ANALYSING REGRESSIVE REASONING AT UNIVERSITY LEVEL

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ABSTRACT:

This paper focuses on the epistemic and cognitive characterization of regressive reasoning in resolving strategic games. It explores the use of the Finer Logic of Inquiry Model as a tool for the analysis of the regressive reasoning. It reports the results of a study carried out on 32 undergraduate students who are studying a Mathematics Degree in a university of Spain.

Key words: *Teaching and learning of specific topics at university level mathematics, Teaching and learning of logic, reasoning and proof, Regressive reasoning, Logic of Inquiry, Strategy games.*

1. INTRODUCTION

The method of analysis has proved to be extremely stimulating in various fields, and has played a crucial role in the emergence of the modern world-view. The combination of the two branches of analysis and synthesis has been applied to several fields of artificial intelligence, theoretical computer science, and in programming methodology (Peckhaus, 2000; Grosholz, Breger, 2000). For many engineering students and mathematics undergraduate students, learning the method of analysis in tertiary education mathematics is a critical issue. They have the challenge of incorporating it in different disciplines related to the design and production of products and services, such as, Project Management, Systems Engineering and Design Science. They have no theoretical and methodical basis (Koskela and Kagioglou, 2006). A conscious integration of regressive reasoning in mathematics university learning raises the need for articulation between epistemological and cognitive aspects. Regressive reasoning is not completely logically determined, but has elements of contingency, creativity and intuition. The purpose of this text is to highlight the potential of Finer Logic of Inquiry Model (Arzarello 2014) as a tool for the didactical analysis of the regressive reasoning. This model has been used at secondary level education, not being used at tertiary level so far.

Here we will report the results of a study carried out on 32 undergraduate students studying a Mathematics Degree at a Spanish University, using strategy games in order to promote the regressive reasoning. The choice of strategy games is justified by antecedents to this study in which they have been shown to be a key tool for teaching problem solving and regressive reasoning (Gómez-Chacón, 1992).

The present research is primarily exploratory for two reasons: 1) Regressive reasoning has been scantly analysed in mathematics and educational psychology; 2) the use of the Finer Logic of Inquiry Model methodology to analyse data from mathematical thought at tertiary education is a new development. The theoretical background and empirical studies related to regressive reasoning needs to be developed.

2. REGRESSIVE REASONING

In mathematics, progressive reasoning alone is not exhaustive to fulfil the tasks of solving problems. Great mathematicians like Pappus, Descartes, Leibniz, in their discussions about analysis and synthesis, emphasize this fact (Peckhaus, 2000). Regressive reasoning is known by different denominations: regressive analysis, backward solution, method of analysis, etc. This process includes different ways of proceeding in problem solving: backward strategy, strategy of assuming the problem solved, Reductio ad Absurdum, beginning at the end of the problem, etc.

Pappus was the mathematician who has contributed substantially to the clarification and exemplification of the method. In the seventh book of his Collection he deals with the topic of Heuristics (methods to solve the problems).Where he exemplifies the method of analysis as the method of synthesis, therefore making the development of this reasoning clearer. Pappus defines the method of analysis as follows: “In analysis, we start from what is required, we take it for granted; and we draw correspondence (ακολουθον) from it and correspondence from the correspondence, till we reach a point that we can use as a starting point in synthesis. That is to say, in analysis we assume what is sought as already found (what we have to prove as true).” (elaboration by Polya, 1965 and by Hintikka and Remes, 1974). Subsequently he points out: “This procedure we call analysis, or solution backward, or regressive reasoning.” (Hintikka and Remes, 1974) And on the Method of Synthesis: “In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. This procedure we call synthesis, or constructive solution, or progressive reasoning.”(Hintikka and Remes, 1974)

In summary, the following was considered backward reasoning: the practice that involves the making of a number of arguments from the bottom of the problem and proceeds through logical correspondences which allow to obtain something known or to be reached through other paths. The analytical method consists of a procedure that starts with the formulation of the problem and ends with the determination of the conditions for its solution.

3. FINER LOGIC OF INQUIRY MODEL (FLIM)

Trying to overcome the static approach of habitual logical mathematical reasoning, Hintikka (1996, 1999) developed what he calls *Logic of Inquiry*. The idea, already elaborated by ancient Greek philosophers, is building knowledge through a questioning process, implicit or explicit. The knowledge is the result of research generated by a specific question. The philosopher introduces it as the "logic of question and answer".

