounds on Du_1 found in Lemma 1.3, Lemma 1.4 and the regularizing property of $\mathcal{F}$, we get

$$\mathcal{V}_\delta(m_2^0) - \mathcal{V}_\delta(m_1^0) \leq$$

$$C \int_0^{h+\tau} \boldsymbol{d}(m_2(s), m_1(s)) ds + \int_h^{h+\tau} C \boldsymbol{d}(m_1^0, m_2^0) K_s + 2\frac{C}{\tau^2} \boldsymbol{d}^2(m_1^0, m_2^0) e^{Q(1+\frac{4}{h})} K_s^2 ds \leq$$

$$C \int_0^{h+\tau} \boldsymbol{d}(m_2(s), m_1(s)) ds + C \boldsymbol{d}(m_1^0, m_2^0) e^{Q(1+\frac{4}{h})} \int_h^{h+\tau} \left(1 + \frac{K_s}{\tau^2} \boldsymbol{d}(m_1^0, m_2^0)\right) K_s ds \leq$$

$$C \int_0^{h+\tau} \boldsymbol{d}(m_2(s), m_1(s)) ds + \frac{C}{\tau^2} \boldsymbol{d}(m_1^0, m_2^0) e^{Q(1+\frac{4}{h})} \int_h^{h+\tau} K_s^2 ds. \tag{23}$$

In the last inequality we neglected the terms which go to infinity slower than K_s^2 and which vanish faster than $\boldsymbol{d}(m_1^0, m_2^0)$. Note that the constant K_s might explode when s goes to 0 but, otherwise, it is bounded. Therefore, as $h > 0$, there is no problem of integrability for the term $\int_h^{h+\tau} K_s^2$.

We now focus on the first term in the above inequality. In order to estimate $\boldsymbol{d}(m_1(s), m_2(s))$, we have to look at the SDEs verified by the stochastic processes whose laws are m_1 and m_2. We first recall that an equivalent formulation of the $1-$Wasserstein distance between two probability measures μ and ν is

$$\boldsymbol{d}(\mu, \nu) = \inf_\gamma \left\{ \int_{\mathbb{T}^d \times \mathbb{T}^d} |x - y| d\gamma(x,y) \text{ s.t. } \pi_1 \gamma = \mu, \, \pi_2 \gamma = \nu \right\}. \tag{24}$$

We consider a standard probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and two random variables Z^1, Z^2 such that $\mathcal{L}(Z^i) = m_i^0$ and $\mathbb{E}\left[\left|Z^2 - Z^1\right|\right] = \boldsymbol{d}(m_1^0, m_2^0)$. Therefore, m_1 and m_2 are the laws of the processes defined by of the following SDEs

$$\begin{cases} dX_s^i = \alpha_i(t, X_s) ds + \sqrt{2} dB_s \\ X_0^i = Z^i. \end{cases}$$

Using the definition of distance in (24), we have

$$\boldsymbol{d}(m_1(s), m_2(s)) \leq \mathbb{E}\left[\left|X_s^2 - X_s^1\right|\right] \leq \mathbb{E}\left[\left|Z^2 - Z^1\right| + \int_0^s \left|\alpha_2(l, X_l^2) - \alpha_1(l, X_l^1)\right| dl\right].$$

We first split $\int_0^{h+\tau} \boldsymbol{d}(m_2(s), m_1(s))ds$ in the sum of the integrals on the intervals $[0, h]$ and $[h, h+\tau]$. For any $s \in [0, h]$, $\alpha_1(l, x) = D_pH(x, Du_1(t, x)) = \alpha_2(l, x)$, then

$$\boldsymbol{d}(m_1(s), m_2(s)) \leq \boldsymbol{d}(m_1^0, m_2^0) + \mathbb{E}\left[\int_0^s \left|D_pH(X_l^2, Du_1(l, X_l^2)) - D_pH(X_l^1, Du_1(l, X_l^1))\right|\right].$$

Hypothesis (1), (2) and Lemma 1.3 ensure that both $p \to D_pH(x, p)$ and $x \to D_pH(x, Du_1(l, x))$ are Lipchitz continuous, hence

$$\boldsymbol{d}(m_1(s), m_2(s)) \leq \boldsymbol{d}(m_1^0, m_2^0) + C\int_0^s \boldsymbol{d}(m_1(l), m_2(l))dl.$$

If we apply Gronwall's inequality, then for any $s \in [0, h]$

$$\boldsymbol{d}(m_1(s), m_2(s)) \leq \boldsymbol{d}(m_1^0, m_2^0)e^{Cs}. \tag{25}$$

We now look at $\int_h^{h+\tau} \boldsymbol{d}(m_1(s), m_2(s))ds$. According to the definition of α_2, for $s \in [h, h+\tau]$, we have

$$\begin{aligned}\boldsymbol{d}(m_1(s), m_2(s)) &\leq \boldsymbol{d}(m_1(h), m_2(h)) + \\ +\mathbb{E}\left[\int_h^s \left|D_pH\left(X_l^2, Du_1(l, X_l^2)\right) + \frac{D\zeta(l, X_l^2)}{\tau m_2(l, X_l^2)} - D_pH(X_l^1, Du_1(l, X_l^1))\right|\right]. &\end{aligned} \tag{26}$$

Using (25) on $\boldsymbol{d}(m_1(h), m_2(h))$ and splitting the last term, we get that (26) is smaller than

$$\boldsymbol{d}(m_1^0, m_2^0)e^{Ch} + \mathbb{E}\left[\int_h^s \left|D_pH\left(X_l^2, Du_1(l, X_l^2)\right) - D_pH(X_l^2, Du_1(l, X_l^1))\right|\right] + \mathbb{E}\left[\int_h^s \left|\frac{D\zeta(l, X_l^2)}{\tau m_2(l, X_l^2)}\right|\right].$$

If we use again that $x \to D_pH(x, Du_1(l, x))$ is Lipschitz continuous, we get

$$\boldsymbol{d}(m_1(s), m_2(s)) \leq \boldsymbol{d}(m_1^0, m_2^0)e^{Ch} + \frac{C}{\tau}\mathbb{E}\left[\int_h^s \left|\frac{D\zeta(l, X_l^2)}{m_2(l, X_l^2)}\right|\right] + \int_h^s \boldsymbol{d}(m_1(l), m_2(l)). \tag{27}$$

Thanks to estimates (22) on m_2 we can use Tonelli's theorem and switch the integral with the expectation. Using Lemma 1.4, we eventually have

$$\mathbb{E}\left[\int_h^s \left|\frac{D\zeta(l, X_l^2)}{m_2(l, X_l^2)}\right|\right] = \int_h^s \int_{\mathbb{T}^d} |D\zeta(l, x)|\, dx \leq \boldsymbol{d}(m_1^0, m_2^0)\int_h^s K_l dl.$$

If we plug the last inequality into (27), we can use again Gronwall's inequality so that for $s \in [h, h+\tau]$

$$\boldsymbol{d}(m_1(s), m_2(s)) \leq \left(e^{Cs} + \frac{e^{s-h}}{\tau}\int_h^s K_l dl\right)\boldsymbol{d}(m_1^0, m_2^0). \tag{28}$$

We can now suppose $h = \tau = 1$ and plugging (25) and (28) into (23), we finally get that, for a given constant C depending on all the other ones

$$\begin{aligned}&\mathcal{V}_\delta(m_2^0) - \mathcal{V}_\delta(m_1^0) \leq \\ &C\boldsymbol{d}(m_1^0, m_2^0)\left(\int_0^2 e^{Cs}ds + \int_1^2 e^{s-1}\int_1^s K_l dlds + e^{5Q}\int_1^2 K_s^2 ds\right).\end{aligned}$$

We recall that the constant K_s of Lemma 1.4 is bounded when h is not close to 0 (Theorem 11.1 in [23]). The infimum in the expression above is finite and none of the constants therein depends on δ. Therefore, $\{\mathcal{V}_\delta\}_\delta$ is uniformly $\bar{K}$-Lipschitz with

$$\bar{K} = C\left(\int_0^2 e^{Cs}ds + \int_1^2 e^{s-1}\int_1^s K_l dlds + e^{5Q}\int_1^2 K_s^2 ds\right).$$

□

Proposition 1.6. *For any fixed $\eta \in \mathcal{P}(\mathbb{T}^d)$ there exists a subsequence $\delta_n \to 0$, such that $\mathcal{V}_{\delta_n}(\cdot) - \mathcal{V}_{\delta_n}(\eta)$ uniformly converges to a function $\chi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ when $n \to +\infty$. Moreover, $\delta_n \mathcal{V}_{\delta_n}$ uniformly converges to a constant $-\lambda$*

Proof. The continuity proved in Theorem 1.5 ensures the boundedness of $\mathcal{V}_\delta(\cdot) - \mathcal{V}_\delta(\eta)$. Indeed we have $|\mathcal{V}_\delta(\cdot) - \mathcal{V}_\delta(\eta)| \leq \bar{K}\boldsymbol{d}(\cdot, \eta)$. As $\mathcal{P}(\mathbb{T}^d)$ is compact, the right hand side is bounded by a constant K. Arzelà-Ascoli theorem ensures that there exists a subsequence $\delta_n \to 0$ such that $\mathcal{V}_{\delta_n}(\cdot) - \mathcal{V}_{\delta_n}(\eta)$ converges to a continuous function χ.

We now want to prove that $\delta\mathcal{V}_\delta$ is a bounded function. We fix a measure $\mu \equiv 1$, then we define the control (m, α) as follows: $m(t) = \mu$ and $\alpha(t) = 0$ for all $t \in [0, +\infty)$. The control is admissible, therefore we have

$$\delta\mathcal{V}_\delta(\mu) \leq \delta(H^*(x, 0) + \mathcal{F}(\mu)) \int_0^\infty e^{-\delta s} ds = H^*(x, 0) + \mathcal{F}(\mu).$$

Given that H^* and $\mathcal{F}$ are bounded from below, then

$$\delta\mathcal{V}_\delta(\mu) \geq \delta \left(\inf_{(x,a) \in \mathbb{T}^d \times \mathbb{R}^d} H^*(x, a) + \inf_{m \in \mathcal{P}(\mathbb{T}^d)} \mathcal{F}(m) \right) \int_0^\infty e^{-\delta s} ds = \inf_{(x,a)} H^*(x, a) + \inf_m \mathcal{F}(m).$$

Therefore, $\delta\mathcal{V}_\delta(\mu)$ is uniformly bounded in δ. If we fix any other measure m_0 we can use again the uniform continuity of $\mathcal{V}_\delta$ to get that $|\delta\mathcal{V}_\delta(m_0) - \delta\mathcal{V}_\delta(\mu)| \leq \delta K$ that in turn tells us that $\delta\mathcal{V}_\delta(\cdot)$ is a sequence of uniformly continuous functions. Using again Arzelà-Ascoli theorem we get that $\delta_n\mathcal{V}_{\delta_n}$ uniformly converges to a function Ψ (we can suppose δ_n to be the same subsequence that we identified earlier). Moreover, we have $|\delta_n\mathcal{V}_{\delta_n}(\cdot) - \delta_n\mathcal{V}_{\delta_n}(\mu)| \leq \delta_n K$. Taking the limit we get $|\Psi(\cdot) - \Psi(\mu)| \leq 0$ so that $\delta_n\mathcal{V}_{\delta_n}$ converges to the constant function $\Psi(\mu) := -\lambda$.

