



Revisiting path-type covering and partitioning problems

Paul Manuel

► To cite this version:

| Paul Manuel. Revisiting path-type covering and partitioning problems. 2018. hal-01849313

HAL Id: hal-01849313

<https://hal.science/hal-01849313>

Preprint submitted on 25 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

This is a survey article which is at the initial stage. The author will appreciate to receive your comments and contributions to improve the quality of the article. The author's contact address is pauldmanuel@gmail.com.

Revisiting path-type covering and partitioning problems

Paul Manuel

July 25, 2018

Department of Information Science,
College of Computing Science and Engineering,
Kuwait University, Kuwait
pauldmanuel@gmail.com

Abstract

Covering problems belong to the foundation of graph theory. There are several types of covering problems in graph theory such as covering the vertex set by stars (domination problem), covering the vertex set by cliques (clique covering problem), covering the vertex set by independent sets (coloring problem), and covering the vertex set by paths or cycles. A similar concept which is partitioning problem is also equally important. Lately research in graph theory has produced unprecedented growth because of its various application in engineering and science. The covering and partitioning problem by paths itself have produced a sizable volume of literatures. The research on these problems is expanding in multiple directions and the volume of research papers is exploding. It is the time to simplify and unify the literature on different types of the covering and partitioning problems. The problems considered in this article are path cover problem, induced path cover problem, isometric path cover problem, path partition problem, induced path partition problem and isometric path partition problem. The objective of this article is to summarize the recent developments on these problems, classify their literatures and correlate the inter-relationship among the related concepts.

Keywords: Survey, covering and partitioning problems, paths, induced paths and isometric paths.

AMS Subj. Class.: 05C12, 05C70, 05C76; 68Q17

1 Introduction

An undirected connected graph is represented by $G(V, E)$ where V is the vertex set and E is the edge set. There are several types of covering problems such as covering the vertex set by stars (domination problem), covering the vertex set by cliques (the clique covering problem), covering the vertex set by independent sets (the coloring problem), and covering the vertex set by path-types. A *path-type* means a path, an induced path or an isometric path. A path P is an *induced path* in G if the subgraph induced by the vertices of P is a path. An induced path is also called a *chordless path* [56]. Let us recall that *isometric path* and *geodesic* are other names for shortest path.

Mathematicians agree that the term “covering” is over-used or inappropriately-used in graph theory. In set theory, there are two different terminologies: set covering problem and set hitting problem. The edge cover problem is a special case of set covering problem while the vertex cover problem is a special case of set hitting problem. While set theorists distinguished the two concepts by “covering” and “hitting”, graph theorists chose to use the same term “covering” to mean two different concepts. Thus, right in the beginning, the term “cover” was over-used. The trend has been continuing. It is not uncommon that researchers use the term “path cover problem” to mean “path partition problem” and use the same term to mean “induced path cover problem”. It is observed that the concepts “cover” & “partition” and the terms “path” & “induced path” are used inappropriately. There is no consistency in the usage of notations too. For example, while some authors use the notation π to represent the path cover number, others use the notation μ to represent the same concept of path cover number. These “inconsistent notation” and “inappropriate terminologies” lead novice readers to confusion. This situation demands an article that classifies and simplifies the notations and terminologies of path-type covering and partitioning problems.

In this article, we divide the “theory of decomposition” into six problems and classify the literatures accordingly. We also bring their related concepts together under one umbrella. This article targets young researchers in graph theory. The benefit of this article is that it exposes the gap and open space in research area under each problem. The article highlights that there is plenty of research to be done in this topic. In addition, this article will save time for young researchers to choose a right research topic in this domain and it may be a reference to apply common notations and appropriate terminologies in their research articles.

In section 3, some relevant problems such as the k -path cover problem, the S -path

cover problem and k -path vertex cover problem are discussed. Since these topics are not our main focus, we do not provide elaborate literature survey for these problems. In section 4, the significance of Gallai-Milgram theorem and Berge path partition conjecture is highlighted. The three cover problems are discussed in section 5 and the three partition problems are studied in section 6.

2 Path-type covering and partitioning problems

Brause and Krivoš-Belluš [13] state that the cover problem and the partition problem are central problems in graph theory. One such well-known concept in graph theory is “clique covering” and “clique partitioning” [23]. A *path cover* of G is a set of paths such that every vertex $v \in V$ belongs to at least one path, whereas a *path partition* of G is a set of paths such that every vertex $v \in V$ belongs to exactly one path [35]. In a path partition, the paths are mutually vertex-disjoint. A few authors use the term “vertex-disjoint path cover” to mean “path partition” [9]. The following six invariants (See Figure 1) are defined here:

- (1) The *path cover number* of G , denoted by $\pi_c(G)$, is the cardinality of a minimum path cover of G . The *path cover problem* is to find a path cover of minimum cardinality in G .
- (2) The *induced path cover number* of G , denoted by $\rho_c(G)$, is the cardinality of a minimum induced path cover of G . The *induced path cover problem* is to find an induced path cover of minimum cardinality in G .
- (3) The *isometric path cover number* of G , denoted by $ip_c(G)$, is the cardinality of a minimum isometric path cover of G . The *isometric path cover problem* is to find an isometric path cover of minimum cardinality in G .
- (4) The *path partition number* of G , denoted by $\pi_p(G)$, is the cardinality of a minimum path partition of G . The *path partition problem* is to find a path partition of minimum cardinality in G .
- (5) The *induced path partition number* of G , denoted by $\rho_p(G)$, is the cardinality of a minimum induced path partition of G . The *induced path partition problem* is to find an induced path partition of minimum cardinality in G .

