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Appendix

Alzheimer’s Disease Modelling and Staging through Independent Gaussian Process Analysis of Spatio-Temporal Brain Changes

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A. Lower bound derivation

In this section we detail the derivation of the lower bound:

\[
\log(p(Y,C|\sigma,\lambda)) = \log\left[\int_A \int_S \int_{S'} p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S|S',\lambda) \, dAdSdS'\right] = \log\left[\int_A \int_S \int_{S'} p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S'|S,\lambda)p(S) \, dAdSdS'\right]
\]

(1)

If we know \(S\) this completely determines \(S'\), thus we have \(\int p(S'|S,\lambda) \, dS' = 1\) which gives us:

\[
\log(p(Y,C|\sigma,\lambda)) = \log\left[\int_A \int_S p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S) \, dAdS\right] = \log\left[\int_A \int_S p(Y|A,S,\sigma)p(C|S',\lambda)p(A)q_1(A)q_2(S) \, dAdS\right]
\]

(2)

This is obtained thanks to Jensen’s inequality. Finally this leads us to:

\[
E_{A \sim q_1, S \sim q_2}\left[\log\left(\frac{p(Y|A,S,\sigma)p(C|S',\lambda)p(A)p(S)}{q_1(A)q_2(S)}\right)\right] = E_{A \sim q_1, S \sim q_2}\left[\log[p(Y|A,S,\sigma)]\right] + E_{S \sim q_2}\left[\log(p(C|S',\lambda))\right]
- D[q_1(A|Y)||p(A)]
- D[q_2(S|Y)||p(S)]
\]

(3)
Independent Gaussian Process Analysis

In the Method section we introduced the approximation \( q_1(A) = \prod_{n=1}^{Ns} \mathcal{N}(\mu_n, \Sigma(\alpha, \beta)) \).

The covariance matrix is shared by all the spatial processes which gives us the set of spatial parameters:

\[
\psi = \{\mu_n, n \in [1, Ns], \alpha, \beta\}
\] (4)

Following [5] we introduce for each GP two vectors, \( \Omega_n \) with a prior \( p(\Omega_n) = \mathcal{N}(0, \frac{1}{\ell_n} I) \) for each element and \( W_n \) with a prior \( p(W_n) = \mathcal{N}(0, I) \), such that \( S_n(t) = \Phi(t \Omega_n)W_n \). Where \( \Phi \) is chosen to obtain a RBF kernel as explained in [5]. We define the approximated distributions \( q_3(W_n) = \prod_j \mathcal{N}(m_{n,j}, s^2_{n,j}) \) and \( q_4(\Omega_n) = \prod_j \mathcal{N}(\alpha_{n,j}, \beta^2_{n,j}) \) of \( p(W_n) \) and \( p(\Omega_n) \). Using these approximations and following [5], we can derive a lower bound for \( S \) with the same technique than above. We have the set of temporal parameters:

\[
\theta = \{m_n, s_n, \alpha_n, \beta_n, I_n, n \in [1, Ns]\}
\] (5)

Now we can obtain every term of [3]. The Kullback-Leibler of a multivariate Gaussian has a closed-from:

\[
\mathcal{D}[q_1(A|X)||p(A)] = \frac{1}{2} \sum_{n=1}^{Ns} Tr(\Sigma) + \mu_n^T \Sigma \mu_n - F - \log|\det(\Sigma)|
\] (6)

Using the factorized form of \( q_2 \) and the fact that the different Gaussian processes are independent from each other we can write:

\[
\mathcal{D}[q_2(S|X)||p(S)] = \sum_{n=1}^{Ns} \mathcal{D}[q_3(W_n)||p(W_n)] + \mathcal{D}[q_4(\Omega_n)||p(\Omega_n)]
\] (7)

Since the approximations \( q_3 \) and \( q_4 \) and their respective priors are normally distributed we have an analytic formula for both Kullback-Leibler divergences.

\[
\mathcal{D}[q_3(W_n)||p(W_n)] = \frac{1}{2} \sum_j s^2_{n,j} + m^2_{n,j} - 1 - \log(s^2_{n,j})
\] (8)

\[
\mathcal{D}[q_4(\Omega_n)||p(\Omega_n)] = \frac{1}{2} \sum_j \beta^2_{n,j}I_n + \alpha^2_{n,j}I_n - 1 - \log(\beta^2_{n,j}I_n)
\] (9)

As in [10] we employ the reparameterization trick to have an efficient way of sampling the expectations of [3]. Thus we have:

\[
\begin{align*}
W_{n,j} &= m_{n,j} + s_{n,j} \epsilon_{n,j} \\
\Omega_{n,j} &= \alpha_{n,j} + \beta_{n,j} \zeta_{n,j} \\
A_n &= \mu_n + \Sigma^\frac{1}{2} \kappa_n
\end{align*}
\]

Which gives us:

\[
\begin{align*}
\mathbb{E}_{A \sim q_1, S \sim q_2} [\log(p(Y|A, S, \sigma))] &= \mathbb{E}_{\epsilon, \zeta, \kappa} [\log(p(Y|m, s, \alpha, \beta, \mu, \Sigma, \sigma))] \quad (10) \\
\mathbb{E}_{S \sim q_2} [\log(p(C|S', \lambda))] &= \mathbb{E}_{\epsilon, \zeta} [\log(p(C|m, s, \alpha, \beta, \lambda))]
\end{align*}
\]

Where \( \epsilon_{n,j} \sim \mathcal{N}(0, 1), \zeta_{n,j} \sim \mathcal{N}(0, 1) \) and \( \kappa_n \sim \mathcal{N}(0, I) \).
B. Kronecker factorization

Here we detail how to split the covariance matrix in a Kronecker product of three matrices along each spatial dimensions. We have:

$$\Sigma_{i,j}(\alpha,\beta) = \alpha \exp\left(-\frac{||u_i - u_j||^2}{2\beta}\right)$$ (12)

We can use the separability properties of the exponential to decompose the covariance between two locations $u_i = (x_i, y_i, z_i)$ and $u_j = (x_j, y_j, z_j)$:

$$\Sigma_{i,j}(\alpha,\beta) = \alpha \exp\left(-\frac{(x_i - x_j)^2}{2\beta}\right) \exp\left(-\frac{(y_i - y_j)^2}{2\beta}\right) \exp\left(-\frac{(z_i - z_j)^2}{2\beta}\right)$$ (13)

So $\Sigma$ can be decomposed into the Kronecker product of 1D processes:

$$\Sigma = \Sigma_x \otimes \Sigma_y \otimes \Sigma_z$$ (14)

Allowing us to deal with large-size matrices.
C. Comparison with ICA

We performed a comparison of our algorithm with ICA on a similar example than in 3.1. However the data was generated in a simplified setting since ICA can’t be applied when the time associated to each image is unknown. To do so we assigned the ground truth parameter $t_p$ beforehand. The goal was to compare the separation performances of both our algorithm and ICA, on data generated with three latent spatio-temporal processes. In Figure 1 we observe that the sources estimated by ICA are more noisy and uncertain than the ones estimated by our method, highlighting the performances of our algorithm in terms of sources separation.

![Figure 1: First Row : Raw sources. Second Row : Sources estimated by our method. Third Row : Sources estimated by ICA.](image-url)
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