From single to multi-variable Calculus: a transition?

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We recently used the notion of praxeology from the Anthropological Theory of the Didactic to model the knowledge that is necessary for students to learn in order to succeed in an undergraduate multivariable Calculus course. We considered the presence and absence of elements of the knowledge to be taught, as proposed by curricular documents, in the knowledge to be learned, as indicated by final exams. Our results indicate that the mathematical activities expected of students at this level align with the activities observed in differential and integral Calculus, where exercise-driven assessments set students' work mainly in the recognition of types of tasks and recollection of appropriate techniques.

Keywords: transition to and across university mathematics, assessment practices in university mathematics education, teaching and learning of analysis and calculus, Anthropological Theory of the Didactic, praxeology.

INTRODUCTION

So far, research on the teaching and learning of Calculus has focused on *single-variable* Calculus. Cognitive and epistemological obstacles have been illustrated against students' learning of Calculus (Tall & Vinner, 1981; Sierpinska, 1994) and an institutional perspective has also been taken to study the influence of institutional practices on students' learning of Calculus (Barbé, Bosch, Espinoza, & Gascón, 2005; Hardy, 2009). There's a pattern that indicates Calculus students mostly engage in procedural work that requires only a superficial grasp of the underlying concepts (Hardy, 2009; Lithner, 2004; Selden, Selden, Hauk, & Mason, 1999).

We recently undertook a study that shifts the focus to *multivariable* Calculus courses (Brandes, 2017). Our goal was to determine the knowledge that is *essential* for students to learn in order to provide acceptable solutions on the final exam of an undergraduate multivariable Calculus course. To this end, we used the notion of *praxeology* from the Anthropological Theory of the Didactic (Chevallard, 2002) to model the knowledge students are expected to learn and the knowledge to be taught. We present our operationalization of this concept in the first part of this paper. In second stage, we discuss a partial result of our study that places this multivariable Calculus course along the transitions that university mathematics students undergo in their engagement with mathematics (Winsløw, Barquero, de Vleeschouwer, & Hardy, 2014).

THE EDUCATIONAL SYSTEM

We studied a 'Multivariable Calculus I' course offered to students in two mathematics programs at a large North-American university. One of the programs is for those who plan to join the workforce after graduation; the other aims to prepare students for graduate studies in mathematics. Students in either stream will have completed one-variable differential and integral Calculus and an introductory Linear Algebra course on matrix and vector algebra. The multivariable Calculus course and its sequel ('Multivariable Calculus II') are prerequisite to most of the courses in the program geared towards graduate studies. Students usually complete Multivariable Calculus I and II within the first year of their degree.

In any given term, the course is split into two sections per program, with about 70 students per section. The course is heavily coordinated across sections and terms through a strict curriculum, course examiner, and common assessments. The course outline specifies what to teach every week along with exercises from the textbook. The course examiner writes common assessments for students in all sections. A student's grade is obtained from the highest of the following: 10% assignments, 30% midterm, 60% final exam, or 10% assignments and 90% final exam. Finals exams are therefore the crux of a student's performance; in turn, the exams are consistent from term to term in both format and content. Past exams are readily available to students, and concern with their reactions prevents changes being made to the final exams.

ROUTINE PROBLEMS IN SINGLE-VARIABLE CALCULUS

We are interested in the mathematical activities with which students of a multivariable Calculus course are expected to engage. We focus on the types of problems that typify the learning of *multivariable* Calculus; a wealth of studies do so for *single-variable* Calculus (Hardy, 2009; Lithner, 2004; Selden et al., 1999). These studies emphasize the exercise-driven quality of the course assessments, in the sense of Selden et al.'s (1999) *routine* problems, which "mimic sample problems found in the text or lectures, except for minor changes in wording, notation, coefficients, constants, or functions" and "can be solved by well-practiced methods" (p.18).

The exercise-driven quality of the course assessments extends to elements of the curricula (Lithner, 2004). Calculus textbooks traditionally adhere to a definition-theorem-example-exercise format, wherein the exercises repeat the problematics of the examples and algorithms outlined in the text. Lithner (2004) measured the extent to which intrinsic mathematical properties play a role in the minimal reasoning required to solve routine tasks in traditional Calculus textbooks. Lithner's classification of reasoning types runs along a scale of how big a role is played by the mathematical properties intrinsic to the problem versus the reapplication of known algorithms; this scale runs parallel to Selden et al.'s (1999) spectrum of problems from *very routine* to *very non-routine*, which vary based on how familiar the solver is with the given problem. The more routine the problem, the less interaction is required of the solver with the mathematics specific to that problem.

