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Appendix

Alzheimer’s Disease Modelling and Staging through Independent Gaussian Process Analysis of Spatio-Temporal Brain Changes

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A. Lower bound derivation

In this section we detail the derivation of the lower bound:

\[
\log(p(Y,C|\sigma,\lambda)) = \log\left[\int_A \int_S \int_{S'} p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S,S'|\lambda)dAdSdS'\right]
= \log\left[\int_A \int_S p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S'|S,\lambda)p(S)dAdSdS'\right]
\]

(1)

If we know S this completely determines S', thus we have

\[\int p(S'|S,\lambda)dS' = 1\]

which gives us:

\[
\log(p(Y,C|\sigma,\lambda)) = \log\left[\int_A \int_S p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S)dAdS\right]
= \log\left[\int_A \int_S p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S)\frac{q_1(A)q_2(S)}{q_1(A)q_2(S)}dAdS\right]
= \log[\mathbb{E}_{A \sim q_1,S \sim q_2}[p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S)]]
\geq \mathbb{E}_{A \sim q_1,S \sim q_2}[\log[p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S)]]
\]

(2)

This is obtained thanks to Jensen’s inequality. Finally this leads us to:

\[
\mathbb{E}_{A \sim q_1,S \sim q_2}[\log[p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S)]] = \mathbb{E}_{A \sim q_1,S \sim q_2}[\log[p(Y|A,S,\sigma)]]
+ \mathbb{E}_{S \sim q_2}[\log(P(C|S',\lambda))]
- D[q_1(A|Y)||p(A)]
- D[q_2(S|Y)||p(S)]
\]

(3)
Independent Gaussian Process Analysis

In the Method section we introduced the approximation \( q_1(A) = \prod_{n=1}^{Ns} \mathcal{N}(\mu_n, \Sigma(\alpha, \beta)) \).

The covariance matrix is shared by all the spatial processes which gives us the set of spatial parameters :

\[
\psi = \{ \mu_n, n \in [1, Ns], \alpha, \beta \}
\]

Following [5] we introduce for each GP two vectors, \( \Omega_n \) with a prior \( p(\Omega_n) = \mathcal{N}(0, \frac{1}{\lambda_n} I) \) for each element and \( W_n \) with a prior \( p(W_n) = \mathcal{N}(0, I) \), such that \( S_n(t) = \Phi(t(\Omega_n)W_n) \). Where \( \Phi \) is chosen to obtain a RBF kernel as explained in [5]. We define the approximated distributions \( q_3(W_n) = \prod_j \mathcal{N}(m_{n,j}, \sigma_{n,j}^2) \) and \( q_4(\Omega_n) = \prod_j \mathcal{N}(\alpha_{n,j}, \beta_{n,j}^2) \) of \( p(W_n) \) and \( p(\Omega_n) \). Using these approximations and following [5], we can derive a lower bound for \( S \) with the same technique than above. We have the set of temporal parameters :

\[
\theta = \{ m_n, s_n, \alpha_n, \beta_n, l_n, n \in [1, Ns] \}
\]

Now we can obtain every term of [3]. The Kullback-Leibler of a multivariate Gaussian has a closed-from :

\[
\mathcal{D}[q_1(A|X)||p(A)] = \frac{1}{2} \sum_{n=1}^{Ns} Tr(\Sigma) + \mu_n^T \mu_n - F - \log|\det(\Sigma)|
\]

Using the factorized form of \( q_2 \) and the fact that the different Gaussian processes are independent from each other we can write :

\[
\mathcal{D}[q_2(S|X)||p(S)] = \sum_{n=1}^{Ns} \mathcal{D}[q_3(W_n)||p(W_n)] + \mathcal{D}[q_4(\Omega_n)||p(\Omega_n)]
\]

Since the approximations \( q_3 \) and \( q_4 \) and their respective priors are normally distributed we have an analytic formula for both Kullback-Leibler divergences.

\[
\mathcal{D}[q_3(W_n)||p(W_n)] = \frac{1}{2} \sum_j \beta_{n,j}^2 l_n + \alpha_{n,j}^2 l_n - 1 - \log(\beta_{n,j}^2 l_n)
\]

As in [II] we employ the reparameterization trick to have an efficient way of sampling the expectations of [3]. Thus we have :

\[
- W_{n,j} = m_{n,j} + s_{n,j} * \epsilon_{n,j}
- \Omega_{n,j} = \alpha_{n,j} + \beta_{n,j} \zeta_{n,j}
- A_n = \mu_n + \Sigma_n^{\frac{1}{2}} * \kappa_n
\]

Which gives us :

\[
\mathbb{E}_{A \sim q_1, S \sim q_2}[\log(p(Y|A, S, \sigma))] = \mathbb{E}_{\epsilon, \zeta, \kappa}[\log(p(Y|m, s, \alpha, \beta, \mu, \Sigma, \sigma))]
\]

\[
\mathbb{E}_{S \sim q_2}[\log(p(C|S', \lambda))] = \mathbb{E}_{\epsilon, \zeta}[\log(p(C|m, s, \alpha, \beta, \lambda))]
\]

Where \( \epsilon_{n,j} \sim \mathcal{N}(0, 1) \), \( \zeta_{n,j} \sim \mathcal{N}(0, 1) \) and \( \kappa_n \sim \mathcal{N}(0, I) \).
B. Kronecker factorization

Here we detail how to split the covariance matrix in a Kronecker product of three matrices along each spatial dimensions. We have:

\[ \Sigma_{i,j}(\alpha, \beta) = \alpha \exp\left(-\frac{\|u_i - u_j\|^2}{2\beta}\right) \] (12)

We can use the separability properties of the exponential to decompose the covariance between two locations \( u_i = (x_i, y_i, z_i) \) and \( u_j = (x_j, y_j, z_j) \):

\[ \Sigma_{i,j}(\alpha, \beta) = \alpha \exp\left(-\frac{(x_i - x_j)^2}{2\beta}\right) \exp\left(-\frac{(y_i - y_j)^2}{2\beta}\right) \exp\left(-\frac{(z_i - z_j)^2}{2\beta}\right) \] (13)

So \( \Sigma \) can be decomposed into the Kronecker product of 1D processes:

\[ \Sigma = \Sigma_x \otimes \Sigma_y \otimes \Sigma_z \] (14)

Allowing us to deal with large-size matrices.
C. Comparison with ICA

We performed a comparison of our algorithm with ICA on a similar example than in 3.1. However the data was generated in a simplified setting since ICA can’t be applied when the time associated to each image is unknown. To do so we assigned the ground truth parameter $t_p$ beforehand. The goal was to compare the separation performances of both our algorithm and ICA, on data generated with three latent spatio-temporal processes. In Figure 1 we observe that the sources estimated by ICA are more noisy and uncertain than the ones estimated by our method, highlighting the performances of our algorithm in terms of sources separation.

![Fig. 1: First Row: Raw sources. Second Row: Sources estimated by our method. Third Row: Sources estimated by ICA.](image-url)
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