- (6) The *isometric path partition number* of G , denoted by $ip_p(G)$, is the cardinality of a minimum isometric path partition of G . The *isometric path partition problem* is to find an isometric path partition of minimum cardinality in G .

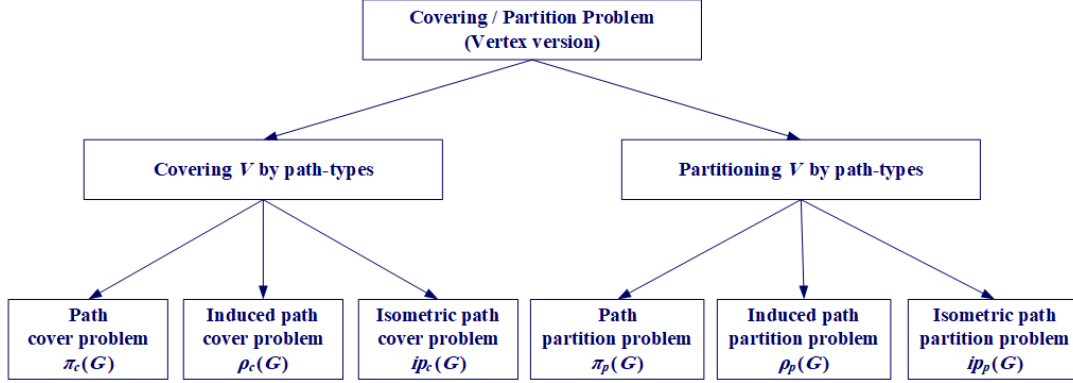


Figure 1: Six types of covering / partitioning problems

While some authors [2, 19, 23, 34, 35, 63, 69, 70, 75, 76] use the term “path partition”, others [6, 9, 16, 20, 38, 53, 58, 59, 67, 73, 79] use the term “path cover” (or vertex-disjoint path cover) in the place of “path partition”. Some authors use the term “decomposition” for “partition” [3, 65].

A simple example in Figure 2 demonstrates the different behaviors of six different problems:

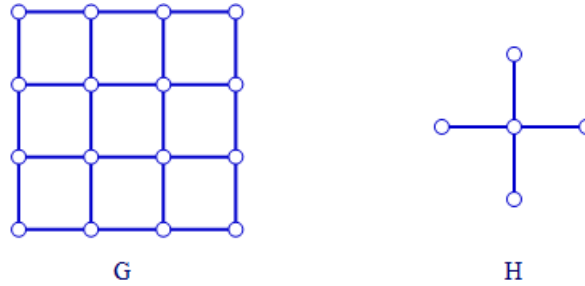


Figure 2: Illustration how the six problems are different

$$\begin{aligned}
 \pi_c(G) &= 1; \rho_c(G) = 2; ip_c(G) = 3; \pi_p(G) = 1; \rho_p(G) = 2; ip_p(G) = 4; \\
 \pi_c(H) &= 2; \rho_c(H) = 2; ip_c(H) = 2; \pi_p(H) = 3; \rho_p(H) = 3; ip_p(H) = 3;
 \end{aligned}$$

The inequalities in Lemma 2.1 are true because of the following facts:

- (1) An isometric path is an induced path. An induced path is a path.
- (2) A path partition is a path cover.

Lemma 2.1 *For a given graph G ,*

- (1) $\pi_c(G) \leq \rho_c(G) \leq ip_c(G); \pi_p(G) \leq \rho_p(G) \leq ip_p(G);$
- (2) $\pi_c(G) \leq \pi_p(G); \rho_c(G) \leq \rho_p(G); ip_c(G) \leq ip_p(G);$

3 Some generalized concepts

In this section, we discuss some problems which are closely related to the six problems.

3.1 The k -path cover problem

A k -path is a path having at most k vertices. Here is a word of caution: While defining a k -path, some authors define k -path as a path of at most k vertices [41, 72] and some authors define k -path as a path of length at most k [45]. Korpelainen [45] states that the 3-path partition problem is NP-complete on comparability graphs. According to his definition of k -path, the correct statement is that the 2-path partition problem is NP-complete on comparability graphs.

A k -path cover is a set of k -paths that cover V . The k -path cover problem is to find a k -path cover of minimum cardinality in G . This is a generalized version of path cover problem [72]. When $k = 2$, the k -path cover problem becomes the edge cover problem. It is straightforward to define other five problems in the same way [45, 57, 72].

The k -path partition problem is solved for trees [77], cacti [41], cographs for a fixed k [71], threshold graphs [71], bipartite permutation graphs [72]. The 3-path partition problem is NP-complete on general graphs [31] and comparability graphs [72]. The k -path partition problem is NP-complete on the class of cographs [71] and chordal bipartite graphs [72] if k is part of the input. The k -path partition problem as well as the induced k -path partition problem remains NP-complete on bipartite graphs of maximum degree three [57]. The k -path partition problem remains NP-complete for a graph class defined by finitely many minimal forbidden induced subgraphs [45]. Korpelainen [45] has nicely analyzed the differences between the k -path cover problem and induced k -path cover problem. The author also provides a good account of literature survey on k -path cover problems and their related topics [45].

3.2 The S -path cover problem

Here is another generalization of the path cover problem. For a set S of vertices in a graph $G = (V, E)$, an S -path cover is a path cover P in which every vertex of S is an endpoint of a path in P . The S -path cover problem is to find an S -path cover of minimum cardinality in G . Some authors [4, 5, 7, 47] call it k -fixed-endpoint path cover problem (or kPC problem) where $k = |S|$. Notice that, when S is empty, the S -path cover problem coincides with the classical path cover problem [78].

