Knowledge of the Practice in Mathematics in University Teachers

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This research is part of the study of university teachers' knowledge that has emerged as a new line of research, which aims to understand the components of this knowledge, its development, and how it is reflected in university teachers' teaching practice. This study, using the Mathematics Teacher's Specialized Knowledge model, seeks to characterize a university teacher's knowledge of practice in mathematics in the content area of mathematical analysis. Based on an instrumental case study, through classroom observation, we provide indicators of the teacher's knowledge of ways of reasoning, validating, and proceeding in mathematics, contributing to the understanding of the nature of this teacher knowledge.

Keywords: teachers' and students' practices at university level, teaching and learning of analysis and calculus, university teachers' knowledge, knowledge of the practices in mathematics, Mathematics Teacher's Specialized Knowledge model.

INTRODUCTION

Research on mathematics teachers' knowledge began to be carried out in the nineties (e.g. Fennema & Franke, 1992; Broome, 1994) and currently continues to be developed with great force. Studies in this line of research have focused on teachers' knowledge of various concepts, such as fractions (Llinares & Sanchez, 1991) and functions (Even & Markovits, 1991), and in different mathematical domains, such as algebra (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012) and geometry (Herbst & Kosko, 2012). We also find proposals of models of teacher knowledge such as the Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005), Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) and, more recently, Mathematics Teacher's Specialized Knowledge (Carrillo, Climent, Contreras, & Muñoz-Catalan, 2013). These models have been used for studying mathematics teachers' knowledge, principally in primary and secondary education, with scarce accounts of studies of university teachers, as noted by Speer, King, & Howell (2014).

Currently, it can be said that research in higher education has gone from being centered on students to having a more balanced interest in both, students and teachers (Artigue, 2016), to the point that research on university teachers' knowledge has emerged as a new line of research from which it is asked what is understood as knowledge, how this knowledge is developed, and how it is reflected in the teaching practice of university teachers (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016).

In agreement with the above and taking into account that the Mathematics Teacher's Specialized Knowledge model (MTSK) has been shown to be useful for studying university teachers' knowledge (e.g. Vasco, 2015), we carry out our research based on the MTSK with the aim of characterizing the knowledge of a university mathematics teacher who teaches content in the area of mathematical analysis. In this text, some results are presented in relation to knowledge of the practice in mathematics, one of the sub-domains of knowledge considered in the model.

MATHEMATICS TEACHER'S SPECIALIZED KNOWLEDGE

Mathematics Teacher's Specialized Knowledge (MTSK) is an analytical model for understanding mathematics teachers' knowledge and at the same time a methodological tool that allows analyzing teachers' teaching practices (Carrillo et al., 2013). The model was developed based on a theoretical, empirical, and reflective work proposed by Shulman (1986) regarding foundational knowledge for teaching and the refining of Mathematical Knowledge for Teaching (Ball et al., 2008).

In MTSK, two domains of teacher knowledge are distinguished, mathematical knowledge (MK) and pedagogical content knowledge (PCK), considering that all of this knowledge is specialized, that is, it derives from the teaching profession and is conditioned by mathematics itself. Consequently, MTSK does not include, for example, general psycho-pedagogical knowledge included in Shulman's works. Also, the model takes into account that teacher's beliefs and conceptions about mathematics, its teaching, and its learning permeate the organization and the use of knowledge (Carrillo et al., 2013).

