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“A function is continuous if and only if you can draw its graph without lifting the pen from the paper” – Concept usage in proofs by students in a topology course

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Many students enter university having learned that the graph of a continuous function is “in one piece” and “can be drawn without lifting the pen from the paper.” Rigorously, a function $\mathbb{R} \to \mathbb{R}$ is continuous if and only if its graph is path-connected. In this article, I examine proofs of this fact by students in a topology course. Based on Moore (1994), concept usage of continuity and path-connectedness is analysed through recognition and building-with of the RBC-model of epistemic actions (Dreyfus & Kidron, 2014) in combination with a refinement of Oerter’s (1982) contextual layers of objects. A “propositional” layer to describe relationships between objects used in proofs is introduced and used to perform case studies of students’ solutions.

Keywords: Teaching and learning of specific topics in university mathematics, teaching and learning of analysis and calculus, topology, continuity, epistemic actions.

INTRODUCTION

The concept image of a continuous real function of one real variable as one whose graph is “in one piece” or “can be drawn without lifting the pen from the paper” (provided that the function is defined on an interval) is held by many students in school or university (Tall & Vinner, 1981; Hanke & Schäfer, 2017).

This piece of research is intended to expand the viewpoint from students’ concept usage of continuity from first-year analysis to higher courses and sensitise for some students’ thinking processes. To a large extent, this paper is of philosophical nature and suggests a refinement of Schäfer’s (2010) approach to describe epistemic actions with the RBC-model, namely by introducing a new specific layer called propositional layer. This is relevant for the specifically mathematical procedure of deduction from theorems about relationships between objects. This refinement is then applied to students’ (partial) proofs of the fact that a real function defined on $\mathbb{R}$ is continuous if and only if it has a path-connected graph. Thus, this study begins to fill a gap in the literature on this highly recurring concept image and the difficulties in finding a rigorous proof which requires some topological knowledge.

The general research question before starting this investigation was which mental images students use in a proof to a prevalently, vividly acceptable theorem. Here, the aim of this article is to display students’ proofs and moot a way to dissect these according to the levels of concreteness of the objects the students used.
The task

The exercise in question was (translation E.H.; see Ross (2013, p. 182)):

“In school, one often says ‘A function is continuous if you can draw its graph without lifting the pen.’ Prove the following exact version of this proposition: A function \( f: \mathbb{R} \to \mathbb{R} \) is continuous if and only if its graph \( \Gamma_f = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2 \) is path-connected.”

The intuition of “without lifting the pen” has to be translated into a valid statement carefully. Path-connectedness really is required instead of connectedness, and the theorem is no longer valid for functions from an arbitrary path-connected space to the real numbers. It is tacitly assumed in this task that the topologies for \( \mathbb{R} \) and \( \mathbb{R}^2 \) are Euclidean and \( \Gamma_f \) inherits the induced topology. Note that the graph \( \Gamma_f \) is the image of the function \( \text{id} \times f: \mathbb{R} \to \mathbb{R}^2, x \mapsto (x, f(x)) \). Thus, the forward implication in the exercise follows from the facts that functions into products of topological spaces are continuous (with respect to the product topology) if their components are continuous, and continuous images of path-connected sets are path-connected. For the reverse direction, continuity of \( f \) at \( p \in \mathbb{R} \) can be proven by contraposition via the \( \varepsilon\)-\( \delta \)-definition using the existence of a path of the form \( \gamma = (\varphi, f \circ \varphi) \) between \( (u, f(u)) \) and \( (v, f(v)) \) in \( \Gamma_f \) for some \( u < p < v \).

Since this paper has theoretical aims next to the empirical investigation and due to page restrictions, I omit an à-priori-analysis of different possibilities of proving this theorem and prerequisites.

FRAMEWORK: THEORY AND METHODOLOGY

Concept usage, object layers and the model of nested epistemic actions

The concept image of a learner for a mathematical object, class of objects or procedures is “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). In contrast, (personal) concept definitions are students’ attempts to specify a concept. Moore (1994) claims that besides “mathematical language and notation” and “getting started” one of the major difficulties in proving is “concept understanding” (mix of concept images, concept definition and concept usage) (p. 249): This means, many students who fail in a proof “lack intuitive understanding of concepts”, “cannot use concept images to write a proof”, “cannot state the definitions [properly, E.H.]” or “do not know how to structure a proof from a definition” (Moore, 1994, p. 253). The term concept usage “refers to the ways one operates with the concept in generating or using examples or in doing proofs” (Moore, 1994, p. 252).