It is shown that the 1-Fixed-Endpoint path cover problem is polynomial on interval graphs [47] and the k -fixed-endpoint path cover problem is polynomial on cographs [4], proper interval graphs [5]. The S -path partition number is polynomially solvable for bipartite distance-hereditary graph [78]. The PhD thesis by Baker [7] is a good source of information on this problem.

3.3 The k -path vertex cover problem

There is another concept in the literature called k -path vertex cover (which is denoted by k -(All-) Path Cover) [30]. A set C of vertices is a k -path vertex cover if every k -path on k vertices contains at least one vertex from C [48]. While the k -path cover problem is an extension of edge cover problem, the k -path vertex cover problem is an extension of vertex cover problem. Brause and Krivoš-Belluš [13] have studied the relationship between the k -path vertex cover problem and the k -path partition problem. Both these problems are completely different from the computational complexity point of view. The k -path vertex cover problem is NP-complete for cubic planar graphs of girth 3 for $k = 3$ [74]. Funke, Nusser and Storandt [30] have provided a list of applications of this problem on different domains.

4 A discussion on the Gallai-Milgram theorem

One of the earliest contributions to the path partition problem is the Gallai-Milgram theorem. While surveying the Gallai-Milgram theorem, Berge states that the Gallai-Milgram theorem has been independently extended by Las Vergnas and Linial [11, 34, 35, 50]. The Gallai-Milgram theorem illustrates the differences among the three path-types such as path, induced path and isometric path. The Gallai-Milgram theorem is true only for path covers and path partitions in undirected graphs. Let $\alpha(G)$ denote the independence number of G .

Theorem 4.1 (Gallai-Milgram) *Let P be a minimal path partition of an undirected graph G . Then there is an independent set S in G such that S contains exactly one vertex from each path in P . The vertex set V of G can be partitioned by at most $\alpha(G)$ paths.*

Theorem 4.2 (Gallai-Milgram) *Let G be an undirected graph. Then $\pi_c(G) \leq \pi_p(G) \leq \alpha(G)$.*

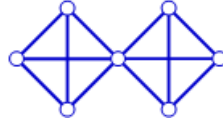


Figure 3: The Gallai-Milgram theorem fails for induced paths and isometric paths in undirected graphs

The Figure 3 demonstrates how the Gallai-Milgram theorem is NOT true for induced path cover and isometric path cover. Let us make some interesting observations on the Gallai-Milgram theorem. While the Gallai-Milgram theorem fails for the induced path cover and the isometric path cover on chordal graphs, it is true on bipartite graphs.

Theorem 4.3 *Let G be an undirected chordal graph without any pendant vertices. Then*

$$\begin{aligned} \alpha(G) &\leq \rho_c(G) \leq \rho_p(G); \\ \alpha(G) &\leq ip_c(G) \leq ip_p(G); \end{aligned}$$

Proof. In chordal graphs, $\rho_c(G) = ip_c(G)$ and $\rho_p(G) = ip_p(G)$. It is enough to prove that $\alpha(G) \leq \rho_c(G)$. The inequality can be proved by induction on the number q of maximal cliques of a chordal graph G . The base case is a single clique. Applying perfect elimination ordering, a few simplicial vertices (a minimum of two vertices) can be removed from G in such a way that the remaining graph G' is a chordal graph with $q - 1$ maximal cliques. Since there are no pendant vertices of G , it is necessary to remove a minimum of two vertices from G to build G' . Since $\alpha(G') \leq \rho_c(G')$ by induction assumption, it follows that $\alpha(G) \leq \rho_c(G)$. $\square$

Notice that the inequalities in Theorem 4.3 are not true for trees. The following result is straightforward and does not require any proof:

Theorem 4.4 *Let G be an undirected bipartite graph. Then*

$$\begin{aligned}\rho_c(G) &\leq \rho_p(G) \leq \alpha(G); \\ ip_c(G) &\leq ip_p(G) \leq \alpha(G)\end{aligned}$$

In undirected graphs, the Gallai-Milgram theorem illustrates how the behavior of isometric and induced paths changes from the dense graphs (such as chordal graphs without pendant vertices) to sparse graphs (such as bipartite graphs).

Following Las Vergnas and Linial, Berge takes the Gallai-Milgram theorem to the next level which is the famous Berge path partition conjecture [11, 34, 35] and weak path partition conjecture [51]. Hartman [35] has provided a survey on the Berge path partition conjectures.

As we have discussed the behavior of the Gallai-Milgram theorem on bipartite graphs and chordal graphs, a similar discussion on the Berge path partition conjecture will be interesting. It will also be useful to study the behavior of the Berge path partition conjecture for different path-types such as paths, induced paths, and isometric paths on special classes of undirected graphs.

5 The three cover problems

In this section, we shall discuss three problems: the path cover problem, the induced path cover problem and the isometric path cover problem.

5.1 The path cover problem

The Hamiltonian path problem, and hence the path cover problem is NP-complete for planar graphs, bipartite graphs, chordal graphs, chordal bipartite graphs, and strongly chordal graphs [78]. Lin, Olariu and Pruesse [49] have presented an optimal algorithm for determining a minimum path cover for cographs. Unfortunately, there is no record of any significant research work on the path cover problem. But there is a variant of path cover problem which is called the *identifying path cover problem*. A path cover $\tilde{P}$ is an *identifying path cover* if for each pair u, v of vertices, there is a path P of $\tilde{P}$ such that exactly either u or v belongs to P . Foucaud and Kovše [29] have given lower and upper bounds on the minimum size of an identifying path cover for general graphs and discuss the tightness of the bounds. They have also illustrated that any connected graph G has an identifying path cover of size at most $\lceil 2|V|/3 \rceil + 5$.