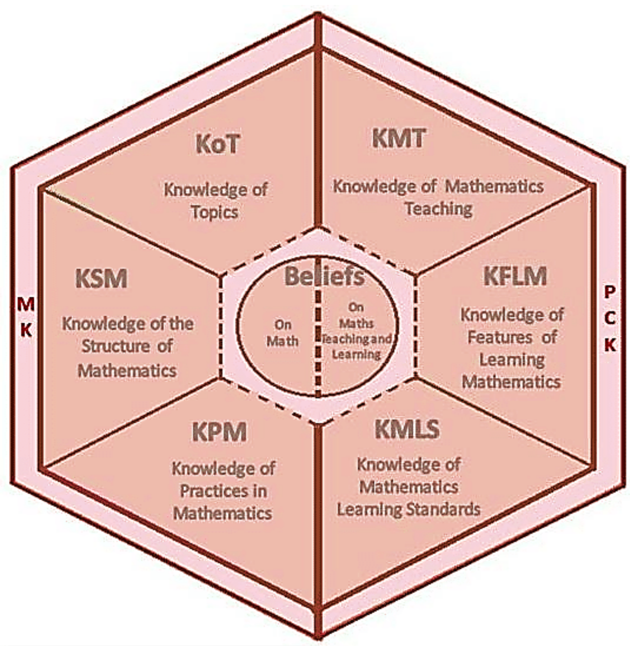

Figure 1: Mathematics Teacher's Specialized Knowledge (Carrillo et al., 2013).

Mathematical Knowledge comprises three sub-domains, knowledge of topics (KoT), knowledge of the structure of mathematics (KSM), and knowledge of the practice in mathematics (KPM).

KoT includes knowledge of concepts, properties, procedures, classifications, formulas, and algorithms, with their respective meanings and foundations. For example, knowing the topological property, density of the rational numbers in **R**, lies in this sub-domain. KSM includes knowledge of the interconceptual connections that can be established among mathematical concepts. So, knowledge of relationships between infinity and the Archimedean property of the real numbers belongs to this sub-domain. KPM includes knowledge of how to proceed, reason, and establish validity in mathematics. Knowledge of how proofs are made using different methods is part of teachers' KPM.

Pedagogical Content Knowledge contains three sub-domains: knowledge of mathematics teaching (KMT), knowledge of features of learning mathematics (KFLM), and knowledge of mathematics learning standards (KMLS).

KMT includes knowledge about didactic resources, strategies, tasks, and examples for making mathematical contents understandable. A teacher using an example to illustrate the meaning of a necessary condition forms part of his or her KMT. KFLM addresses the teacher's knowledge about mathematical contents as an object of learning, for example, teacher's knowledge about students' difficulties in understanding proofs belongs to this sub-domain. KMLS describes what students should achieve in a given course, conceptual and procedural capacities and mathematical reasoning that are promoted in given educational moments. The sequencing of the topics completeness theorem, characterization of the greatest element, and the Archimedean property of the real numbers is an example of KMLS.

KNOWLEDGE OF THE PRACTICE IN MATHEMATICS

The idea of knowledge of the practice in mathematics (KPM) comes from the works of Schwab (1978), Ball (1990), and Ball & Bass (2009) regarding syntactic knowledge of mathematics, which implies that the teacher should know how to reason mathematically, know different kinds of reasoning, and know in which mathematical contexts a particular kind of reasoning is more adequate than others. In this regard, within KPM, the importance of the teacher not only knowing established mathematical results, but also how to proceed and think in mathematics to arrive at these results is highlighted.

Knowledge that makes up part of this sub-domain is propositional logic, mathematical language and its precision, how definitions are made and used in the construction of mathematical knowledge, knowledge about different kinds of proof and their internal logics, the role that examples and counterexamples play in proofs, different kinds of heuristic reasoning, how knowledge is created in mathematics, how it is validated, reasoned, and generalized, and the role of mathematical conventions and symbols.

Although this knowledge is important, research carried out in MTSK with teachers at different educational levels reports scarce evidence of KPM in their mathematics classes (e.g. Montes, 2014; Vasco, 2015). In this regard, this investigation contributes to the understanding of the nature of this knowledge in university teachers.

METHODOLOGY

In this investigation, based on an interpretive paradigm and a qualitative methodology (e.g. Denzin & Lincoln, 2000), we carry out an exploratory study such that our results correspond to a first approximation for characterizing a university teacher's knowledge of the practice in mathematics (KPM). Our work is supported by an instrumental case study (Stake, 1995) for which we chose as an informant a teacher who taught a real analysis course for prospective teachers of mathematics during one semester in a Chilean university.

The teacher, who we will call Diego, is a mathematics researcher with a Ph.D., with more than 20 years of teaching experience, and this is the sixth time in recent years that he has taught the real analysis course. These academic characteristics of Diego make it likely that he possesses plentiful knowledge in elements of KPM.