The Abstraction in Context Methodology (AiC) (Dreyfus & Kidron, 2014) offers a theory about learning, originating in the need for a new mathematical construct, its construction of knowledge and its consolidation taking into account a specific form of context. In this setting, construction of knowledge means performing nested epistemic actions (RBC-model): Recognising, building-with and constructing. Recognising is
“seeing the relevance of a specific previous knowledge construct to the problem at hand”, building-with “comprises the combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification, or the solution of a problem” and construction “consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct” (Dreyfus & Kidron, 2014, p. 89). The actions are nested in the sense that building-with something requires its recognition and construction requires building-with. Hence, the word construction is meant globally within the AiC methodology and locally in the RBC-model. It is noteworthy that flexibility and availability of a construct does not stem from construction itself but consolidation.

Based on Oerter’s (1982) theory of activity, Schäfer (2010) differentiated between three layers which help to describe recognition processes of objects more precisely. Next to objectification [orig. Vergegenständlichung] which is the creation of objects, objectual concern [orig. Gegenstandsbezug] towards previously constructed objects can be classified on three layers (Oerter, 1982): On the singular layer objects are not distinguishable from the action of an individual itself (e.g. recognising numbers in a table); for the actor, the objects are not yet seen as objects and do not need to have names. On the contextual layer objects are characterised by their usage – not anymore restricted to an individual but shared within a community – and the usage is performed within a specific contentual context and similarity of situations; the objects gain persistence beyond singular action. Finally, on the formal layer objects are disengaged from specific actions or context (Oerter, 1982; Schäfer, 2010). The objectual concern of previously constructed objects (of a learning process) is reflected in the way someone can use this object. In this article, this activity theory oriented standpoint is specified to mathematics practice in students’ attempts to prove a topological fact.

The approach to consider object layers and actions on them seems related to Sfard’s (1991) idea of the duality of operational and structural conceptions (views of an individual of a concept) within the process of concept formation; concept here means a “mathematical idea […] in its ‘official’ form” (p. 5). Structural conception considers concepts as “abstract objects” and its dual form, the operational conception, is about “processes, algorithms and actions”, not the notion as an object itself (Sfard, 1991, p. 4; emph. orig.). Both of these views of conception can be seen in mathematical practice: In the action of formal recognition as well as propositional recognition and building-with (see below) objects are conceptualised as structural and through operational conceptions a conclusion is achieved. Propositions themselves are structural, and their scope is reflected in their use, allowing a deduction or justification (not necessarily mathematically correct though).

Founded in the Anthropological Theory of the Didactic, Hausberger (2018) described “structuralist praxeologies” which characterise mathematical justification practices that are oriented towards replacing a statement about the particular with one about the general: “Structuralist thinking is characterized by reasoning in terms of classes of objects, relationships between these classes and (structural) stability of properties
under operations on structures” (pp. 82f.; emph. orig.). What I describe here as propositional building-with action would fall most likely under his levels 2 and 3 of “structuralist dimension” of a proof (Hausberger, 2018, p. 81; emph. orig.) which describe the application of theorems to a task at hand, therefore reasoning on structuralist rather concrete object level. For example, the task of showing that an object O (e. g. ℤ) possesses property A (e. g. unique factorisation domain) can be changed to the task of showing that O belongs to a class of objects C (e. g. Euclidean rings) in order to apply a theorem which states that each member of C possesses A (e. g. every Euclidean ring is a unique factorisation domain). The identification of O’s membership to C resembles in the proofs in this paper to recognition actions (of different layers) that O possesses property C. This procedure is “illuminating as to the ‘root causes’ behind the result” (Hausberger, 2018, p. 83).

