A study of transitions in an undergraduate mathematics program

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In this paper, we introduce an in-progress study of the transitions students face as they advance in their mathematics courses. Previous work has discussed the changes that occur in the transition from high school to university. With regards to the knowledge students are expected to learn, however, significant similarities have been noted: to do well in introductory university courses, students can learn to solve a particular subset of tasks through routinized techniques, with limited awareness of the supporting mathematical theory. In contrast, students in advanced courses are required to work with and on that theory. The first stage of our project aims to better understand this transition by building praxeological models of the knowledge to be learned in a succession of two introductory analysis courses.

Keywords: Transition to and across university mathematics, teachers' and students' practices at university level, teaching and learning of analysis and calculus.

INTRODUCTION

Several studies have discussed the specific knowledge taught and learned in pre-calculus, calculus, and analysis courses, from different perspectives: for example, concept image and concept definition (e.g., O'Shea, 2016), APOS theory (e.g., Martínez-Planell, Trigueros Gaisman, & Mcgee, 2016), and the Anthropological Theory of the Didactic (ATD; e.g., Bergé, 2016). Our starting point is the general and relatively vague question of when in an undergraduate degree in mathematics does a student need (*need* in the sense of *to succeed in the course*) to engage in *mathematical activities* that may substantially, or meaningfully, lead to developing *mathematical practices*. We consider and frame this question within the ATD (Chevallard, 1999), which provides theoretical tools for modelling any human activity or practice. The semantic distinction between these two words is essential to us. Our hypothesis is that the kinds of didactic constructs to which professors and students are exposed are decisive in fostering the emergence of practices out of collections of local, particular, and relatively short-lived activities. From the theoretical stance we take, this means the development of mathematical knowledge out of local, particular, and relatively short-lived mathematical activities.

Previous research has found that the activities proposed to students in introductory calculus courses do not necessarily encourage the development of mathematical practices. Lithner's (2004) study of the exercises in undergraduate calculus textbooks used in Sweden led to the conclusion that the majority of tasks students encounter can be solved by mathematically superficial techniques such as finding and copying a similar solution outlined somewhere in the same section of the book. When working in Spanish high school calculus classes, Barbé, Bosch, Espinoza, and

Gascón (2005) observed teachers implementing mathematically incomplete practices: they solved numerous tasks in hopes of guiding students in developing solid mathematical techniques, but struggled to introduce any lasting rational discourse (i.e., theoretical block) that produced or explained the techniques. Hardy (2009), who conducted task-based interviews with students in North American college calculus courses, showed that in the absence of such a theoretical block, students construct non-mathematical reasoning to support the highly routinized practical block they develop (i.e., the techniques and corresponding types of tasks).

To describe the kinds of transitions students are expected to go through as they progress in their university mathematics coursework, Winsløw (2008) introduced the model depicted in Figure 1 below. The conjecture is that students encounter at least two types of transitions in the practices they are supposed to develop. The first requires them to gain some level of awareness of the theoretical block that was once absent from their exclusively practical work; the second occurs when elements of that theoretical block become part of the practical block with which they must engage autonomously. Think, for instance, of how some early university courses spend a significant amount of time in lectures elaborating previously scarce definitions, theorems, and proofs, which students may be expected to understand enough to quote in assignments or reproduce on exams. In contrast, more advanced coursework requires students to develop their own proofs, often involving the more abstract objects that were part of the theoretical block constructed in earlier courses.

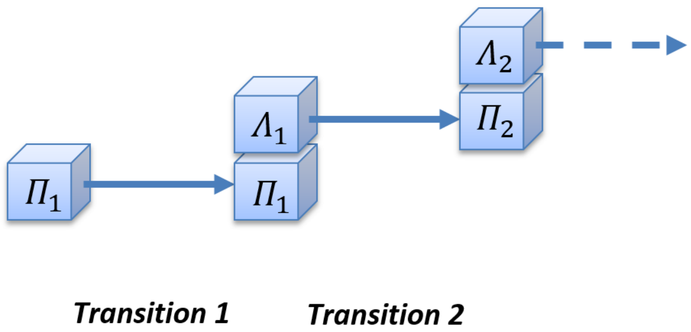

Figure 1: Transitions in university mathematics coursework (from Winsløw, 2008)

A recent study suggests that in the context of undergraduate multivariable calculus courses, students are not yet required to go through Transition 1: the models of the knowledge to be learned in these courses show that students are exposed to a limited practical block (Π), with no need to work with or on the corresponding theoretical block (Λ; Brandes, 2017). This said, students are indeed expected to work with and on mathematical theory when they take advanced courses later on.