5.2 The induced path cover problem

The induced path cover problem is NP-complete for general graphs [31]. There is no record of any significant research work on the induced path cover problem too.

5.3 The isometric path cover problem

There may not be vast difference between paths and induced paths. But there is distinct difference between induced paths and isometric paths. The following problems illustrate the difference:

- (1) The problem of finding an induced path between a vertex s and a vertex t containing a vertex v remains NP-complete in bipartite graphs [56]. However, the problem of finding an isometric path between a vertex s and a vertex t containing a vertex v is polynomially solvable for general graphs.
- (2) Given a positive integer k , the problem of finding an induced path of length at least k is NP-complete [31]. However, the problem of finding an isometric path of length at least k is a trivial problem.
- (3) The problem of deciding, given a graph G and two vertices s and t , whether there exists an induced path of given parity between s and t in G is NP-complete [43]. But the corresponding isometric problem is polynomially solvable.

Fitzpatrick [25] has introduced the concept of the isometric path cover problem in her PhD thesis [61]. The isometric path cover number is computed for trees, cycles, complete bipartite graphs, the Cartesian product of paths (including hypercubes) under some restricted cases [24, 25, 26, 28]. Fisher and Fitzpatrick [24, 27] have derived a lower bound that $ip_c(G) \geq \lceil |V|/(\text{diam}(G) + 1) \rceil$ and have shown that the isometric path cover number of an $r \times r$ grid is $\lceil 2r/3 \rceil$ and the isometric path cover number of a tree with ℓ leaves is $\lceil \ell/2 \rceil$. Fitzpatrick et al. [28] have shown that the isometric path number of hypercube Q_r is at least $2r/(r+1)$. In addition, they have also shown that $ip_c(Q_r) = 2^{r-\log_2(r+1)}$ when $r+1$ is a power of 2. Pan and Chang have given a linear-time algorithm to solve the isometric path cover problem on block graphs [61], complete r -partite graphs and Cartesian products of 2 or 3 complete graphs [62].

The strong geodetic problem is a variation of the geodetic problem which is to find a minimum set S of vertices of a graph G such that each vertex of G lies on some isometric path between a pair of vertices from S [33]. A *strong geodetic set* of a graph G is a set

S of vertices such that each vertex of G lies on a fixed isometric path between a pair of vertices from S [55]. For a graph G , its *strong geodetic number* $sg(G)$ is the cardinality of a minimum strong geodetic set of G . The relationship between the isometric path cover number and the strong geodetic number is given below:

Theorem 5.1 ([55]) *Let G be a graph.*

- (1) *When G is an undirected graph, $\frac{1+\sqrt{(8 \times ip_c(G)+1)}}{2} \leq sg(G) \leq 2 \times ip_c(G)$.*
- (2) *When G is a tree or block graph, $ip_c(G) = \lceil sg(G)/2 \rceil$.*

Even though the isometric path cover problem is a fundamental concept in graph theory, to our knowledge, there are no literatures on the computational complexity status of the isometric path cover problem for general graphs.

6 The three partition problems

In this section, we discuss the other set of three problems: the path partition problem, the induced path partition problem and the isometric path partition problem.

6.1 The path partition problem

The path partition problem was introduced by Ore in 1961 [59]. The Hamiltonian path problem is NP-complete and hence the path partition problem is NP-complete for planar graphs, bipartite graphs, chordal graphs, chordal bipartite graphs, and strongly chordal graphs [78]. Nakano, Olariu and Zomaya [58] call this problem as notoriously difficult and have reported an NC-algorithm to solve the path partition problem for cographs. Dinh [22] has studied the path partition problem for graphs with toughness greater than or equal to 1. Asdre, Nikolopoulos and Papadopoulos [6] have presented an optimal parallel algorithm for the path partition problem on P_4 -sparse graphs. Pan and Chang [60] have designed a linear-time algorithm for the path-partition problem in graphs whose blocks are complete graphs, cycles or complete bipartite graphs. The path-partition problem is polynomially solvable for forests [69], interval graphs [2, 18, 40], circular arc graphs [38], distance-hereditary graphs [39], bipartite permutation graphs [70], block graphs [75, 76], cographs [17], bipartite distance-hereditary graphs [78], co-comparability graphs [20, 44]. Given a connected graph G , there is a spanning tree T of G such that $\pi_p(G) = \pi_p(T)$ [12]. The relationship between the path partition

number and $L(2, 1)$ -labeling number of graphs have been studied by several authors [17, 52].

6.1.1 Graffiti.pc conjecture

The number of edges in a maximum matching of a graph G is called the matching number of G , denoted by $\alpha'(G)$. The total domination number of G , denoted by $\gamma_t(G)$, is a minimum set S of vertices such that every vertex of G is adjacent to some vertex in S . The Graffiti.pc conjectures are well-known for total domination number [21]. Some of the conjectures are related to path partition number. DeLaViña et al. [21] have positively answered to some Graffiti.pc conjectures which are related to path partition numbers.

Theorem 6.1 ([21]) *For every graph G with no isolated vertex, $\gamma_t(G) \leq \alpha'(G) + \pi_p(G)$. When G is a connected 3-regular graph, $\gamma_t(G) \geq 2\pi_p(G)$.*

Henning and Wash [36] have improved Graffiti.pc conjectures and shown that the bounds are tight.