Mathematical notions such as “continuous real function” are on the formal layer for experienced students and mathematicians, and instances thereof can be recognised as having the general properties of elements of the class they belong to. However, mathematical theorems can also be seen as objects on the formal layer. If they come into use, e. g. by specialisation to a concrete situation in an exercise, they become an object one builds-with. In fact, since the usage of objects is of particular interest here, I claim that a new object layer should be included, the propositional layer: On the propositional layer we find theorems as objects that describe properties of objects on the formal layer. Thus, it does not merely contain abstract mathematical objects but relationships between objects as own objects. These relationships can then be applied in a propositional building-with action towards objects on the formal or the contextual layer. [1]

In practice, the decision for which object layer occurs at which place can be guided by the instantiation of objects (“let f be given by f(x) = x^2 + 1”), which mostly suggests the singular or contextual layer, or their declaration (“let f be a function such that …”), which highlights an object rather on formal layer. In proofs, the identification of propositional recognition and building-with actions may be assisted by the writer, e. g. referencing the lecture, the number of a theorem etc. However, somebody’s “personal mathematical toolkit” may determine which layer really is in use. The analyses below reflect an interpretation of the written product, not the way of finding the proof.

Examples for the identification of the object layers

If one wants to show that the unit circle S^1 is compact, one can directly use the definition of compactness by taking any open cover of the circle, assuming there was no finite subcover and, using concrete, contextual properties of S^1, evoking a contradiction. This way, propositional objects are not necessarily involved (except for the definition of course, and depending on the particular argumentation). On the other hand, identifying S^1 as continuous image of [0,1] → ℝ^2, t ↦ (cos(2πt), sin(2πt)), is a contextual recognition of S^1 to the context of the map, and together with the propositional recognition of the fact that continuous images of compact spaces are
compact the propositional building-with action of this fact to the situation at hand yields the compactness of $S^1$. Structurally, these two proofs are completely different.

Next, consider the above proof that the graph of a continuous function is path-connected. First, the graph is recognised as the image of a certain map $\text{id} \times f$ (contextual layer), and this map is recognised as a product of continuous maps (contextual layer). The theorem that products of continuous maps are continuous is recognised on the propositional layer and used to deduce the continuity of the map written down in the proof (building-with on the propositional layer to further recognise the continuity of $\text{id} \times f$ on formal layer: The concrete map is no longer important, simply its continuity). Finally, the theorem that continuous images of path-connected spaces are path-connected (recognition on the propositional layer) is used in a building-with action on the propositional layer yielding that the graph of $f$ is path-connected by recognising that the theorem is applicable to the situation at hand. Even though this proof is very short, many recognition and building-with actions had to be completed, brought into order and were compressed in only two sentences.

The “if”-direction of the given task can be proved in the formal/propositional manner as described and is a special case of the theorem “If $F: X \to Y$ is any function between a path-connected topological space $X$ and any topological space $Y$, then the graph of $F$ is path-connected if $F$ is continuous.” The main ingredient is the fact that for any path-connected space $U$ and any function $g: U \to V$ between topological spaces, continuity of $g$ implies the path-connectedness of $g(U)$. Since the reverse directions of these two statements are not true [2], the reverse direction of the students’ task cannot be proved (completely) in the formal/propositional manner considering solely (path-connected) topological spaces, continuous maps and their properties as above. Nevertheless, it is surely possible to argue with propositional objects, e.g. using the intermediate value theorem or the intermediate value property of functions (Ross, 2013, p. 182f.).

DATA COLLECTION

The data collection for this study took place during the spring semester 2017 at the University of Bremen within the topology class for Bachelor students in pure mathematics. I was not involved in this class but was informed by the lecturer about the contents. During the third week of the semester the students had to solve (besides others) the task presented above. Construction is not directly observable in the analyses below because all notions involved are not new to the students. The context of the contextual layer is understood very locally, depending on the objects already available in the particular solution, for instance those that have been declared before.

RESULTS

The following cases are supposed to illustrate the work with the above framework (for groups 2 and 3 only one implication is shown). Overall, the solutions were very different regarding the approaches used. More details cannot be included here. All of the following transcripts were translated from German respecting (unusual) syntax.
Notational and language errors are mostly ignored. Abbreviations of German words were often not abbreviated. Small notational errors like forgetting a closing bracket were corrected. An open ball of radius $\omega$ centred at $a$ is denoted by $U_\omega(a)$.

**Case study: Group 1**

The following is a transcript of the solution of group 1 with a sketch redrawn by myself.

1. “⇐” Let $\varepsilon > 0$
2. Since $\Gamma_f$ is path-connected, there exists $\gamma: [0,1] \to \Gamma_f$
3. with $\gamma(0) = (x - \delta_1, x - \varepsilon)^T$ and $\gamma(1) = (x + \delta_2, x + \varepsilon)^T$
4. Choose $\delta = \max\{\delta_1, \delta_2\}$, then it holds that
5. $|x - y| < \delta: |f(x) - f(y)| < \varepsilon$ $y \in \mathbb{R}$
6. and thus $f$ is continuous because $\varepsilon$ arbitrary.