Assessments in North-American Calculus courses are largely drawn from the course textbook, which Lithner (2004) showed to be steeped in routine problems. Accordingly, he found students' strategies to be anchored in what they recall superficially rather than in the mathematics specific to a problem. This correlates with Calculus students' failure to complete non-routine problems (Selden et al., 1999; Hardy, 2009). If textbook exercises can mostly be solved by identifying superficial similarities with a known example (Lithner, 2004), then students' non-reliance on intrinsic mathematical properties and over-reliance on the recall of algorithms may have roots in their learning environment. We follow this view by framing our study within the Anthropological Theory of the Didactic and focusing on elements of students' learning environment: curricular and assessment documents.

ANTHROPOLOGICAL THEORY OF THE DIDACTIC (ATD)

Framework

From the perspective of the ATD, knowledge does not exist in a vacuum, rather, it is bound to the institution in which it is shared and somehow connected to the knowledge shared in related institutions; such connection is called *transposition* and is of a didactic nature in the context of educational institutions (Chevallard, 1985). Didactic transpositions take place along a spectrum of knowledge in which *scholarly mathematics* (the knowledge developed, shared, and used by the experts – the mathematicians) is transposed into knowledge to be taught in a given institution, up to a transposition into knowledge actually learned by the students. This transformation of mathematical knowledge takes form in several stages: scholarly knowledge, knowledge to be taught, knowledge actually taught, knowledge to be learned, and knowledge actually learned.

An essential feature of the ATD is an epistemological model called *praxeology*. It allows the researcher to model knowledge at any stage of a didactic transposition. The notion of *praxeology* is based in the assumption that any human activity consists of a practical block (*praxis*) and a theoretical block (*logos*). The *praxis* is made up of tasks T to be accomplished and techniques τ with which to accomplish them; the *logos* is the discourse that produces, justifies, and explains the techniques in the practical block. Chevallard (1999) specifies two components of a theoretical block: technology θ, the discourse that produces and justifies the techniques in the practical block, and theory Θ that justifies the technology.

In light of these theoretical considerations, and given our goal of finding the minimal core of knowledge that students must learn in order to succeed in their multivariable Calculus course, we treated three instances of didactic transposition. We created a model of the knowledge to be learned, as determined by the final examinations; to this end, we needed a model of the knowledge to be taught, as indicated by the curricular documents. In order to familiarize ourselves with the mathematics prior to these two instances of didactic transposition, we also created a reference model based on the scholarly multivariable Calculus knowledge to be transposed. Before we present our praxeological models of the knowledge to be taught and to be learned, we review some of the literature about mathematics students' praxeologies.

Transitions in students' praxeologies

Winsløw et al. (2014) explain that students, at the pre-university level and in some cases at the university level, tend to have a praxeology defined mostly by practice. This is especially the case in differential and integral Calculus courses where assessment is concerned mostly with the practical block and does not address the ways in which the theoretical maintains the practical. This may have a precedent in the way knowledge is taught in the classroom, as teachers may not have time to justify tasks and techniques, given often-hefty curricula to deliver. Students, for their part, tacitly accept the existence of a theoretical discourse supporting the practical without concerning themselves with it (Hardy, 2009; Winsløw et al., 2014). Their work is mainly in recognizing types of tasks and identifying a suitable technique (Hardy, 2009; Winsløw et al., 2014), much as in Lithner's *identification of similarities* reasoning (2004) and Selden et al.'s *routine problems* (1999).

As students progress in university mathematics, they undergo two transitions. Where once they might have ignored theoretical blocks and worked exclusively within the practical block of a praxeology, they increasingly have to engage with theory and technology in their completion of tasks. Winsløw et al. (2014) call the transition from praxeologies that are purely practical to praxeologies that include a theoretical *and* a practical block a *first transition* of university mathematical praxeologies (p.101). For example, prior to the first transition, students complete tasks such as using derivative rules to find the derivative of a function. Here, differentiability is an always-met *condition* of the functions upon which students act in the tasks they do. Prior to the *first transition*, it is sufficient for students to attend only to the practical block of the mathematical knowledge; at the other end of this transition, students are required to acknowledge the theoretical block as the justification for the techniques they use for accomplishing a task. For instance, students may have to address the differentiability condition of a function before engaging in finding its derivative.