The data was obtained through video recording while Diego taught the system of real numbers. The video recordings were transcribed and later subjected to content analysis (Bardin, 1997), identifying the units of analysis associated with the KPM sub-domain and considering the differentiation between evidence and indication of knowledge (Moriel-Junior & Carrillo, 2014). An evidence is an element that supports the presence of teacher knowledge, while an indication provides suspicion of the existence of knowledge but requires additional information in order to be confirmed as evidence.

RESULTS

In this section, we present episodes from one of Diego's classes on the properties of real numbers that allow us to observe his knowledge of the practice in mathematics (KPM).

Diego begins the class enunciating the property of the density of $\mathbf{Q}$ and $\mathbf{R}\backslash\mathbf{Q}$ in $\mathbf{R}$. For the proof of this property, Diego takes an interval $[a, b]$ with a and b in $\mathbf{R}$ and considers the case $a = 0$, then he generalizes it for any positive a. In fact, he takes a and b positive numbers and indicates that with this supposition there is no loss of generality.

Teacher: We are going to assume that a and b are two positive numbers, which is not a great assumption. If they were negative, for example, I work with $-a$ and $-b$.
If I have a and b here [indicating on the number line], If I take $-a$ and $-b$ then they are on the other side of the zero.

And if I find a rational number here [indicating on the number line between a and b], then its opposite, will also be rational.
And if I find an irrational number, its opposite will be irrational.
And if one number is positive and the other is negative, then zero is rational, and it is between them, and the irrational will come from a proposition.

In the previous episode, Diego's knowledge of the different cases that should be taken into account to make a proof whose hypothesis possesses an implicit or explicit disjunction is observed, as is his knowledge that a particular case can be sufficient for showing the behavior of other cases in the proposition. Following this, the teacher's knowledge about the consideration of cases to particularize or generalize, is a *way of proceeding* in mathematics and can be considered as an indicator of KPM. The teacher's knowledge that the proof goes beyond the example of a concrete case and addresses all the possible cases the statement can include (e.g. Brodie, 2010; Montes, 2014) is part of his KPM.

Additionally, Diego emphasizes that in mathematics for a fact to be considered valid it must be proven:

Teacher: Now, a question, do I know of any irrational number that is less than 1 and greater than 0?
Student: The square root of square root of 2.
Teacher: Ah! Why is it less than 1? We have to prove that it exists, and, wait, we have to prove that the root exists and that it is a number less than 1. And prove something else, because they didn't ask us for a number between 0 and 1, but rather for an irrational number between 0 and 1.
So, you can take a real number, prove that its root exits, but you still don't know if it is irrational or not.

In the prior episode, Diego's knowledge of proof as a *way of validating* in mathematics is shown. As Brodie (2010) maintains, in addition to knowing that a kind of proof exists that confirms the truth of a statement, it is necessary to know how this type of proof works. Furthermore, Diego points out the importance of this role of proofs (de Villiers, 1990), as he is in front of a course for prospective mathematics teachers and after proving the density property he says:

Teacher: So, now you can say to your students with certainty that between any two rational numbers there is always an irrational number.

The comment above shows as indicator of KPM, the teacher´s knowledge of the necessity and importance of proofs as a *way of validating* in mathematics (e.g. Balacheff, 2000).

Continuing with the density proof, Diego discusses with the students about taking the maximum or minimum of a set of natural numbers. In the following episode, we observe Diego's knowledge about the convenience of selecting a certain element to develop an argument in a proof.

Teacher: The issue is that if I take either of the two [the maximum or minimum] I can argue that there is a rational number in between them.
What I have to guarantee before choosing one of the two is that in fact there is something to choose, so, which is easier to guarantee? The smallest or the largest?
Student: The smallest.
Teacher: Why?
Student: Because of the well-ordered principle.
Teacher: Of course, the minimum exists by the well-order principle.

