Tasks for enriching the understanding of the concept of linear span

Evi Papadaki, Christos Kourouniotis

Tasks for enriching the understanding of the concept of linear span

Evi Papadaki[1] and Christos Kourouniotis[2]

Department of Mathematics and Applied Mathematics, University of Crete, Greece, [1]evi.papadaki@math.uoc.gr; [2]chrisk@uoc.gr

The concept of linear span is one of the first abstract notions that students encounter in a course on Linear Algebra. Using the theoretical construct of concept image and concept definition (Tall & Vinner, 1981) along with observations about teaching and learning Linear Algebra, we present two tasks designed to enrich students' concept image regarding linear span. These tasks could be included in a problem workshop of an introductory university course on Linear Algebra. Each task is carefully created and/or selected so as to foster the ground for potential conflict factors to arise and be confronted. A preliminary evaluation shows that the tasks are well received by students and succeed in addressing certain conflicting factors.

Keywords: Teaching and learning of linear and abstract algebra; Teachers' and students' practices at university level; Linear span; Task-design.

INTRODUCTION

Linear Algebra is a subject with many applications in Mathematics and other sciences, but its teaching and learning proves to be demanding both for lecturers and students. The difficulties encountered are partly attributed to the way the subject is usually taught, as well as to students' lack of familiarity with proofs and limited knowledge of Logic and Set Theory. (Dorier et al., 2000; Hillel, 2000). Sierpinska (2000) attributes students' difficulties in Linear Algebra to their practical rather than theoretical way of thinking.

The concept of linear span seems to be quite difficult for students. Carlson (1993) states that difficulties in the notions of subspace, linear span and linear dependence / independence, if they are not addressed in time, create barriers for students. The analysis of Stewart and Thomas (2009) showed that students who were taught these concepts through formal definitions faced significant difficulties in understanding the concept of span compared to a group who were taught with emphasis on embodiment (Tall, 2004) and geometry. Moreover, they report that students have experienced several difficulties in linking the concept of span to the concept of a base. Finally, Wawro et al. (2012) propose teaching the concept through the solution of systems of linear equations and present a teaching approach through a series of realistic mathematical activities.

The main purpose of this paper is to investigate students' understanding of the concept of linear span and to use tasks to help resolve conflict factors in the students' concept image (Tall & Vinner, 1981). Based on a study of first year Mathematics undergraduates in a Greek university, we identify the misconception many students have that in a linearly dependent set each vector is in the span of the others. We use a set of design principles based on Sierpinska's (2000) remarks about theoretical thinking and Harel's (2000) principles of teaching and learning Linear Algebra, to create a set of tasks, and we present results of a preliminary evaluation of the tasks which indicate their potential to address the above misconception.

The work presented in this paper is part of the first writer's Master's thesis.

THE SETTING

The course "Geometry and Linear Algebra" is a first year mandatory course for students following the degrees in Mathematics or in Applied Mathematics at a Greek University. The course is typically taught through 4 hours of lectures and a two-hour problem workshop per week. Problem workshops are an important part in the teaching of the mandatory courses in the department. In the workshops the students are encouraged to work in groups of 5 or 6 students, on selected problems on the topics taught that week with guidance from the lecturer and a number of postgraduate or senior undergraduate students. The role of the latter is to discuss with students about the problems and the key mathematical ideas that may come up in the process. Promoting mathematical discussion among the students is a promindent element of the workshops of this course. During the semester of the study, the second writer was the lecturer of the course and the first one of the postgraduate students involved in the workshops.

During the first part of the course, students experiment with the idea of linear span in Euclidean 2- and 3-space, as an intuitive introduction to the concept. Later on, students are given a slightly modified version of the formal definition, limited to the spaces $\mathbf{R}^n$. The notion of linear span is usually described as the "subspace generated by the set S of vectors in $\mathbf{R}^n$". In relation to the general goals of the course, students are expected to familiarize with the concept of linear span in subspaces of $\mathbf{R}^n$, to be able to identify its geometrical representation in the case of $\mathbf{R}^2$ and $\mathbf{R}^3$ and to determine if a vector is in the span of a fixed set of vectors. We note the most important aspects of the concept. Firstly, linear span is a subspace, hence it is *closed* under the operations of a vector space. Secondly, *every* element in this subspace is a linear combination of some of the vectors in S. The final aspect is also very important but sometimes overlooked. In contrast to the concept of basis, there is *no limitation* in the choice of the set of generators S.