Theorem 6.2 ([36]) *If G is a graph of order n , then $\alpha'(G) + (1/2)\pi_p(G) \geq (n/2)$. Moreover, when $\delta(G) \geq 3$, then $\gamma_t(G) \leq \alpha'(G) + (1/2)(\pi_p(G) - 1)$.*

6.1.2 Some bounds on the path partition number $\pi_p(G)$

While introducing the path partition problem, Ore [59] has produced the following bound:

Theorem 6.3 ([59]) *Given a graph G of order n , the path partition number π_p satisfies $\pi_p(G) \leq n - \sigma_2(G)$, where $\sigma_2(G)$ is the minimum sum of degrees of two nonadjacent vertices.*

According to Magnant and Martin [53], the above bound by Ore is sharp. There is another strong bound on the path partition number due to Hartman [34]. The Gallai-Milgram theorem with respect to the path partition number which is $\pi_p(G) \leq \alpha(G)$ is improved by Hartman [34] under certain conditions:

Theorem 6.4 ([34]) *Let G be a graph with connectivity k . If $\alpha(G) > k$, then the path partition number satisfies $\pi_p(G) \leq \alpha(G) - k$.*

Magnant and Martin [53] have posed the following conjecture:

Conjecture 6.5 ([53]) *Given a k -regular graph G of order n , the path partition number satisfies $\pi_p(G) \leq n/(k+1)$.*

The bound in the Conjecture 6.5 is sharp [53]. In support of their conjecture, they have provided the following result:

Theorem 6.6 ([53]) *Given a k -regular graph G of order n with $0 \leq k \leq 5$, the path partition number satisfies $\pi_p(G) \leq n/(k+1)$.*

Magnant, Wang and Yuan [54] have recently revised the Conjecture 6.5 as follows:

Conjecture 6.7 ([54]) *Given positive integers δ and Δ with $\delta \leq \Delta$, if G is a graph of order n with $\delta \leq \delta(G) \leq \Delta(G) \leq \Delta$, then $\pi_p(G) \leq \max\{n/(\delta+1), (\Delta-\delta)n/(\Delta+\delta)\}$.*

They have also proved this conjecture when $\delta = 1, 2$. They have also provided another bound as follows:

Theorem 6.8 ([54]) *Suppose G is a graph of order n with $\Delta(G) = \Delta < 2\delta = 2\delta(G)$, then $\pi_p(G) \leq ((\Delta-2)n)/(2(\Delta+\delta-4))$.*

While pointing out that Conjecture 6.5 is true even for $k \geq (n-1)/2$, Han [32] has provided a tight bound for bipartite graphs as follows:

Theorem 6.9 ([32]) *Given a bipartite k -regular graph G of order n , the path partition number satisfies $\pi_p(G) \leq n/2k$.*

For 3-regular graphs, Reed [66] has provided a bound which is sharper than the bound in Conjecture 6.5.

Theorem 6.10 ([66]) *Given a connected 3-regular graph G of order n , the path partition number satisfies $\pi_p(G) \leq n/9$.*

Reed [66] also demonstrates that these results are tight. Furthermore, Reed has conjectured in the same paper that

Conjecture 6.11 ([66]) *Given a 2-connected 3-regular graph G of order n , the path partition number satisfies $\pi_p(G) \leq \lceil n/10 \rceil$.*

This conjecture has been recently settled by Yu [79]. In addition, Yu has given an interesting lower bound for $\pi_p(G)$:

Theorem 6.12 ([79]) *There are infinitely many 2-connected 3-regular n -vertex graphs whose path cover numbers are at least $n/14$.*

6.2 The induced path partition problem

Chartrand et al. [19] have introduced the concept of induced path partition problem and have produced the induced-path number of complete bipartite graphs, complete binary trees, 2-dimensional meshes, butterflies, and general trees. They have shown that $\rho_p(P_m \times P_n) = 2$. In addition, they have also derived upper bounds for hypercubes and conjectured that $\rho_p(Q_r) \leq r$ for the r -dimensional hypercube Q_r with $r \geq 2$ [19]. Alsardary [1] has answered to the conjecture that $\rho_p(Q_r) \leq 16$ [63]. The induced path partition problem is concluded to be NP-complete by showing that it is NP-complete to determine whether the vertex set of a graph can be partitioned into two induced paths [46]. Pan and Chang [63] have presented a linear-time algorithm for the induced path partition problem on graphs whose blocks are complete graphs, cycles or complete bipartite graphs. Broere, Jonck and Domke [14] have determined $\rho_p(G)$ for $G = K_m \times K_n$ and have shown that $\rho_p(C_m \times C_n) < 3$.

Broere et al. [15] have studied the induced path partition problem on the complements of some graphs such as cycles, paths, product of complete graphs, and product of cycles. They have also produced an elegant Nordhaus-Gaddum type relation in terms of induced path partition number ρ_p as follows:

Theorem 6.13 ([15]) *Given an undirected graph on n vertices,*

- (1) $\sqrt{n} \leq \rho_p(G) + \rho_p(\overline{G}) \leq \lceil 3n/2 \rceil$ for arbitrary graph G
- (2) $1 + \lceil n/4 \rceil \leq \rho_p(G) + \rho_p(\overline{G}) \leq \lceil 3n/2 \rceil$ for bipartite graph G .
- (3) $|\ell| \leq \rho_p(T) + \rho_p(\overline{T}) \leq \lceil 3(n-1)/2 \rceil$ for trees T where ℓ is the # of leaves of T .

Several authors [8, 37, 64, 67, 73] have studied the relationship between the zero-forcing number $z(G)$ and the induced path partition number $\rho_p(G)$. Hogben [37] have shown that $\rho_p(G) \leq z(G)$ for general graphs. It is shown that $\rho_p(G) = z(G)$ when G is a block-cycle graph, unicyclic graph, double path, and series of double paths [73], a tree [8] and a cactus [67].

Catral et al. [16] have studied the induced path partition problem for edge subdivision graphs. Let $e = uv$ be an edge of G . An edge subdivision graph G_e is obtained from G by inserting a new vertex w into V , deleting the edge e and inserting edges uw and wv .