As a first step, the students declare a positive $\varepsilon$ (line 1) which indicates that they would like to test the $\varepsilon$-$\delta$-definition of continuity. It looks like a recognition on the formal layer, but it is not stated explicitly at which point they want to check continuity; most probably it is “$x$”. Afterwards, the students assume that there exists a function (most likely a path, even though not stated) $[0,1] \to \Gamma_f$ connecting the points $(x - \delta_1, x - \varepsilon)^T$ and $(x + \delta_2, x + \varepsilon)^T$. It is not clear why these points should lie on the graph of $f$ but this should be the case since the path lies in $\Gamma_f$ by their assumption (recognition on contextual layer) (lines 2-3). Interpreting the students’ sketch of the graph, which they do not refer to, I hypothesise that they actually mean the points $(x - \delta_1, f(x) - \varepsilon)^T$ and $(x + \delta_2, f(x) + \varepsilon)^T$, and $\delta_1$ and $\delta_2$ are chosen such that $x - \delta_1$ and $x + \delta_2$ are preimages of the corresponding second components of the points on the graph under $f$. Thus, the path is recognised on contextual layer using false assumptions. Then, the students choose the minimal of these $\delta$s to implicate that $f$ fulfils the $\varepsilon$-$\delta$-definition of continuity in line 5 (lines 4-6). This is a building-with action on contextual layer since taking the minimum of the $\delta$s is quite a standard technique in analysis. However, if the function is not “nice enough” between $x - \delta_1$ and $x + \delta_2$, for example monotonically increasing on $[x - \delta_1, x + \delta_2]$ as indicated by the students’ figure, then the interval $(x - \delta, x + \delta)$ is not necessarily mapped into the interval $(f(x) - \varepsilon, f(x) + \varepsilon)$. Hence, the building-with action does not lead to the needed conclusion in line 5. Even if there were preimages of $f(x) \pm \varepsilon$ (only to the left or right of $x$ according to the sign of $\varepsilon$), one had to choose $\delta_1 = \inf\{\delta > 0: f(x - \delta) = f(x) - \varepsilon\}$ and $\delta_2 = \inf\{\delta > 0: f(x + \delta) = f(x) + \varepsilon\}$, and would need to show that these are different from 0. Since functions often encountered are “nice enough” or monotonically increasing, this wrong argumentation might originate in students’ “met-befores” (McGowen & Tall, 2010).

7. “⇒” Let $\Gamma_f = \{(x, f(x)): x \in \mathbb{R}\} \subset \mathbb{R}^2$ be the graph of the continuous function $f$.
8. Let $(x, f(x))^T, (y, f(y))^T \in \Gamma_f$ (wlog $x < y$)
9. To show $\exists \gamma: [0,1] \to \Gamma_f$ path.
10. Let $z \in [x, y]$. Since $f$ is continuous, it holds that
11. $\forall \varepsilon > 0 \exists \delta > 0: |z - a| < \delta: |f(z) - f(a)| < \varepsilon$ $\quad (a \in \mathbb{R})$
No matter how small the neighbourhood $U_{\varepsilon}(f(z))$ is chosen, the values of $f(a) \in U_{\varepsilon}(f(z))$ for $a \in U_\delta(z)$.

$\Rightarrow \Gamma_f' = \{(a, f(a)) : a \in (z - \delta, z + \delta) \subset U_\delta(z) \times U_\varepsilon(f(z))\}$

Since $\mathbb{R}$ is connected, the neighbourhoods are also connected.