□

Lemma 1.7. *Dynamic programming principle for χ: for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and $t > 0$,*

$$\chi(m_0) = \inf_{(m,\alpha)} \left(\int_0^t H^*(x, \alpha) dm(s) + \mathcal{F}(m(s))ds + \chi(m(t)) \right) + \lambda t \tag{29}$$

where $m \in C^0([0, t], \mathcal{P}(\mathbb{T}^d))$, $\alpha \in L^2_m([0, t] \times \mathbb{T}^d, \mathbb{R}^d)$ and the pair (m, α) solves in sense of distribution $-\partial_t m_+ \triangle m + \operatorname{div}(m\alpha) = 0$ with initial condition m_0.

Proof. In Proposition 1.6 we proved the convergence of $\mathcal{V}_\delta(\cdot) - \mathcal{V}_\delta(\eta)$ to $\chi(\cdot)$ along the subsequence $\{\delta_n\}_n$, for a fixed measure η. Hereafter, $\{\delta_n\}_n$ and η will be the ones identified in that proposition.

We know from Proposition 1.1 that, for any $\delta > 0$, there exists a solution (u_δ, m_δ) to (7) such that

$$\mathcal{V}_\delta(m_0) = \int_0^t e^{-\delta s} \int_{\mathbb{T}^d} H^*(x, \alpha_\delta) dm_\delta(s) + \mathcal{F}(m_\delta(s))ds + e^{-\delta t} V_\delta(m_\delta(t)),$$

where $\alpha_\delta = D_p H(x, Du_\delta)$. If we take the expansion of $e^{-\delta t}$ and we subtract on both sides $\mathcal{V}_\delta(\eta)$ we get

$$\mathcal{V}_\delta(m_0) - \mathcal{V}_\delta(\eta) = \int_0^t e^{-\delta s} \int_{\mathbb{T}^d} H^*(x, \alpha_\delta) dm_\delta(s) + \mathcal{F}(m_\delta(s))ds + (1 - t\delta + o(t\delta))V_\delta(m_\delta(t)) - V_\delta(\eta). \tag{30}$$

We recall that the estimates in Lemma 1.3 are uniform in δ hence $D_pH(x, Du_{\delta_n})$ converges uniformly to a function α. We can now take the limit $n \to +\infty$ and using that $\mathcal{V}_{\delta_n}(\cdot) - \mathcal{V}_{\delta_n}(\eta) \to \chi(\cdot)$ and $\delta_n \mathcal{V}_{\delta_n}(\cdot) \to -\lambda$, we get

$$\chi(m_0) = \int_0^t \int_{\mathbb{T}^d} H^*(x, \alpha) dm(s) + \mathcal{F}(m(s))ds + \lambda t + \chi(m_t)$$

In order to show that (α, m) is optimal, we fix a competitor (β, μ). According to the dynamic programming principle of $\mathcal{V}_\delta$, if we plug (β, μ) into (30), we get

$$\mathcal{V}_\delta(m_0) - \mathcal{V}_\delta(\eta) \leq \int_0^t e^{-\delta s} \int_{\mathbb{T}^d} H^*(x, \beta) d\mu(s) + \mathcal{F}(\mu(s))ds + (1 - t\delta + o(t\delta))V_\delta(\mu(t)) - V_\delta(\eta).$$

Taking again the limit on the subsequence $\{\delta_n\}_n$ we eventually have that

$$\chi(m_0) \leq \int_0^t \int_{\mathbb{T}^d} H^*(x, \beta) d\mu(s) + \mathcal{F}(\mu(s))ds + \lambda t + \chi(\mu_t),$$

which proves the result.

□

1.3 Convergence of $\mathcal{U}(t, \cdot)/t$ and $\delta\mathcal{V}_\delta(\cdot)$

In this section we propose a Tauberian-type result where we prove that the limit of $\delta\mathcal{V}_\delta(\cdot)$ coincides with the one of $\mathcal{U}(t, \cdot)/t$ when $t \to +\infty$.

Theorem 1.8. *The limit value $-\lambda$ is uniquely defined and $\delta\mathcal{V}_\delta(\cdot) \to -\lambda$ does not depend on a subsequence. Moreover, $\frac{1}{T}\mathcal{U}(T, \cdot)$ uniformly converges to $-\lambda$ when T goes to $+\infty$.*

Proof. Let $\{\delta_n\}_n$ such that $\delta_n \mathcal{V}_{\delta_n} \to -\lambda$ and $\mathcal{V}_{\delta_n}(\cdot) - \mathcal{V}_{\delta_n}(\eta) \to \chi(\cdot)$. As χ is a continuous function on the compact set $\mathcal{P}(\mathbb{T}^d)$, there exists a constant $C > 0$ such that $0 \leq \chi(m) + C$ for any $m \in \mathcal{P}(\mathbb{T}^d)$. If $(m(t), w(t)) \in \mathcal{E}_2^T(m_0)$, then

$$\int_0^T \int_{\mathbb{T}^d} H^*\left(x, -\frac{dw(t)}{dm(t)}(x)\right) dm(t) + \mathcal{F}(m(t))dt$$

$$\leq \int_0^T \int_{\mathbb{T}^d} H^*\left(x, -\frac{dw(t)}{dm(t)}(x)\right) dm(t) + \mathcal{F}(m(t))dt + \chi(m(T)) + \lambda T - \lambda T + C.$$

Taking the infimum over $\mathcal{E}_2^T$, the definition of $\mathcal{U}(T, m_0)$ and the dynamic programming principle of χ lead to

$$\mathcal{U}(T, m_0) \leq \chi(m_0) - \lambda T + C.$$

As the constant C does not depend on m_0 and T, if we divide on both sides by T and we take the limit $T \to +\infty$, we get

$$\lim_{T \to +\infty} \frac{1}{T}\mathcal{U}(T, m_0) \leq -\lambda.$$

The other inequality is analogous. We just need to take a $C_2 > 0$ such that $0 \geq \chi(m) - C_2$ and repeat the same computation.

Note that the limit $\mathcal{U}(T, \cdot)/T \to -\lambda$ is uniform and does not depend on the subsequence δ_n or the function χ. Therefore, the limit $\delta\mathcal{V}_\delta$ is uniquely defined. □

We conclude the section showing that our limit value λ is never lower than the ergodic one $\bar{\lambda}$ defined in (5).

Proposition 1.9. *Under the above assumptions, $\lambda \geq \bar{\lambda}$.*

Proof. We know that the convergence of $\mathcal{U}(\cdot, T)/T$ is uniform, therefore, if (m, α) is an admissible couple for the static problem, we can use it as competitor for $\mathcal{U}(m, T)$. So,

$$\frac{1}{T}\mathcal{U}(T, m) \leq \frac{1}{T}\int_0^T \int_{\mathbb{T}^d} H^*(x, \alpha)dm + \mathcal{F}(m)dt = \int_{\mathbb{T}^d} H^*(x, \alpha)dm + \mathcal{F}(m).$$

If we take the infimum over all admissible static (m, α) we get

$$\frac{1}{T}\mathcal{U}(T, m) \leq -\bar{\lambda}.$$

Letting T go to $+\infty$, we get the result. □

1.4 An other representation for λ

We can now introduce a third representation for λ, inspired again by classic results on weak KAM theory (see for instance [16]), which consists in minimizing over paths with fixed endpoints.

Let $\Pi_T(m_0, m_1)$ be the set of $(m, \alpha) \in C^0([0, +\infty), \mathcal{P}(\mathbb{T}^d)) \times L^1_{m,loc}([0, +\infty) \times \mathbb{T}^d, \mathbb{R}^d)$ such that (m, α) solves the usual Fokker-Plank equation $-\partial_t m + \triangle m - \operatorname{div}(m\alpha) = 0$ with the extra constraint $m(0) = m_0$ and $m(T) = m_1$. Note that, due to the smoothing property of the parabolic constraint, not for every m_1 we can find such a path, so $\Pi_T(m_0, m_1)$ might be the empty set.

Proposition 1.10. *Let m_0, $m_1 \in \mathcal{P}(\mathbb{T}^d)$. If m_1 has a density in $H^1(\mathbb{T}^d)$ and there exists an $\varepsilon > 0$ such that $m_1 > \varepsilon$ almost everywhere then*

$$-\lambda = \lim_{T\to\infty} \frac{1}{T} \inf_{\Pi_T(m_0,m_1)} \int_0^T \int_{\mathbb{T}^d} H^*(x, \alpha)dm(s) + \mathcal{F}(m(s))ds.$$

Proof. Let m_0 and m_1 be as above and $(\bar{m}, \bar{\alpha})$ be optimal for $\mathcal{U}(T, m_0)$. We extend $(\bar{m}, \bar{w})$ in $[0, T+1]$ as follows: for any $t \in [T, T+1]$ we define $\bar{\alpha}(t, x) = \bar{\alpha}(T, x)$ and $\bar{m}(t, x)$ as the solution of $-\partial_t m + \triangle m + \operatorname{div}(\bar{\alpha} m) = 0$ with $m(T, x) = \bar{m}(T, x)$. Note that $\bar{\alpha}$ is continuous and bounded in $[0, T+1]$, therefore, the estimates (22) still apply.