Let us recall that $M(G)$ denotes the maximum nullity of G . Johnson and Duarte [42] have shown that $M(G) = \rho_p(G)$ when G is a tree. Barioli, Fallat and Hogben [10]

have proved that $M(G) = \rho_p(G)$ or $M(G) = \rho_p(G) - 1$ for unicyclic graphs. Sinkovic [68] have derived that $M(G) \leq \rho_p(G)$ for any outerplanar graph G and $M(G) = \rho_p(G)$ for any partial 2-path G . Barioli, Fallat and Hogben [9] have studied the relationship between minimum rank and the induced path partition number and have derived that $\Delta(G) \leq \rho_p(G)$ where $\Delta(G) = \max\{p - q / \text{there are } q \text{ vertices of } G \text{ whose deletion leaves } p \text{ induced paths}\}$.

6.3 The isometric path partition problem

It is quite a big surprise that there is no literature on the isometric path partition problem including the computational complexity status of isometric path partition problem for general graphs.

7 Conclusion

This article is meant for young researchers in graph theory. It identifies the gaps and open space in “covering and partitioning by path-types” as potential research topics. Among all the six problems, the most attractive and interesting problem is path partition problem. The literatures on the path partition problem are larger than those of the other five problems. The reason may be due to the Gallai-Milgram theorem and Berge's path partition conjecture. There are a very few literatures in path cover problem, induced path problem and isometric partition problem. Between the covering and partitioning problems, it is observed that researchers are more active on path-type partition problems than path-type cover problems.

In addition to the six problems, we have also considered several problems which are related to these six problems. We have also discussed the correlation among these six problems and their related problems. We have covered the literatures for the last 30 years.

Chang and his teammates have carried out significant research on paths [17, 60, 76, 77, 78], induced paths [63] and isometric paths [61, 62]. Young researchers are encouraged to read these papers to understand how Chang and his teammates apply different techniques to solve different path-type problems.

We have discussed only vertex version of the covering and partitioning problems. The edge version of these problems is equally popular among the researchers. In the same way, covering and partitioning by cycles or trees are other well-known subtopics

of research in graph theory.

Acknowledgment

The author expresses his deep gratitude to Prof Sandi Klavžar for his valuable contribution and encouragement. The author also thanks Prof Michael Henning and Prof Gexin Yu for their comments.

References

- [1] Salar Alsardary, *The induced path number of the hypercube*, Congr. Numer. **128** (1997), 5–18.
- [2] Srinivasa Rao Arikati and Pandu Rangan, *Linear algorithm for optimal path problem on interval graphs*, Information Processing Letter **35** (1990), 149–153.
- [3] S. Arumugam, Ismail Sahul Hamid, and V. M. Abraham, *Decomposition of graphs into paths and cycles*, Journal of Discrete Mathematics, Article ID 721051 (2013).
- [4] Katerina Asdre and Stavros D. Nikolopoulos, *A linear-time algorithm for the k -fixed-endpoint path cover problem on cographs*, Networks: An International Journal **50** (2007), no. 4, 231–240.
- [5] ———, *A polynomial solution to the k -fixed-endpoint path cover problem on proper interval graphs*, Theoretical Computer Science **411** (2010), 967–975.
- [6] Katerina Asdre, Stavros D. Nikolopoulos, and Charis Papadopoulos, *An optimal parallel solution for the path cover problem on P_4 -sparse graphs*, Journal of Parallel and Distributed Computing **67** (2007), no. 1, 63–76.
- [7] Breeanne Alyse Baker, *The k -fixed-endpoint path partition problem*, PhD thesis, Department of Mathematics, Lehigh University, PA 18015 USA, 2013.
- [8] AIM Minimum Rank Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioaba, D. Cvetkovic, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanovic, H. van der Holst, K. Vander Meulen, and A. W. Wehe), *Zero forcing sets and the minimum rank of graphs*, Linear Algebra and its Applications **428** (2008), no. 7, 1628–1648.

- [9] Francesco Barioli, Shaun Fallat, and Leslie Hogben, *Computation of minimal rank and path cover number for certain graphs*, Linear Algebra and its Applications **392** (2004), 289–303.
- [10] ———, *On the difference between the maximum multiplicity and path cover number for tree-like graphs*, Linear Algebra and its Applications **409** (2005), 13–31.
- [11] Claude Berge, *Path partitions in directed graphs*, North-Holland Mathematics Studies **75** (1983), 59–63.
- [12] F. T. Boesch, S. Chen, and N. A. M. McHugh, *On covering the points of a graph with point disjoint paths*, Graphs and Combinatorics (A. Dold and B. Eckman, eds.), Lecture Notes in Mathematics, vol. 406, Springer, Berlin, 1974, pp. 201–212.
- [13] Christoph Brause and Rastislav Krivoš-Belluš, *On a relation between k -path partition and k -path vertex cover*, Discrete Applied Mathematics **223** (2017), 28–38.
- [14] Izak Broere, Elizabeth Jonck, and Gayla Domke, *The induced path number of the cartesian product of some graphs*, The Australasian Journal of Combinatorics **36** (2006), 3–12.
- [15] Izak Broere, Elizabeth Jonck, Gayla Domke, and Lisa Markus, *The induced path number of the complements of some graphs*, The Australasian Journal of Combinatorics **33** (2005), 15–32.
- [16] Minerva Catral, Anna Cepek, Leslie Hogben, My Huynh, Kirill Lazebnik, Travis Peters, and Michael Young, *Zero forcing number, maximum nullity, and path cover number of subdivided graphs*, Electronic Journal of Linear Algebra **23** (2012).
- [17] Gerard J. Chang and David Kuo, *The $L(2, 1)$ -labeling problem on graphs*, SIAM J. Discrete Math **9** (1996), no. 2, 309–316.
- [18] Maw-Shang Chang, *A certifying algorithm for the path cover problem on interval graphs*, The 25th Workshop on Combinatorial Mathematics and Computation Theory, 2008.
- [19] Gary Chartrand, Joseph McCanna, Naveed Sherwani, Moazzem Hossain, and Jangir Hashmi, *The induced path number of bipartite graphs*, Ars Combin. **37** (1994), 191–208.