In his approach he considers Game Theory and game semantics to support formal epistemic logic. Hintikka overcomes the limitations and excessive abstractions of Tarski's Definitions of Truth (Sher, 1999), which leave the process used to reach the truth unexplained. He introduces a top-down definition of truth (Hintikka, 1995) unlike the classical and tarskian bottom-up view, highlighting the regressive way of proceeding in problem solving from an epistemological point of view. Hintikka (1995) retakes the idea of Wittgenstein's language games and some aspects of Game Theory, elaborating on a theory where the centre is "a path towards the formulation of a truth that, instead of proceeding recursively from atomic to complex formulas, reverses the approach and proceeds from the more complex ones to their simplest constituents". In this research, the study of games will try to explain this interlacing between game theory and strategic rules that allows the student to win.

The FLIM elaborated on by Arzarello (2014) sought to propose a concretion of Hintikka`s proposal to be used in the Didactics of Mathematics. More specifically, he explained the elements needed to analyse the interactions between strategic and deductive components of students' resolution protocols. This model allows for the structuring of the resolution in two components: Inquiry Component (IC) and Deductive Component (DC).

In the Inquiry Component the subject alternates a series of questions, answers and explorations, according to Hintikka's *Logic of Inquiry*. Its purpose is to meet the aim of the problem, solving conjectures that gradually rise from results of two explorations:

- *Exploration:* in order to analyse and understand the situation in which the subject is involved
- *Control:* in order to verify the ideas or conjectures that came out during the development of the activity.

In the above Component, the cognitive dimension of reasoning is necessary. From a cognitive point of view, the progressive-regressive reasoning movement has been highlighted by studies such as those of Saada-Robert (1989). The psychological model for solving mathematical problems focuses on the distinction between two phases of the resolution: investigate why things are like this (backwards, until reach a plausible hypothesis -abduction- or a known fact) and verify this investigation (forward, codified by the classical logic). Based on Saada-Robert's model, Arzarello (2014) and Soldano (2017) characterized this cognitive dimension through the sequence of actions in three different modalities: ascending, neutral and descending.

Ascending modality (A) refers to the path towards the formation of ideas and conjectures after a phase of exploration. Descending modality (D) characterizes the transition from a conjecture to an investigation. The purpose of descending modality is to find an equivalence between the object of thought (the conjecture, the idea) and the object of work (the problem and its resolution). Neutral modality (N) marks the change between the ascendant and the descendent; it is the moment in which a conjecture is formulated. Observable actions in the subjects are: formulations (of questions, of resolutions plan, of conjectures), affirmations, explorations and controls.

In the Deductive Component the subject is not directly involved in the investigation and verification of conjectures and uses a language with a logical nature to formally formulate the truth. Three specific modalities are added: detached modality, logical control and deductive modality (Arzarello, 2014; Soldano, 2017). Detached modality is the moment in which a conjecture, which has not arisen immediately after an exploration, is formulated. Logical control is the time when an exploration-control is done without using instruments. It is characterised by the use of formal language. Deductive modality characterises control phases where instruments are involved. Deductive Steps and Logical Chains are added to the Inquiry Component actions.

Inquiry and Deductive components are not often well differentiated during problem resolution where the subject passes from one component to another, even more than once. We can say that the typical components structure is nested in this way: (IC ~ (DC ~ (IC ...))) with "~" that expresses the passage from one component to the other.

Observable actions		**Modalities**
General	*Specific*	
Verbal Handwritten Gestures Others (gaze, …) Silent	Question Affirmation Conjecture Exploration Control Plan formulation Deductive step Logical chain	Ascendant Neutral Descendant Detached Logical Control Deductive

Table 1

Table 1 summarizes some observable actions and their modalities according to the definitions given and that will be considered in the analysis.

4. AIM AND METHODOLOGY

Aim

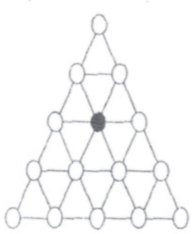
The aim of this paper is to show an evaluation tool for examining how regressive reasoning develops in university students. In particular, how the FLIM can be a valid tool to analyse the interplay between cognitive and epistemic in the regressive reasoning.