Nevertheless, on occasions, rather than choose, is necessary construct an element that allows developing an argument. Diego also gives evidence of this knowledge when discusses with the students about irrationality of a number that belongs to the interval they are working on. He uses $\sqrt{2}$ to construct $\sqrt{2}/m$ as shown below.

Teacher: I know that $\sqrt{2}$ does not belong to the rational numbers. We proved it. It is real, it exists, and it is not rational.
If I now divide this number by a whole number m, will it continue to be not rational? will it become a rational number?
That is, this is not rational, but is this [pointing to $\sqrt{2}/m$] not rational either?

According to above episodes, an indicator of KPM is the teacher´s knowledge of the construction or selection of elements for developing an argument in a proof as a *way of proceeding* in mathematics.

Regarding to $\sqrt{2}/m$, Diego prove its irrationality and establishes this affirmation as a lemma that he uses in different moments in the proof of the density property:

Teacher: So, if I take the Archimedean property for $\delta = \varepsilon/\sqrt{2}$, then exists an $m \in N$ such that $0 < 1/m < \delta$, and this implies that, $\sqrt{2}/m$ is smaller than ε and does not belong to the rational numbers.
So, between 0 and ε, no matter how small ε is, there is always a rational number, and there is always an irrational number.
Ok, so this affirmation that I just wrote, we're going to write it as a lemma.

This episode gives us evidence that the teacher knows that establishing preliminary results is a *way of proceeding* in mathematics that facilitates the development and the communication of a long and/or complex proof.

On the other hand, regarding the existential quantifier present in the Archimedean property and the well-ordered principle, Diego expresses that the existential quantifier only gives information about the characteristics or properties of an element, but does not say what the element is like or which element it is, only that it exists

Teacher: When the *for all* quantifier is given, I can always choose an element that works best for me, but when it is *existence*, it's "whatever you've gotten", no more. Then as *exists* gives you "whatever you've gotten", you have to do some work so that what you get, is what you need.

An indicator of KPM in this episode is the teacher's knowledge of the role and the meaning of the quantifiers when they are found in the hypothesis or in the conclusion of a proposition.

Summarizing, indicators of the teacher's knowledge of the practice in mathematics related to ways of reasoning, validating, and proceeding in mathematics has been evidenced. In the exposed episodes, the teacher teaches mathematical reasoning in order to give meaning to the mathematical activity (e.g. Brodie, 2010), not only for students to understand and acquire sensibility regarding how to establish truth in mathematics, but also, they find meaning in the need to do so (Montes, 2014).

CONCLUSIONS

A mathematics teacher is a professional whose knowledge of the discipline he or she teaches has a level of deepening, organization, and structure that is greater than what the students are going to receive (Ma, 1999). In this regard, KPM is necessary knowledge for the teacher, as it provides logical thinking structures that help to understand the function of diverse aspects of mathematics (Flores-Medrano, Escudero-Avila, Montes, Aguilar, & Carrillo, 2014). As observed in the case studied, the teacher's knowledge of processes of particularization-generalization, of the necessity and importance of proof for validation, and the knowledge of different ways of proceeding in mathematics are closely linked to the transition to advanced knowledge (e.g. Pino-Fan, Godino, Castro, & Font, 2012) and are related to the particular way of mathematical work.

In line with the ideas above, we consider that KPM allows the teacher to promote in students the construction of mathematical knowledge and the acquisition of abilities for reasoning, proof, and problem solving that are considered important for learning mathematics at all levels of education (e.g. Flores-Medrano et al., 2014).

With reference to the indicators of KPM obtained in this investigation regarding ways of reasoning, validating, and proceeding in mathematics, we agree with Sosa, Flores-Medrano, & Carrillo (2015) that indicators shown directly in empirical data can contribute to identifying, understanding, and analyzing teachers' knowledge in their discipline. Given that teacher knowledge is a complex and multidimensional construct, more studies are necessary to deepen the understanding of its different components, in order to advance in interrogations proposed regarding how this knowledge is developed and how it is reflected in the teaching practice of university teachers (e.g. Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016).

ACKNOWLEDGEMENTS Beca de Doctorado año 2017 CONICYT, folio 21170442.

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