We now define a path from m_0 to m_1 as follows:

$$m_2(s, x) = \begin{cases} \bar{m}(s, x) & s \in [0, T] \\ (T + 1 - s)\bar{m}(s, x) + (s - T)m_1(x) & s \in [T, T + 1] \end{cases}.$$

Let also $\zeta(s, x)$ be solution of $-\triangle\zeta(s, x) = m_1(x) - \bar{m}(s, x)$ with $\int_{T^d} \zeta = 0$. We can define the control

$$\alpha_2(s, x) = \begin{cases} \bar{\alpha}(s, x) & s \in [0, T] \\ \bar{\alpha}(s, x) - \frac{(s-T)\bar{\alpha}(T,x)m_1(x) + D\zeta(s,x) + (s-T)Dm_1(x)}{(T+1-s)\bar{m}(s,x) + (s-T)m_1(x)} & s \in [T, T + 1]. \end{cases}$$

The couple (m_2, α_2) belongs to $\Pi_T(m_0, m_1)$. From the definition of $\mathcal{U}$ we deduce that

$$\frac{1}{T+1} \inf_{m\in\mathcal{P}(\mathbb{T}^d)} \mathcal{U}(T + 1, m) \leq \frac{1}{T+1} \inf_{\Pi_{T+1}(m_0,m_1)} \int_0^{T+1} \int_{\mathbb{T}^d} H^*(x, \alpha)dm(s) + \mathcal{F}(m(s))ds$$

$$\leq \frac{1}{T+1}\int_0^{T+1}\int_{\mathbb{T}^d} H^*(x,\alpha_2)dm_2(s)+\mathcal{F}(m_2(s))ds$$

$$=\frac{T}{T+1}\left(\frac{1}{T}\mathcal{U}(T,m_0)\right)+\frac{1}{T+1}\int_T^{T+1}\int_{\mathbb{T}^d} H^*(x,\alpha_2)dm_2(s)+\mathcal{F}(m_2(s))ds. \tag{31}$$

If we prove that $\frac{1}{T+1}\int_T^{T+1}\int_{\mathbb{T}^d} H^*(x,\alpha_2)dm_2(s)+\mathcal{F}(m_2(s))ds$ converges to zero we have the result. Indeed, if we let T go to $+\infty$, according to Theorem 1.8, we have

$$-\lambda \leq \lim_{T\to\infty}\frac{1}{T+1}\,\Pi_{T+1}(m_0,m_1)\int_0^{T+1}\int_{\mathbb{T}^d} H^*(x,\alpha)dm(s)+\mathcal{F}(m(s))ds \leq -\lambda.$$

We now focus on the last part in (31). Given that $\mathcal{F}(m_2)$ is uniformly bounded, we look at the first term.

$$\int_T^{T+1}\int_{\mathbb{T}^d} H^*(x,\alpha_2)dm_2(s)$$

$$\leq C\int_T^{T+1}\int_{\mathbb{T}^d}\frac{|\bar\alpha(T,x)m_2(s)+(s-T)\bar\alpha(T,x)m_1(s)+D\zeta(s,x)+(s-T)Dm_1(s)|^2}{m_2^2(s)}+1\,dm_2(s)ds$$

$$\leq C\int_T^{T+1}\int_{\mathbb{T}^d}\frac{\left(|\bar\alpha(T,x)m_2|+|\bar\alpha(T,x)m_1|+|D\zeta(s,x)|+|Dm_1|\right)^2}{m_2}dxds+C \tag{32}$$

If we use the hypothesis on m_1 and the estimates (22) on $\bar m$ with $h=T+1$ and $t=T-1$, we get that $m_2\geq\tau$ for a certain $\tau>0$ independent of T. Lemma 1.3 ensures that $\bar\alpha$ is uniformly bounded by a constant K independent of T. Therefore, (32) is lower than

$$\frac{1}{\tau}\left(K\|m_2\|_{L^2(\mathbb{T}^d\times[T,T+1])}+K\|m_1\|_{L^2(\mathbb{T}^d)}+\|D\zeta\|_{L^2(\mathbb{T}^d\times[T,T+1])}+\|Dm_1\|_{L^2(\mathbb{T}^d)}\right)^2+C.$$

Thanks again to the boundedness of $\bar\alpha$, standard result on parabolic equations tell us that $\bar m(s)$ (which is defined at the beginning of the proof) is uniformly bounded from above in $[T,T+1]$. Hence, $\|D\zeta\|_{L^2}\leq C\|m_1\|_{L^2}$ and $\|m_2\|_{L^2(\mathbb{T}^d\times[T,T+1])}\leq C\|m_1\|_{L^2(\mathbb{T}^d)}+C_2$. Thus

$$\int_T^{T+1}\int_{\mathbb{T}^d} H^*(x,\alpha_2)dm_2(s)\leq M\|m_1\|^2_{H^1(\mathbb{T}^d)}+M_2$$

where neither M nor M_2 depends on T. Dividing by $T+1$ and taking the limit completes the proof. □

2 Projected Mather set and Calibrated curves

2.1 Calibrated Curves

We borrow again some tools and some notations from the weak KAM theory (see Chapter 4 of [19]) and in particular we will focus on the notion of calibrated curve. Before introducing this notion, we look back to the dynamic programming principle verified by corrector functions, which reads

$$\chi(m_0)=\inf_{(m,w)\in\mathcal{E}_2^t(m_0)}\left(\int_0^t H^*\left(x,-\frac{w}{m}\right)dm(s)+\mathcal{F}(m(s))ds+\chi(m(t))\right)+\lambda t.$$

As the function χ is continuous, standard arguments show that, for any fixed $m_0\in\mathcal{P}(\mathbb{T}^d)$ and $t>0$, there exists a solution $(\bar m,\bar w)\in\mathcal{E}_2^t(m_0)$ to the minimization problem described above.

Therefore, extending $(\bar m, \bar w)$ from $\bar m(t)$, it is easy to construct a new trajectory $(\bar m, \bar w)$, defined on $[0, +\infty)$ such that, for any $\tau > 0$, it verifies

$$\chi(m_0) = \lambda\tau + \int_0^\tau \int_{\mathbb{T}^d} H^*\left(x, -\frac{\bar w(s)}{\bar m(s)}\right) d\bar m(s) + \mathcal{F}(\bar m(s))ds + \chi(\bar m(\tau)).$$

We now prove that any of these trajectories is associated to a MFG system.

Proposition 2.1. *Let $m_0 \in \mathcal{P}(\mathbb{T}^d)$, χ be a corrector function and $(\bar m, \bar w)$ be a minimizing trajectory on $[0, +\infty)$ defined as above. Then, $\bar m \in C^{1,2}((0+\infty) \times \mathbb{T}^d)$ and there exists a function $\bar u \in C^{1,2}([0, +\infty) \times \mathbb{T}^d)$ such that $\bar w = -\bar m D_p H(x, D\bar u)$ where $(\bar m, \bar u)$ solves*

$$\begin{cases} -\partial_t u - \triangle u + H(x, Du) = F(x, m) & \text{in } [0, +\infty) \times \mathbb{T}^d \\ -\partial_t m + \triangle m + \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } [0, +\infty) \times \mathbb{T}^d \\ m(0) = m_0. \end{cases}$$

Remark 2.2. *Due to the lack of regularity of χ we can not derive the MFG system as optimal condition for the minimization problem (33). Indeed, if χ were C^1 we would derive typical MFG system with terminal condition $u(t) = \delta\chi(m(t))/\delta m$ but this latter term is not well defined.*

Proof. The proof relies on the same arguments as in Proposition 1.1. Let $(\bar m, \bar w)$ be as in the hypothesis. Then it verifies the Fokker-Plank equation and it is a minimizer of the problem

$$\inf_{(m,w)\in\mathcal{E}_2^t(m_0)} \int_0^t \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \mathcal{F}(m(s))ds + \chi(m(t)). \tag{33}$$

As $(\bar m, \bar w)$ is optimal for the minimization problem above, then it must be also optimal for the following MFG planning problem

$$\inf_{(m,w)\in\Pi(m_0,\bar m(t))} \int_0^t \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \mathcal{F}(m(s))ds,$$

where $\Pi(m_0, \bar m(t))$ is the set of $(m, w) \in \mathcal{E}_2^t(m_0)$ that solves the usual Fokker-Plank equation on $[0, t]$ with the constraints $m(0) = m_0$ and $m(t) = \bar m(t)$. We want to prove that $\bar w = -\bar m D_p H(x, D\bar u)$ where $(\bar m, \bar u)$ solves in classical sense

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x, m) & \text{in } [0, t] \times \mathbb{T}^d \\ \partial_t m - \Delta m - \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } [0, t] \times \mathbb{T}^d \\ m(0) = m_0,\ m(t) = \bar m(t). \end{cases}$$

We argue again as in Proposition 1.1. According to Proposition 3.1 in [4], $(\bar m, \bar w)$ minimizes also the following convex problem

$$\inf_{(m,w)\in\Pi(m_0),\bar m(t))} \int_0^t \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \int_{\mathbb{T}^d} F(x, \bar m(s))dm(s)ds. \tag{34}$$

This problem admits a dual formulation, in the sense of Fenchel Rokafellar Theorem, which reads

$$\inf_{\psi\in\bar{\mathcal{K}}} \left\{ \int_{T^d} \psi(x, t)d\bar m(t) - \int_{T^d} \psi(x, 0)dm_0) \right\}$$

where $\bar{\mathcal{K}}$ is the set of $\psi \in C^{1,2}([0, t] \times \mathbb{T}^d)$ such that $-\partial_t\psi - \triangle\psi + H(x, D\psi) \le F(x, \bar m)$. A full justification of the result above can be found again in [9].