- [20] Derek G. Corneil, Barnaby Dalton, and Michel Habib, *LDFS-based certifying algorithm for the minimum path cover problem on cocomparability graphs*, SIAM Journal on Computing **42** (2013), no. 3, 792–807.
- [21] Ermelinda DeLaViña, Qi Liu, Ryan Pepper, Bill Waller, and Douglas West, *Some conjectures of graffiti.pc on total domination*, Congressus Numer. **185** (2007), 81–95.
- [22] Hoa Vu Dinh, *Path partition number in tough graphs*, Discrete Mathematics **164** (1997), no. 1-3, 291–294.
- [23] Paul Erdős, Ralph Faudree, and Edward T. Ordman, *Clique partitions and clique coverings*, Discrete Mathematics **72** (1988), 93–101.
- [24] David C. Fisher and Shannon L. Fitzpatrick, *The isometric number of a graph*, Journal of Combinatorial Mathematics and Combinatorial Computing **38** (2001), 97–110.
- [25] Shannon L. Fitzpatrick, PhD thesis, Department of Mathematics, Dalhousie University, Nova Scotia, Canada, 1997.
- [26] ———, *The isometric path number of the cartesian product of paths*, Congr. Numer. **137** (1999), 109–119.
- [27] Shannon L. Fitzpatrick, Jeannette Janssen, and Richard J. Nowakowski, *Fractional isometric path number*, Australian Journal of Combinatorics **34** (2006), 281–298.
- [28] Shannon L. Fitzpatrick, Richard J. Nowakowski, Derek A. Holton, and Ian Caines, *Covering hypercubes by isometric paths*, Discrete Mathematics **240** (2001), 253–260.
- [29] Florent Foucaud and Matjaž Kovše, *Identifying path covers in graphs*, Journal of Discrete Algorithms **23** (2013), 21–34.
- [30] Stefan Funke, André Nusser, and Sabine Storandt, *On k -path covers and their applications*, Proceedings of the VLDB Endowment **7** (2014), no. 10, 893–902.
- [31] Michael Garey and David S. Johnson, *Computers and Intractability: A guide to the theory of NP-completeness*, Freeman, New York, 1979.

- [32] Jie Han, *On vertex-disjoint paths in regular graphs*, The Electronic Journal of Combinatorics **25** (2018), no. 2.
- [33] Frank Harary, Emmanuel Loukakis, and Constantine Tsouros, *The geodetic number of a graph*, Mathematical and Computer Modelling **17** (1993), no. 11, 89–95.
- [34] Ben-Arroyo Hartman, *Variations on the Gallai-Milgram theorem*, Discrete Mathematics **71** (1988), 95–105.
- [35] ———, *Berge’s conjecture on directed path partitions - a survey*, Discrete Mathematics **306** (2006), 2498–2514.
- [36] Michael A. Henning and Kirsti Wash, *Matchings, path covers and domination*, Discrete Mathematics **340** (2017), no. 1, 3207–3216.
- [37] Leslie Hogben, *Minimum rank problems*, Linear Algebra and its Applications **432** (2010), no. 8, 1961–1974.
- [38] Ruo-Wei Hung and Maw-Shang Chang, *Solving the path cover problem on circular-arc graphs by using an approximation algorithm*, Discrete Applied Mathematics **154** (2006), no. 1, 76–105.
- [39] ———, *Finding a minimum path cover of a distance-hereditary graph in polynomial time*, Discrete Applied Mathematics **155** (2007), no. 17, 2242–2256.
- [40] ———, *Linear-time certifying algorithms for the path cover and Hamiltonian cycle problems on interval graphs*, Applied Mathematics Letters **24** (2011), no. 5, 648–652.
- [41] Zemin Jin and Xueliang Li, *On the k -path cover problem for cacti*, Theoretical Computer Science **355** (2006), no. 3, 354–363.
- [42] Charles R. Johnson and António Leal Duarte, *The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree*, Linear and Multilinear Algebra **46** (1999), no. 1-2, 139–144.
- [43] Marcin Kaminski and Naomi Nishimura, *Finding an induced path of given parity in planar graphs in polynomial time*, SODA - 12 Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete algorithms, Kyoto, Japan, January 17 - 19, vol. 1450, 2012, pp. 656–670.