A few questions arise from this:

1. What does this “work with and on a theoretical block” look like in comparison to the routinized, principally practical activity in which students seem to be engaging in introductory courses?

2. If the practices students develop in advanced mathematics courses can be modelled by the third stage shown in Figure 1, when, if ever, do students' practices reflect the second stage, and what are the mathematical activities proposed to them in such contexts?

The purpose of our study is to contribute to addressing these questions, and therefore, to the discussion of the transitions students face. To do so, we propose to model the knowledge at different stages in the didactic transposition process in two courses contained in what we will call the "analysis path" in a typical undergraduate mathematics program in North America (US and Canada). Ultimately, the goal is to reflect on the general question mentioned above: Can the activities in which students are obliged to engage lead to the development of mathematical practices (i.e., mathematical knowledge)?

THEORETICAL FRAMEWORK

"Activity and practice"

As mentioned above, we have come to see the semantic difference between *activity* and *practice* as pertinent to our work. The ATD's notion of *praxeology* provides a fundamental model for defining mathematical practice, which, in the context of the theory, is equated to mathematical *knowledge*. According to the model, any practice (or piece of knowledge) can be represented by a quadruplet [T, τ, θ, Θ] involving four interconnected components: a type of *tasks* T, which generates the practice, the corresponding collection of *techniques* τ developed to accomplish T, the discourse used to describe, justify, explain, and produce the techniques (i.e., their *technologies* θ), and the underlying *theories* Θ that serve as a foundation of the technological discourse. As students progress in their studies of mathematics, they engage in numerous activities, which progressively determine the practices they develop.

As a strictly hypothetical example, we could imagine students in an introductory calculus course being asked to engage in the following activities, inspired by a commonly used calculus textbook (Stewart, 2008):

a_1: Estimate the area under the graph of $f(x) = \frac{1}{x^2}$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate? What happens if you repeat the exercise with left endpoints? (Areas and Distances, Section 5.1)

a_2: Evaluate $\int_0^1 (3t - 1)^{50} dt$. (The Substitution Rule, Section 5.5)

a_3: Determine if $\int_0^9 \frac{1}{\sqrt[3]{x-1}} dx$ is convergent or divergent. If it is convergent, evaluate it. (Improper Integrals, Section 7.8)

If these were the first activities completed by students in the corresponding sections, we could expect their actions to be localized and particular. In other words, the solutions students produce would likely be the result of their engagement in a relatively isolated act of figuring out how to solve the specific given problem. As the students participate in more activities, however, they may be exposed to tasks of the same type, and may consequently begin to develop a related practice. By the end of a calculus course, for example, students will have typically solved a large number of problems involving the calculation of definite integrals by way of various integration techniques. From this, they may have learned to recognize other activities (e.g., $\int_0^{\pi/6} \frac{\sin x}{\cos^2 x} dx$ or $\int_e^{e^4} \frac{dy}{y\sqrt{\ln y}} dy$) as forming a type of task with a_2, and therefore as requiring the same technique: making a substitution (not forgetting to change the bounds!), determining the anti-derivative of the new function, and calculating the difference of this anti-derivative evaluated at the bounds. In comparison, certain activities may be encountered by students only in insignificant (e.g., unevaluated), rare, and/or disconnected situations. The action of accomplishing those tasks may hence remain isolated and particular, never contributing to the development of practices. Activities like a_1 or a_3, for instance, might never be encountered beyond a few recommended exercises at isolated, unique moments in the course.