In the definition of the dual problem we can replace $\bar{\mathcal{K}}$ with $\mathcal{K}$, where $\mathcal{K}$ is the set of $u \in C^{1,2}([0,t]\times\mathbb{T}^d)$ such that $-\partial_t u - \Delta u + H(x,Du) = F(x,\bar{m})$. Indeed, if ψ verifies $-\partial_t\psi - \triangle\psi + H(x,D\psi) \leq F(x,\bar{m})$, we can alway consider $u \in C^{1,2}([0,t]\times\mathbb{T}^d)$ solution of

$$\begin{cases} -\partial_t u - \Delta u + H(x,Du) = F(x,\bar{m}) & \text{in } [0,t]\times\mathbb{T}^d \\ u(x,t) = \psi(x,t) & \text{in } \mathbb{T}^d \end{cases}$$

Thanks to the comparison principle we have that $u(0,x) \geq \psi(0,x)$, thus

$$\inf_{\psi\in\bar{\mathcal{K}}} \left\{ \int_{T^d} \psi(x,0) d\bar{m}(t) - \int_{T^d} \psi(x,0) dm_0 \right\} \geq \inf_{u\in\mathcal{K}} \left\{ \int_{T^d} u(x,t) d\bar{m}(t) - \int_{T^d} u(x,0) dm_0 \right\} \quad (35)$$

The opposite inequality holds by inclusion. Lemma 3.2 in [9] and Proposition 3.1 in [4], which rely on the Fenchel-Rokafellar Theorem, ensure that, if $(\bar{m},\bar{w})$ is a minimizer of (34) and $\bar{u}\in\mathcal{K}$ is a minimizer of the dual problem, then

$$\int_{T^d} u(x,t) d\bar{m}(t) - \int_{T^d} u(x,0) dm_0 + \int_0^t \int_{\mathbb{T}^d} H^*\left(x, -\frac{\bar{w}}{\bar{m}}\right) d\bar{m}(s) + \int_{\mathbb{T}^d} F(x,\bar{m}(s)) d\bar{m}(s) ds = 0.$$

This implies that $\bar{w} = -\bar{m} D_p H(x,\bar{u})$. As a consequence, we have that $\bar{m}$ is driven by a smooth drift and so, by Schauder theory, $\bar{m} \in C^{1,2}((0,t]\times\mathbb{T}^d)$. In particular, given that t is arbitrary, then $\bar{m} \in C^{1,2}((0,+\infty)\times\mathbb{T}^d)$.

We assumed that the minimization problem (35) admits a solution. The proof of this result is developed in Lemma 4.1 in appendix. □

Remark 2.3. *Note that the convex duality between the minimization problems*

$$\inf_{(m,w)\in\Pi(m_0,\bar{m}(t))} \int_0^t \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \int_{\mathbb{T}^d} F(x,\bar{m}(s)) dm(s) ds$$

and

$$\inf_{u\in\mathcal{K}} \left\{ \int_{T^d} u(x,t) d\bar{m}(t) - \int_{T^d} u(x,0) dm_0 \right\}$$

holds true independently from the existence of minimizers for the latter one and, therefore, independently from Lemma 4.1.

We can now introduce the notion of calibrated curve. Let $\mathcal{E}_2^\infty$ be the set of $(m(t),w(t)) \in \mathcal{P}(\mathbb{T}^d)\times\mathcal{M}(\mathbb{T}^d;\mathbb{R}^d)$ such that $m\in C^0(\mathbb{R},\mathcal{P}(\mathbb{T}^d))$, w is absolutely continuous with respect to m, its density dw/dm belongs to $L^2_{m,loc}(\mathbb{R}\times\mathbb{T}^d)$ and $-\partial_t m + \triangle m - \mathrm{div} w = 0$ is verified in sense of distribution.

Definition 2.4. *We say that $(\bar{m},\bar{w}) \in \mathcal{E}_2^\infty$ is a calibrated curve if there exists a continuous function $\chi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ which verifies the dynamic programming principle* (29) *and $(\bar{m},\bar{w})$ is optimal for χ: for any $t_1 < t_2 \in \mathbb{R}$*

$$\chi(\bar{m}(t_1)) = \lambda(t_2 - t_1) + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^*\left(x, -\frac{\bar{w}(s)}{\bar{m}(s)}\right) d\bar{m}(s) + \mathcal{F}(\bar{m}(s)) ds + \chi(\bar{m}(t_2)).$$

A direct consequence of Proposition 2.1 is the following result which tells that calibrated curves are smooth and associated to MFG systems defined for any time $t\in\mathbb{R}$.

Proposition 2.5. *If $(m,w)\in\mathcal{E}^\infty$ is a calibrated curve, then $m\in C^{1,2}(\mathbb{R}\times\mathbb{T}^d)$ and there exists a function $u\in C^{1,2}(\mathbb{R}\times\mathbb{T}^d)$ such that $w = -mD_pH(x,Du)$ where (m,u) solves*

$$\begin{cases} -\partial_t u - \triangle u + H(x,Du) = F(x,m) & \text{in } \mathbb{R}\times\mathbb{T}^d \\ -\partial_t m + \triangle m + \mathrm{div}(mD_pH(x,Du)) = 0 & \text{in } \mathbb{R}\times\mathbb{T}^d \end{cases}$$

2.2 The projected Mather set

Definition 2.6. *We say that $m_0 \in \mathcal{P}(\mathbb{T}^d)$ belongs to the projected Mather set $\mathcal{M} \subset \mathcal{P}(\mathbb{T}^d)$ if there exists a calibrated curve $(m(t), w(t))$ such that $m(0) = m_0$.*

Note that, if from m_0 starts a calibrated curve $m(t)$, then, by translation, $m(t) \in \mathcal{M}$ for any $t \in \mathbb{R}$.

Proposition 2.7. *There exists a calibrated curve and, consequently, the projected Mather set $\mathcal{M}$ is not empty.*

Proof. We fix a smooth density m_0 and we look at the δ-discounted problem (4) which reads

$$\mathcal{V}_\delta(m_0) = \inf_{(m,w)} \int_0^\infty e^{-\delta t} \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(t) + \mathcal{F}(m(t))dt.$$

We recall that $\mathcal{V}_\delta$ satisfies the dynamic programming principle

$$\mathcal{V}_\delta(m_0) = \inf_{m,w} \int_0^T e^{-\delta s} \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \mathcal{F}(m(s))ds + e^{-\delta T}\mathcal{V}_\delta(m(t)),$$

where the infimum is taken over $(m, w) \in \mathcal{E}_2^\delta(m_0)$. We already know that the solution of the minimization problem corresponds to a couple $(\bar{m}_\delta^T, -\bar{m}_\delta^T D_p H(x, D\bar{u}_\delta^T))$ where $(\bar{m}_\delta^T, \bar{u}_\delta^T)$ solves

$$\begin{cases} -\partial_t u - \triangle u + \delta u + H(x, Du) = F(x, m) & \text{in } \mathbb{T}^d \times [0, +\infty) \\ -\partial_t m + \triangle m + \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } \mathbb{T}^d \times [0, +\infty) \\ m(0) = m_0 & \text{in } \mathbb{T}^d. \end{cases} \tag{36}$$

Note that, as the initial condition is smooth, the solution $(\bar{m}_\delta^T, \bar{u}_\delta^T)$ is smooth as well.

We define the new couple (m_δ^T, w_δ^T) as $m_\delta^T(t, x) = \bar{m}_\delta^T(t + T, x)$ and $w_\delta^T(t, x) = \bar{w}_\delta^T(t + T, x)$ so that our problem is set on $[-T, +\infty)$. We now want to prove that, when we take the limit $T \to +\infty$, our sequence converges to a couple (m_δ, w_δ) defined on $\mathbb{R} \times \mathbb{T}^d$ such that the Fokker-Plank equation is still verified. We proved in Lemma 1.3 that the drift $D_p H(x, Du_\delta^T)$ is uniformly bounded in T, therefore, m_δ^T is the solution of a Fokker-Plank equation with bounded and smooth drift. This means that m_δ^T is uniformly bounded in $C^{1,2}([-T + 1, +\infty) \times \mathbb{T}^d)$.

This implies that, at least on compact subsets of $\mathbb{R} \times \mathbb{T}^d$, when we take the limit $T \to +\infty$, we have, up to a subsequence, uniform convergence of m_δ^T to a limit m_δ.

The same convergence holds true also for w_δ^T. Indeed, in Lemma 1.3 we proved also the uniform boundedness of $D^2 u_\delta^T$ and $\partial_t u_\delta^T$ that implies the uniform continuity and the uniform boundedness of w_δ^T. The convergence (m_δ^T, w_δ^T) to (m_δ, w_δ) ensures that the couple (m_δ, w_δ) verifies the Fokker-Plank equation on $\mathbb{R} \times \mathbb{T}^d$.

We fix two different times $t_1 < t_2$. For sufficiently large T, the interval $[t_1, t_2]$ is included in $[-T, +\infty)$. If we apply the dynamic programming principle for $\mathcal{V}_\delta$, we get

$$\mathcal{V}_\delta(m_\delta^T(t_1)) = \int_{t_1}^{t_2} e^{-\delta(s-t_1)} \int_{\mathbb{T}^d} H^*\left(x, -\frac{w_\delta^T}{m_\delta^T}\right) dm_\delta^T(s) + \mathcal{F}(m_\delta^T(s))ds + e^{-\delta(t_2-t_1)}\mathcal{V}_\delta(m_\delta^T(t_2)).$$

We can now take the limit of $T \to +\infty$ in the above expression and we find that (m_δ, w_δ) verifies

$$\mathcal{V}_\delta(m_\delta(t_1)) = \int_{t_1}^{t_2} e^{-\delta(s-t_1)} \int_{\mathbb{T}^d} H^*\left(x, -\frac{w_\delta}{m_\delta}\right) dm_\delta(s) + \mathcal{F}(m_\delta(s))ds + e^{-\delta(t_2-t_1)}\mathcal{V}_\delta(m_\delta(t_2)).$$

for any $t_1 < t_2 \in \mathbb{R}$.