A starting point for this work was a study of the written answers given by students in response to a question in the final examination for the "Geometry and Linear Algebra" course, asking them to determine whether a vector belongs to the subspace spanned by two other vectors. The findings suggested that some students may have the misconception that in a linearly dependent set of vectors, every vector can be expressed as a linear combination of the others (see Papadaki, 2017). This misconception was found to affect students' understanding of linear span and to be a potential conflict factor (Tall & Vinner, 1981). We believe that examining the notion of linear span through tasks may offer the opportunity to confront such difficulties in a meaningful way.

THEORETICAL FRAMEWORK & DESIGN PRINCIPLES

Tall & Vinner's (1981) cognitive model of concept image and concept definition is used in the development of the task and to account for students' responses. According to them *concept image* is "the total cognitive structure that is associated with the concept" (p. 152). For each individual a concept image includes all the mental pictures (graphs, symbols, formulas etc) generated about the concept, associated properties and processes. The concept image is unique for each student and is changing over time when the student meets new stimuli. The term *evoked concept image* (Tall & Vinner, 1981) is used to describe the part of a concept image which is evoked at a specific time. Different parts of the concept image which contain conflicting aspects are called *potential conflict factors* (Tall & Vinner, 1981) and they are not evident to the individual until a stimulus causes the conflicting images to be evoked simultaneously and create confusion, in which case they are referred to as *conflict factors*.

The term *concept definition* is referring to "the form of words used to specify that concept" (Tall & Vinner, 1981: 152). The concept definition might be a reflection of an evoked concept image associated with the definition or a rote memorization of a given definition with little or no meaning to the student. We adopt Tall & Vinner's (1981) differentiation between the *personal concept definition,* constructed by the individual, and the *formal definition* of a concept, the definition accepted by the mathematical community as a whole. The personal concept definition might contain aspects not included in the formal definition and/or ignore others. Finally, the (personal) concept definition creates its own concept image, which is part of the concept image as a whole, called *concept definition image*. Tall & Vinner (1981) argue that potential conflict factors can be an obstacle in understanding the formal theory, especially the ones that are in contrast with the formal concept definition. Warwo et al. (2011) investigated students' concept images of subspace and the links students create with the formal definition of a linear subspace.

Bingolbali & Monaghan (2008) demonstrated how the construct of concept image – concept definition can be used in socio-cultural research. They argued that although concept image is unique to the individual there are aspects that are shared among students. They link these aspects to teaching and shared experiences in the department they are studying.

In this paper we adopt the original concept image – concept definition framework (Tall & Vinner, 1981) along with its more recent developments (Bingolbali & Monaghan, 2008) to design tasks that can enrich the understanding of linear span of undergraduate Mathematics students when used in situations which encourage interaction among students and tutors. We believe that this framework can be easily understood and used by mathematicians. Nardi (2006) presents evidence from discussions with mathematicians which support this idea. Therefore, we find this framework useful as a means to communicate our design and findings both to Mathematics lecturers and researchers in Mathematics Education.

In designing the tasks, we take into account Sierpinska's (2000) remarks about theoretical thinking. To be more specific, the task should have characteristics that correspond to theoretical thinking, such as opportunities for conscious reflection, connections between related concepts or different representations and attention to contradictory thoughts. Harel (2000) emphasizes the need for curricula tailored to students' needs which aid the understanding of abstract concepts in Linear Algebra. He proposes three principles that we take into account in designing the tasks. That is, the tasks should include familiar concepts that allow connection with prior knowledge and language (concreteness principle), they should justify the need of linear span (necessity principle) and allow generalization of the key ideas (generalizability principle).

We identify the following principles based on the theoretical framework, the concept of linear span as thought in the course "Geometry and Linear Algebra" as well as the needs of our students.

1. <u>Include key aspects of linear span:</u> Closure under the operations of a vector space; Every vector is a linear combination of the set of generators; No limitation in the choice of the set of generators
2. <u>Tackle potential conflict factors:</u> The difference between linear combination and linear dependence; Modes of representation (Hillel, 2000)
3. <u>Promote theoretical thinking (Sierpinska, 2000):</u> Reflection; Connections between different representations; Attention to contradictory thoughts
4. <u>The three principles of teaching and learning Linear Algebra (Harel, 2000):</u> Concreteness principle; Necessity principle; Generalizability principle
5. <u>Promote discussion:</u> among the students; between the students and the tutor

METHODOLOGY

The aim of this work is to investigate the conflict factor identified earlier through tasks that are designed to foster the ground for this conflict to emerge and to be discussed with the students. We present data collected during a preliminary evaluation of the tasks through semi-structured interviews with seven students who had attended the course "Geometry and Linear Algebra" the previous semester. The analysis of this preliminary evaluation is expected to answer the following questions: Can the tasks tackle this potential conflict factor? What are the roots of this conflict factor? Does the discussion around the task help students resolve their misconceptions? Do students find the tasks interesting and/or useful?