Participants and instrument

Data were collected in 2014 from 32 (19 women and 13 men, aged between 21 and 23) Caucasian undergraduates working toward a BSc. in mathematics. All of the participants were in their last year of academic studies. They were following advanced courses in several areas of geometry, algebra, probability and analysis. With regard to solving problems, the students had been introduced to the problem solving heuristics. They had not received any special training about backtracking heuristics.

The work dynamic started with individuals being given paper and pencil with which they need to resolve two games, each lasting one and a half hours. Figure 1 shows the problem which we will analyse in the results section. Strategy games allow for the natural development of regressive reasoning. These games are disconnected from the mathematical content which forces the student to use their mathematical knowledge acquired in their university degree.

The Triangular Solitaire (Gómez-Chacón, 1992) is a game for a single person that requires a board with 15 boxes as the figure shows.



These are the rules:

1. Place the pegs in all boxes, except in the one marked in black.

2. The player can move as many pegs as they like as long as they are able to jump over an adjacent peg and onto an empty space (along the line). At the same time, he "eats" the peg that was jumped over and that peg gets taken out of the game. All pegs move in the same way. Pegs can move around the table in any direction.

Objective: The player wins when there is only one peg on the table.

Figure 1

Students were given the game and asked to describe their approaches to solving the problem on protocols including: thought processes in the resolution, explanations of the difficulties they might face, and strategies they would use in order to solve with paper and pencil. A qualitative analysis was chosen to examine the resolution protocols of the students through the "Finer Logic of Inquiry Model" (Arzarello 2014). A general analysis of 32 students took place before a case study was carried out. In this paper we describe an individual student case in order to show a deep understanding of the tendencies of the behaviour related to the sequences of actions and movement between modalities of reasoning. The protocols analysis, at a macroscopic level of this case, provides the identification of reasoning difficulties and way of using backward reasoning that determined success or failure in the resolution. It's worth noting that Student M (see section 5) is a key informant of the group because he belongs to 60% of students that use the backward strategy and incorporates graphical representations to achieve the transition between modalities.

5. REGRESSIVE REASONING USE (CASE STUDY)

Regressive reasoning use varies among the group of students. Let us examine a case study. A student (Student-M) has combined regressive reasoning with different strategies and auxiliary constructions: drawings, graphical representations. Student-M indicates difficulties in creating the solution because of the actions which are needed for discovering the solution and because of the recognition of representational equivalences. The visualisation and representations which are used help during the resolution process; Student-M performs continuous control over its own resolution process. She is able to slightly modify the strategy or even change it completely to reach the solution. For analysis purposes, Student-M's protocol has been divided into the following phases: familiarisation, exploring and carrying out the strategy, results verification. According to the Finer Logic of Inquiry Model, this student's protocol is mainly characterised by the inquiry component. This begins with the first part of the protocol, corresponding to the *familiarization phase*. The entire protocol has been translated highlighting the parts where student M uses backward reasoning (in *Italics).*

Student M protocol

1 To accomplish the exercise, I'm going to number the holes on the board in order to leave a trace of the movements I'm doing. At the beginning, all the holes are filled except number 5.

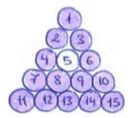


Figure 3

2 I observe that you can only start with two movements 14-9-5 or 12-8-5.

3 Since this is an equilateral triangle, I think it does not matter what the starting movement is because they should lead to "symmetrical" solutions.

4 I'll start to do it roughly.

5 The steps I'll take are: 14-9-5; 7-8-9; 12-13-14; 2-4-7; 11-7-4; 10-9-8; 3-6-10.

6 *At this point, I note that the only way to eliminate 1 would be to move 8-5-3.*

7 *Here I notice that [with these movements] the game cannot be solved because the 4 cannot be eliminated and the remaining pegs cannot eliminate each other.*

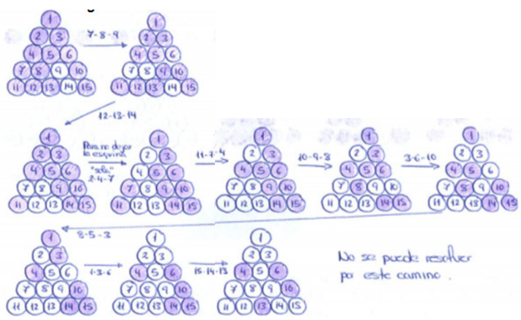


Figure 4

8 I realise that I can try to go backwards, that is, starting with just one peg in one position and undo the jumps trying to fill the table with the exception of a hole.