As the function u_δ^T is uniformly bounded in T we have also uniform convergence of u_δ^T to a function u_δ. We can then pass to the limit in the MFG system (36) and the couple (u_δ, m_δ) solves

$$\begin{cases} -\partial_t u_\delta - \triangle u_\delta + \delta u_\delta + H(x, Du_\delta) = F(x, m_\delta) & \text{in } \mathbb{T}^d \times \mathbb{R} \\ -\partial_t m_\delta + \triangle m_\delta + \operatorname{div}(m_\delta D_p H(x, Du_\delta)) = 0 & \text{in } \mathbb{T}^d \times \mathbb{R}. \end{cases}$$

As in [1], in order to let $\delta \to 0$, we need to define $\bar{u}_\delta(t,x) = u_\delta(t,x) - u_\delta(0,0)$ and $\theta_\delta = u_\delta(0,0)$. The couple $(\bar{u}_\delta, m_\delta)$ solves

$$\begin{cases} -\partial_t \bar{u}_\delta - \triangle \bar{u}_\delta + \delta \bar{u}_\delta + \delta\theta_\delta + H(x, D\bar{u}_\delta) = F(x, m_\delta) & \text{in } \mathbb{T}^d \times \mathbb{R} \\ -\partial_t m_\delta + \triangle m_\delta + \operatorname{div}(m_\delta D_p H(x, D\bar{u}_\delta)) = 0 & \text{in } \mathbb{T}^d \times \mathbb{R} \\ \bar{u}_\delta(0,x) = u_\delta(0,x) - u_\delta(0,0) & \text{in } \mathbb{T}^d. \end{cases}$$

We restrict ourselves to the subsequence $\{\delta_n\}_n$ identified in the proof of Lemma 1.7. Using again the uniform estimates on $D\bar{u}_\delta$, we have that $\bar{u}_\delta(0,x)$ is uniformly bounded which implies the boundedness of $\delta\theta_\delta$. Moreover, thanks to the bounds on $D^2 u_\delta$ and $\partial_t u_\delta$, $\bar{u}_\delta$ is also uniformly continuous and the same holds true for m_δ. We can then pass to the limit on any compact set and $\bar{u}_{\delta_n} \to u$, $m_{\delta_n} \to m$ and $\delta_n \theta_{\delta_n} \to \theta$ where (u, m, θ) solves

$$\begin{cases} -\partial_t u - \triangle u + \theta + H(x, Du) = F(x, m) & \text{in } \mathbb{T}^d \times \mathbb{R} \\ -\partial_t m + \triangle m + \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } \mathbb{T}^d \times \mathbb{R} \end{cases}$$

As we can always replace $u(t,x)$ with $u(t,x) - \theta t$ we can suppose $\theta = 0$. The convergences above give us also the uniform convergence on compact sets (up to subsequence) of the couple $(m_{\delta_n}, w_{\delta_n}) = (m_{\delta_n}, -m_{\delta_n} D_p H(x, D\bar{u}_{\delta_n}))$ to $(m, w) = (m, -m D_p H(x, Du))$ which solves the usual Fokker-Plank equation.

Let now $\eta \in \mathcal{P}(\mathbb{T}^d)$ be the measure identified in the proof of Lemma 1.7. Then

$$\mathcal{V}_{\delta_n}(m_{\delta_n}(t_1)) - \mathcal{V}_{\delta_n}(\eta) =$$

$$\int_{t_1}^{t_2} e^{-\delta_n(s-t_1)} \int_{\mathbb{T}^d} H^*\left(x, -\frac{w_{\delta_n}}{m_{\delta_n}}\right) dm_{\delta_n}(s) + \mathcal{F}(m_{\delta_n}(s))ds + e^{-\delta_n(t_2-t_1)} \mathcal{V}_{\delta_n}(m_{\delta_n}(t_2)) - \mathcal{V}_{\delta_n}(\eta).$$

Given the continuity of $\mathcal{V}_\delta$, the uniform convergence of m_δ and w_δ, we can pass to the limit in n and we finally get that for any interval $[t_1, t_2]$ the couple (m, w) verifies the Fokker-Plank equation on $\mathbb{R}$ and

$$\chi(m(t_1)) = \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \mathcal{F}(m(s))ds + \chi(m(t_2)) + \lambda(t_2 - t_1).$$

In particular we found a calibrated curve and, for any $t \in \mathbb{R}$, $m(t)$ belongs to the projected Mather set $\mathcal{M}$.

□

2.3 Compactness of the projected Mather set

In Proposition 2.1 we proved that, if χ is a corrector function and (m, w) is a trajectory starting from $m_0 \in \mathcal{P}(\mathbb{T}^d)$ which is optimal for the dynamic programming principle of χ, then (m, w) is associated to a MFG system which enjoys the estimates we proved in Lemma 1.3. Therefore, a completely analogous proof to the one proposed in Theorem 1.5 gives the following result.

Proposition 2.8. *The set of corrector functions is uniformly Lipschitz continuous.*

We can now prove the compactness of the projected Mather set $\mathcal{M}$

Proposition 2.9. *The projected Mather set $\mathcal{M}$ is compact*

Proof. Let $m_n \in \mathcal{M}$ such that $m_n^0 \to m_0$. Let $(m_n(t), w_n(t))$ be the calibrated curve starting from m_n^0. For any t_1, t_2 we know that $(m_n(t), w_n(t))$ verifies

$$\chi_n(m_n(t_1)) \geq \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w_n}{m_n} \right) dm_n(s) + \mathcal{F}(m_n(s)) + \lambda(t_2 - t_1) + \chi_n(m_n(t_2)). \tag{37}$$

We know from Proposition 2.8 that the set $\{\chi_n\}_n$ is uniformly Lipschitz. If we replace χ_n with $\chi_n(\cdot) - \chi_n(\eta)$, then $\{\chi_n\}_n$ is also bounded and thus compact. Therefore, we can pick a subsequence such that χ_n converges to a function χ.

Given that χ_n are uniformly bounded, then

$$\int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w_n}{m_n} \right) dm_n(s) + \mathcal{F}(m_n(s)) ds \leq C.$$

The constant C does not depend on n and, therefore, $\int_{t_1}^{t_2} |w_n|$ is uniformly bounded as well. As we argued in Proposition 1.1, this implies that $\{m_n\}_n$ is uniformly bounded in $C^{1/2}([t_1, t_2], \mathcal{P}(\mathbb{T}^d))$ for any t_1, t_2. We have then that m_n converges uniformly on any compact set to a limit $m \in C^0(\mathbb{R}, \mathcal{P}(\mathbb{T}^d))$ and the same holds true for w_n in $\mathcal{M}(\mathbb{R} \times \mathbb{T}^d; \mathbb{R}^d)$, therefore (m, w) solves in sense of distribution the usual FP equation on $\mathbb{R}$. By weak lower semicontinuity of the integral part in (37) and the uniform convergence of χ_n we get that

$$\chi(m(t_1)) \geq \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w}{m} \right) dm(s) + \mathcal{F}(m(s)) ds + \chi(m(t_2)) + \lambda(t_2 - t_1)$$

with $m(0) = m_0$ because $m_n^0 \to m_0$. The opposite inequality is true by dynamic programming principle and so this proves that $m_0 \in \mathcal{M}$ and, eventually, that $\mathcal{M}$ is closed.

□

2.4 Minimal invariant set and Ergodicity

We say that a closed subset $\mathcal{C}$ of $\mathcal{M}$ is invariant if, for any $m_0 \in \mathcal{C}$, there exists a calibrated curve (m, w) such that $m(0) = m_0$ and $m(t) \in \mathcal{C}$ for any $t \in \mathbb{R}$. We say that an invariant set $\mathcal{C}$ is minimal if $\mathcal{C}$ does not contains any proper closed invariant subset.

Lemma 2.10. *There exists a minimal set $\mathcal{N}$.*

We do not present the proof which is a standard application of Zorn's Lemma (see for instance [15]).

Proposition 2.11. *Let $\mathcal{N}$ be a minimal invariant set. If $m_0 \in \mathcal{N}$ and $\{m(t),\ t \in \mathbb{R}\}$ is a calibrated curve such that $m(0) = m_0$, then $\{m(t),\ t \in \mathbb{R}\}$ is dense in $\mathcal{N}$.*

Proof. The proof is analogous to the one we used to prove that $\mathcal{M}$ is closed. We define $\mathcal{C}$ the closure of the trajectory $\{m(t)\ t \in \mathbb{R}\}$. $\mathcal{C}$ is a closed subset of $\mathcal{N}$, in order to prove that it coincides with $\mathcal{N}$ we just need to prove that it is invariant or, in other words, that, if $\bar{m} \in \mathcal{C}$, then also a calibrated curve passing through $\bar{m}$ belongs to $\mathcal{C}$.

Let $\bar{m}$ be the limit of $m_n = m(t_n) \in \{m(t),\ t \in \mathbb{R}\}$ and $\{m_n(t)\}$ their corresponding calibrated curves. If $w_n(t)$ are the control associated to the calibrated curve $m_n(t)$ then, as in Lemma 2.10, we get that $\|w_n\|_{L^1}$ is uniformly bounded on any compact set. As we already pointed out, it implies the uniform convergence of $m_n(t)$ on compact sets. If $\bar{m}(t)$ is the trajectory to which $m_n(t)$ converges, then it must be a calibrated curve starting from $\bar{m}$ because we imposed that $m_n \to \bar{m}$. This means that for any $s \in \mathbb{R}$, $\bar{m}(s)$ is the limit of $m_n(s)$. As $\mathcal{C}$ is closed and $m_n(s) \in \mathcal{C}$, then $\bar{m}(s) \in \mathcal{C}$.

We proved that $\mathcal{C}$ is a not empty, invariant, closed subset of $\mathcal{N}$. By the minimality of $\mathcal{N}$ the two sets must conicide. □

3 The role of Monotonicity

So far, the hypothesis on $\mathcal{F}$ were mostly about its regularity and no structural assumptions were imposed. On the other hand, when we are interested in understanding whether the limit value λ coincides with $\bar{\lambda}$, the ergodic one, the structure of $\mathcal{F}$ does actually play a fundamental role. In the next section we impose convexity and, as it was already proved in [10], we get that $\lambda = \bar{\lambda}$. More interestingly, in Section 3.2 we provide a class of explicit examples where $\lambda > \bar{\lambda}$ and, therefore, there is not convergence of the time dependent MFG system to the ergodic one.

3.1 The convex case: $\lambda = \bar{\lambda}$

In this section we will show that under convexity those two values are the same.

We assume the following further assumptions on F:

$$\int_{\mathbb{T}^d} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2) \geq 0 \tag{38}$$

We introduce the functional $J^T(m_0, \cdot, \cdot) : \mathcal{E}^T(m_0) \to \mathbb{R}$ defined by

$$J^T(m_0, m, w) = \int_0^T \int_{\mathbb{T}^d} H^* \left(x, -\frac{dw(t)}{dm(t)}(x) \right) dm(t) + \mathcal{F}(m(t))dt,$$

so that

$$\mathcal{U}(T, m_0) = \inf_{(m,w) \in \mathcal{E}_2^T(m_0)} J^T(m_0, m, w).$$

Under the monotonicity assumption (38), the functional J^T is convex, therefore we can easily prove that $\lambda = \bar{\lambda}$. We recall that

$$-\bar{\lambda} = \inf_{(m,w) \in \mathcal{E}} \int_{\mathbb{T}^d} H^* \left(x, -\frac{dw}{dm}(x) \right) dm(x) + \mathcal{F}(m).$$

Proposition 3.1. *Under the above assumptions $\lambda = \bar{\lambda}$.*

Proof. In order to prove the proposition we use the representation of λ that we discussed in Proposition 1.10:

$$-\lambda = \lim_{T\to\infty} \frac{1}{T} \inf_{\Pi_T(m_0,m_1)} \int_0^T \int_{\mathbb{T}^d} H^*(x,\alpha) dm(s) + \mathcal{F}(m(s)) ds.$$

Given that λ does not depend on the initial value m_0, we take $m_0 = m_1$. We can also suppose that m_0 is smooth and bounded from below by a positive constant, so that we can apply Proposition 1.10. We now consider any admissible (m^T, w^T) for $\Pi(m_0, m_0)$ and we define $\bar{m}^T = \frac{1}{T}\int_0^T m^T dt$ and $\bar{w}^T = \frac{1}{T}\int_0^T w^T dt$. Given that the $m^T(0) = m^T(T) = m_0$, the couple $(\bar{m}^T, \bar{w}^T)$ verifies, in sense of distributions, $-\triangle m + \mathrm{div}(w) = 0$ and it is an admissible competitor for the stationary problem.