Research confirms that the collection of activities given (and not given) to students play a crucial role in determining the kinds of practices they develop (and do not develop). Although students may seem to be learning mathematical practices, they may in fact be engaging in isolated activities or developing practices of a non-mathematical nature. In her research, Hardy (2009) noticed that when first-year calculus students are given activities related to slightly non-routine tasks, they often apply techniques in a mathematically unjust way. For example, when asked to compute $\lim_{x \to 1} \frac{x-1}{x^2+x}$, 20 out of 28 students factored, seven of which did direct substitution first. Her analysis of students' discourse during task-based interviews led her to conclude that the students tended to justify their techniques through perceived norms. She specifies, for example, that "it seems that students were doing substitution not to find the limit or to characterize an indetermination, but because that is 'what you do first'" (p. 351). To explain her observations, Hardy (2009) discusses how the kinds of activities to which the students were exposed led them to develop such practices, composed of a limited practical block and non-mathematical technologies. The activities in which students participated did not only relate to sets of highly routinized tasks, they also required no form of mathematical justification. Engaging in such activities, students observed patterns that led to the construction of techniques based on arbitrary lists of steps that just seemed to work; at least enough to do well on assignments and exams.

In a similar sense, we could imagine a student in our hypothetical example justifying their solution to the activity "integrate $f(x) = 10x^2 + 3x^4 - 1$ over $[-1,1]$" by saying something like: "first, find the antiderivative, then find the difference between the value at 1 and the value at -1, because that's how we always do it!" An activity such as “integrate $f(x) = 1/x^2$ over $[-1,1]$”, might therefore elicit the following erroneous response:

$$\int_{-1}^{1} \frac{1}{x^2} dx = -\frac{1}{x}\Big]_{-1}^{1} = -\frac{1}{(1)} + \frac{1}{(-1)} = -2 \,.$$

Unless of course the isolated activities, a_1 and a_3, were eventually, substantially, and meaningfully incorporated into developing the above non-mathematical practice into a practice more mathematical in nature.

“Undergraduate mathematics coursework”

As illustrated in the previous section, an anthropological perspective does not interpret students’ non-mathematical practices (or knowledge) as reflecting a common misconception inspired by difficulties inherent to a given mathematical concept. Rather, it sees such practices as resulting from a concrete situation within which the student finds themselves, under the influence of institutions (Douglas, 1986). In the ATD, the word “institution” is taken in a wide sense. For example, mathematicians work within an overarching institution that we could call Mathematical Research (MR), where their praxeologies are shaped by various shared criteria (concerning consistency, beauty, explanatory power, efficiency, etc.), but survive only if they follow the strict rules of mathematical reasoning. The students of interest to us, in contrast, are subjects of the institution Undergraduate Mathematics Coursework (UMC), which was in large part created to train potential participants of MR. This said, various conditions and constraints within UMC can require and enable a network of praxeologies that is fundamentally different from that built and recognized by MR. The non-mathematical praxeologies described in the previous section provide some examples.

To capture the transposition of knowledge as it moves from MR into UMC, Chevallard, and others (e.g., Bosch, Chevallard, & Gascón, 2005), have introduced a distinction between different types of knowledge (i.e. practice):

- Scholarly Knowledge, produced and used by mathematicians;
- Knowledge to be Taught, as determined by curricula, textbooks, and professors’ teaching plans;
- Knowledge Actually Taught, according to professors’ actual interactions with students, e.g., in lectures;
- Knowledge to be Learned, i.e., the knowledge students are expected to develop, which is often a transposed subset of the knowledge to be taught and actually taught, with the minimal core indicated by assessment tools;
- Knowledge Actually Learned, which can only be predicted, through analyses of student work, in-class observations of students, or other specially-designed interactions with students, such as interviews or problem-solving situations.

Although a lot can happen in university lectures, the minimal knowledge students are obliged to learn to pass their courses is determined by their assignments and exams. It is not surprising that the knowledge actually learned by students is often only a transposed subset of this minimal core. Hence, if we want to know what kind of knowledge students are or could possibly be developing in UMC, then we cannot restrict our exploration to curricula, textbooks, and teachers' lecturing practices: we need to pay careful attention to the way in which students are assessed.