$\Rightarrow$ It exists a path $\gamma: [0,1] \to \Gamma_f$

with $\gamma(0) = (x, f(x))^T \quad \gamma(1) = (y, f(y))^T$

The solution starts with the definition of the graph of $f$ and the students choose two points on the graph (lines 7-8). They want to show the existence of a path in $\Gamma_f$ (line 9), presumably that links the two points, although they do not state it. This is contextual recognition since the path is adjusted to the concrete setting and formal recognition would be hypothetical because the definition of path-connectedness is not completely adapted correctly to the given problem. Afterwards, they recognise the definition of continuity of $f$ at some point $z$ between the first components of the given points on the graph (lines 10-11) on the formal layer (independent of a concrete $f$). Next, they recognise on the formal layer a topological version of continuity via neighbourhoods (line 12) and built-with on the contextual layer the implication that a part of the graph lies inside the product of the neighbourhoods $U_\delta(z)$ and $U_\varepsilon(f(x))$ (nevertheless, one has to mention that the neighbourhoods have never been instantiated because $\varepsilon$ and $\delta$ only appear within quantifiers) (line 13). Afterwards, the recognition of $\mathbb{R}$ as a connected space is formal (proven property in the lecture) but the deduction of the connectedness of the neighbourhoods (likely those in lines 12-13), or their product, is not justified (line 14) (possibly, the students also mixed up connectedness with path-connectedness here). As a last step, the group now directly concludes that there exists a path in $\Gamma_f$ joining the points on the graph chosen in the beginning (lines 15-16). It can be interpreted that the students believe that subspaces of path-connected sets are path-connected and apply this to $\Gamma_f' \subset U_\delta(z) \times U_\varepsilon(f(z))$ without making clear that the neighbourhoods are path-connected, not only connected. Under this interpretation, the building-with action would be propositional, but since the group’s claimed implication does not seem to be logically connected to their previous proof steps, their thinking cannot be ascertained.

**Case study: Group 2**

“$\Rightarrow$” Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Note that $\mathbb{R}$ is path-connected.

Now let $(a, f(a))$ and $(b, f(b)) \in \Gamma_f$.

Since $\mathbb{R}$ is connected, there is a path $\gamma$ such that $\gamma(0) = a, \gamma(1) = b$.

Then, $g = (\gamma, f \circ \gamma)$ is continuous because the composition of continuous maps is continuous.

$g$ is also a path from $(a, f(a))$ to $(b, f(b))$ because:

$g(0) = (\gamma(0), (f \circ \gamma)(0)) = (a, f(a))$

$g(1) = (\gamma(1), (f \circ \gamma)(1)) = (b, f(b))$

thus, $\Gamma_f$ is path-connected since $g$ is continuous.

On the formal layer, the students recognise that $\mathbb{R}$ is path-connected (note that no
additional information of $\mathbb{R}$ is used) (line 17) and that they have to find a path between two arbitrary points in the graph which is indicated by the declaration of two points in $\Gamma_f$ (line 18). On the propositional layer, they build (which means they state its existence) a path $\gamma$ between the first components of the chosen points using that $\mathbb{R}$ is connected (in fact, they should have used that it is path-connected; it is not clear whether this is just a notational error since they recognised before that $\mathbb{R}$ is path-connected) (line 19). Next, the students recognise on the propositional layer that compositions of continuous functions are continuous and build-with this fact on the propositional layer that the product function $g = (\gamma, f \circ \gamma)$, contextually recognised as a product of continuous maps, is continuous (their argument lacks the fact that the continuity of the components implies the continuity of the product map) (line 20). Recognising that they have to plug in 0 and 1 for verification (formal layer of a part of the definition of path-connectedness), they conclude that $g$ is in fact a path between the initial points (lines 21-23) (singular/contextual layer in lines 22-23 because the concrete form of $g$ is used). The last line contains again a propositional building-with action since the graph of $f$ is (presumably) shown to have the property that any two of its points can be linked with a path (even though not explicitly stated).

Case study: Group 3

25 “$\Leftarrow$” Approach: If one can show, let it be supposed that there exists a path between two points of the graph of a function, then there also exists an injective path, it follows (*).

26 Suppose, $f$ not continuous at $x \in \mathbb{R} \Rightarrow \exists \varepsilon > 0: \forall \delta > 0 \exists \xi \in \mathbb{R}: \xi \in U_\delta(x), f(\xi) \notin U_\varepsilon(f(x))$

27 By assumption there is a path $\mu: [0,1] \to \Gamma_f$ with $\mu(0) = (\bar{x}, f(\bar{x})), \mu(1) = (x', f(x'))$.

28 Wlog $\bar{x} < x < x'$.

29 By (*) there exists an injective path $\bar{\mu}$. Now let $\tau: [a, b] \to [0,1], \tau(x) := (x - a)/(b - a)$, is continuous!

30 $\Rightarrow \pi_2 \circ \bar{\mu} \circ \tau = f|_{[a,b]}$, where $\pi_2$ is the projection of the second component.