Now we just need to apply Jensen's inequality to get

$$J^T(m_0, m^T, w^T) = \frac{1}{T}\int_0^T \int_{\mathbb{T}^d} H^*(x, w^T/m^T) dm^T + \mathcal{F}(m^T) dt \geq \int_{\mathbb{T}^d} H^*(x, \bar{w}^T/\bar{m}^T) d\bar{m}^T + \mathcal{F}(\bar{m}^T).$$

If we take the infimum over $(m^T, w^T) \in \Pi_T(m_0, m_0)$ and we take the limit in T, we end up with

$$-\lambda \geq \lim_{T\to\infty} \inf_{\Pi_T(m_0,m_0)} \int_{\mathbb{T}^d} H^*(x, \bar{w}^T/\bar{m}^T) d\bar{m}^T + \mathcal{F}(\bar{m}^T) \geq -\bar{\lambda}.$$

We already proved the opposite inequality in Proposition 1.9, which we recall that it holds true also outside the monotonicity assumption. □

As $\lambda = \bar{\lambda}$, the dynamic programming principle for χ reads

$$\chi(m(t_1)) = \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \mathcal{F}(m(s)) ds + \chi(m(t_2)) + \bar{\lambda}(t_2 - t_1).$$

If $(\bar{m}, \bar{w})$ is a minimizer for the static MFG problem then, if we define $(m(t), w(t)) = (\bar{m}, \bar{w})$ for any $t \in \mathbb{R}$, the relation above holds true. This means that the constant trajectory $(\bar{m}, \bar{w})$ is a calibrated curve so that $\bar{m} \in \mathcal{M}$. Moreover, as the calibrated curve is stationary, the singleton $\mathcal{N} = \{\bar{m}\}$ is a minimal invariant set because it is closed, invariant and it cannot contain any proper subset.

3.2 A non convex example where $-\lambda < -\bar{\lambda}$

We now present an example where the non convexity of $\mathcal{F}$ leads to an ergodic configuration where the limit value $-\lambda$ is strictly lower then the ergodic one $-\bar{\lambda}$. A straightforward consequence will be that there cannot be any stationary calibrated curve, which in turn implies that any calibrated curve in a minimal invariant set has to be either periodic or chaotic. We say that a calibrated curve $m(t)$ in a minimal invariant set $\mathcal{N}$ has a chaotic behavior if its trajectory is strictly included in $\mathcal{N}$.

We fix $e_d \in \mathbb{R}^d \setminus \{0\}$ a unit vector parallel to one of the axes and we identify $\mathbb{T}^d$ with $\mathbb{T}^{d-1} \times \mathbb{T}$ where $\mathbb{T}$ is the torus identified by the direction e_d. We fix also H such that H^* verifies the following conditions: $H^*(x,p) > 0$ for any $x \in \mathbb{T}^d$, $p \neq -e_d$ and $H^*(x, -e_d) = 0$ for any $x \in \mathbb{T}^d$. The assumption (1) on the hamiltonian H implies that there exist two constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1 I_d \leq D_{pp} H^*(x,p) \leq C_2 I_d \qquad \forall x \in \mathbb{T}^d, \forall p \in \mathbb{R}^d. \tag{39}$$

Let us define the set $\mathcal{A} \subset \mathcal{P}(\mathbb{T}^d)$ as the set of $\mu \in \mathcal{P}(\mathbb{T}^d)$ for which there exits $\mu' \in \mathcal{P}(\mathbb{T}^{d-1})$ such that $\mu = \mu' \otimes dx_d$, where dx_d si the Lebesgue measure on $\mathbb{T}$. Note that $\mu \in \mathcal{A}$ if and only if $\mathrm{div}(e_d\mu) = 0$.

We fix $m_0 : \mathbb{T}^d \to \mathbb{R}$ a smooth, strictly positive density such that $m_0 \notin \mathcal{A}$. A measure m belongs to the set $\mathcal{B}$ if there exists $z \in \mathbb{T}^d$ such that $m(\cdot) = m_0(\cdot + z)$. As $\mathcal{A}$ and $\mathcal{B}$ are closed and disjoint, they are separated by a positive distance $\varepsilon > 0$.

We choose $\mathcal{F} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that $\mathcal{F} \geq 0$, $\mathcal{F} \equiv 2$ in $\mathcal{A}$ and $\mathcal{F} \equiv 0$ in $\mathcal{B}$. The existence of such a function is ensured by Lemma 4.2 and Lemma 4.3 in Appendix. They also guarantee that we can choose $\mathcal{F}$ such that it verifies the regularity assumptions that were in place in the previous sections.

We recall that the functional $J^T(\mu, \cdot, \cdot)$ is defined on $\mathcal{E}_2^T(\mu)$ by

$$J^T(\mu, m, w) = \int_0^T \int_{\mathbb{T}^d} H^*\left(x, -\frac{dw(t)}{dm(t)}(x)\right) dm(t) + \mathcal{F}(m(t))dt.$$

In this section we add in the definition of $\mathcal{E}_2^T(\mu)$ a viscosity constant $\sigma > 0$ so that (m, w) verifies $-\partial_t m + \sigma \triangle m - \mathrm{div}(w) = 0$.

We also define the ergodic functional $J : \mathcal{E} \to \mathbb{R}$ as follows

$$J(m, w) = \int_{\mathbb{T}} H^*\left(x, -\frac{dw(t)}{dm(t)}(x)\right) dm(t) + \mathcal{F}(m(t))dt, \tag{40}$$

where in this framework, (m, w) verifies $\sigma \triangle m - \mathrm{div} w = 0$. According to the definition of $\bar{\lambda}$ in (6), we have

$$-\bar{\lambda} = \inf_{(m,w)\in\mathcal{E}} J(m, w) \tag{41}$$

Proposition 3.2. *There exists a $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0]$ we have $-\lambda < -\bar{\lambda}$.*

Proof. We define

$$m(t, x) = m_0(x - e_d t),$$

and

$$w(t, x) = e_d m(t, x) + \sigma D m_0(x - e_d t).$$

The couple (m, w) belongs to $\mathcal{E}_2^T(m_0)$, so $-\lambda \leq J^T(m, w)$. By definition of $\mathcal{F}$, we know that $\mathcal{F}(m(t)) = 0$ for any time t. Moreover, since $D_{pp}L \leq C_2 I_d$ with $0 = H^*(x, -e_d) \leq L(x, \alpha)$, we have $H^*(x, \alpha) \leq \frac{1}{2}C_2|\alpha + e_d|^2$. Thus

$$-\lambda \leq \frac{1}{T}\int_0^T \int_{\mathbb{T}^d} \frac{C_2}{2}\left|\frac{w(t, x)}{m(t, x)} - e_d\right|^2 m(t, x)dxdt = \frac{C_2}{2}\sigma^2 \int_{\mathbb{T}^d} \frac{|Dm_0(x)|^2}{m_0(x)}dx = \sigma^2 I,$$

where $I = \frac{C_2}{2}\int_{\mathbb{T}^d} \frac{|Dm_0(x)|^2}{m_0(x)}dx$.

We now focus on the static case. We recall that the differential constraint on J is $-\sigma \triangle m + \mathrm{div} w = 0$. By standard arguments we have that there exists a minimizer $(\bar{m}, \bar{w})$ of (40).

As in the proof of Proposition 1.1 we can define a dual problem which reads

$$\inf_{(u,\lambda)\in C^2(\mathbb{T}^d)\times\mathbb{R}} \left\{\lambda \text{ s.t. } -\lambda - \triangle u + H(x, Du) \leq F(x, \bar{m})\right\}.$$

Thanks to the regularity of $F(\cdot,\bar{m})$ we have a smooth solution $(\bar{u},\bar{\lambda})$ which solves $-\bar{\lambda}+\triangle\bar{u}+H(x,D\bar{u})=F(x,\bar{m})$. By duality, if we argue as in Proposition 1.1 (see [4]) we get that $\bar{w}=-\bar{m}D_pH(x,D\bar{u})$, so that, by Schauder theory, $\bar{m}$ is smooth and bounded from below.