The following table summarizes the information about the seven participants.

	Mathematics			Applied Mathematics	
	1st Year	*2nd Year*	*3rd Year*	*1st Year*	*2nd Year*
Male	0	1	0	1	0
Female	3	0	1	0	1

Prior to the interviews each student was given a folder including the task and other necessary information. The students had one week to attempt and review the tasks before the interviews. All interviews were videotaped. To ensure confidentiality each student was assigned and referred to with an alias.

ANALYSIS

The first task is based on an exercise from the book "Linear Algebra: Concepts and Methods" by Antony and Harvey (2012). Its structure was slightly altered to fit that of the course notes (Kourouniotis, 2014). It aims to create connections with prior knowledge, known processes and language under the new context and introduce to students basic ideas linked with the concept through algebraic and geometric representations of the notion. The task is divided into three interconnected sub-tasks as a scaffolding strategy to support students.

Task 1: Consider the vectors:

$$v_1 = (-1, 0, 1),\ v_2 = (1, 2, 3),\ w_1 = (-1, 2, 5),\ w_2 = (1, 2, 5)$$

i) Show that w_1 can be expressed as a linear combination of v_1 and v_2, but w_2 cannot be expressed as a linear combination of v_1 and v_2.

ii) Explain what subspace of $\mathbf{R}^3$ is spanned by v_1, v_2 and w_1. Explain what subspace of $\mathbf{R}^3$ is spanned by v_1, v_2 and w_2. What do you observe?

iii) Show that the vectors v_1, v_2, w_1 and w_2 span $\mathbf{R}^3$, that is for every u = (x, y, z) there are a, b, c, d such that:

$$u = av_1 + bv_2 + cw_1 + dw_2$$

Show also that every vector $u \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways.

The first, introductory, sub-task aims to support students' theoretical thinking in the following sub-task by limiting its focus on calculations. This task was completed by all the participants without difficulty prior to the interview. The second sub-task is expected to enrich students' image of linear span by making connections between the algebraic and geometrical representations of the concept in $\mathbf{R}^3$. It may also motivate students to seek a deeper connection between Analytic Geometry and Linear Algebra. This sub-task was completed by 5 students. Finally, the third sub-task aims to create a link between the relation of the given vectors and the number of ways arbitrary vectors can be expressed as a linear combination of the elements in the set.

Sub-task (iii) proved to be the most difficult for the participants, being completed by only 2 students before the interviews.

In more detail, the students who did not complete (ii) appeared to have trouble with methodology. The students are expected to know from the first part of the course what the geometric representation of a 1-, 2- or 3-dimensional subspace of $\mathbf{R}^3$ is, therefore one has to connect this idea with the notion of linear span and check if the given vectors are linearly dependent. In both cases the students did not make this connection beforehand but the problem was quickly resolved through discussion. Apart from that, six out of the seven students found the question *"what do you observe?"* useful. This question was added to the task as an encouragement for reflection on the effect that different choices of vectors have on the outcome and to promote discussion. In particular, three of the students indicated that they might not have given a second thought to their result if it wasn't for this question. One of the students found the question stressful, although she had successfully answered it. Her reaction is significant to us at this point. Clute (1984) found that students with higher anxiety levels can benefit more from instrumental approaches. Open questions, such as the above, are not frequent in Greek secondary education. It is therefore reasonable to assume that some students would have difficulty (and in some cases anxiety) answering this question in a problem workshop.

While discussing sub-task (ii) an unexpected observation was made by two of the students. These students interestingly replied that the span of the vectors v_1, v_2 and w_1 is the vector space $\mathbf{R}^2$. This conflict factor is called by Wawro et al. (2011: p. 13) the "nested subspaces". Based on their evidence they hypothesized that this confusion has roots in students identifying any 2-dimensional subspace of $\mathbf{R}^n$ with $\mathbf{R}^2$ and suggested that lecturers must be aware of this as a potential conflict factor. Their hypothesis was confirmed in these cases too.