Students' learned knowledge in UMC may also be characterized as a progression through various sub-institutions: from secondary school to early university courses (e.g., in single and multivariable calculus), through to more advanced university courses (e.g., in real analysis, metric spaces, measure theory, and functional analysis), which may eventually lead to graduate studies and beyond. Programs can vary from school to school and from country to country. However, a common phenomenon in secondary schools seems to be that assessments focus solely on the practical block of mathematical knowledge. The teacher may be expected to know the theoretical block for explaining the material to students; but the students are typically not obliged or even invited to develop an awareness of the technology or theory, let alone how it is linked to the practical block (Barbé et al., 2005; Winsløw, Barquero, De Vleeschouwer, & Hardy, 2014). One observed result is that many students interpret mathematical knowledge (practice) as equivalent to identifying a type of task and applying the corresponding technique (Bergqvist, Lithner, & Sumpter, 2008). Several studies confirm that this same kind of situation can arise in early university coursework (e.g., Lithner, 2003; Hardy, 2009; Brandes, 2017).

Indeed, over multiple years of coursework, students not only gain a particular view of what mathematical knowledge is, but they also develop knowledge that, when judged against the scholarly knowledge produced and used by mathematicians, is evidently non-mathematical – from the strategies they develop to identify tasks, to the discourses they use to justify these strategies and the techniques they choose. Nevertheless, as conjectured in the schema shown in Figure 1, a transition is expected to occur at some point: students are eventually required to develop knowledge that is completely and coherently mathematical. These circumstances lead Winsløw et al. (2014) to wonder about how teachers could help students accomplish such transitions. In parallel, we are inspired to validate, specify, and extend these researchers' claims by constructing praxeological models of how the different kinds of knowledge produced in the didactic transposition process progress throughout an entire undergraduate degree. In other words, we are inspired to investigate more closely the nature of the mathematical training being received by future mathematicians in the progression of their undergraduate coursework.

Of course, developing praxeological models to represent the knowledge (to be) taught and (to be) learned throughout an entire undergraduate degree is a hefty task. Within the context of our PhD project, we propose to accomplish a first stage, based on a subset of courses in one coursework path. Like in the Mathematical Research institution, Undergraduate Mathematics Coursework is divided into several sub-institutions according to domain – e.g., algebra, geometry and topology, analysis, statistics, mathematical physics, or probability – each of which contain a grouping of courses, which can be placed in some chronological order according to their prerequisites. Having already carried out research in the early courses of an "analysis path", this is the context that seemed most appropriate for our work.

METHODOLOGY

Although our project aims at modelling different levels of knowledge that can be identified in the didactic transposition process, in this paper we discuss only the modelling of *the knowledge to be learned* (KTL).

Our research is conducted at a large, urban, Canadian university. The mandatory courses in the analysis path of an Honours Bachelor of Science in Mathematics include multivariable calculus (MVC) I and II, and mathematical analysis (MA) I, II, and III. Since previous work (Brandes, 2017) suggests that the KTL in MVC I and II is similar in nature to the KTL in calculus and pre-calculus courses, we decided to start by focussing our attention on the two courses that come next and are likely candidates for housing the transitions of interest to us: MA I and II.

As mentioned above, the KTL represents the knowledge that students are expected to develop, which can be gleaned from the various activities in which they are invited to engage (lectures, assignments, and exams), as well as the materials that frame and support the activities (course outlines and textbooks). Since the minimal core of the KTL is represented in the assessment activities students must complete on their own, we have decided to ignore what happens in lectures and focus on the activities that comprise assignments and (practice) exams.