31 Namely, for $x \in [a, b]$ it holds:

32 $\pi_2(\bar{\mu}(\tau(x))) = \pi_2((x - a)/(b - a)) = \pi_2(x, f(x))$, since there is only one possibility for an injective continuous map in the first component of $\bar{\mu}$ in question. $\pi_2(x, f(x)) = f(x)$. // $\Leftarrow$ to $f$ not continuous

The group begins a proof by contradiction in line 26 and therefore the students recognise the negation of the $\varepsilon$-$\delta$-definition of continuity of $f$ at some $x$ on the formal layer, relying on the notation with neighbourhoods from the lecture. The students built-with on the contextual layer a path from two points on the graph whose first coordinates $\bar{x}$ and $x'$ surround $x$, the point where the function is assumed to be discontinuous (lines 26-28) (however, $x'$ is not explicitly declared). Taken for granted that one can construct an injective path linking two points given any path between these two (line 25) – admittedly, the group does neither argue how this may work nor state explicitly that this injective path has to have the same start and end point or domain – the students use
such a path $\tilde{\mu}$ and compose it with the above $\tau$ which is recognised on contextual level as continuous (line 29), to perform a building-with action deducing that $\pi_2 \circ \tilde{\mu} \circ \tau$ is equal to $f$ restricted to the interval $[a, b]$ (lines 29-30). The map $\tau$ is used here to transform the path defined on the unit interval to a path $\tilde{\mu} \circ \tau$ in $\Gamma_f$ defined on $[a, b]$ which shall function as the domain where $f$ can be applied (likely, the students actually meant $a = \tilde{x}$ and $b = x'$). In lines 30ff., the students try to justify that the second component of $\tilde{\mu} \circ \tau$ is $f^{[\mu]}_{\mu}$. Here, the students try to recognise the first component of $\tilde{\mu} \circ \tau$ as the identity on $[a, b]$ because there shall be only one possibility for an injective path between two real numbers (lines 32-33). This is wrong. However, this mistake could formally be resolved by performing a “velocity change of paths” which makes the first component of $\tilde{\mu} \circ \tau$ equal to the identity on $[a, b]$. The second equal sign in line 32 is then only justified by this erroneous recognition of the only injection $[a, b] \to [a, b]$ being the identity. Obtaining $\pi_2(x, f(x))$ as solution of the calculation in line 32 is a building-with action on the singular/contextual layer; singular here refers to the special situation – the assumption in line 25 – the students find themselves in. As a last step of the proof, I hypothesise that the students recognise $f^{[\mu]}_{\mu}$ (they write $f$ in line 33 though, which clearly agrees with $f$ on $[a, b]$) to be continuous (at $x$) (a contradiction to their assumption in line 26). Their justification is however not directly observable; they may have used the composition of the continuous maps $\tau, \tilde{\mu}$ and $\pi_2$. This is recognised as a contradiction to the assumption of discontinuity of $f$ at $x$ on formal layer (line 33). Finally, the end of the proof is obtained as the result of the propositional building-with action that finding a contradiction to the hypothesis of the contraposition of the statement to prove is equivalent to the original statement.

DISCUSSION & CONCLUSION

The notion of propositional layer of objects refines the three layers of objects Schäfer (2010) used to analyse the epistemic action of recognising. In particular, this new layer describes the building-with action of applying a proposition about relationships between objects on the formal layer. In the case studies, it turned out that the recognition and building-with actions usually succeeded when the prerequisites of a definition or theorem in use had been successfully recognised. However, the recognition of the non-satisfaction of necessary conditions for the application of a theorem failed several times because of insufficient mental imagery of continuity (e. g. “local niceness” in the “only if”-proof of Group 1) or paths (e. g. injectivity of paths with group 3) and wrong properties attributed to the objects to be acted on. Nevertheless, subsequent building-with actions on the propositional layer were often carried out coherently based on these wrong assumptions.

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NOTES

1. Of course, in definitions previously defined concepts are usually specialised, thus definitions may also be seen as objects on propositional layer. Seeing what makes up a definition or whether something satisfies a definition is regarded as a recognition action here, and objects which are recognised to satisfy a definition will be on formal layer.

2. The graph and the image of arg: \( S^1 \to \mathbb{R}, e^{i\theta} \to \theta \) (with \( 0 \leq \theta < 2\pi \)) are path-connected but arg is not continuous.

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