In trying to answer sub-task (iii) the biggest pitfall was following the same reasoning used in subtask (ii). This approach will not help answering the second part which requires from students to solve a system of linear equations. Despite the instructions included in the Task, four out of the five students who didn't complete (iii), tried to use the same approach as in (ii). Additionally, three of them faced a difficulty making use of the proposition "*for every $u = (x, y, z)$ there are a, b, c, d such that $u = av_1 + bv_2 + cw_1 + dw_2$*" and did not manage to recognize the random vector $u = (x, y, z)$ as a parameter of the problem. Instead they identified it as another variable. In each case the task was completed with the help of the interviewer but we find that subtask (iii) required more guidance from the part of the interviewer compared to subtask (ii). The fifth student managed to solve the required linear system but she could not make a connection between the infinite number of solutions and the fact that the four vectors are more than enough to describe any vector in $\mathbf{R}^3$.

The second task was created to address potential conflict factors in relation to the notions of linear combination and linear dependence in the context of linear span.

The idea for this task was based on our goal to promote theoretical thinking and discussion. The conflict is given to the student as a statement - challenge and the goal is to find an example to support the given proposition. It is expected that students will first use a trial and error approach by reaching for appropriate vectors in their example space (Mason & Watson, 2008). This approach will probably fail if students are not able to identify what are the key relations between v_1, v_2 and w in the proposition. If one's concept image includes conflicting ideas about the status of vectors in a set of generators, it might be difficult to find an example without careful prompting and discussion. Because of the nature of the problem, we believe that students would want to cross-examine their findings or get some guidance.

> ***Task 2:*** Let v_1, v_2 and w be linearly dependent vectors in $\mathbf{R}^3$. It is possible for w not to be in the space spanned by v_1 and v_2 although v_1, v_2 and w are linearly dependent. Give an example. Why do you think this can happen?

Moving on to the interviews, only one student had found an example of three vectors fulfilling the requirements of the task before the interview. In four of the seven cases clear signs of conflicting images emerged. This reinforces our preliminary hypothesis that students struggle with identifying the difference between the notions of linear combination and linear dependence. Furthermore, it might be an indication that Task 2 can help potential conflict factors to emerge and be resolved in a controlled environment. The following quotations capture these observations.

> Minos: So, what I thought was that I can have two vectors... which will be linearly independent that will span a plane in $\mathbf{R}^3$. I can of course... I am sure that I can find another third vector that will not belong in the plane but the relationship to be true... these three vectors to be linearly dependent.

Minos' evoked concept image of the linear span is geometric. He thinks of the span of the two vectors as a plane and he tries to find an example by checking vectors that are not on that plane. Of course, if the two vectors are linearly independent, adding a third vector that does not belong in their span will result in a linearly independent set. It seems that either this fact is not part of his concept image or his evoked concept image does not include this information because of the phrasing of the task.

In the following two quotations, the conflict can be directly connected to our preliminary findings in Papadaki (2017). The students seem to struggle with the idea of three vectors being linearly dependent and at the same time one of them not being able to be expressed as a linear combination of the others.

> Interviewer: Well, so for w not to belong in the span of the two other vectors it could not be written as a linear combination of them…

> Pasiphae: Yes… yes… well… But then how can they be linearly dependent? They are all together linearly dependent…

The student thinks of the two notions as equivalent. She later justifies her thinking by stating that if they are linearly dependent she can solve the algebraic equation $av_1 + bv_2 + cw = 0$ for any of the three vectors. Similarly, Ariadne describes her own experience with the task. It is worth mentioning that later in the interview Ariadne successfully refers to the (personal) definitions for both concepts.

> Ariadne: To begin with, to me it seemed absurd at first... because… what does it tell me? It tells me that they are linearly dependent, so if I solve for w, I will find a linear combination, so based on the theory it belongs to the subspace spanned by v_1 and v_2.

In Ariadne's case, it can be assumed that although her concept definition for linear dependence includes the information that the coefficients a, b and c are not all zero, in her evoked concept image this statement is replaced by none of them being zero.

The quotations depict two possible roots of students' difficulties with the task. That is, thinking of the linear span of two vectors as necessarily a 2-dimensional subspace or thinking of the algebraic representations of linear dependence and linear combination as equivalent.

Task 2 was thoroughly discussed with the students using different approaches based on the line of thinking of the students, but also influenced by the interviewer. The ideas portrayed in this task were discussed using an algebraic approach with four of the students and geometrically with two of them. In each interview the final example was found by the students using an informed trial and error approach. All six students reported that the discussion was very useful and Task 2 is important for understanding the concept. Three of them also said that this was the task that made them the biggest impression and four of the students suggested that it would be better if this task was presented to them in a problem workshop after a sequence of related more instrumental tasks.