MA I and II are institutions in themselves in that they enjoy some sort of stability. For various reasons, course outlines and assigned textbooks tend to remain the same from year to year. The courses also maintain the same assessment structure: students complete assignments on a regular basis during the term, a midterm exam halfway through, and a final exam, with most of their mark (90% or more) concentrated in the examinations. This said, the actual activities proposed on assignments, midterms, and finals have less stability in that they can reflect personal choices of the professors assigned to teach the course in a given term. On top of this, our approach to modelling the knowledge actually learned will involve task-based interviews with students after they have passed MA I or II. Hence, we have collected the assignments and (practice) exams proposed only by the professor(s) who would be teaching those students. For instance, from the two MA I professors teaching in Fall 2017, we collected eleven weekly assignments, seven practice midterms, six practice final exams, and the actual examinations they gave to their students (these professors worked together in that they gave the same set of activities to their students).

To analyse such a collection of activities, we think about whether each activity is "isolated" or part of a "path to a practice". An "isolated" activity may occur only once in the sense that no other activities engage students in accomplishing the same type of task. Since such activities are unlikely to contribute directly to the development of a practice, we reflect on why they are proposed. In comparison, the activities that belong to a "path to a practice" typically combine with other activities to expose students to a type of task. Our goal in studying these activities is to extract a theoretical model of the praxeologies that the *ideal* student (i.e., the student that receives a good passing grade) is expected to develop in the course. We start by constructing punctual praxeologies related to groups of non-isolated activities. Looking at the problem statements, we can establish the types of tasks (T) that generate the praxeologies. Determining the technologico-theoretical blocks ([τ, θ, Θ]), however, requires more data. We rely on the solutions a professor makes available to students to uncover the intended techniques, as well as portions of the expected theoretical blocks; and we complete the latter by checking the course outline and reading the relevant textbook chapters. The resulting collection of punctual praxeologies then becomes part of our data, which we use to construct more generalized praxeologies, think about how they are related to one another, and reflect on the nature of the KTL.

Eventually, we plan to put the models of the KTL for MA I and II together and compare our results with what previous researchers have found in calculus and pre-calculus courses. This, we hope, will allow us to discuss how the ideal student is expected to progress in the early stages of the analysis path. At the time of the INDRUM 2018 conference, we will have completed this initial theoretical stage of our project and will thus be able to share our results.

SUMMARY AND EXPECTATIONS

For a long time, mathematics students *survive* their courses based on developing a transformed version of a practical block, where they learn to recognize routine tasks and apply techniques to solve them in a sort of mechanical, naturalized, or normalized way, void of a mathematical theoretical block. At some point throughout an undergraduate degree in mathematics, however, the conditions for students' survivability change dramatically and possibly abruptly: they are faced with activities that require them not only to fill the void of a mathematical theoretical block, but also to develop techniques for accomplishing tasks (e.g., proofs) that involve the abstract theoretical objects that have come out of hiding. Through modelling the knowledge the ideal student is expected to develop, as well as, eventually, the knowledge students actually develop in early analysis courses, we expect our project to bring about a more detailed and concrete understanding of praxeological "transitions" that have been theorized to occur, and give us some insight into how (or if) students adapt to them.

Returning to the vague and general question that originally inspired our project, we ultimately hope to learn more about what students are actually learning throughout an undergraduate degree in mathematics. The empirical data we collect will be a contribution to largely anecdotal discussions about when (if at all) students' knowledge is invited to become, and actually becomes, coherently, completely, and complexly mathematical, just like the scholarly knowledge produced and used in the institution of Mathematical Research. The significant difference between elementary and advanced courses, as professors gain more freedom and teach topics more closely related to their field of study, leads us to predict that students are eventually required to develop mathematical practices. After all, in spite of the apparent disconnection that is often observed between university mathematics courses and mathematical research (cf. Broley, Caron, & Saint-Aubin, 2017), the field of mathematics continues to live on, with new mathematicians emerging from the coursework that made up their mandatory professional education. In any case, through studying the principal conditions that currently shape the activities in which undergraduate mathematics students engage, we feel that we will be in a better position to discuss realistic and meaningful ways of encouraging these students to develop practices that are truly "mathematical", within the confines of educational institutions. This, we hope, will serve as complementary to the recent surge of studies (cf. Barquero, Serrano, & Ruiz-Munzon, 2016) aiming to explore innovative teaching approaches that question not the nature of the knowledge developed, but the dynamics of the knowledge development.

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