We can now estimate $-\bar{\lambda}$. Thanks to the regularity of $(\bar{m},\bar{w})$, the parabolic constraint ensures that $\bar{w}=\sigma D\bar{m}+\zeta$ where ζ is a smooth, divergence free vector field. If $(\bar{m},\bar{w})=(\bar{m},\sigma D\bar{m}+\zeta)$ is a minimizer of (40) and we use the growth assumption (39), we have

$$\int_{\mathbb{T}^d} H^*\left(x,\frac{\sigma D\bar{m}+\zeta}{\bar{m}}\right)\bar{m}(dx)+\mathcal{F}(\bar{m})\geq\frac{C_1}{2}\int_{\mathbb{T}^d}\left|\frac{\sigma\bar{D}m+\zeta}{\bar{m}}+e_d\right|^2\bar{m}(dx)+\mathcal{F}(\bar{m}). \quad (42)$$

Note that, as $\mathrm{div}\zeta=0$ and $\bar{m}$ is smooth and bounded from below, $\int_{\mathbb{T}^d}\zeta\cdot D\bar{m}/\bar{m}=0$. Indeed,

$$\int_{\mathbb{T}^d}\frac{D\bar{m}}{\bar{m}}\zeta dx=\int_{\mathbb{T}^d}D(\ln(\bar{m}))\zeta dx=-\int_{\mathbb{T}^d}\ln(\bar{m})\mathrm{div}(\zeta)dx=0.$$

Therefore, if we expand the square in (42), we get

$$\int_{\mathbb{T}} H^*\left(x,\frac{\sigma D\bar{m}+\zeta}{\bar{m}}\right)\bar{m}(dx)+\mathcal{F}(\bar{m})\geq\frac{C_1}{2}\int_{T^d}\sigma^2\frac{|D\bar{m}|^2}{\bar{m}}+\left|\frac{\zeta}{\bar{m}}+e_d\right|^2\bar{m}dx\geq\frac{C_1}{2}\int_{T^d}\left|\frac{\zeta}{\bar{m}}+e_d\right|^2\bar{m}dx. \quad (43)$$

Plugging (43) into (41) we eventually find that

$$-\bar{\lambda}\geq\frac{C_1}{2}\int_{T^d}\left|\frac{\zeta}{\bar{m}}+e_d\right|^2\bar{m}dx+\mathcal{F}(\bar{m}).$$

The righthand side of the above inequality is bounded from below by a positive constant independent of σ. Indeed, we know that, for any σ, $\bar{m}>0$, so

$$-\bar{\lambda}\geq\inf_{(m,\xi)}\int_{T^d}\left|\frac{\xi}{m}+e_d\right|^2 mdx+\mathcal{F}(m)$$

where the infimum is taken over all the probability densities $m>0$ and the free divergence vectors ξ. Here, m and ξ do not verify the elliptic constraint, therefore we lose the dependence on σ.

Let (m_n,ξ_n) be a minimizing sequence and $m\in\mathcal{P}(\mathbb{T}^d)$ a the limit of m_n (the existence of m is guaranteed by the compactness of $\mathcal{P}(\mathbb{T}^d)$). If the infimum were achieved at zero then $\mathrm{div}(m_ne_d)\to\mathrm{div}(me_d)=0$. Indeed, as both the addends should converge to zero, for any test function φ, we have

$$\begin{aligned}\left|\int_{\mathbb{T}^d}\mathrm{div}(m_ne_d)\varphi dx\right|&=\left|\int_{\mathbb{T}^d}m_ne_d\cdot D\varphi dx\right|=\left|\int_{\mathbb{T}^d}(m_ne_d-\xi_n)\cdot D\varphi dx\right|\\&\leq\left(\int_{\mathbb{T}^d}\frac{|m_ne_d-\xi|^2}{m_n}dx\right)^{1/2}\left(\int_{\mathbb{T}^d}|D\varphi|^2m_ndx\right)^{1/2}\to 0\end{aligned}$$

On the other hand, if $\mathrm{div}(me_d)=0$, then, by construction of $\mathcal{F}$, we have $\mathcal{F}(m_n)\to\mathcal{F}(m)=2$. Therefore, there exists a constant $K>0$ independent of σ such that $-\bar{\lambda}>K$.

We can conclude the proof choosing σ small enough such that

$$0\leq-\lambda\leq\sigma^2I<K\leq-\bar{\lambda}.$$

□

Proposition 3.3. *Under the hypothesis of Subsection 3.2 the projected Mather set $\mathcal{M}$ does not contain any stationary calibrated curve. Moreover, if m_0 belongs to a minimal invariant set $\mathcal{N}$ and $m(t)$ is a calibrated curve starting from m_0 then $m(t)$ is either periodic or it has a chaotic behavior.*

Proof. We recall that if $m(t)$ is a calibrated curve then

$$\chi(m(t_1)) - \chi(m(t_2)) = \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w}{m}\right) dm(s) + \mathcal{F}(m(s))ds + \lambda(t_2 - t_1).$$

If $m(t)$ is constantly equal to $\bar{m}$ then we have that $\int_{\mathbb{T}} L\left(x, -\bar{w}/\bar{m}\right) dm + \mathcal{F}(\bar{m}) = -\bar{\lambda}$. As $-\lambda < -\bar{\lambda}$, it implies

$$\chi(\bar{m}) - \chi(\bar{m}) = \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{\bar{w}}{\bar{m}}\right) d\bar{m}(s) + \mathcal{F}(\bar{m})ds + \lambda(t_2 - t_1) = (t_2 - t_1)(-\bar{\lambda} + \lambda) > 0,$$

so the contradiction.

Moreover, if $m(t)$ is a calibrated curve in $\mathcal{N}$, we proved in Proposition 2.11 that its trajectory has to be dense in $\mathcal{N}$. If the trajectory is closed then the curve is periodic and $\{m(t)\ t \in \mathbb{R}\} = \mathcal{N}$. Otherwise $\{m(t)\ t \in \mathbb{R}\} \subsetneq \mathcal{N}$ and $m(t)$ is chaotic. □

4 Appendix

We present here the result used in Proposition 2.5, which is part of an on-going work with Marco Cirant.

Lemma 4.1. *Let $(\bar{m}, \bar{w})$ be as in Proposition 2.5. For any $0 \leq t_1 < t_2$, the minimization problem*

$$\bar{A}_{t_1}^{t_2} = \inf_{u \in \mathcal{K}} A_{t_1}^{t_2}(u) = \inf_{u \in \mathcal{K}} \left\{ \int_{T^d} u(x, t_2) d\bar{m}(t_2) - \int_{T^d} u(x, t_1) d\bar{m}(t_1) \right\}$$

where $\mathcal{K}$ is the set of $u \in C^{1,2}([t_1, t_2] \times \mathbb{T}^d)$ such that $-\partial_t u - \Delta u + H(x, Du) = F(x, \bar{m})$ and $\int_{\mathbb{T}^d} u(t_1, x)dx = 0$, admits a solution.

Proof. The difficulties of this minimization problem comes from the fact that, a priori, we have no regularity on the measure $\bar{m}$, which does not allow us to directly get the compactness of the minimizing sequence that we need. On the other hand the dynamic programming principle of χ and some local in time semiconcavity estimates help to overcome this obstacle.

Let us recall that the dynamic programming principle for χ reads

$$\chi(m_0) = \inf_{(m,w)} \left(\int_0^t H^* \left(x, -\frac{w}{m}\right) dm(s) + \mathcal{F}(m(s))ds + \chi(m(t)) \right) + \lambda t$$

and that the following convex duality holds true (see Remark 2.3)

$$\inf_{(m,w) \in \Pi(\bar{m}(t_1), \bar{m}(t_2))} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} H^* \left(x, -\frac{w}{m}\right) dm(s) + \int_{\mathbb{T}^d} F(x, \bar{m}(s)) dm(s) ds$$

$$= - \inf_{u \in \mathcal{K}} \left\{ \int_{T^d} u(x, t_2) d\bar{m}(t_2) - \int_{T^d} u(x, t_1) d\bar{m}(t_1) \right\}.$$

Fist of all we prove that for any $t_1 < t_2 < t_3$ we have that $\bar{A}^{t_3}_{t_1} = \bar{A}^{t_2}_{t_1} + \bar{A}^{t_3}_{t_2}$. Indeed, using the duality between the two minimization problems, we have

$$\bar{A}^{t_2}_{t_1} + \bar{A}^{t_3}_{t_2} = \inf_{(m,w)\in\Pi(\bar{m}(t_1),\bar{m}(t_2))} \int_{t_1}^{t_2}\int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \int_{\mathbb{T}^d} F(x,\bar{m}(s))dm(s)ds$$

$$+ \inf_{(m,w)\in\Pi(\bar{m}(t_2),\bar{m}(t_3))} \int_{t_2}^{t_3}\int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \int_{\mathbb{T}^d} F(x,\bar{m}(s))dm(s)ds.$$

If we use the dynamic programming principle of χ and the fact that $(\bar{m},\bar{w})$ is optimal, the expression above is equal to

$$\int_{t_1}^{t_3}\int_{\mathbb{T}^d} H^*\left(x, -\frac{\bar{w}}{\bar{m}}\right) d\bar{m}(s) + \int_{\mathbb{T}^d} F(x,\bar{m}(s))d\bar{m}(s)ds =$$

$$\inf_{(m,w)\in\Pi(\bar{m}(t_1),\bar{m}(t_3))} \int_{t_1}^{t_3}\int_{\mathbb{T}^d} H^*\left(x, -\frac{w}{m}\right) dm(s) + \int_{\mathbb{T}^d} F(x,\bar{m}(s))dm(s)ds = \bar{A}^{t_3}_{t_1}$$

We claim now and we prove later that, if $\{u_n\}_n$ is a minimizing sequence for $\bar{A}^{t_3}_{t_1}$, then u_n uniformly convergences to a function $u \in C^{1,2}([t_1,t_3)\times\mathbb{T}^d)$ on any $[t_1,t]$ with $t < t_3$ and u is admissible for $\bar{A}^t_{t_1}$. This implies that the function u is a minimizer for $\bar{A}^t_{t_1}$ and in particular for $t = t_2$. If we suppose that $A^{t_2}_{t_1}(u) = \bar{A}^{t_2}_{t_1} + \varepsilon$, then we have

$$A^{t_3}_{t_1}(u_n) = A^{t_2}_{t_1}(u_n) + A^{t_3}_{t_2}(u_n) \geq A^{t_2}_{t_1}(u_n) + \bar{A}^{t_3}_{t_2}.$$

If we take the limit $n\to\infty$ on both side, the uniform convergence of u_n on $[t_1,t_2]$ and the fact that u_n is a minimizing sequence for $\bar{A}^{t_3}_{t_1}$ give us

$$\bar{A}^{t_3}_{t_1} \geq A^{t_2}_{t_1}(u_n) + \bar{A}^{t_3}_{t_2} = A^{t_2}_{t_1}(u) = \bar{A}^{t_2}_{t_1} + \varepsilon + \bar{A}^{t_3}_{t_2} = \bar{A}^{t_3}_{t_1} + \varepsilon$$

which is impossible and so u has to be a minimizer for $\bar{A}^{t_2}_{t_1}$.

We now prove our claim and we show that the set of functions $u \in C^{1,2}([t_1,t_3]\times\mathbb{T}^d)$ which solves

$$\begin{cases} -\partial_t u - \triangle u + H(x,Du) = F(x,\bar{m}) \quad \text{in } [t_1,t_3]\times\mathbb{T}^d \\ \int_{\mathbb{T}^d} u(t_1,x)dx = 0 \end{cases} \tag{44}$$

is uniformly bounded in $C^1([t_1,\tau]\times\mathbb{T}^d)$ for any $\tau < t_3$. This gives the local convergence that we used earlier. Without loss of generality we can suppose $t_1 = 0$ and $t_3 = T$. As in Lemma 1.3 we argue by semiconcavity.