9 *Looking at the board, I think that maybe the fact that the last piece stays on the board (the peg from which I start to move backwards), in a position that you can come up with many jumps, facilitates the strategy.* These places are positions 4, 6 and 13 because you can get to them with 4 jumps.

10 To fill up the game table I will have to do 13 moves, because there are 15 holes, an initial peg and an empty final hole.

11 Let's start only with peg 13.

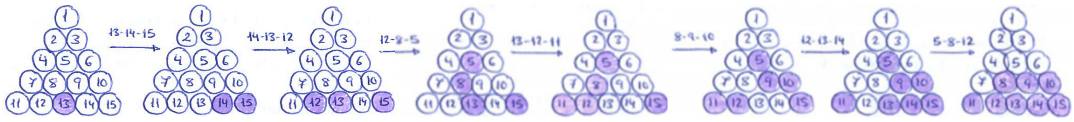


Figure 5

12 Here I already notice that *I do not reach the solution because I will never fill the top corner due to the absence of a peg in the 3rd row*; I should do 11-7-4 leaving corner 11 without a peg [so that the top corner will be filled].

13 Let's start with the reason for the various steps:

13-14-15: *I want to start filling the corners as soon as possible because these holes are the hardest to fill up* (the peg is in hole 15 and I will not move it anymore).

14-13-12: Random movement.

12-8-5: *I want to leave hole 12 free to get to the next step at corner 11.*

8-9-10: I want to leave hole 8 free to retrieve peg 12 (to fill 13 and 14) in the next step, so I can complete it later [the row].

12-13-14: I want to complete the row below.

5-8-12: I want to complete the row below.

14 I think trying to fill the centre was not a good strategy...

15 … so now I'm going to try to fill the outside of the triangle, that is, [I'll try to] undo the jumps to the corners and sides. (Playing normally would involve jumping to the centre avoiding corners and sides if possible.).

16 I also get stuck [on the fact] that by eating pegs or undoing the jumps, the movements that are made are triangular. So I will try to fill the smaller triangles contained in the big triangle.

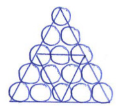


Figure 6

17 First, I will fill the lower right triangle.

18 *Now I'm going to fill the upper triangle; to do so (Since i do not want to remove the peg I placed in position 1), I have to get some pegs in the 4th row that, undoing the jump fills the 2nd and 3rd row. I undo the jump with the 9.*