A *second transition* occurs when students reach courses whose curricula and assessment prioritize what once may have been the theoretical block of a praxeology; as students transition into proof-making and validating, theoretical blocks of the past become their practical blocks. For instance, the second transition will have occurred in a student who knows to use the definition or theorems about continuity to prove that, *if a function is continuous*, then some property of that function is true. The characteristics of a second transition are that students explicitly acknowledge and use the theoretical block to *generate* a technique for achieving a task.

KNOWLEDGE TO BE TAUGHT

The textbook of the multivariable Calculus course is typical of those used in North-American Calculus courses and follows the usual definition-theorem-example-exercise format. The course outline lists the textbook sections to be covered each week and a choice of end-of-section exercises. By *knowledge to be taught* (KT) we mean the mathematical knowledge in the sections and exercises listed on the outline. To model the KT, we identified the praxeologies of which it consists.

In the case of the knowledge to be delivered in this course, we found that technology and theory can be taken as one. There is no clear distinction between the two in the textbook; the discourse throughout is set in the geometry and algebra of three-dimensional space organized in the Cartesian system, and at times in Euclidean metric spaces. However, the theory is not made explicit and tends to be woven into the technology. Further, we found that the focus of the KT is mainly in the practical blocks. For the purpose of this study, then, it was sufficient to compile a list of items (definitions, theorems, etc.) that form the theoretical blocks of the praxeologies of KT without distinguishing theory from technology.

This tended to the theoretical block of the praxeology that modelled each section of the textbook on the course outline. To identify the tasks to be accomplished, we considered the examples and the end-of-section exercises listed in the outline. To describe the associated techniques, we consulted the examples and discussion portions (theorems, explanations) of the text. To account for the build-up of knowledge between sections (e.g. the notion of derivative of a vector function is defined in one section and reused in later sections), we cross-referenced across theoretical blocks and across and within practical blocks.

KNOWLEDGE TO BE LEARNED

In an operational sense, we define *knowledge to be learned* (KL) as the subset of the KT which students need to know in order to provide solutions on final exams. This operationalization was necessary from a methodological perspective: while the questions in the final exams indicate the tasks to be accomplished, in most cases there is no indication as to the expected technique or theoretical justifications. The model of KT was therefore necessary to identify these elements of a mathematical activity. In this sense, the main purpose of the model of KT was to model the KL. Our operationalization, although useful to describe and characterize the KL, does not properly reflect the fact that a transposition takes place and that some of the praxeological elements (likely, elements of the theoretical block) are more likely ill-defined than well-defined subsets of the praxeological elements of the KT. While the KL may borrow elements of the KT praxeologies, the discourse that unifies the two blocks of a praxeology might be distorted in the transposition.

Our model of KL is based on twelve final exams given recently within a span of three years. We described the solution to each exam question in terms of KT task-technique pairs that occur in the solution. Here is an instance of this work. Consider the following item from one of the exams:

> Find the tangent plane T that touches S at $(x, y) = (2,1)$, where the surface S is given by $z = f(x, y) = 1 - e^{-\left(\frac{1}{4}x^2+y^2\right)}$.

We recorded this as "to find the tangent plane to a surface at a point." This task corresponds identically to task $T_{19.1}$ from the KT model; in turn, the technique for this task requires the completion of $T_{18.2}$: to find the value of the partial derivative of a function at a point. Thus, we associated to this task the KT sequence $[(T_{18.2}, \tau_{18.2}), (T_{19.1}, \tau_{19.1})]$. This particular exam task corresponded identically to a KT task; this was not always the case. Nevertheless, apart from a handful of cases, we were able to identify sequences of task-technique pairs that would form complete solutions to the exam questions; this methodological affordance may attest to the routine quality (Selden et al., 1999) of the tasks students are expected to accomplish.