We consider $\xi\in\mathbb{R}^d$, $|\xi|\leq 1$ and we look at the equation solved by $w(t,x) = D^2u^T(t,x)\xi\cdot\xi$. We now define $\bar{w}(t,x) = w(t,x)\eta(t)$, where η is the cutoff function $\eta(t) = (t-T)^2$. We choose ξ such that it maximizes $\sup_{t,x}\bar{w}(t,x)$.

If we derive twice in space the HJB equation in (44), then $\bar{w}$ solves

$$-\partial_t\bar{w} - w\eta' - \triangle\bar{w} + D_{\xi\xi}H(x,Du)\eta + 2D_{\xi p}H(x,Du)\cdot D^2u\xi\eta$$
$$+D_{pp}H(x,Du)D^2u\xi\cdot D^2u\xi\eta + D_pH(x,Du)\cdot D\bar{w} = D^2_{\xi\xi}F(x,m)\eta.$$

The cutoff function ensures the existence of a positive interior maximum of $\bar{w}$. At the maximum, using also the boundedness of $D^2_{pp}H$, the above equation implies

$$-w\eta' - K + 2D_{\xi p}H(x,Du)\cdot D^2u\xi\eta + \bar{C}^{-1}|D^2u\xi|^2\eta \le D^2_{\xi\xi}F(x,m)\eta.$$

Rearranging the terms and using the boundedness of $D^2_{\xi\xi}F$ we get

$$|D^2u\xi|^2\eta \le w\eta' + C + 2C|D^2u\xi|\eta.$$

As $\eta' = 2\eta^{1/2}$ we can apply the Young's inequality so that $\eta' w \le \eta/2|D^2u\xi|^2 + 4$ and

$$\frac{1}{2}|D^2u\xi|^2\eta \le C + 2C|D^2u\xi|\eta,$$

which in turn gives

$$|D^2u\xi|^2\eta \le C.$$

If w^+ and $\bar{w}^+$ are the positive parts of w and $\bar{w}$, then we have our semiconcavity estimates because in $[0,\tau]\times\mathbb{T}^d$

$$(w^+\eta)^2 \le (\bar{w}^+)^2 \le |D^2u\xi|^2\eta^2 \le M.$$

Note that $M = M(\tau)$ and it diverges when $\tau \to T$. On the other hand the estimates above, along with (11), gives uniform bounds on $\|Du\|_\infty$ on $[0,\tau]$ with $\tau < T$.

Integrating in space the HJB equation we get

$$|\partial_t \int_{\mathbb{T}^d} u dx| \le \int_{\mathbb{T}^d} |H(x,Du)| + |F(x,\bar{m})| dx \le C(\tau).$$

As $\int_{\mathbb{T}^d} u(0,x)dx = 0$, the above inequality ensures that $|\int_{\mathbb{T}^d} u(t,x)dx| \le C(\tau)$ for any $t \le \tau$. This gives us $\mathrm{osc}(u(t,\cdot)) \le |\int_{\mathbb{T}^d} u(t,x)dx| + C\sup_x |Du(t,x)| \le C(\tau)$.

As in Lemma 1.3, the boundedness of space derivatives implies also that $|\partial_t u(t)| \le C(\tau)$ for any $t \le \tau$ and so the claim.

□

Here we propose the proof of the existence of the smooth functions that we used in Subsection 3.2.

Lemma 4.2. *For any $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and any $\epsilon > 0$, there exists $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ of class C^1 such that $\Phi(m_0) = 1$ and $\Phi = 0$ on $B^c_\varepsilon(m_0)$. Moreover, we can choose Φ such that*

$$\|D_m\Phi\|_\infty \le 10/\epsilon.$$

and with $D_xD_m\Phi$ bounded.

Proof. Let E be the compact set of $1-$Lipschitz continuous maps on $\mathbb{T}^d$ vanishing at 0. Let (ϕ_n) be a dense family in E consisting of smooth maps. For N large, we consider

$$\Psi_N(m) = \sup_{n=1,\dots,N} \int_{\mathbb{T}^d} \phi_n(m - m_0).$$

Then (Ψ_N) is a family which is uniformly Lipschitz continuous in $\mathcal{P}(\mathbb{T}^d)$ and converges to $\mathbf{d}_1(\cdot, m_0)$. So, for $\eta > 0$ small there exists N large enough such that

$$\|\Psi_N - \mathbf{d}_1(\cdot, m_0)\|_\infty \le \eta.$$

Next we approximate the sup in the definition of Ψ_N. We consider

$$\Psi_N^\delta(m) = \delta \log\left(\sum_{n=1}^N \exp\left\{\delta^{-1}\int_{\mathbb{T}^d}\phi_n(m-m_0)\right\}\right).$$

Recall that

$$\Psi_N(m) \le \Psi_N^\delta(m) \le \delta\ln(N) + \Psi_N(m).$$

Note that Ψ_N^δ is C^1, with

$$D_m\Psi_N^\delta(m,x) = \left(\sum_{n=1}^N \exp\left\{\delta^{-1}\int_{\mathbb{T}^d}\phi_n(m-m_0)\right\}\right)^{-1}\sum_{n=1}^N \exp\left\{\delta^{-1}\int_{\mathbb{T}^d}\phi_n(m-m_0)\right\} D\phi_n(x). \tag{45}$$

Note that $D_m\Psi_N^\delta(m,x)$ is a convex combination of $D\phi_n(x)$, so that

$$|D_m\Psi_N^\delta(m,x)| \le \sup_n |D\phi_n(x)| \le 1.$$

For $\delta > 0$ small (depending on N), we have

$$\|\Psi_N^\delta - \mathbf{d}_1(\cdot, m_0)\|_\infty \le 2\eta.$$

In particular, for $\epsilon > 0$, choose $\eta = \epsilon/5$: then

$$\inf_{m\in B_\epsilon^c(m_0)} \Psi_N^\delta(m) \ge \epsilon - 2\eta = 3\epsilon/5 \qquad \text{and} \qquad \Psi_N^\delta(m_0) \le 2\eta = 2\epsilon/5.$$

Moreover, if we derive (45) in space we get

$$D_xD_m\Psi_N^\delta(m,x) = \left(\sum_{n=1}^N \exp\left\{\delta^{-1}\int_{\mathbb{T}^d}\phi_n(m-m_0)\right\}\right)^{-1}\sum_{n=1}^N \exp\left\{\delta^{-1}\int_{\mathbb{T}^d}\phi_n(m-m_0)\right\} D^2\phi_n(x).$$

Note that $D_xD_m\Psi_N^\delta(m,x)$ is a convex combination of $D^2\phi_n(x)$. Therefore, there exists a constant C_N such that

$$|D_xD_m\Psi_N^\delta(m,x)| \le \sup_n |D^2\phi_n(x)| \le C_N.$$

To complete the result, define a map $\zeta_\epsilon = \mathbb{R} \to [0,1]$ smooth and nonincreasing, with $\zeta_\epsilon(s) = 0$ if $s \ge 3\epsilon/5$ and $\zeta_\epsilon(s) = 1$ for $s \ge 2\epsilon/5$. We can choose $\|\zeta_\epsilon'\|_\infty \le 10/\epsilon$. The map $\Phi = \zeta_\epsilon \circ \Psi_N^\epsilon$ satisfies the claim. □

Lemma 4.3. *Let A and B be closed subsets of $\mathcal{P}(\mathbb{T}^d)$ with an empty intersection. Then there exists a C^1 map $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that $\Phi = 1$ on A, $\Phi = 0$, B and $D_xD_m\Phi$ bounded.*

Proof. Let $\epsilon > 0$ be the minimal distance between A and B:

$$\epsilon := \inf_{m\in A,\ m'\in B} \mathbf{d}_1(m,m') > 0.$$

Let (m_n) be dense in A and $\Phi_n : \mathcal{P}(\mathbb{T}^d) \to [0,1]$ be associated with m_n as in Lemma 4.2: $\Phi_n(m_n) = 1$, $\Phi_n = 0$ in $B_\epsilon^c(m_n)$ and $\|D_xD_m\Phi_n\|$ bounded. For $\delta > 0$ small and N large, let us set

$$\Psi_N^\delta(m) = \delta\log\left(\sum_{n=1}^N \exp\left\{\delta^{-1}\Phi_n(m)\right\}\right).$$

Note that Ψ_N^δ is C^1 with

$$D_x D_m \Psi_N^\delta(m,y) = \left(\sum_{n=1}^N \exp\left\{\delta^{-1}\Phi_n(m)\right\}\right)^{-1} \sum_{n=1}^N \exp\left\{\delta^{-1}\Phi_n(m)\right\} D_x D_m \Phi_n(m,y).$$

In particular,

$$|D_x D_m \Psi_N^\delta(m,x)| \le \sup_n |D^2\phi_n(x)| \le C_N.$$

For $m \in B$ we have $\Phi_n(m) = 0$, so that $\Psi_N^\delta(m) = \delta \ln(N)$. As (m_n) is dense, we can choose, for $\eta > 0$, N large enough so that

$$\max_{m\in A} \min_{n=1,\dots,N} \mathbf{d}_1(m,m_n) \le \eta.$$

Then, for $m \in A$, there exists $n \in \{1,\dots,N\}$ with $\mathbf{d}_1(m,m_n) \le \eta$, so that (by Lipschitz continuity of Φ_n)

$$\Phi_n(m) \ge \Phi_n(m_n) - 10\epsilon^{-1}\mathbf{d}_1(m,m_n) \ge 1 - 10\epsilon^{-1}\eta.$$

Thus

$$\Psi_N^\delta(m) \ge \delta \log\left(\exp\left\{\delta^{-1}\Phi_n(m)\right\}\right) \ge 1 - 10\epsilon^{-1}\eta.$$

We now choose $\eta > 0$ such that $1 - 10\epsilon^{-1}\eta = 2/3$ (which in turns fixes N), and then $\delta > 0$ small such that $\delta \ln(N) \le 1/3$. Then we have

$$\inf_{m\in A} \Psi_N^\delta(m) \ge 2/3 \qquad \text{and} \qquad \sup_{m\in B} \Psi_N^\delta(m) \le 1/3.$$

Then conclusion follows easily. □

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