19 *Now you have to fill the lower left triangle.*

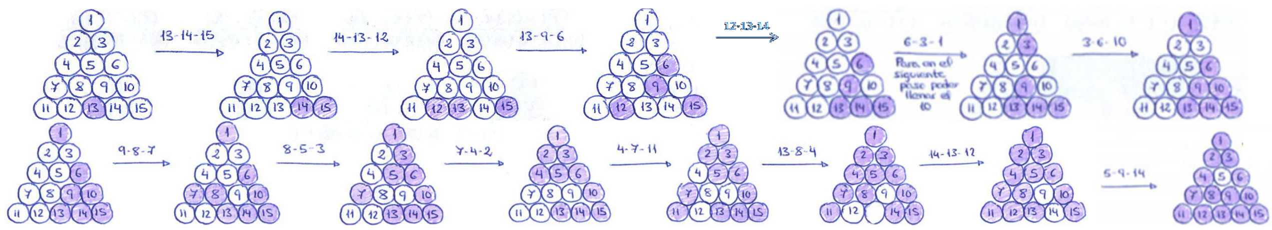


Figure 7

20 Now I just have to write the jumps in the correct order

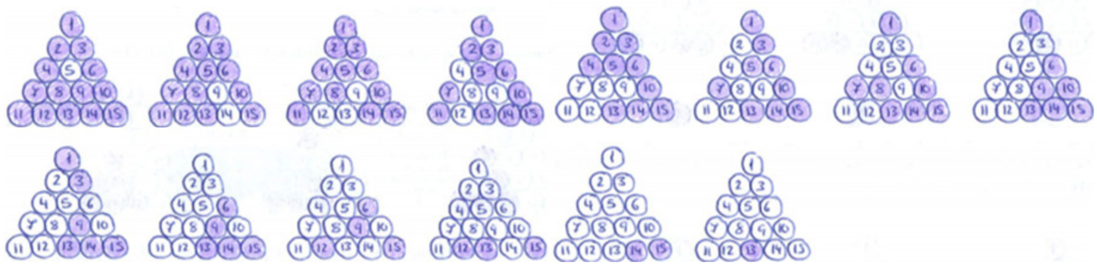


Figure 8

The following table shows actions and cognitive modalities associated with each protocol line and figure; a check (✓) indicates the lines where the regressive reasoning is used. The last column of the table shows different strategies involved.

Familiarization phase				
Protocol parts	***Action***	***Modality***	***R.R***	***Strategy***
Lines 1-4 y Fig. 3	Exploration	Descendant		
Line 5	Affirmation	Neutral		
L. 6-7 and Fig. 4	Exploration	Ascendant	✓	
Explore and carry out the strategy				
Line 8	Plan	Neutral		Backward
Line 9	Exploration	Ascendant	✓	Begin from the end
Line 10	Affirmation	Descendant		
Line 11 y Fig. 5	Exploration	Descendant		
Line 12	Affirmation	Ascendant	✓	
Line 13	Exploration	Ascend/Descen	✓	
Line 14	Affirmation	Ascendant		
L. 15-16 and Fig. 6	Plan	Neutral		Auxiliary construction
Lines 17 y Fig. 7	Exploration	Descendant		
Line 18	Exploration	Ascendant	✓	
Line 19 y Fig. 7	Exploration	Ascendant	✓	
Results verification				
Line 20 y Fig. 8	Control	Detached		

Table 2

This cognitive analysis shows that the first two resolution phases are characterised by a continuous alternation of explorations and plan formulations together with an alternation of descending and ascending modalities. The second resolution phase involves the continuous use of the going backward strategy. Subdivision of the board into rows and then into triangles is fundamental to reach the solution. Student-M modifies the strategy slightly by adding new elements in the resolution (board subdivision into rows and triangles) typical of problem solving using regressive reasoning. Crucial points of backward reasoning are reached in the ascending modality (see ✓ in Table 2) where ideations occur. A routine that can be established regarding the use of modalities is A~N~D~(A~N~D~(A~…)). The neutral modality marks the transition between A and D and it is characterised by the incorporation of auxiliary constructions as generating tools of new knowledge (epistemic transaction).

In the third phase of the resolution, by writing and graphically representing the steps taken to reach the solution, Student-M (in detached modality) checks the result obtained by going backwards.

6. CONCLUSION

Analysis with the FLIM model allows to model student's cognitive movement in a logical concatenated way. The strategic aspects are more dominant in the ascending and descending modality, while the epistemic ones are prevailing in the neutral modality. Our study confirms results obtained by Soldano (2017) (with upper secondary school students in geometry): the ascending modality characterises the backward way of thinking, while descending is the cognitive modality that characterises the progressive way of reasoning. However, most likely, abductive reasoning has been used in the formulation of conjectures in ascending modality, but we cannot be sure of it by only analysing the protocol, we need to complete this information by interview. This is an open question for further research.

At a phenomenological level, this method allows us to analyse the development of strategic aspects within the cognitive modality movement to reach the solution. But it mainly focuses on cognitive modalities while it doesn't distinguish between the strategic principles that are used. Through this tool it's possible to emphasise that regressive reasoning involves auxiliary intuition elements that are necessary to achieve the solution; these aspects are developed by looking at the consequence and looking for the premises. A larger sample size with two different tasks, find the winning strategy and mathematically solving the game, would allow us to advance in the development of the tools for evaluating regressive reasoning.

7. ACKNOWLEDGMENT

Supported by NILS Program (ES07) (007-ABEL-IMI-2013), Project "Mathematical Working Space" (UCM-CmdeGuzman-2015-01), University of Torino doctoral scholarship and by special action IMI-INVEDUMAT_uni.

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