Next, we grouped tasks of the same type so as to reflect praxeologies that occur in the KT. For example, the following tasks come up in solutions to exam questions:

> To find the first partial derivatives of a function $(T_{18.1}, \tau_{18.1})$
>
> To find the first partial derivatives of a two-variable function defined implicitly $[(T_{18.1}, \tau_{18.1}), (T_{18.1}, \tau_{18.1.1})]$
>
> To verify that a two-variable function satisfies a partial differential equation $[(T_{18.1}, \tau_{18.1}), (T_{18.5}, \tau_{18.5})$

This cluster of tasks is drawn from the praxeology of KT specific to partial derivatives. Altogether, we partitioned the model that captures KL about partial derivatives and surfaces into groups of tasks that match up with these praxeologies of KT: the above cluster specific to partial derivatives, along with tasks that draw from KT praxeologies specific to functions of several variables, the chain rule, tangent planes and linear approximations, directional derivatives and the gradient vector, extreme values, and Lagrange multipliers. Organizing the model of KL in parallel to the model of KT facilitated our analysis of the structure of the KL.

STRUCTURE OF THE KNOWLEDGE TO BE LEARNED

The KL has to do with partial derivatives and surfaces; space curves and vector functions; equations of lines and planes and distance in R^3; limits of rational functions; polar curves; and Taylor Series. Let's call 'ideal student' one who has the requisite knowledge to write acceptable solutions in a final exam. How might we characterize the praxeologies of an ideal student in this course? Below, we consider which parts of the KT praxeologies are to be learned and characterize them in the language of Lithner (2004) and Selden et al. (1999).

Knowledge from all KT praxeologies occurs as knowledge to be learned. Thus, the KL is not necessarily a subset of KT in the sense that some praxeologies are to be learned while others are not. Rather, we found that the KL is a subset in the sense of what's *left* of the KT praxeologies after the didactic transposition of KT into KL.

First, the practical blocks of the KT praxeologies are downsized in this transposition. For instance, consider the praxeology of KT about polar coordinates. The ideal student can convert polar equations into Cartesian equations and sketch the curve – *given the following curves* (up to a change in constants and functions sine or cosine):

$$\gamma_1:\ r = 2 + \sin\theta, \quad 0 \leq \theta \leq 2\pi; \quad \gamma_2:\ r = 3\sin\theta, \quad 0 \leq \theta \leq \pi$$

The algebraic manipulations specific to converting these types of polar equations into Cartesian equations are in examples from the textbook, as is the technique for sketching them. The ideal student's *topos* ('action space') (Chevallard, 2002) does not need to extend beyond the point praxeology (a praxeology of knowledge that is particular to a single type of task) specific to these functions. We found many of the practical blocks of KT praxeologies to be reduced in this way to point praxeologies.

Most of the praxeologies of KT are downsized in another sense: their theoretical block is removed following the transposition from KT to KL. For instance, consider the praxeologies that constitute the knowledge to be taught about partial derivatives. We found that the practical blocks are reduced to computational tasks where the ideal student needs to apply the appropriate differentiation algorithm; the geometric interpretation of partial derivatives as slopes is unneeded and the ideal student does not need to know any of the theory or technology at the backbone of the procedures. The ideal student does not need to know the limit-based definition of partial derivative nor the definitions and roles of limits, continuity, and differentiability in the concepts of derivative, gradient, and extrema of a function. The theoretical blocks of these praxeologies vanish in the transposition from KT to KL. In general, it seems that the ideal student needs to be fluent in the algorithms prescribed by praxeologies of the KT but doesn't need to justify or explain them.

The absence of theoretical blocks in the ideal student's praxeology is manifested in several ways: first, the student needn't justify the validity or choice of technique (e.g. by verifying or stating that the chain rule is applicable, since the functions in the exams are always differentiable); second, the exam questions do not require students to interpret any results (e.g. by making a sketch of a surface near a point where some geometric properties of the surface were computed); and finally, it suffices to have a superficial grasp of the concepts in the theoretical blocks in order to accomplish the types of tasks in the final exams. We expand on this point.

In general, the ideal student can recognize task types and identify the appropriate technique, in reasoning similar to Lithner's (2004) *identification of similarities* (IS), whereby a strategy for tackling a problem is chosen based on the similarities of certain surface properties between the new problem and a known problem (e.g. given a limit-finding problem, note whether the limit is taken at a numerical value or infinity and identify the type of function involved). For instance, the exam questions specific to limits of multivariable functions are as follows: find the limit of a function $f(x, y)$ at the origin, if it exists, or show that it does not exist. The function f given in the exams is either an odd rational function with no limit at the origin or it involves a trigonometric component which could be rid of to reduce f to a rational function in the process of an $\varepsilon - \delta$ argument (in these cases, the exam functions invariably have limit 0). This task occurs in examples in the textbook and exercises in these students' assignments. In general, tasks required by the exam questions were similar to those in the KT, so that students could rely on IS reasoning rather than on the underlying mathematics in order to choose the appropriate technique. This course is therefore in line with students' pre-university mathematics, where much of their responsibility is in recognizing types of tasks and choosing an appropriate known technique (Winsløw et al., 2004).