Concluding, four of the students reported that they understand a notion better through examples and tasks. The way that students' concept image is formed through model examples and experience, is of course well known. What is important is the fact that the students are aware of this happening. This last observation is an indication why it is crucial to pay attention to the examples and tasks used in any course. There are students who are consciously depending on them and expect to understand the "mysterious" concepts that the lecturer is talking about through them.

RESULTS & DISCUSSION

The analysis of the interviews gave us very important information about how tasks can be improved and used in a problem workshop for an introductory course on Linear Algebra. Although all students indicated that they found the tasks useful they gave us opportunities to reflect upon their design and experiment with different tactics which can be used by tutors in an attempt to make the most out of these tasks.

Beginning with the first task, students appeared to have particular difficulty in subtask (iii). One reason might be that (iii) requires a shift in thinking and cannot be fully answered by using the same approach as in subtask (ii). In an attempt to resolve this issue we are also considering a slightly different version of this part of the task that forces students to begin with the shifted approach as follows:

> Show that for every u = (x, y, z) there exist a, b, c, d such that:
>
> $$u = av_1 + bv_2 + cw_1 + dw_2$$
>
> Conclude that v_1, v_2, w_1 and w_2 span $\mathbf{R}^3$. Moreover, show that every vector $u \in \mathbf{R}^3$ can be expressed as a linear combination of v_1, v_2, w_1 and w_2 in infinitely many ways.

Another observation we made while discussing Task 1 with the students was that of “nested subspaces”. This is another conflict factor we didn’t take into account at first and realized it only during the interviews with the students. Our observation is in line with the hypothesis of Warwo et al. (2011).

Task 2 was fruitful both in terms of meaningful discussion and reflection. Students found Task 2 important for understanding the concept of span. We also observed manifestations of cognitive conflict which indicates that the task can be used as a means to resolve potential conflict factors. Different approaches can be used to discuss these conflicts with students (algebraically, geometrically or by trial and error). A useful tactic might be to discuss the conflicting factors using more than one representation of vectors with the same group of students.

In addition, the indications about the need of examples and tasks made by the students were of great importance. This fact depicts the necessity of well thought examples and tasks in order to help students create a coherent concept image.

This paper presents an approach on how lecturers can design tasks inspired by their observations on students’ misconceptions and taking advantage of the research in Mathematics Education. The framework could be used as guidelines for tutors that are interested in developing tasks for a Linear Algebra course based on their students needs and related research. Finally, the tasks need to be tested in a problem workshop and be compared to other tasks aiming to familiarize first year Mathematics undergraduates with the concept of linear span.

References

Anthony, M., & Harvey, M. (2012). *Linear algebra: concepts and methods*. Cambridge University Press.

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, *68*(1), 19-35.

Clute, P. S. (1984). Mathematics anxiety, instructional method, and achievement in a survey course in college mathematics. *Journal for Research in Mathematics Education*, 50-58.

Dorier, J. L., Robert, A., Robinet, J., & Rogalski, M. (2000). The obstacle of formalism in linear algebra. In *On the teaching of linear algebra* (pp. 85-124). Springer Netherlands.

Harel, G. (2000). Three principles of learning and teaching mathematics. In *On the teaching of linear algebra* (pp. 177-189). Springer Netherlands.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In *On the teaching of linear algebra* (pp. 191-207). Springer Netherlands.

Kourouniotis, C. (2014). *Geometry and Linear Algebra* [lecture notes]. University of Crete, Heraklion. http://math.uoc.gr/~chrisk/GLA-Notes.pdf

Mason, J., & Watson, A. (2008). Mathematics as a constructive activity: Exploiting dimensions of possible variation. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 191-204). Washington, DC: MAA.

Nardi, E. (2006). Mathematicians and conceptual frameworks in mathematics education… or: Why do mathematicians' eyes glint at the sight of concept image/concept definition. In *Retirement as Process and concept; A festschrift for Eddie Gray and David Tall*, (pp. 181-189). Charles University, Prague.

Papadaki, E. (2017) *Design of tasks for the understanding of the concept of linear span* (Master's thesis). University of Crete, Heraklion, Greece.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209-246). Springer Netherlands.

Stewart, S., & Thomas, M. O. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology*, *40(7)*, 951-961.

Tall, D. (2004). Building theories: The three worlds of mathematics. For the learning of mathematics, 24(1), 29-32.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.

Wawro, M., Sweeney, G. F., & Rabin, J. M. (2011). Subspace in linear algebra: investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics*, *78*(1), 1-19.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS*, *22*(8), 577-599.