- [44] Ekkehard Köhler and Lalla Mouatadid, *Linear time LexDFS on cocomparability graphs*, Scandinavian Workshop on Algorithm Theory, SWAT 2014 - Algorithm Theory, 2014, pp. 319–330.
- [45] Nicholas Korpelainen, *A boundary class for the k -path partition problem*, Electronic Notes in Discrete Mathematics **67** (2018), 49–56.
- [46] Hoáng-Oanh Le, Van Bang Le, and Haiko Müller, *Splitting a graph into disjoint induced paths or cycles*, Discrete Applied Mathematics **131** (2003), 199–212.
- [47] Peng Li and Yaokun Wu, *A linear time algorithm for the 1-fixed-endpoint path cover problem on interval graphs*, SIAM Journal on Discrete Mathematics **31** (2017), no. 1, 210–239.
- [48] Zhao Li and Liancui Zuo, *The k -path vertex cover in Cartesian product graphs and complete bipartite graphs*, Applied Mathematics and Computation **331** (2018), 69–79.
- [49] Raullinoux Lin, Stephan Olariu, and Gara Pruesse, *An optimal path cover algorithm for cographs*, Computers and Mathematics with Applications **30** (1995), no. 8, 75–83.
- [50] Nathan Linial, *Covering digraphs by paths*, Discrete Mathematics **23** (1978), no. 3, 257–272.
- [51] ———, *Extending the greene-kleitman theorem to directed graphs*, J. Combin. Theory Ser. A **30** (1981), 331–334.
- [52] Changhong Lu and Qing Zhou, *Path covering number and $L(2, 1)$ -labeling number of graphs*, Discrete Applied Mathematics **161** (2013), no. 13-14, 2062–2074.
- [53] Colton Magnant and Daniel M. Martin, *A note on the path cover number of regular graphs*, Australasian journal of combinatorics **43** (2009), 211–217.
- [54] Colton Magnant, Hua Wang, and Shuai Yuan, *Path partitions of almost regular graphs*, Australasian Journal of Combinatorics **64** (2016), no. 2, 334–340.
- [55] Paul Manuel, Sandi Klavžar, Antony Xavier, Andrew Arokiaraj, and Elizabeth Thomas, *Strong geodetic problem in networks*, Discussiones Mathematicae Graph Theory (2018), 1–15.

- [56] Mauro Mezzini, *On the complexity of finding chordless paths in bipartite graphs and some interval operators in graphs and hypergraphs*, Theoretical Computer Science **411** (2010), 1212–1220.
- [57] Jérôme Monnot and Sophie Toulouse, *The path partition problem and related problems in bipartite graphs*, Journal Operations Research Letters **35** (2007), no. 5, 677–684.
- [58] Koji Nakano, Stephan Olariu, and Albert Y. Zomaya, *A time-optimal solution for the path cover problem on cographs*, Theoretical Computer Science **290** (2003), 1541–1556.
- [59] Oystein Ore, *Arc coverings of graphs*, Ann. Mat. Pura Appl **55** (1961), no. 4, 315–321.
- [60] Jun-Jie Pan and Gerard Chang, *Path partition for graphs with special blocks*, Discrete Applied Mathematics **145** (2005), 429–436.
- [61] Jun-Jie Pan and Gerard J. Chang, *Isometric path numbers of block graphs*, Information Processing Letters **93** (2005), 99–102.
- [62] ———, *Isometric path numbers of graphs*, Discrete Mathematics **306** (2006), no. 17, 2091–2096.
- [63] ———, *Induced-path partition on graphs with special blocks*, Theoretical Computer Science **370** (2007), no. 1-3, 121–130.
- [64] Cecília Perdigão and Amélia Fonseca, *Minimum rank and path cover number for generalized and double generalized cycle star graphs*, Linear Algebra and its Applications **554** (2018), 146–169.
- [65] Nirmala Rani, Indra Rajasingh, and Jennifer Rajkumari, *Induced path decomposition and hole in sierpinsky graph*, International Journal of Pure and Applied Mathematics **86** (2013), no. 6, 965–973.
- [66] Bruce Reed, *Paths, stars and the number three*, Combinatorics, Probability and Computing **5** (1996), no. 3, 277–295.
- [67] Darren Row, *Zero forcing number, path cover number, and maximum nullity of cacti*, Involve **4** (2011), no. 3, 277–291.

- [68] John Sinkovic, *Maximum nullity of outerplanar graphs and the path cover number*, Linear Algebra and its Applications **432** (2010), 2052–2060.
- [69] Zdzislaw Skupień, *Path partitions for vertices and Hamiltonicity of graphs*, Recent Advances in Graph Theory, Proceedings of the Symposium held in Prague, June 1974, Academia, Praha, 1975, pp. 481–491.
- [70] R. Srikant, Ravi Sundaram, Karan Sher Singh, and Pandu Rangan, *Optimal path cover problem on block graphs and bipartite permutation graphs*, Theoretical Computer Science **115** (1993), no. 2, 351–357.
- [71] George Steiner, *On the k -path partition problem in cographs*, Cong. Numer. **147** (2000), 89–96.
- [72] ———, *On the k -path partition of graphs*, Theoretical Computer Science **290** (2003), 2147–2155.
- [73] Fatemeh Alinaghipour Taklimi, Shaun Fallat, and Karen Meagher, *On the relationships between zero forcing numbers and certain graph coverings*, Special Matrices **2** (2014), 30–45.
- [74] Jianhua Tu and Fengmei Yang, *The vertex cover P_3 -problem in cubic graphs*, Information Processing Letters **113** (2013), 481–485.
- [75] Pak-Ken Wong, *Optimal path cover problem on block graphs*, Theoretical Computer Science **225** (1999), 163–169.
- [76] Jing-Ho Yan and Gerard J. Chang, *The path-partition problem in block graphs*, Information Processing Letter **52** (1994), 317–322.
- [77] Jing-Ho Yan, Gerard J. Chang, Sandra M. Hedetniemi, and Stephen T. Hedetniemi, *k -path partitions in trees*, Discrete Applied Mathematics **78** (1997), 227–233.
- [78] Hong-Gwa Yeh and Gerard J. Chang, *The path-partition problem in bipartite distance-hereditary graphs*, Taiwanese journal of mathematics **2** (1998), no. 3, 353–360.
- [79] Gexin Yu, *Covering 2-connected 3-regular graphs with disjoint paths*, Journal of Graph Theory **88** (2018), no. 3, 385–401.