IS reasoning is characterized as requiring little reflection on the intrinsic mathematical properties of the problem at hand (Lithner, 2004). To successfully implement IS, the ideal student needs to recognize terms in the question statements (arc length, curvature, normal plane, binormal vector…) and the formulas for deriving them. But the ideal student is not tested on the meaning of these quantities and geometric properties as they relate to a curve at a point (e.g. a student might need to find the equation for an osculating plane, but does not need to explain what the osculating plane describes). The irrelevance of intrinsic mathematical properties to the tasks students need to achieve suggests that the theoretical block of KT praxeologies need not be present in the ideal student of either course.

Theoretical blocks are missing from the ideal student's *topos* in a few senses: the student is not required to justify or explain the techniques chosen to complete a task, and at times is even told which technique to use (e.g. via instructions to 'use Lagrange Multipliers' or 'use the chain rule'). The ideal student is not required to interpret the numerical or algebraic results of their calculus in any way; and it suffices to learn the components of the theoretical blocks only superficially. In all, this multivariable Calculus course seems to follow in the pre-university mathematics tradition whereby students need not link the practical and theoretical blocks of a praxeology (Winsløw et al., 2014). Further, the components of the practical blocks themselves are discrete, as the ideal student does not need to combine tasks in any way – for instance, the ideal student must know how to find invariant quantities of a curve, but needn't provide a local description of a curve based on its invariant quantities. This may be called the "*compartmentalization* of knowledge in calculus courses" described by Winsløw et al. (2014, p.104).

On the whole, it appears that only a surface version of the KT theoretical blocks is *essential* for the ideal student to learn: they need to know terms and associated formulas, in some cases have some intuitive image of certain concepts, and be fluent in the algorithms described by the technologies. This surface acquisition of the theoretical block serves to recognize routine tasks and identify a suitable technique.

In light of the absence of theoretical blocks in the minimal core of knowledge that is *essential* for students to learn in order to provide solutions to exam questions, we conclude that the KL cannot be described by actual praxeologies (made up of a *praxis* and a *logos*). Rather, the KL is an amalgamation of practical blocks. This places this university-level multivariable Calculus course in the stage prior to the *first transition* in university mathematics education previously discussed:

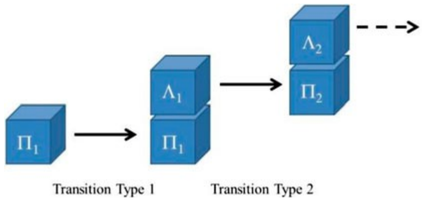

Figure 1. Transitions in university mathematics education (Winsløw et al., 2014, p.101)

where Π refers to the practical block of a praxeology and Λ to its theoretical block. Winsløw et al. (2014) explain that this *first transition* occurs when students no longer work strictly within the practical block of a praxeology and begin to incorporate a theoretical block, perhaps by using it to justify or produce a technique; a *second transition* occurs when students' past theoretical blocks turn into their current practical blocks, as when they start making and validating proofs in Analysis.

CONCLUSIONS

The aim of our study was to determine the minimal core of knowledge that is necessary for students to learn in a multivariable Calculus course in order to provide acceptable solutions on their final exam. We found that the exercise-driven quality of the course assessments makes it *essential* for students to recognize certain types of tasks and to identify the appropriate technique, but does not require students to learn the theoretical block that maintains these tasks and techniques.

Historically, the studied educational system introduced the multivariable Calculus course as a prerequisite to Analysis in an effort to help students adapt to university mathematics in the first year of their studies. It seems, however, that the mathematical activities expected of students in this bridge between pre-university and university courses are of the type expected in past Calculus courses: students' action space is fully within the practical block of the praxeologies that model the knowledge to be taught. As a result, students are no more required to engage with the theoretical in this Calculus course than they previously were. Meanwhile, the mathematical activities in Analysis courses are two steps ahead, after the *second transition* described by Winsløw et al. (2014), where students must work within what once were the theoretical blocks that backed the practical of Calculus. The question therefore remains: what course would make it *essential* for students to incorporate the theoretical blocks of a praxeology into the work they do in a practical block?

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